AN ASYMPTOTIC FORMULA IN ADELE DIOPHANTINE APPROXIMATIONS

BY

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ABSTRACT. In this paper an asymptotic formula is found for the number of solutions of a system of linear Diophantine inequalities defined over the ring of adeles of an algebraic number field. The theorem proved is a generalization of results of S. Lang and W. Adams.

1. Introduction. Serge Lang [5] defines a number to have type \( \leq g \) if \( g \) is a positive increasing function for which \( |q b - p| \geq 1/\psi(q) \) for all \( q \) sufficiently large. Lang then shows that the number \( \Lambda(N, b) \) of solutions of \( |q b - p| \leq \psi(q) \) with \( q \leq N \) is asymptotic to \( S_N = \sum_{q=1}^{N} 2\psi(q) \) if \( b \) has type \( \leq g \) and \( \psi \) decreases so slowly that \( \psi(q)qg(q)^{-1} \) increases to infinity with \( q \). W. Adams [1] has extended this result of Lang to the simultaneous approximation of real numbers by rationals. I have also shown in [8] how these results may be extended to linear forms. The purpose of this paper is to show that the Lang-Adams theorem holds for the approximation of linear forms in the ring of adeles over a number field \( k \). A \( p \)-adic theorem, as well as some of the results in [8], could be stated as corollaries to the theorem proved here. The theorem proved is probably not the best possible such theorem. This is suggested by a metric example I will give later.

Diophantine approximations over the adeles have previously been considered by David Cantor in [2]. In his paper Cantor shows adele analogues of some of the basic theorems. To some extent, I have followed Cantor in notation and setting up the problem in the ring of adeles.

I wish to thank Professor W. Adams for his help and encouragement in my work.

2. Notation. We use \( k \) to denote an algebraic number field of degree \( n \) with ring of integers \( \mathcal{O} \). Let \( P \) be the set of all primes of \( k \). We write \( P_\infty \) for the set of all infinite primes, and \( P_0 \) for the set of all finite primes. When \( P_0 \) and \( P_\infty \) are used as subscripts, we will replace them by 0 and \( \infty \) respectively. For \( \mathfrak{p} \in P \), we let \( k_\mathfrak{p} \supset k \) denote the completion of \( k \) with respect to \( \mathfrak{p} \).

We may assume \( P_0 \) is the set of all prime ideals of \( \mathcal{O} \). For \( \mathfrak{p} \in P_0, x \in k \), let \( \nu = \nu_\mathfrak{p}(x) \) be the \( \mathfrak{p} \)-order of \( x \). We normalize the absolute value \( |\cdot|_\mathfrak{p} \) associated with \( \mathfrak{p} \) so that \( |x|_\mathfrak{p} = N\mathfrak{p}^{-\nu} \), where \( N\mathfrak{p} \) is the norm of the ideal \( \mathfrak{p} \).

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Let \( x \rightarrow x^{(i)}, i = 1, \ldots, n \), be the embeddings of \( k \) into \( \mathbb{C} \), the complex numbers. We arrange the notation so that the first \( R_1 \) embeddings map into the real numbers \( \mathbb{R} \) and the remaining maps consist of \( R_2 \) pairs of complex conjugate mappings listed so that

\[
x^{(R_1 + R_2 + i)} = x^{(R_1 + i)} \quad \text{for } i = 1, \ldots, R_2.
\]

The infinite primes of \( k \) can be identified with the first \( R = R_1 + R_2 \) of these mappings. We use \( | \cdot | \) to stand for the ordinary absolute value on \( \mathbb{C} \). If \( \mathfrak{p} \) is the infinite prime corresponding to \( x \rightarrow x^{(i)} \), then we set \( |x|^\mathfrak{p} = |x^{(i)}| \) if \( k^{(i)} \) is real, otherwise we set \( |x|^\mathfrak{p} = |x^{(i)}|^2 \). The infinite prime \( \mathfrak{p} \) is called real when \( k^{(i)} \subseteq \mathbb{R} \) and complex otherwise. If \( \mathfrak{p} \) is real, then \( k_\mathfrak{p} = \mathbb{R} \) and we will often identify \( k \) with a subfield of \( \mathbb{R} \) by means of \( x \rightarrow x^{(i)} \). A similar statement can be made when \( \mathfrak{p} \) is complex, in which case \( k_\mathfrak{p} = \mathbb{C} \). Hence, if we write \( | \cdot | _\mathfrak{p} \) for the extension of \( | \cdot | \) to \( k_\mathfrak{p} \), we may think of \( | \cdot | _\mathfrak{p} \) as the ordinary absolute value when \( k_\mathfrak{p} = \mathbb{R} \) and the square of the ordinary absolute value when \( k_\mathfrak{p} = \mathbb{C} \).

For \( \mathfrak{p} \in \mathcal{P}_0 \), the set \( \mathcal{O}_\mathfrak{p} \) of all \( x \) in \( k_\mathfrak{p} \) for which \( |x|^\mathfrak{p} < 1 \) is the ring of \( \mathfrak{p} \)-adic integers of \( k_\mathfrak{p} \). For \( \mathfrak{p} \in \mathcal{P}_\infty \), we set \( \mathcal{O}_\mathfrak{p} = k_\mathfrak{p} \).

Let \( S \) be any subset of \( \mathcal{P} \). Consider the product \( \prod k_\mathfrak{p} \) over all \( \mathfrak{p} \in S \), with componentwise algebraic operations. For any \( a \) in this product we use \( a_\mathfrak{p} \) to stand for the \( \mathfrak{p} \)-th component of \( a \). We define the ring \( k_S \) of \( S \)-adeles to be the subset of this product consisting of all \( a \) with \( a_\mathfrak{p} \in \mathcal{O}_\mathfrak{p} \) for all but a finite number of \( \mathfrak{p} \). Note that this is not the ring usually referred to as the \( S \)-adele ring. We embed \( k \) in \( k_S \) by identifying \( a \in k \) with the element in \( k_S \), also denoted by \( a \), for which \( a_\mathfrak{p} = a \in \mathfrak{p} \) for all \( \mathfrak{p} \in S \). We let \( S_\infty = S \cap \mathcal{P}_\infty \) and \( S_0 = S \cap \mathcal{P}_0 \). Then we can write \( k_S = k_\infty \times \times k_{S_0} \). For \( a \in k_S \) we write \( a_\infty \) for the \( k_\infty \) component of \( a \), and we write \( a_0 \) for the \( k_{S_0} \) component of \( a \).

We denote the multiplicative group of units of \( k_S \) by \( k_S^* \), and call this the group of \( S \)-ideles. Clearly, \( a \in k_S \) is an idele if and only if \( a_\mathfrak{p} \) is nonzero for all \( \mathfrak{p} \in S \) and \( |a_\mathfrak{p}| = 1 \) for all but a finite number of \( \mathfrak{p} \in S \).

We extend \( | \cdot |_\mathfrak{p} \) to \( k_S \) by defining \( |a|^\mathfrak{p} = |a_\mathfrak{p}|_\mathfrak{p} \) for \( a \) in \( k_S \). For \( T \subseteq S \) and \( a \in k_T \), put \( |a|_T = \prod_{\mathfrak{p} \in T} |a_\mathfrak{p}|_\mathfrak{p} \), if this product converges; and otherwise set \( |a|_T = 0 \). So, if \( a \in k_S^* \), then \( |a|_T \neq 0 \). For \( a, b \in k_S \), write \( a \leq b \) if \( |a_\mathfrak{p}| \leq |b_\mathfrak{p}| \) for all \( \mathfrak{p} \in S_\infty \) and write \( a < b \) if \( a \leq b \) and \( |a_\mathfrak{p}| < |b_\mathfrak{p}| \) for all infinite primes in \( S \).

If \( S \supseteq \mathcal{P}_\infty \) and \( x = (x_1, \ldots, x_m) \in k_S^m \), we write \( |x|_\mathfrak{p} = 1/|x_1|_\mathfrak{p} \) where \( n_\mathfrak{p} \) is the local degree of \( \mathfrak{p} \) and the max is taken over all \( \mathfrak{p} \in \mathcal{P}_\infty \) and all \( i \) satisfying \( 1 \leq i \leq m \).

We topologize \( k_S \) in the usual way by requiring that the sets \( \{x \in k_S: x - b \leq a, a \in k_S^* \} \) form a neighborhood basis at \( b \) in \( k_S \). This makes \( k_S \) into a locally compact additive topological group.
It is well known that $\mathfrak{o}$ is a discrete subset of $k_\infty$ and $k_\infty/\mathfrak{o}$ is compact. If $S \subseteq P_0$, by the strong approximation theorem, $k$ is dense in $k_S$.

We now define some measures, all of which will be denoted by $\mu$ when there is no ambiguity. Let $\mu_\mathfrak{p}$ be the Haar measure on $k_\mathfrak{p}$ normalized so that $\mu_\mathfrak{p}(\mathfrak{o}_\mathfrak{p}) = 1$ when $\mathfrak{p} \in P_0$, and so that $\mu_\mathfrak{p}$ is ordinary Lebesgue measure when $\mathfrak{p} \in P_\infty$. The Haar measure $\mu_S$ on $k_S$ is normalized by requiring that this measure agree with the product measure on

$$k_S(T) = \prod_{\mathfrak{p} \in T} k_\mathfrak{p} \times \prod_{\mathfrak{p} \in S - T} \mathfrak{o}_\mathfrak{p}$$

where $T$ is any finite subset of $S$. So

$$\mu_S(x \in k_S: x \leq a) = 2^{-mR^2} a^{m/2}$$

Whenever we talk about a measure on $k_S^m$ we mean the product measure $\mu_S^m$. If $G$ is a discrete subgroup of $k_S^m$ we will always take the counting measure. Furthermore, if $k_S^m/G$ is compact we normalize the measure $\mu$ on this group so that the measure of the group is just the $\mu_S^m$ measure of any measurable set of representatives in $k_S^m$ of the cosets of $G$. So $\mu(k_\infty^m/G^m) = 2^{-mR^2} |d|^{m/2}$ where $d$ will always stand for the discriminant of $k$.

If $\sigma$ is a topological automorphism of $k_S^m$ the modulus of $\sigma$ is defined by $\text{mod } \sigma = \mu(\sigma X)/\mu(X)$ where $X$ is any measurable set in $k_S$. If $\sigma$ is a $k_S$ module automorphism of $k_S^m$ with determinant $\det \sigma$, then $\text{mod } \sigma = |\det \sigma|_S$.

3. Statement of the theorem. Let $L$ be the system

$$L_i(x) = \sum_{j=1}^{s} a_{ij}x_j, \quad i = 1, \ldots, r,$$

of linear forms with coefficients in $k_S$. Set $m = r + s$. We will suppose $z = (z_1, \ldots, z_m)$, $x = (x_1, \ldots, x_s)$, and $y = (y_1, \ldots, y_r)$ are related by $z = (x, y)$. Suppose $A_\mathfrak{p}$ is the $i$th component of the coefficient matrix of the system

$$L_i^0(z) = \sum_{j=1}^{s} a_{ij}^0 z_j - z_{i+s}, \quad 1 \leq i \leq r.$$

Write $\delta_\mathfrak{p}$ for the determinant of the $r \times r$ submatrix of $A_\mathfrak{p}$ with the $\mathfrak{p}$-adic absolute value of its determinant maximal. We define $\delta = \delta(L) = (\delta_\mathfrak{p}) \in k_S$. For simplicity, we will assume that $S \supseteq P_\infty$, except when we specifically state otherwise.

We let $\psi$ be a mapping from the positive reals $\mathbb{R}_+$ to $k_S^s$. We would like to count the number $\lambda(N)$ of solutions $x \in \mathfrak{o}^s$, $y \in \mathfrak{o}^r$ of

$$L_i(x) - y_i < \psi(|\overline{z}|), \quad 1 \leq i \leq r,$$

$$|\overline{z}| \leq N.$$
We will show how to do this when $|\psi(l)|_S$ does not decrease too fast. Note, there are only finitely many $x \in \mathcal{O}^S$ with $|x| \leq N$, because $|x| \leq N$ defines a bounded region in $k^S = \mathbb{R}^{s_n}$ which therefore contains a finite number of points of the lattice $\mathcal{O}^S$. Also, in the same way the number of $y$ corresponding to a given $x$ in (2) is finite. In fact, if $|\psi(l)|_\infty < 2^{-n}$, then $y$ is uniquely determined; for, if $y'$ and $y''$ both correspond to the same $x$, then

$$y_i = y_i' - y_i'' \leq H \max \{L_i(x) - y_i', L_i(x) - y_i''\} \leq H\psi(|x|),$$

so

$$|\text{Norm } y_i| = |y_i' - y_i''|_\infty \leq |H\psi(|x|)|_\infty < 1$$

and thus $y = 0$.

We use $M$ to denote the transpose system

$$M_j(y) = \sum_{i=1}^r a_{ij}y_i, \quad 1 \leq j \leq s.$$ 

Let $g: \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function. We say $L$ has type $\leq g$ if

$$(3) \max_j |M_j(y) - x_j|_S \leq g(|x|)^{-1}|x|^{-rn/s}$$

has only finitely many solutions $z = (x, y) \in \mathcal{O}^m$. The motivation for the right-hand side of (3) is the following version of Dirichlet's theorem.

Proposition. If $S \supseteq P_\infty$ then there are infinitely many $z = (x, y) \in \mathcal{O}^m$ such that

$$|L_i(x) - y_i|_S \leq c|x|^{-zn/r}, \quad 1 \leq i \leq r.$$ 

If $S \subseteq P_0$ then there are infinitely many $z \in \mathcal{O}^m$ such that

$$|L_i(x) - y_i|_S \leq c|x|^{-mn/r}, \quad 1 \leq i \leq r.$$ 

Here $c$ is some constant depending on $k$ and $L$.

A proof of a slightly different version of this adele theorem may be found in [2, Theorem 2.3].

We prove the following theorem.

Theorem. Assume the following:

(i) $L$ has type $\leq g$.

(ii) $\psi(l)$ is decreasing.
(iii) $\psi(t) = t^{sn}g(s/\gamma)^{-s}$ increases to $\infty$.

(iv) $\psi(t) \leq 1$, i.e., $|\psi(t)|_p \leq 1$ for all finite primes $p \in S$.

(v) $|\psi(t)|_p^{-1} \leq C$ for all pairs of infinite primes $p_1, p_2$, where $C$ is a constant independent of $t$.

Then the number $\lambda(N)$ of solutions of (2) is

$$\lambda(N) = \gamma \int_1^N t^{s_n-1} |\psi(t)|^s dt + O \left( \int_1^N \frac{t^{s_n-1} |\psi(t)|^s}{F(t)} dt \right)$$

with $\gamma = nsR^m m^R \frac{1}{s_0} |d|^{-m/2}$.

Remark. If we specialize the type theorem to the case $k = Q$, $S = P_\infty$, we get the homogeneous version of the theorem in [8].

Remark. If we assume $S \subseteq P_0$, delete condition (v), and replace the right-hand side in condition (iii) by $|\psi(t)|_p t^{sn} g(s/\gamma)^{-s}$, then we can show, by making only minor changes in the proof of the above theorem, that the number of solutions of (2) and $|\gamma| \leq N$ satisfies

$$\lambda(N) \sim \gamma \int_1^N t^{s_n-1} |\psi(t)|^s dt$$

for some constant $\gamma$. This specializes to a $p$-adic theorem when $k = Q$. A similar result may be proved when $S$ includes some but not all primes of $P_\infty$.

In §4 I develop some results from the geometry of numbers which I will need when I prove the above theorem in §5. In §6 I will show how a metric result follows from this theorem.

4. The geometry of numbers over $k$. We call $\Lambda$ an $m$-dimensional $\omega$-lattice if $\Lambda$ is a discrete $\omega$ submodule of $k^m$ and $k^m/\Lambda$ is compact; this last condition is the same as requiring that $\Lambda$ contain $m$ independent elements. We call $\mu(k^m/\Lambda)$ the determinant of $\Lambda$ and denote this by $\det \Lambda$. Note that $\omega^m$ is a lattice with $\det = 2^{-mR^2} |d|^{-m/2}$. From our identification of $k^m$ with $R_{n_1}^1 \times R_{n_2}^2 \cong R^n$, it is clear that an $\omega$-lattice is just an ordinary $R^m$ lattice with the same determinant. Note that not every lattice in $R^m$ is an $\omega$-lattice.

If $a$ is an ideal of $k$, we let $a\Lambda$ be the set of all sums $\sum a_i x_i$ with $a_i$ in $a$ and $x_i$ in $\Lambda$. It has been shown by K. Rogers and H. P. F. Swinnerton-Dyer [7, Theorem 1] that

**Proposition 1.** If $\Lambda$ is an $\omega$-lattice in $k^m$, there exist $m$ distinct points $P_1, \ldots, P_m$ in $\Lambda$ and an ideal $b \supseteq a$ in $k$ such that

$$\Lambda = aP_1 + \cdots + aP_{m-1} + bP_m$$

where the ideal class of $b$ depends only on $\Lambda$. 
We may now state the following:

**Proposition 2.** \(\alpha \Lambda\) is an \(\alpha\)-lattice with \(\det \alpha \Lambda = N\alpha^m \det \Lambda\).

**Proof.** The first assertion follows from the expression

\[
\alpha \Lambda = \alpha P_1 + \cdots + \alpha P_{m-1} + \alpha bP_m.
\]

To prove the second assertion we may suppose \(\alpha\) is integral. Then \(\alpha \Lambda \subseteq \Lambda\) and

\[
\Lambda/\alpha \Lambda = \frac{\alpha P_1 + \cdots + \alpha P_{m-1} + \beta P_m}{\alpha P_1 + \cdots + \alpha P_{m-1} + \alpha \beta P_m}
\]

\[
\cong (\alpha/\alpha)^{m-1} \times \beta/\alpha \cong (\alpha/\alpha)^m
\]

so the order of \(\Lambda/\alpha \Lambda\) is \((N\alpha)^m\). The proposition now follows.

For \(x = (x_1, \ldots, x_m), \ y = (y_1, \ldots, y_m)\) we denote the dot product by \(x \cdot y = \sum_i x_i y_i\). Also, let \(\text{Tr}\) denote the trace function extended to \(k_\infty\). We define

\[
\Lambda^{-1} = \{x \in k_\infty^m; x \cdot y \in \alpha \text{ for all } y \in \Lambda\},
\]

\[
\Lambda^* = \{x \in k_\infty^m; \text{Tr}(x \cdot y) \in \mathbb{Z} \text{ for all } y \in \Lambda\}.
\]

It is straightforward to show

**Proposition 3.** \(\Lambda^* = D^{-1}\Lambda^{-1}\), where \(D\) is the different of \(k\), i.e. \(D^{-1}\) is the fractional ideal consisting of all \(x \in k\) such that \(\text{Tr}(ax) \in \mathbb{Z}\) for all \(a \in \alpha\).

If \(P_1, \ldots, P_m\) are the independent points in Proposition 1, we can find points \(P_1', \ldots, P_m'\) such that

\[
P_i \cdot P_j' = \begin{cases} 1, & i = j, \\ 0, & \text{otherwise.} \end{cases}
\]

So, \(\Lambda^{-1} = \alpha P_1' + \cdots + \alpha P_{m-1}' + \beta^{-1} P_m'\), and hence \(\Lambda^{-1}\), and therefore, also, \(\Lambda^*\), is an \(\alpha\)-lattice. We call \(\Lambda^*\) the polar lattice of \(\Lambda\). This is just the ordinary polar lattice in \(R^m\) with respect to the bilinear form \((x, y) = \text{Tr}(x \cdot y)\).

We now give some examples of \(\alpha\)-lattices we will need later.

**Example 1.** Let \(L\) be the independent system \(L_i(z) = \sum_{j=1}^m a_{ij} z_j^i, 1 \leq i \leq m,\) with \(a_{ij} \in k_\infty^m\). The coefficient matrix \(A\) of this system has determinant in \(k_\infty^m\). So, \(L\) determines an automorphism \(L: z \rightarrow L(z) = z A\) of \(k_\infty^m\) with mod \(L = |\det A|_\infty^m\). If \(\Lambda\) is an \(\alpha\)-lattice, then so is \(L(\Lambda)\). It is clear that

\[
\det L(\Lambda) = \text{mod } L \det \Lambda = |\det A|_\infty^m \det \Lambda.
\]

Now, let \(M\) be the system with coefficient matrix \(tA^{-1}\) (the \(t\) stands for transpose). Then
\[ L(z) \cdot M(w) = (zA) \cdot (w^t A^{-1}) = zAA^{-1}(w') = z \cdot w. \]

Hence \( L(\Lambda)^{-1} = M(\Lambda^{-1}) \) and therefore also
\[ L(\Lambda)^* = D^{-1} L(\Lambda)^{-1} = D^{-1} M(\Lambda^{-1}) = M(D^{-1} \Lambda^{-1}) = M(\Lambda^*). \]

**Example 2.** Assume \( S \subseteq P_0 \) and let \( L \) be the system of independent linear forms \( L.(z) = \sum_{j=1}^{m} a_{ij} z_j, \quad 1 \leq i \leq m, \) with coefficients \( a_{ij} \in k_S. \) Let \( \epsilon \) be an idele \( \leq 1 \) in \( k_S \) and define
\[
\Lambda = \Lambda_{L, \epsilon} = \{ z \in k^m : L.(z) < \epsilon, \ 1 \leq i \leq r \}.
\]

Since all \( p \in S \) are nonarchimedean the set \( \Lambda_{L, \epsilon} \) is an \( \sigma \)-module. The set is discrete because \( \Lambda_{L, \epsilon} \subseteq \sigma^m. \) Also, it contains the \( m \) \( k_\infty \)-independent elements \( a e_i \) where \( a \) is an appropriately chosen element of \( \sigma \) and \( e_i \) is the \( m \)-tuple with 1 in the \( i \)th position and 0 elsewhere. Hence \( \Lambda_{L, \epsilon} \) is an \( \sigma \)-lattice.

We compute the determinant of \( \Lambda_{L, \epsilon} \). Let \( A \) be the \( r \times m \) coefficient matrix of the system \( L \) and let \( A_p \) be the \( p \)-th component of this matrix. Write \( \delta_p \) for the determinant of the \( r \times r \) submatrix of \( A_p \) with the \( p \)-adic absolute value of its determinant maximal. Also, write \( \delta'_p \) for the determinant of the submatrix of \( A_p \) with the \( p \)-adic absolute value of its determinant maximum; this last submatrix may be of any size \( i \times i \) with \( 0 \leq i \leq r \), and by convention we take the determinant of a \( 0 \times 0 \) matrix to be 1. We define \( \delta = (\delta_p) \in k_S, \delta' = (\delta'_p) \in k_S. \)

**Proposition 4.** If \( \delta, \delta' \) are ideles and \( \epsilon < \delta/\delta' \), then
\[
\det \Lambda_{L, \epsilon} = 2^{-mR_2} |d|^{m/2} |\epsilon^{-1} \delta|_S.
\]

**Proof.** It suffices to prove the order of \( \sigma^m/\Lambda \) is \( |\epsilon^{-1} \delta|_S. \) Set
\[
E = \left\{ z \in k^m_S : z_i \leq 1, \ 1 \leq i \leq m \right\},
\]
\[
E' = \left\{ z \in k^m_S : L.(z) \leq \epsilon, \ z_i \leq 1, \ 1 \leq i \leq r, \ 1 \leq j \leq m \right\}.
\]

Since all \( p \) in \( S \) are nonarchimedean, \( E \) and \( E' \) are groups with \( E' \subseteq E. \) Because \( \sigma^m \) is dense in \( E, \) each coset of \( E' \) in \( E \) contains an element of \( \sigma^m \) and therefore the injection \( \sigma^m \to E \) induces an isomorphism \( \sigma^m/\Lambda \cong E/E'. \) Thus, we need to find the order \( #(E/E') \) of \( E/E'. \) But \( \mu(E) = 1. \) So \( #(E/E') = \mu(E')^{-1}, \) and therefore it suffices to prove \( \mu(E') = |\epsilon \delta^{-1}|_S. \)

Consider the inequalities
\[
\epsilon^{-1} L_i(z) \leq 1, \quad 1 \leq i \leq r, \quad z_j \leq 1, \quad 1 \leq j \leq m.
\]
Let $B$ be the coefficient matrix of the left-hand side of (5), and let $B_p$ be the $p$th component of $B$. Let $C_p$ denote the $m \times m$ submatrix of $B_p$ with the $p$th absolute value of its determinant maximum. Clearly, $\det C_p = \epsilon^{-j} \det D_p$ where $D_p$ is a $j \times j$ submatrix of $A_p$. I claim that $j = r$, and therefore, clearly, $\det D_p = \delta_p$. Suppose that $j < r$. The submatrix of $A_p$ with determinant $\delta_p$ yields a submatrix of $B_p$ with determinant $\epsilon^{-j} \delta_p$; so $|\epsilon^{-r} \delta_p| < |\epsilon^{-j} \det D_p|_p$ and therefore
\[
|\epsilon|_p \geq |\epsilon|_p^{r-j} > |\delta_p/\det D_p|_p \geq |\delta/\delta'|_p
\]
which is a contradiction.

We may assume $C_p$ appears in the same rows of $B_p$ for each $p$. We denote the submatrix of $B$ in these $m$ rows by $C$. The other rows of $B$ may be represented as linear combinations of the $m$ rows of $C$. By Cramer's rule, the coefficients in these combinations will be of the form $\det C'/\det C$ where $C'$ is some submatrix of $B$. But, by the choice of $C$, $\det C'/\det C \leq 1$. Hence, because all $p \in S$ are nonarchimedean, the inequalities (5) hold if and only if the inequalities hold for the rows of $C$. Hence, $E' = C^{-1}E$ and therefore
\[
\mu(E') = \mu(C^{-1}E) = (\bmod C^{-1})\mu(E) = |\det C^{-1}|_S = |\epsilon \delta^{-1}|_S.
\]
This proves the proposition.

A theorem similar to Proposition 4 may be found in [6].

Suppose $L$ has the form

\[
L_i(x) = \sum_{j=1}^{s} a_{ij} z - z_{s+i} 1 \leq i \leq r,
\]
with $m = r + s$. Then $\delta = \delta'$ and both are ideles.

We now compute the polar lattice of $\Lambda_{L,\epsilon}$ when $L$ has the special form (6). Let $M$ be the transposed system

\[
M_j(w) = w_j + \sum_{i=1}^{r} a_{ij} w_{i+s}, 1 \leq j \leq s,
\]
so that

\[
z \cdot w = -\sum_{i=1}^{r} L_i(z) w_{i+s} + \sum_{j=1}^{s} M_j(w) z_j.
\]
Define $a_L = a$ to be the integral ideal of $k$ consisting of all $a$ in $\mathfrak{o}$ for which
\[
a a_{ij} \leq 1, 1 \leq i \leq r, 1 \leq j \leq s.
\]
Also, define $\delta = \delta_0$ to be the ideal $\delta_0 = \prod_{p \in S} p^{\nu_p(\epsilon)}$; so $a \in \mathfrak{o}$ is such that $a \leq \epsilon$ if and only if $a \in \delta$. We now prove

Proposition 5. $\delta_0 a_L \Lambda_{L,\epsilon}^{-1} \subseteq \Lambda_{M,\epsilon}$ if all the $a_{ij}$ satisfy $a_{ij} \leq 1$, then equality holds.
Proof. Let \( e_i \) be the \( m \)-tuple with 1 in the \( i \)th position and 0 elsewhere. It is clear \( \beta a e_i \subseteq \Lambda_L \). So \( (\beta a e_i) \cdot \Lambda_{L}^{-1} \subseteq \alpha \), and therefore \( \beta a \Lambda_{L}^{-1} \subseteq \alpha^m \). Since \( k \) is dense in \( k_S \), we can replace the \( a_{ij} \) by elements of \( k \) and still get the same lattices \( \Lambda_L, \epsilon^* \Lambda_M, \epsilon^* \). So assume \( a_{ij} \in k \) and set

\[
a_j = (0, \ldots, 0, 1, 0, \ldots, 0, a_{1j}, \ldots, a_{rj}) \in k^m
\]

where the 1 is in the \( j \)th position. Because \( aa_j \in \alpha^m \) and \( L_i(aa_j) = 0 \), then \( aa_j \subseteq \Lambda_L \), so \( (aa_j) \cdot \Lambda_{L}^{-1} \subseteq \alpha \). By (7), with \( w \in \alpha \beta \Lambda_{L}^{-1} \) and \( z \in aa_j \), we get

\[
M_j(aa_jL_{L}^{-1})a = (aa_jL_{L}^{-1}) \cdot aa_j = a \beta (\Lambda_{L}^{-1}) \cdot aa_j \subseteq \alpha
\]

so canceling the \( a \)'s we have \( \alpha \beta \Lambda_{L}^{-1} \subseteq \Lambda_M \), as desired.

Now assume \( a_{ij} < 1 \) for all \( i \) and \( j \). So \( a = 0 \), and we can assume \( a_{ij} \in 0 \). Let \( w \in \Lambda_M \) and \( z \in \Lambda_L \). Then \( M_j(w) \in \delta \) and \( L_i(z) \in \delta \). So, by equation (7), we see that \( z \cdot w \in \delta \), and therefore \( z \cdot (\delta^{-1}w) \in 0 \). This shows that \( \delta^{-1} \Lambda_M \subseteq \Lambda_{L}^{-1} \), as desired.

It is easy to produce an example to show that equality does not in general hold in Proposition 5.

5. Proof of the theorem. Let \( \epsilon \) be an idele with \( \psi(0) \geq \epsilon \geq \psi(N) \) and satisfying

(\( v' \)) \( |\epsilon|_{p_1} |\epsilon|_{p_2}^{-1} \leq C \) for all infinite primes \( p_1, p_2 \) where \( C \) is the constant of condition (\( v' \)). Set \( l_N = N/F(N) \) and note \( 1 \leq l_N \leq N \) if \( N \) is sufficiently large. We first find an estimate of the number \( \alpha(N, \epsilon) \) of solutions \( x \in \alpha^s \), and \( y \in \alpha' \) of the inequalities

\[
L_i(x) - y_i \leq \epsilon, \quad N - l_N \leq |x| \leq N.
\]

Define systems \( \Lambda \) and \( \Lambda^* \) by the formulas

\[
\Lambda_i(z) = \begin{cases} 
    z_i & \text{for } 1 \leq i \leq s, \\
    -l_N/\epsilon^\infty \left( \sum_{j=1}^{s} a_{i-s}^\infty z_{j-i} - z_i \right) & \text{for } s+1 \leq i \leq m,
\end{cases}
\]

\[
\Lambda_i^*(z) = \begin{cases} 
    z_j + \sum_{i=1}^{s} a_{i-s}^\infty z_{j+i} & \text{for } i \leq j \leq s, \\
    \epsilon^\infty z_j/\epsilon N & \text{for } s+1 \leq j \leq m,
\end{cases}
\]

where for \( a \in k_S \), as usual, \( a^\infty \) denotes the \( k_{\infty} \) component of \( a \). Note, we are assuming real numbers such as \( l_N \) are embedded along the diagonal in \( k_{\infty} \). Let
$L^0$ be as in (1) and define $M^0$ by

$$M_j^0(x) = z_j + \sum_{i=1}^r a_{ij}^0 z_{s+i}, \quad 1 \leq j \leq s.$$ 

In Example 2 of §4 we used $L^0$ and $\epsilon^0$ to define an $\alpha$-lattice $\Lambda_{L^0, \epsilon^0}$ with determinant

$$\det \Lambda_{L^0, \epsilon^0} = 2^{m/2} |\epsilon|^{m/2} |\delta(L)|_{S_0}.$$ 

Then, by Example 1 of §4, $\Lambda = \overline{L(\Lambda_{L^0, \epsilon^0})}$ is an $\alpha$-lattice with determinant

$$\det \Lambda = \left( \frac{f_N}{\epsilon^m} \right) \det \Lambda_{L^0, \epsilon^0} = \frac{\gamma_1^r m^2}{|\epsilon|}$$

with $\gamma_1 = 2^{-mR^2} |d|^m/|\delta(L)|_{S_0}$. We see $\alpha(N, \epsilon)$ is just the number of points of $\Lambda$ in the region $T$ of $k_m^\infty$ consisting of all $z \in k_m^\infty$ satisfying

$$N - l_N \leq \|z\| \leq N, \quad x = (z_1, \ldots, z_s),$$

$$z_i \leq l_i, \quad i = s+1, \ldots, m.$$ 

Let $B_b$ be the boundary of $T$ expanded by the diameter $b$ of some fundamental parallelepiped of $\Lambda \subseteq R^m$. Then, if $\mu$ is Lebesgue measure on $R^m$, we see that

$$\alpha(N, \epsilon) = \frac{\mu T}{\det \Lambda} + O\left( \frac{\mu B_b}{\det \Lambda} \right).$$

We have

$$\mu T = \left( (2^1 \pi^1 R^1)^m s - (2^1 \pi^1 R^2 (N - l_N)^m s) (2^1 \pi^1 R^2 m)^s \right)$$

$$= (2^1 \pi^1 R^2)^m l_N \int_{N-l_N}^N t^{ns-1} dt$$

and

$$\mu B_b = O(N^{ns-1} l_N^r b)$$

if

$$b \ll l_N.$$ 

Using in (9) the value for $\det \Lambda$ given in (8), we get

$$\alpha(N, \epsilon) = \gamma |\epsilon| \int_{N-l_N}^N t^{ns-1} dt + O(N^{ns-1} |\epsilon| b)$$

provided that (10) holds, where
\[ y = n s \left( 2^R \pi \right)^m / \gamma_1 = 2^m \pi R^2 |d(L)|_{\frac{1}{2}} \left| d \right|^{-m/2} n s. \]

We now find an upper bound for \( b \). Let \( \mu_1, \ldots, \mu_{mn} \) be the successive minimum of \( \Lambda \) with respect to the distance function \( f^* \) polar to the distance function \( f: \mathbb{R}_\infty \rightarrow \mathbb{R}, f(z) = |z| \). It can be shown (see [3, Chapter V, Lemma 8]) there is a basis \( c_1, \ldots, c_n \) of \( \Lambda \) satisfying \( f^*(c_i) \leq \frac{1}{2} n m \mu_i \). So, if we choose \( b \) to be the diameter of the fundamental parallelepiped determined by this basis, we see that

\[ b \leq \sum |c_i| \ll \sum f^*(c_i) \ll \mu_{nm}. \]

By Mahler's theorem (see [3]), if \( \mu_{i}^* \) is the first minimum of \( \Lambda^* \) with respect to \( f \), then

\[ \mu_{i}^* \ll 1. \]

So, we can find an upper bound for \( \mu_{nm} \) and hence for \( b \) by finding a lower bound for \( \mu_{i}^* \). This is where we use the type condition.

If \( a = a_{L0} \), \( b = b_{0} \), and \( \Lambda_{M0} = \Lambda_{M0, e0} \) are defined as in §4, we know

\[ \Lambda_{L0}^* \subseteq c^{-1} \Lambda_{M0}^* \]

where \( c = D b a \subseteq a \). Now \( \bar{M} \) and \( \bar{L} \) are such that

\[ \bar{M}(z') \cdot \bar{L}(z') = z' \cdot z''; \]

so, as in Example 1 of §4, the lattices \( \Lambda \) and \( \Lambda^* = \bar{M}(\Lambda_{L0}^*) \) are polar. Define

\[ \bar{\Lambda} = \bar{M}(c^{-1} \Lambda_{M0}^*). \]

Then, by (13), \( \Lambda^* \subseteq \bar{\Lambda} \). So, if \( \bar{\mu}_1 \) is the first minimum of \( \bar{\Lambda} \) with respect to \( f \), then \( \bar{\mu}_1 \leq \mu_{i}^* \). Hence, we will find a lower bound for \( \bar{\mu}_1 \).

Choose \( z' \in c^{-1} \Lambda_{M0} \) such that \( f(z') = |z'| = \bar{\mu}_1 \). By a simple application of Minkowski's theorem, there is \( c \in c \) such that

\[ |c| \leq (2^R \pi \left| d \right|^{1/2} |\text{Norm } c|)^{1/n}. \]

By the definition of \( b \) given in §4, we have

\[ |\text{Norm } b| = \prod_{p \leq S_0} N^p \nu_p(c) = |c| S_0^{-1}; \]

so \( |c| \ll |c| S_0^{-1/n} \) and therefore, also, \( |\text{Norm } c| \ll |c| S_0^{-1} \) where the constants implied by \( \ll \) do not depend on \( N \).

We have \( z = cz' \in \Lambda_{M0} \subseteq a^m \). Hence, with this \( z = (x, y) \), we have
\[ x_j + \sum_{i=1}^{r} a_{ij}^0 y_i \leq \epsilon^0, \quad 1 \leq j \leq s. \]

From the definition of \( f \) and \( \overline{\Lambda} \) we see that

\[ x_j + \sum_{i=1}^{r} a_{ij}^0 y_i \leq c\overline{\mu}_1, \quad 1 \leq j \leq s, \]

(14)

\[ y_i \leq \frac{l_N}{\epsilon^0} c\overline{\mu}_1, \quad 1 \leq i \leq n. \]

Hence \( \max_j |x_j + M_j(y)| \leq |\text{Norm } c|\mu_1^n|\epsilon|_{S_0} \leq \overline{\mu}_1^n \). By the type condition, this implies

\[ g(\overline{\mu})^{-1} |\overline{\mu}|^{-n/s} \ll \overline{\mu}_1^n. \]

By (14) and condition (v') for \( \epsilon \),

\[ |\overline{\mu}| \ll l_N \overline{\mu}_1 |\epsilon|_{S_0}^{-1/n} \ll l_N \overline{\mu}_1 |\epsilon|_{\infty}^{-1/n} = l_N \overline{\mu}_1 |\epsilon|_{S}^{-1/n}. \]

We also have \( |\overline{\mu}| \ll l_N \overline{\mu}_1 |\epsilon|_{S}^{-1/n} \) from (14), since \( \epsilon \leq \psi(0) \) implies that \( c\overline{\mu}_1 \leq (l_N/\epsilon^0)c\overline{\mu}_1 \) for large \( N \). Therefore \( |\overline{\mu}| \ll l_N \overline{\mu}_1 |\epsilon|_{S}^{-1/n} \), and then by (15)

\[ |\epsilon|_{S}^{r/s} l_N^{1-r/m/s} \overline{\mu}_1^{-1/m/s} g(\overline{\mu})^{-1} \ll \overline{\mu}_1^n. \]

Solving for \( \overline{\mu}_1 \) we get

\[ (|\epsilon|_{S}^{r-s} l_N^{1-r/m/s} g(\overline{\mu})^{-s})^{1/mn} \ll \overline{\mu}_1^n. \]

Minkowski’s convex body theorem says \( \overline{\mu}_1^{nm} \leq 2^{nm} \det(\overline{\Lambda})/V_f \) where \( V_f \) is the volume of the region defined by \( f(z) \leq 1 \). It is easy to see (in the same way we got (8)) that

\[ \overline{\mu}_1^{nm} \ll \det(\overline{\Lambda}) = \text{Norm } c^{-m}(|\epsilon|_{\infty}^{r-1} l_N^{-r/m/2} |A|^{m/2}|\epsilon^{-s} g|_{S_0}))^{r} l_N^{-r/m}. \]

So, by our bound for \( |\overline{\mu}| \), we have

\[ |\overline{\mu}|^{mn} \ll l_N^{mn} |\epsilon|^{-m} |\epsilon|^{n} \ll l_N^{sn} |\epsilon|^{-s} = N^{sn}/F(N)^{sn} |\epsilon|_{S}^{s}. \]

From condition (iii), it is now easy to see that \( |\overline{\mu}| \leq N^{s/n} \) if \( N \) is large. Hence by (16)

\[ (|\epsilon|_{S}^{r-s} l_N^{1-r/m} g(\overline{\mu})^{-s})^{1/mn} \ll \overline{\mu}_1^{n} \leq \mu_1^{*}, \]

and therefore from (12) and condition (iii)

\[ b \ll \mu_{nm} \ll (g(N^{s/n}) s l_N |\epsilon|_{S}^{r})^{1/mn} \ll l_N F(N)^{-1}. \]
Now (10) is clearly satisfied, so (11) now reads

\[
\alpha(N, \epsilon) = y|\epsilon|_S^r \int_{N-I_N}^N t^{n-1} dt + O(N^{n-1}|\epsilon_1|_S F(N) - 1),
\]

The rest of the proof follows Lang [5]. We apply formula (17) to \(\epsilon = \psi(N)\) and \(\epsilon = \psi(N - I_N)\) to get the theorem. Since \(\psi\) is decreasing we see

\[
\alpha(N, \psi(N)) \leq \lambda(N) - \lambda(N - I_N) \leq \alpha(N, \psi(N - I_N)).
\]

Then, by (17) with \(\epsilon = \psi(N)\) and \(\epsilon = \psi(N - I_N)\),

\[
\lambda(N) - \lambda(N - I_N) = y|\psi(N)|_S^r \int_{N-I_N}^N t^{n-1} dt
\]

\[
+ O \left( \left| \psi(N-I_N) \right|_S^r - \left| \psi(N) \right|_S^r N^{n-1} I_N + \frac{|\psi(N - I_N)|_S^{N^{n-1}} I_N}{F(N)} \right).
\]

Note, \(F\) increasing implies \(\left| \psi(t) \right|_S^t t^{sn}\) is also increasing. Hence

\[
\left| \psi(N-I_N) \right|_S^t (N-I_N)^{sn} \leq \left| \psi(N) \right|_S^t N^{sn} \leq \left| \psi(N) \right|_S^t ((N-I_N)^{sn} + snN^{sn-1} I_N),
\]

so

\[
\left| \psi(N-I_N) \right|_S^t - \left| \psi(N) \right|_S^t \leq \frac{snN^{sn-1} \left| \psi(N) \right|_S^t I_N}{(N-I_N)^{sn}} \leq \frac{I_N \left| \psi(N) \right|_S^t I_N}{N} = \frac{\left| \psi(N) \right|_S^t}{F(N)},
\]

and therefore also \(\left| \psi(N-I_N) \right|_S^t \ll \left| \psi(N) \right|_S^t\). Using these estimates in (18) we get

\[
\lambda(N) - \lambda(N - I_N) = y|\psi(N)|_S^r \int_{N-I_N}^N t^{n-1} dt + O \left( \left| \psi(N) \right|_S^{N^{n-1}} I_N \right).
\]

Now \(F(t) \to \infty\). So if \(N\) is large enough \(N - I_N \geq N(1 - 1/F(N)) \geq N/2\) and therefore, because \(\psi(t)\) and \(1/F(t)\) are both decreasing,

\[
\frac{\left| \psi(N) \right|_S^r (N-I_N)^{sn-1} I_N}{F(N)} \ll \frac{\left| \psi(N) \right|_S^r (N-I_N)^{sn-1} I_N}{F(t)} \leq \int_{N-I_N}^N \frac{|\psi(t)|_S^r t^{sn-1}}{F(t)} dt.
\]

Also, because \(\psi\) is decreasing, we get

\[
\left| \psi(N) \right|_S^r \int_{N-I_N}^N t^{n-1} dt = \int_{N-I_N}^N \left| \psi(t) \right|_S^r t^{n-1} dt
\]

\[
+ O \left( \left| \psi(N-I_N) \right|_S^r - \left| \psi(N) \right|_S^r N^{n-1} I_N \right).
\]

We have already estimated the error term in this last expression. Hence (19), (20), and (21) yield
6. A metric theorem. We put a measure on the space of all systems $L$ of $r$ linear forms in $s$ variables by identifying the form $L$ with an $rs$-tuple in $k_s^r$ made up of the coefficients of $L$. We will determine a type for almost all systems $L$. For simplicity, we restrict ourselves to the case when $S \supseteq P_\infty$. As preparation we state the following adele version of the convergence theorem:

**Proposition 6.** Let $\epsilon: R_+ \to k_\infty^r$. If $\sum_{x \in o^s} |\epsilon([x])|^r_S < \infty$ then, for almost all systems $L$, there are only finitely many solutions $x \in o^s$, $y \in o^r$ of

$$L_i(x) - y_i \leq \epsilon([x]), \quad 1 \leq i \leq r. \tag{22}$$

This is the easy part of the Khinchin metric theorem; the other part asserts that, if the above sum diverges, then, under certain conditions, for almost all systems $L$ (22) will have infinitely many solutions. A proof of this theorem for the adeles, in the case $s = 1$, may be found in [2].

If $k = Q$ and $S = P_\infty$, the above proposition gives a type for almost all systems $L$. However, in the general case, type is defined in terms of an inequality on the volume $|L|$ and not by simultaneous inequalities such as in (22), so the proposition does not apply directly. By modifying the proof of a theorem in [4, p. 96] we can get what we need, if the set of primes $S$ is finite.

**Proposition 7.** Let $S \supseteq P_\infty$ be a finite set of primes, and let $\epsilon: R_+ \to R_\infty$. If

$$\epsilon(t) < 1 \quad \text{and} \quad \int_1^\infty t^{\eta m - 1} \epsilon(t) t^{\eta(1 - \eta)} dt < \infty, \quad 1 > \eta > 0,$$

then for almost all $L$, there are only finitely many $x \in o^s$, $y \in o^r$ satisfying

$$\max_i |L_i(x) - y_i|_S \leq \epsilon([z]), \quad z = (x, y). \tag{23}$$

**Proof.** It is easy to see, if we replace (22) by

$$\inf \{1, L_i(x) - y_i \leq \epsilon([z]), \quad z = (x, y), \quad 1 \leq i \leq r, \tag{24}$$

then the proof of Proposition 6 shows that for almost all systems $L$ the inequalities (24) have only a finite number of solutions when $\int_1^\infty t^{\eta m - 1} |\epsilon(t)|_S^{r(1 - \eta)} dt < \infty$ (the $\epsilon$ in (24) is as in Proposition 6, i.e., $\epsilon: R_+ \to k_\infty^r$).

For the proof of Proposition 7, we assume, for the sake of simplicity, that $r = 1$. Let $F$ be the set of all $L$ for which (23) has infinitely many solutions. Suppose (23) holds for $z = (x, y)$. If we put

$$\inf \{1, |L_1(x) - y_1|_S = \epsilon([z])^{r_S(x)}, \tag{25}$$

Equation (4) now follows by induction.
then $r_p = r_p(z) \geq 0$ and $\Sigma_p r_p \geq 1$. Let $\nu$ be the number of elements in $S$, and choose a positive integer $A$ so large that $\nu/A < \eta$. We have $A \leq [\Sigma_p A r_p] \leq \Sigma_p [A r_p] + \nu$; and therefore, if $B = A - \nu > 0$, then $B \leq \Sigma_p [A r_p(z)]$. So there exists $b_p = b_p(z)$ such that $b_p$ is an integer and

$$0 \leq b_p \leq [A r_p(z)] \leq A r_p(z), \quad \Sigma b_p = B.$$  

There are only a finite number of possibilities for each $b_p$. So, if $L \in F$, we may assume, for each $\wp \in S$, $b_p = b_p(z)$ takes on the same value for infinitely many solutions $z = (x, y)$ of (23); i.e., we may assume $b_p$ takes on a value depending only on $L$ and not on $z$. By (26), if we set $l_p = b_p/A$, then

$$0 \leq l_p \leq r_p, \quad \Sigma l_p = B/A = (A - \nu)/A > 1 - \eta.$$  

Then (25) implies there are infinitely many solutions of

$$\inf \{1, |L_1 - y_1|_p \leq \epsilon(|z|)l_p. \}

Now $\Pi_p \epsilon(t)|b_p \leq \epsilon(t)^{1-\eta}$. Therefore, since $\int_1^\infty t^{m-1} \epsilon(t)^{1-\eta} dt$ converges, we see that the set $E(b)$, $b = (b_p)_{\wp \in S}$, for which (27) has infinitely many solutions, has measure zero. But $F \subseteq \bigcup E(b)$ where the union is over all tuples $b = (b_p)$ with $b_p \geq 0$ and $\Sigma b_p = B$. So the measure of $F$ is also zero. This proves Proposition 7.

If we apply Proposition 7 to the transposed system $M$ of $s$ forms in $r$ variables, we find that by taking $g(t)$ so that

$$\int_1^\infty \frac{t^{m-1}}{(g(t)t^{m/2})^{s(1-\eta)}} dt \text{ converges,}$$

then almost all $L$ have type $\leq g$. So, for a $g$ satisfying (28) and a $\psi$ satisfying conditions (ii)-(v), we have that formula (4) holds for almost all $L$.

It may be possible that Proposition 7 can be refined, and therefore a better metric theorem would result. For example, in the case $k = Q$, $S = \mathbb{P}_\infty$, almost all systems have type $\leq \log^{1+\eta} t$, while Proposition 7 can never give a type any better than $O(t^\omega)$. Also, in the case $k = Q$ and $S$ consists of one $p$-adic prime, one can show almost all systems $L$ have type $\leq \log^{1+\eta} t$ (see the Khinchin metric theorem in [6] where it is shown that almost all $p$-adic systems

$$|L_i(x) - y_i|_p \leq \epsilon(t), \quad t = \max \{ |x_i|, |y_i| \}

have only a finite number of solutions, if $t \epsilon(t)$ is decreasing and $\sum t^{m-1} \epsilon(t)^r < \infty)$. However, if $S$ contains more than one infinite prime it seems unlikely the integral in Proposition 7 can be improved to anything better than
\[
\int_1^\infty t^{n-1} \epsilon(t)^r \log \epsilon(t)^{-1} \, dt
\]

since, for example, the measure of the set

\[\{(a, b) \in \mathbb{R}^2: \inf |a| \inf |b| \leq \epsilon\}\]

is of the form \(2\epsilon(1 + 2\log \epsilon^{-1})\).

In the case \(s < r\) our theorem will still hold if we replace the definition of type with the following definition of \(\psi\)-type:

**Definition.** Let \(g: \mathbb{R}_+ \to \mathbb{R}_+\) be an increasing function, and let \(\psi: \mathbb{R}_+ \to k^*_S\).

Define \(\epsilon(t)\) by the formulas

\[
e\psi(t) = \psi_p(t) \quad \text{for } \eta \in S_0,
\]

\[
e\psi(t) = (g(t)^{n/s} |\psi(t)|_{S_0^1})^{-1/n} \quad \text{for } \eta \in P_\infty.
\]

Then we say the system \(L\) has \(\psi\)-type \(\leq g\), if \(M_j(y) - x_j \leq \epsilon(|y|), 1 \leq j \leq s\), has only finitely many solutions \(y \in \mathcal{O}\) and \(x \in \mathcal{O}^s\).

In this case we may apply the Khinchin convergence theorem (Proposition 6) directly to obtain the following metric corollary to the type theorem:

**Proposition 8.** Assume \(s < r\). If \(\int_1^\infty g(t)^{-s} t^{-1} \, dt\) converges and conditions (ii) through (v) of the type theorem hold, then

\[
\lambda(N) \sim \gamma \int_1^N t^{n-1} |\psi(t)|_{S_0^1}^r \, dt
\]

for almost all systems \(L\).

**REFERENCES**


