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ALGEBRAS OVER ABSOLUTELY FLAT COMMUTATIVE RINGS

BY

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ABSTRACT. Let A be a finitely generated algebra over an absolutely flat commutative ring. Using sheaf-theoretic techniques, it is shown that the weak Hochschild dimension of A is equal to the supremum of the Hochschild dimension of A_x for x in the decomposition space of R. Using this fact, relations are obtained among the weak Hochschild dimension of A and the weak global dimensions of A and A^e.

It is also shown that a central separable algebra is a biregular ring which is finitely generated over its center. A result of S. Eilenberg concerning the separability of A modulo its Jacobson radical is extended. Finally, it is shown that every homomorphic image of an algebra of weak Hochschild dimension 1 is a type of triangular matrix algebra.

W. C. Brown has shown in [3, Theorem 1, p. 369] that every finitely generated algebra A over a commutative absolutely flat ring R with the property that A modulo its Jacobson radical N is R-separable contains a separable subalgebra S such that $A = S \oplus N$ as R-modules. The result was proved using the sheaf-theoretic techniques of R. S. Pierce [13].

In the first section, these techniques are applied to algebras of finite Hochschild dimension over an absolutely flat ring to relate the Hochschild dimension and the weak global dimension of such algebras.

In the second section, it is shown that the natural analogue of the central simple algebra over a field is the central algebra over an absolutely flat ring which is a biregular ring in the sense of [5]. Moreover, we show for every finitely generated algebra A of finite Hochschild dimension that A modulo its Jacobson radical N is separable over R. This allows us to obtain the characterization theorem [6, Corollary 2 to Theorem 3, p. 311] in case R is absolutely flat.

In the final section, a structure theorem is obtained for algebras of Hochschild dimension 1 and their homomorphic images in terms of their idempotent structure. These results are a generalization of those of S. U. Chase in [4].

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Conventions. All rings will have one, subrings will contain the identity of the original ring, all modules and homomorphisms are unitary. R shall always denote a commutative ring. By an absolutely flat ring, we shall mean a ring all of whose modules are flat (such rings are also called von Neumann regular). By a finitely generated or projective algebra A over R we shall mean that A considered as an R-module is finitely generated or projective. Unless otherwise specified all (weak) homological dimensions will be taken as LEFT dimensions. By the (weak) Hochschild dimension of an R-algebra A, we shall mean the (weak) projective dimension of A considered as a module over its enveloping algebra $A \otimes_R A^{op}$ and we shall denote it (w) R-dim A. An algebra A is said to be separable over R if $R\text{-dim } A = 0$.

N shall denote the Jacobson radical of the algebra A; otherwise, the Jacobson radical shall be denoted $J(\_)$. Ideal shall mean two-sided ideal unless otherwise specified. $Z(A)$ shall denote the center of the algebra A.

All undefined sheaf-theoretic terminology shall be as in Pierce [13]. $X(R)$ shall denote the decomposition space of R—the space of maximal ideals of the Boolean ring of idempotents of R—and if x is in $X(R)$, then for any R-module, $M_x = M/xM$. It is known that $\otimes_R R_x$ is an exact functor.

1. The dimension of algebras over absolutely flat rings.

Proposition 1.1. *Let A be a finitely generated algebra over the absolutely flat ring R and let M be a finitely generated A-module. Then* w $\mathrm{pd}_A(M) = \sup \mathrm{pd}_A(M) = \sup \mathrm{pd}_{A_x}(M_x)$ *where the supremum is taken over all x in $X(R)$.*

Proof. Since R_x is a field for each x in $X(R)$, $\otimes_R R_x$ is an exact functor. Hence if

$$(*) \qquad 0 \to K_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$$

is a flat resolution of M as an A-module with the F_i A-free, then

$$0 \to (K_n)_x \to (F_{n-1})_x \to \cdots \to (F_0)_x \to M_x \to 0$$

is a flat resolution of M_x as an A_x-module. Therefore, w $\mathrm{pd}_A(M) \geq \sup \mathrm{pd}_{A_x}(M_x)$, since for finitely generated modules over the noetherian ring A_x, the weak projective dimension agrees with the projective dimension (thus, $(K_n)_x$ is in fact A_x-projective).

Suppose on the other hand that the supremum is finite. Then there is a y in $X(R)$ such that $n = \mathrm{pd}_{A_y}(M_y) = \sup \mathrm{pd}_{A_x}(M_x)$. Let $(*)$ be an exact sequence such that $F_0, \cdots, F_{n-1}$ are all free A-modules. By the exactness of $\otimes_R R_y$, we have that $(*) \otimes_R R_y$ is a flat resolution of M_y. Now by [2, Corollary 1, p. 43], it is known that a module K is left flat over A if and only if for each finite indexing set I, whenever k_i is in K and b_i is in A with the property that $\Sigma_i b_i k_i = 0$, then there are elements x_j in K and a_{ji} in A (j in some finite indexing set J) such that $\Sigma_i b_i a_{ji} = 0$ and $k_i = \Sigma_j a_{ji} x_j$.

Now suppose that k_i in K and b_i in A are a finite set satisfying $\Sigma_i b_i k_i = 0$. Then $\Sigma_i (b_i)_x (k_i)_x = 0$ for each x in $X(R)$. Now since we are assuming that $(K_n)_x$ is A_x-flat for each x in $X(R)$, there exist elements $(a_{ji})_x$ in A_x and $(x_j)_x$ in $(K_n)_x$ such that $\Sigma_i (b_i)_x (a_{ji})_x = 0$ for all j and $(k_i)_x = \Sigma_j (a_{ji})_x (x_j)_x$ for every i in I. We may now proceed exactly as in Brown's paper [3] to obtain the desired preimages in A. For the sake of completeness, we include the details of this argument.

For each x in $X(R)$, there are elements a^x_{ji} and x^x_j, j in $J(x)$, $J(x)$ finite, such that

(1) $\Sigma_i (b_i)_x (a^x_{ji})_x = 0$ for all j in $J(x)$,

(2) $[\Sigma_j (a^x_{ji})(x^x_j)_x] - (k_i)_x = 0$ for all i in I.

Thus $\Sigma_i (b_i)(a^x_{ji})$ and $k_i - \Sigma_j (a^x_{ji})(x^x_j)$ may be viewed as sections on the sheaf $\mathfrak{A}(A)$ [13, p. 18] over $X(R)$. These sections are zero at x and hence are zero on some open set U of $X(R)$ containing x. But $\{U_x: x \text{ is in } X(R)\}$ is an open covering of $X(R)$. Hence by the partition property, [13, pp. 12–13] there exists a finite number of open and closed, pairwise disjoint subsets $N_1, \cdots, N_q$ of $X(R)$ such that $U_\lambda N_\lambda = X(R)$ and each N_λ is contained in some $U(x_\lambda)$. On each N_λ we may restrict the sections and then by the pairwise disjointness, we may piece these sections together on each clopen set N_λ to obtain global sections a_{ji} and x_j—where j runs through the largest of the $J(x_\lambda)$—and $\Sigma_i b_i a_{ji} = 0$ and $\Sigma_j a_{ji} x_j = k_i$ in K_n. Hence, K_n is flat.

Corollary 1.1.1. *If A is a finitely generated algebra over an absolutely flat ring R, then the following hold*:

(a) w gl dim A = sup gl dim A_x,

(b) w R-dim A = sup R_x-dim A_x.

If, in addition, A is R-projective,

(c) R-dim A = sup R_x-dim A_x,

where the supremum is taken over all x in $X(R)$.

Since each R_x is a field, we may apply Proposition 1.1 and its corollary together with the techniques of [15, Theorem D, p. 78] to obtain

Proposition 1.2. *Let A be a finitely generated algebra over an absolutely flat ring R. If A has finite Hochschild dimension, then*:

(a) w R-dim A = w gl dim A.

(b) w gl dim $A^e = 2$(w R-dim A).

(c) *If* A/N *is* R*-separable, then* w R-dim A = w $\mathrm{hd}_A(A/N)$.

(d) A *is separable over* R *if and only if* A^e *is absolutely flat.*

If A *is also* R*-projective, then* w R-dim A *may be replaced by* R-dim A.

The following proposition may be viewed as an extension of [14, Theorem 2.4, p. 130] and [1, Proposition 14, p. 75].

Proposition 1.3. *Let* A *be a finitely generated, commutative algebra over an absolutely flat ring* R. R-dim $A = 0$ *or* R-dim $A = \infty$.

Proof. Suppose R-dim $A = n < \infty$. Then R_x-dim $A_x \leq n < \infty$ for each x in $X(R)$. But A_x is a commutative algebra of finite Hochschild dimension over a field. Therefore by [1, Proposition 14, p. 75], R_x-dim $A_x = 0$ for each x in $X(R)$. Therefore w R-dim $A = 0$. By [9, Proposition 1.1, p. 233], since A is finitely generated, R-dim $A = 0$ if and only if w R-dim $A = 0$.

The sheaf-theoretic technique also enables us to obtain a condition under which R-dim A = w R-dim A in the case of $n = 1$ and thus to answer partially the conjecture of [14, p. 134].

Proposition 1.4. *Let* A *be a finitely generated algebra over an absolutely flat ring* R. R-dim $A = 1$ *if and only if* w R-dim $A = 1$.

Proof. In case R-dim $A = 1$, that w R-dim $A = 1$ is an easy consequence of [9, Proposition 1.1, p. 233]. Thus suppose that w R-dim $A = 1$. Then using the canonical sequence $0 \to J \to A^e \to A \to 0$ of [8, pp. 39–40] and Proposition 1.1, we have that J_x is a projective left ideal of A^e_x for each x in $X(R)$. If we let $x_1, \cdots, x_n$ be the A^e-generators of J, then there are elements f_{iy} in $\mathrm{Hom}_{A^e_y}(J_y, A^e_y)$ such that $\Sigma_i f_{iy}(a_y)(x_i)_y = a_y$ for each a_y in J_y and for each y in $X(R)$. Therefore, for every y in $X(R)$ there are elements $b_{ij}(y)$ in A^e such that $b_{ij}(y)_y = f_{iy}(x_j)_y$. Consider the following sections, where $a_1, \cdots, a_k$ are the R-generators of A^e: $\Sigma_j\, b_{ij}(y)x_j - x_i$ and $\Sigma_j\, a_k b_{ij}(y)x_j - a_k x_i$ for all i and k. We may find a neighborhood of y, $N(y)$ such that each of these sections is zero for each z in $N(y)$. Using the partition property we may find a finite number of clopen neighborhoods $N^*(y_\lambda)$ each contained in $N(y_\lambda)$ covering $X(R)$. Piecing the sections together, we obtain global sections so that we have b_{ij} in A^e such that (1) $\Sigma_j\, b_{ij}x_j = x_i$ and (2) $\Sigma_j\, a_k b_{ij}x_j = a_k x_i$. We may now define f_j in $\mathrm{Hom}_{A^e}(J, A^e)$ by $f_j(x_i) = b_{ij}$. The x_i and the f_j form a dual basis of J over A^e. Hence J is projective as an A^e-module. By [9, Proposition 1.1, p. 233] we have that R-dim $A \neq 0$ and hence R-dim $A = 1$.

The final proposition of this section is a simple consequence of 1.1.1 and [7, Proposition 3, p. 78].

Proposition 1.5. *Let A be a finitely generated R-algebra such that $Z(A) =$ the center of A is a finitely generated R-algebra, where R is absolutely flat; suppose further that A is finitely generated and projective over $Z(A)$.*

(a) *R-dim $A = \infty$ if and only if R-dim $Z(A) = \infty$ or $Z(A)$-dim $A = \infty$.*

(b) *If R-dim A is finite, then R-dim $Z(A) = 0$ and R-dim $A = Z(A)$-dim A.*

2. Separable algebras over an absolutely flat ring.

The structure theory of separable algebras over an absolutely flat ring closely parallels that of the structure theory of central simple algebras over a field. We will show that every biregular ring A which is finitely generated over its center is central separable; and conversely that every central separable algebra over an absolutely flat ring is biregular. (Recall that a biregular ring is a ring in which every principal ideal is generated by a central idempotent (cf. [5, p. 143], [13, p. 43]).)

We begin with a simple structural result and a lemma on the relation between the Jacobson radical of the algebra and the Jacobson radical of homomorphic images modulo x's in $X(R)$.

Proposition 2.1. *A finitely generated, separable algebra A over an absolutely flat ring R is an absolutely flat, semiprime ring in which every prime ideal is maximal.*

Proof. According to [12, Exercise 12, p. 63], if R is a commutative absolutely flat ring, then every prime ideal is maximal. According to the Morita theory [8, pp. 54–55] there is a one-to-one correspondence between the two-sided ideals of A and the ideals of R.

The author wishes to thank Professor E. C. Ingraham who pointed out to him the proof of the nilpotence of the Jacobson radical which appears in the following proposition.

Proposition 2.2. *Let A be a finitely generated R-algebra. Then $J(A) = N$ is nilpotent and finitely generated and for each x in $X(R)$, $J(A_x) = N_x$.*

Proof. Whenever an algebra is generated by n elements over a field F, the nth power of the Jacobson radical is zero. For every maximal ideal m of an absolutely flat ring R, $R_m = R/m$ [2, Exercise 9, p. 168] whence $[J(A)^n]_m = [J(A)_m]^n = 0$ for all maximal ideals m of R. Therefore $J(A)^n = 0$.

Since $J(A)$ is nilpotent, one may use a standard sheaf-theoretic argument to obtain that $J(A)$ is finitely generated as an R-module.

To show that $N_x = J(A_x)$, we need only show that $J(A_x) \subseteq N_x$ for each x in

$X(R)$. Let a_y be in $J(A_y)$. We will show that there is a preimage a in A which generates a nilpotent ideal in A and therefore is contained in N. We know that there must be an $a(y)$ in A such that $a(y)_y = a_y$. If $a_1, \cdots, a_n$ denote the generators of A over R, then there is an integer $t(y)$ such that the product of $t(y)$ elements of the form $(a_i)_y a_y$ and/or $a_y(a_i)_y$ is zero in A_y. Hence there is a neighborhood of y such that the product of $t(y)$ elements of the form $(a_i)_x a(y)_x$ and/or $a(y)_x(a_i)_x$ is zero in A_x for each x in the neighborhood. Therefore, we may find a clopen set $U(y) = U$ such that the above equations hold. If we set $X - U = V$, there exists an a in A such that $a_x = 0$ for every x in V and $a_x = a(y)_x$ for each x in U. Hence a generates a nilpotent ideal in A and so a is in N.

Theorem 2.3. *Let A be an algebra over the absolutely flat ring R with center of A being R. A is central separable over R if and only if A is a biregular ring which is finitely generated over R.*

Proof. Let A be a biregular ring which is finitely generated over its center. By [5, 3.4, p. 157], we know that A_x is central simple for every x in $X(R)$ and hence A_x is central separable [8, Proposition 1.2, p. 132]. Therefore, by Proposition 1.1, w R-dim $A = 0$ and so by [9, Proposition 1.1, p. 234] it follows that A is R-separable.

Conversely, if A is central separable over R, A_x is central simple for every x in $X(R)$. Therefore, by [5, 3.4, p. 157], A is biregular. That A is finitely generated over R follows from [8, Proposition 2.1, p. 47].

Corollary 2.3.1. *A finitely generated R-algebra A which is biregular is inseparable over R if and only if the center of A is inseparable over R. If A is inseparable over R, then* R-dim $A = \infty$.

Proof. The corollary is an immediate consequence of 2.3 and Propositions 1.3 and 1.5.

The lemma which preceded Theorem 2.3 enables us to prove the absolutely flat ring analogue of a theorem of S. Eilenberg which gives both necessary and sufficient conditions for an algebra to have a given finite Hochschild dimension in terms of the separability of the algebra modulo its Jacobson radical [6, Corollary 2 to Theorem 3, p. 311]. The only other situation providing for the full generalization is for projective algebras over a regular local ring [15, Theorem A, p. 76]. See also [16, Theorem 5, p. 73].

Theorem 2.4. *Let A be a finitely generated algebra of finite Hochschild dimension over an absolutely flat ring R.*

w R-dim $A = n$ *if and only if A/N is R-separable and* w $\mathrm{hd}_A(A/N) = n$.

If, in addition, A *is also* R*-projective,*

R-dim $A = n$ *if and only if* A/N *is* R*-separable and* $\mathrm{hd}_A(A/N) = n$.

Proof. By Proposition 1.1, since w R-dim A is finite, it follows that R_x-dim A_x = w R_x-dim A_x is finite for each x in $X(R)$. Then by the theorem of Eilenberg cited above, we have that R_x-dim $A_x/J(A_x) = 0$. By Proposition 2.2, it follows that R_x-dim $(A/N)_x = 0$ for each x in $X(R)$ and also that $\mathrm{hd}_{A_x}(A/N)_x = R_x$-dim A_x. Thus by Proposition 1.1, we have the result.

In case A is also projective over R, we may additionally use the argument of [14, Theorem 2.1, p. 129] to guarantee that each element of a flat resolution as an A- (A^e)-module can be chosen to be finitely presented over A (A^e) and hence that the weak and projective dimensions are equal.

3. The structure of algebras of Hochschild dimension one. An algebra B is said to be a *maximal algebra* for A if A is an epimorphic image of B, R-dim $B \leq 1$, and $B/J(B)^2 = A/J(A)^2$. It is the purpose of this section to determine the structure of algebras all of whose homomorphic images have finite Hochschild dimension. This can be done by showing that every such algebra possesses a maximal algebra.

We shall say that an idempotent e of a finitely generated algebra A over an absolutely flat ring R is *primitive for each* x *in* $X(R)$ if for each x in $X(R)$ either e_x is a primitive idempotent in A_x or $e_x = 0$ in A_x.

Proposition 3.1. *Let* A *be a finitely generated algebra over an absolutely flat ring* R. *Let* R-dim $A \leq 1$. *Then* A *contains a finite set of mutually orthogonal idempotents* $e_1, \cdots, e_n$ *indexed so that:*

(a) $1 = e_1 + \cdots + e_n$,

(b) $e_i N e_j = 0$ *whenever* $1 \geq j$,

(c) $e_i A e_j = 0$ *whenever* $i > j$.

(d) e_i *is primitive for each* x *in* $X(R)$ *and for each* i.

Proof. Let x be in $X(R)$. Then by [4, Theorem 4.1, p. 21] there exists a complete set of mutually orthogonal primitive idempotents $e_{1x}, \cdots, e_{r(x)x}$ indexed so that the conditions (a) through (d) hold in A_x. Now by Theorem 2.4, A/N is separable and so by [3, Theorem 1, p. 370] we may write $A = S \oplus N$ where S is a separable subalgebra of A. We may now use these facts to write the conditions (a) through (d) in the form of sections. If $f_i(x)$ denote preimages of the e_{ix}, then we have the sections: $f_i f_j - \delta_{ij} f_j$, $\Sigma_i f_i - 1$, $f_i n_k f_j$ for $i \geq j$, $f_i s_k f_j$ for $i > j$

for all k, where δ_{ij} is the Kronecker delta, n_h generate N over R, and s_k generate S over R.

We may apply the arguments of Proposition 2.1 to obtain global sections. Thus the desired idempotents may be obtained.

The argument of the theorem may in general be applied to any finitely generated algebra to obtain idempotents satisfying (a) and (d) of the theorem.

A set of idempotents satisfying conditions (a) and (d) of Theorem 3.1 is said to be refined by $f_1, \cdots, f_r$ if the f_i satisfy the conditions (a) and (d) of Theorem 3.1 and for some idempotent e in the original set $f_j e = e f_j = f_j$ while $f_i e' = e' f_i = 0$ for every other e' in the original set.

In view of Theorem 3.1, we define an algebra A over an absolutely flat ring R to be *triangular* if (a) A/N is biregular and (b) every finite set of mutually orthogonal idempotents $e_1, \cdots, e_n$ such that $1 = e_1 + \cdots + e_n$ and e_i is primitive for each x in $X(R)$ and for each i can be refined and/or re-indexed to a set of idempotents $f_1, \cdots, f_t$ such that $f_i N f_j = 0$ whenever $i \geq j$.

Proposition 3.2. *Let e and f be idempotents in a finitely generated algebra A over an absolutely flat ring R. $e \sim f$ if and only if $e_x \sim f_x$ for every x in $X(R)$. ($e \sim f$ denotes that Ae is isomorphic to Af.)*

Proof. If $e \sim f$ under the map ϕ, then $\phi_x = \phi \otimes 1_x$ is also an isomorphism since $\otimes_R R_x$ is exact for each x in $X(R)$. Conversely, suppose $e_x \sim f_x$ for every x in $X(R)$. By [10, Proposition 4, p. 51] there are elements g_x and h_x such that $f_x g_x e_x = g_x$; $g_x h_x = f_x$; $e_x h_x f_x = h_x$; and $h_x g_x = e_x$. Applying the sectional arguments, we obtain elements $g_{f,e}$ and $h_{e,f}$ in A such that

$$f g_{f,e} e = g_{f,e}; \quad g_{f,e} h_{e,f} = f; \quad e h_{e,f} f = h_{e,f}; \quad h_{e,f} g_{f,e} = e.$$

By the converse of [10, Proposition 4, p. 51], $e \sim f$.

If A and B are two algebras over R and M is a left A- and a right B-bimodule (denoted: (A, B)-bimodule), we define as in [4, p. 16] the R-algebra $\mathfrak{T}(A, B, M)$ to be the ring whose elements are $\left(\begin{smallmatrix} a & m \\ 0 & b \end{smallmatrix}\right)$ where a is in A, b is in B, and m is in M. Addition is componentwise and multiplication is defined by

$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \begin{pmatrix} a' & m' \\ 0 & b' \end{pmatrix} = \begin{pmatrix} aa' & am' + mb' \\ 0 & bb' \end{pmatrix}.$$

Proposition 3.3. *Let R be an absolutely flat commutative ring, A' be a finitely generated R-algebra with Jacobson radical $J(A')$, S a finitely generated, separable R-algebra and M a finitely generated (A', S)-bimodule. Then $A = \mathfrak{T}(A', S, M)$ is a finitely generated R-algebra with $J(A) = \{\left(\begin{smallmatrix} a' & m \\ 0 & 0 \end{smallmatrix}\right)$: a' is in A', m in $M\}$ as its Jacobson radical. If* w R-dim $A' < \infty$, *then*

$$\text{w } R\text{-dim } A = \max\{\text{w } R\text{-dim } A', 1 + \text{w hd}_{A'}(M)\} \leq \text{w } R\text{-dim } A' + 1.$$

Proof. We may apply Proposition 1.1, Corollary 1.1.1, and 1.2(a) together with [4, Lemma 4.1, p. 19] to obtain the following string of equalities where the supremum is taken over all x in $X(R)$:

$$\begin{aligned} \text{w } R\text{-dim } A &= \sup\{R_x\text{-dim } \mathfrak{T}(A'_x, S_x, M_x)\} = \sup\{\text{gl dim } A_x\} \\ &= \sup\{\max(\text{gl dim } A'_x, 1 + \text{pd}_{A_x}(M_x))\} \\ &= \max(\text{w } R\text{-dim } A', 1 + \text{w pd}_A(M)) \\ &\leq 1 + \text{w } R\text{-dim } A'. \end{aligned}$$

If f': $A' \to A'/J(A')$ is the canonical map, define $f\{\begin{pmatrix} a' & m \\ 0 & s \end{pmatrix}\} = (f'(a'), s)$. Clearly, f is an epimorphism from A to $A'/J(A') \oplus S$ with kernel $J(A)$. Clearly, $J(A)$ contains the Jacobson radical N of A since the image is semiprimitive. Equality will hold if we show that for every element x of $J(A)$ and every a in A, $1 - ax$ is left invertible.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} a & m_1 \\ 0 & s \end{pmatrix}\begin{pmatrix} n & m \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 - an & -am \\ 0 & 1 \end{pmatrix}$$

has left inverse $\begin{pmatrix} u & uam \\ 0 & 1 \end{pmatrix}$ where u is the left inverse of $1 - an$ in A'.

Proposition 3.4. *Let A be a finitely generated algebra over an absolutely flat ring R such that A/N is separable. Suppose $e_1, \cdots, e_n$ are mutually orthogonal idempotents such that e_i is primitive for every x in $X(R)$ and $e_1 + \cdots + e_n = 1$. Suppose there is an $s < n$ such that $e_iN = 0$ for $i > s$ and $e_jN \neq 0$ for $j \leq s$. Then there is a refinement f_j, $j = 1, \cdots, n^*$ such that if*

$$e = f_{s+1} + \cdots + f_{n^*}, \quad e' = 1 - e, \quad A' = e'Ae', \quad S = eAe, \quad \text{and} \quad M = e'Ae,$$

then $eAe' = 0$ and $A = \mathfrak{T}(A', S, M)$.

Proof. Suppose there is an x such that $(e_i)_x$ is isomorphic to $(e_j)_x$ while $e_iN = 0$ and $e_jN \neq 0$. Then by the argument of Proposition 3.2, we have that $U = \{x: (e_i)_x$ is isomorphic to $(e_j)_x\}$ is an open set. On the other hand, since A/N is separable, one may readily verify that

$$\begin{aligned} X(R) - U &= \{x: (e_i)_x \text{ is not isomorphic to } (e_j)_x\} \\ &= \{x: (e_i)_x(a_k)_x(e_j)_x = 0 \text{ for each } k = 1, \cdots, v\} \end{aligned}$$

where the a_k are the R-generators of A.

Hence $X - U$ is also an open set. Therefore U is a clopen set. There is an r_i in R such that $(r_i)_x = 0_x$ for each x in U and $(r_i)_x = 1_x$ for each x in $X - U$. For each fixed j, carry out this procedure for each $i > s$, yielding a set of idempotents $r_{s+1}, \cdots, r_n$ of R. Set $f_{1j} = re$ and $f_{2j} = (1 - r)e$ where r is the product of the r_i. Then $(e_i)_x A_x r_x (e_j)_x = 0_x$ for every x in $X(R)$. Moreover, $f_{2j}N = 0$ and $f_{1j}N \neq 0$.

Performing the above procedure for $j = 1$, we reorder as follows: $f_{11}, e_2, \cdots, e_x$; $f_{21}, e_{s+1}, \cdots, e_n$. Perform the operation now, using the reordered set, for $j = 2, \cdots, s$, to obtain a final refinement:

$$f_{11}, f_{12}, \cdots, f_{1s}; \quad f_{21}, \cdots, f_{2s}, e_{s+1}, \cdots, e_n.$$

Then for each idempotent f_a among the f_{1j} and for each f_b among the remaining idempotents in the set, $f_b A f_a = 0$. Now an application of [4, Theorem 2.3, p. 16] completes the proof.

Theorem 3.5. *Let A be a finitely generated algebra over an absolutely flat ring R. Suppose further that A/N is separable over R. The following are equivalent:*

(a) *A is triangular.*

(b) *There is a set of mutually orthogonal idempotents $f_1, \cdots, f_n$ which add to 1 and which are primitive for every x in $X(R)$ indexed so that $f_i N f_j = 0$ whenever $i \geq j$.*

(c) w *R-dim A/I is finite for every ideal I of A.*

(d) w *R-dim A/N^2 is finite.*

Proof. That (a) implies (b) is obvious. Assume that (b) is true. Since every homomorphic image of A contains a set of idempotents with the same property, it suffices to show that A has finite weak Hochschild dimension. Let r be the number of isomorphism classes of the f_j. In the case that $r = 1$, then $N = \sum_{i,j} f_i N f_j = 0$. Hence A is separable by hypothesis.

Assume the result is true if $t < r$. By hypothesis the set of idempotents satisfies the property that $e_i N e_j = 0$ whenever $i \geq j$. Therefore, $e_n N = \sum_j e_n N e_j = 0$. Therefore, we may assume that there is a $k < n$ such that $e_{k+1}N = \cdots = e_n N = 0$, but $e_i N \neq 0$ for $k \leq i$. By Proposition 3.4, we may assume that the idempotents have already been refined so that if $e = e_{k+1} + \cdots + e_n$, $e' = 1 - e$, then $eAe' = 0$, $S = eAe$ is separable and $A = \mathcal{T}(A', S, M)$ where $A' = e'Ae'$ and $M = e'Ae$. A' satisfies (b) and has fewer than r isomorphism classes and hence has finite Hochschild dimension. By 3.3, A also has finite Hochschild dimension. Again (c) implies (d) is obvious.

We must show that (d) implies (a). By the remark following Proposition 3.1, A contains a set of mutually orthogonal idempotents $e_1, \cdots, e_n$ whose sum is 1 and such that e_i is primitive for every x in $X(R)$ and for every i.

It follows from [11, Lemma 1, p. 68] that if $N_x e_x \neq 0$, $e_x N_x f_x \neq 0$, then $N_x f_x = \oplus \Sigma\, S_x(e_j)_x$ for some e_j's with repeats being allowed. Therefore $e_x N_x f_x = \oplus\, \Sigma\, e_x S_x (e_j)_x \neq 0$. Therefore $e_x = (e_j)_x$ for some x. This gives rise to the equation:

$$\mathrm{pd}_{A_x} N_x f_x \geq \mathrm{pd}_{A_x} S e_x = 1 + \mathrm{pd}_{A_x} N_x e_x > \mathrm{pd}_{A_x} N_x e_x \geq 0.$$

If $N_x e_x = 0$, then $\mathrm{pd}_{A_x} N_x f_x \geq \mathrm{pd}_{A_x} N_x e_x = -1$. Thus

$$\mathrm{w\ pd}_A\, Nf \geq \mathrm{w\ pd}_A\, Se = 1 + \mathrm{w\ pd}_A\, Ne > \mathrm{w\ pd}_A\, Ne \geq 0.$$

Therefore, $\mathrm{w\ pd}_A\, Sf > \mathrm{w\ pd}_A\, Se$ provided $eNf \neq 0$. Order the idempotents so that $\mathrm{hd}_A\, Se_i \geq \mathrm{hd}_A\, Se_j$ if $i \geq j$. Then $e_i N e_j = 0$ whenever $i \geq j$. The proof now follows by an induction on the nilpotence degree of N. Assume that $e_i N e_j \subset N^s$ whenever $i \geq j$. Then $e_i N e_j = e_i N^s e_j$ if $i \geq j$; and so

$$e_i N e_j = e_i N^{s-1} \sum_{k=1}^{n} e_k \;\; N e_j = \sum_{k=1}^{n} (e_i N^{s-1} e_k)(e_k N e_j).$$

If $i \geq k$, then $e_i N^{s-1} e_k \subseteq N^s$ and if $k > i$, then $k > j$ and so $e_k N e_j \subseteq N^2$. Hence $e_i N e_j \subseteq N^{s+1}$. Hence $e_i N e_j = 0$. Therefore A is triangular.

Theorem 3.6. *Let A be a finitely generated, triangular algebra over an absolutely flat ring R with A/N separable over R. There is a maximal algebra B for A.*

Proof. We may apply the arguments of 1.1 again to show that $N = N^2 \oplus P$ where P is an S-submodule of N. Define $B = S \oplus P \oplus P^{(2)} \oplus \cdots$ where $P^{(n)}$ is the n-fold tensor product of P with itself over S. Using the middle-four interchange, one may readily verify that B is finitely generated over R. But since $B_x = S_x \oplus P_x \oplus P_x^{(2)} \oplus \cdots$ is the maximal algebra for A_x, we have by 1.4 that $R\text{-dim}\, B \leq 1$.

If we replace $\mathrm{w}\ R\text{-dim}\, A$ by $\mathrm{w\ gl\ dim}\, A$ in 3.3 through 3.5, the results remain true with only the assumption that A/N is biregular. However 3.6 does require the separability of A/N in order to obtain the one-dimensional overring (cf. [4, Remark 2, p. 22]). The results using $\mathrm{w}\ R\text{-dim}\, A$ and $\mathrm{w\ gl\ dim}\, A$ are equivalent in case the absolutely flat ring R is perfect, i.e., R_x is a perfect field for each x in $X(R)$.

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