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ON THE ZEROS OF DIRICHLET L-FUNCTIONS. I

BY

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ABSTRACT. A mean value theorem for $\arg L(\frac{1}{2} + i(t+h), \chi) - \arg L(\frac{1}{2} + it, \chi)$ is established. This yields mean estimates for the number of zeros of $L(s,\chi)$ in small boxes.

1. Statement of results. As is customary we let $S(t, \chi) = \pi^{-1} \arg L(\frac{1}{2} + it, \chi)$, where the argument is determined by continuous variation on the half-line $\sigma + it$, $\sigma \geq \frac{1}{2}$. If t is the ordinate of a zero then we put

$$S(t, \chi) = \tfrac{1}{2}(S(t+0, \chi) + S(t-0, \chi)).$$

We assume throughout that χ is a primitive character mod q. The zeta function arises from the principal character with $q = 1$; thus $\zeta(s) = L(s, \chi_0)$, and we write $S(t) = S(t, \chi_0)$.

In 1944 A. Selberg [2] demonstrated that

$$\int_T^{T+H} S(t)^{2k}\, dt = c_k H(\log\log T)^k + O_k(H(\log\log T)^{k-1/2})$$

provided that $T^{1/2+a} \leq H \leq T$, $0 < a \leq \frac{1}{2}$, where

$$c_k = \frac{(2k)!}{(2\pi)^{2k} k!}. \tag{1}$$

We modify A. Selberg's approach to obtain our

Main Theorem. *Let a_1, a_2 be fixed, $0 < a_i \leq \frac{1}{2}$ for $i = 1, 2$. Let χ be a primitive character* (mod q), *and suppose that $q \leq T^{1/4-a_1}$, $T^{1/2+a_2} \leq H \leq T$, and that $0 < h \leq (H - (H/\sqrt{T})^{1/8})$. Then*

$$\int_T^{T+H} (S(t+h, \chi) - S(t, \chi))^{2k}\, dt = c_k H(2\log(2 + h\log T))^k \cdot (1 + O((Ak)^{3k}(\log(2 + h\log T))^{-1/2})),$$

where $A > 0$ is a suitable absolute constant, and c_k is given by (1).

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Let $N(T, \chi)$ denote the number of zeros $\rho = \beta + i\gamma$ of $L(s, \chi)$ with $0 < \beta < 1$, $0 \le \gamma \le T$, zeros on the boundary being counted with weight one-half. It is well known (see [3]) that if $T \ge 0$ then

$$(2) \qquad N(T, \chi) = \frac{T}{2\pi} \log \frac{qT}{2\pi} - \frac{T}{2\pi} - \frac{\chi(-1)}{8} + S(T, \chi) - S(0, \chi) + O\left(\frac{1}{T+1}\right).$$

Thus we easily derive the following

Corollary. *Under the above hypothesis,*

$$\int_T^{T+H} (N(t+h, \chi) - N(t, \chi))^{2k}\, dt$$

$$= H\left(\frac{h}{2\pi} \log qT\right)^{2k} \left(1 + O\left((Ak)^{4k} \frac{(\log(2 + h \log T))^{1/2}}{2 + h \log T}\right)\right)$$

unless $h \log T \to 0$ *as* $T \to \infty$,

$$= O((Ak)^{4k} H) \text{ if } h \log T \to 0 \text{ as } T \to \infty$$

where $A > 0$ *is a suitable absolute constant.*

We may remark here that implicit constants involved in O or $\ll$ in this paper do not depend on q, k and h unless it is written explicitly like O_k.

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2. Lemmas. The condition $q \le T^{1/4 - a_1}$ in the Main Theorem comes from

Lemma 1. *Let* a_1, a_2 *be fixed,* $0 < a_i \le ½$ *for* $i = 1, 2$. *Let* χ *be a primitive character* (mod q), *and suppose that* $q \le T^{1/4 - a_1}$, $T^{1/2 + a_2} \le H \le T$. *Then*

$$N_\chi(\sigma, T + H) - N_\chi(\sigma, T) = O(H(H/\sqrt{T})^{(1-2\sigma)/4} \log T)$$

uniformly for σ *in* $½ \le \sigma \le 1$, *where* $N_\chi(\sigma, T)$ *denotes the number of zeros* $\beta + i\gamma$ *of* $L(s, \chi)$ *with* $\sigma < \beta$ *and* $0 < \gamma < T$.

Proof. We will follow the argument of A. Selberg (cf. [2]) and we will state only the results in each step.

(i) We use A. F. Lavrik's approximate functional equation in the following form (cf. [1])

$$L(s, \chi) = \sum_{n \le x} \frac{\chi(n)}{n^s} + \epsilon(\chi) \left(\frac{q}{\pi}\right)^{1/2 - s} \frac{\Gamma(½(1 - s + a))}{\Gamma(½(s + a))} \sum_{n \le y} \frac{\overline{\chi(n)}}{n^{1-s}} + R_{xy},$$

where

$s = \sigma + it,\ 0 < \sigma < 1,\ a = \frac{1}{2}(1 - \chi(-1)),\ x = \Delta\sqrt{q|t|/2\pi},\ y = \frac{1}{\Delta}\sqrt{q|t|/2\pi},$

$\epsilon(\chi) = (-1)^a \frac{1}{\sqrt{q}} \sum_{n=1}^{q} \chi(n) e^{2\pi i(n/q)}$ and $R_{xy} \ll \sqrt{q}(y^{-\sigma} + x^{\sigma-1}(qt')^{1/2-\sigma}) \log^2|t|$

with $t' = \max(|t|, 1)$.

(ii) Using (i) if $T > 10$, $\sqrt{T} \le H \le T$, μ_1 and μ_2 are positive coprime integers less than $T/10$, we have for $\frac{1}{2} < \sigma \le 1$,

$$\int_T^{T+H} |L(\sigma + it, \chi)|^2 \left(\frac{\mu_1}{\mu_2}\right)^{it} dt$$

$$= \chi(\mu_1)\overline{\chi}(\mu_2) \int_T^{T+H} \{(\mu_1\mu_2)^{-\sigma}\zeta(2\sigma) + (\mu_1\mu_2)^{\sigma-1}\tau^{2-4\sigma}\zeta(2-2\sigma)\}\, dt$$

$$+ O((qT)^{1-\sigma}\mu_1\mu_2 \log^2(qT))$$

$$+ O\left(q^{1-\sigma}HT^{-2}\left(\left(\frac{\mu_1}{\mu_2}\right)^{1/2} + \left(\frac{\mu_2}{\mu_1}\right)^{1/2}\right)\log(qT) \cdot \log T\right),$$

where $\tau = \sqrt{qt/2\pi}$.

(iii) For $T^{1/2+a_2} \le H \le T$, $0 < a_2 \le \frac{1}{2}$, $\xi = (H/\sqrt{T})^{1/4}$, $\frac{1}{2} + 1/\log \xi \le \sigma \le 1$ and $q \le T^{1/4-a_1}$,

$$\int_T^{T+H} |L(\sigma + it, \chi)\psi(\sigma + it, \chi)|^2\, dt < H + O(H(H/\sqrt{T})^{(1-2\sigma)/4}),$$

where $\psi(s, \chi) = \sum_{\nu<\xi}(\lambda_\nu/\nu^s)\overline{\chi}(\nu)$ and λ_ν is defined as follows: for $\xi > 1$ and $\frac{1}{2} < \sigma \le 1$ and $1 \le \nu < \xi$

$$\lambda_\nu = \frac{\nu^{2\sigma}}{\sum_{\rho<\xi} \mu^2(\rho)/\phi_{2\sigma}(\rho)} \sum_{\rho<\xi/\nu} \frac{\mu(\rho\nu)\mu(\rho)}{\phi_{2\sigma}(\rho\nu)}$$

with $\phi_\gamma(\rho) = \rho^\gamma \sum_{d|\rho} \mu(d)/d^\gamma$ for a positive integer ρ and $\mu(d)$ is the Möbius function.

Proof. Same computation as A. Selberg (cf. [2]) leads to the following.

$$\int_T^{T+H} |L(\sigma + it, \chi)\psi(\sigma + it, \chi)|^2\, dt$$

$$< H + O(H\xi^{1-2\sigma}) + O((qT)^{1-\sigma}\xi^{4-2\sigma}\log^2(qT) \cdot \log^2 T)$$

$$+ O(q^{1-\sigma}HT^{-\sigma/2}\log(qT) \cdot \log^4 T)$$

$$< H + O(H(H/\sqrt{T})^{(1-2\sigma)/4}) + O(q^{1-\sigma}HT^{1/2-\sigma-(2a_1-1)/8} \cdot \log^2 T \cdot \log^2(qT))$$

$$+ O(q^{1-\sigma}HT^{1/8-\sigma/2}\log^4 T \cdot \log(qT)).$$

Hence if $q \leq T^{(1/4-a_1)}$ with $a_1 > 0$, the last two remainder terms become $O(H(H/\sqrt{T})^{(1-2\sigma)/4})$ and we get the conclusion.

(iv) Using Littlewood's lemma (cf. p. 106 in [2]) we get our conclusion. Q.E.D.

Let $a(p)$ be a real or complex valued function of prime numbers p. For simplicity we write

$$F_\alpha(x) = \sum_{p\leq x} \frac{|a(p)|^{2\alpha}}{p^\alpha}$$

for a real number α.

Lemma 2. *Assume that $F_\alpha(x) \ll 1$ for $\alpha \geq 2$. Then*

$$\sum_{p_i<x}^{*} \frac{a(p_1)a(p_2)\cdots a(p_k)\overline{a(p_{k+1})}\cdots\overline{a(p_{2k})}}{\sqrt{p_1p_2\cdots p_kp_{k+1}\cdots p_{2k}}}$$

$$= \begin{cases} k!F_1(x)^k + O(k!F_1(x)^{ov(k-2)}) & \text{if } F_1(x) \to \infty \text{ as } x \to \infty, \\ O(k!) & \text{if } F_1(x) \ll 1, \end{cases}$$

where in $\sum^*_{p_i<x}$ we sum over all primes $p_1, \cdots, p_{2k}$ satisfying $p_1 \cdots p_k = p_{k+1} \cdots p_{2k}$ and $p_i < x$ for $i = 1, 2, \cdots, 2k$, and $ov(k-2) = \max\{0, k-2\}$.

Proof. The expressions of k by the sum of n natural numbers can be ordered lexicographically when we identify two expressions up to permutations. Let $\sigma_k(n)$ be the number of such expressions. Let $\Phi(k-b, \nu_b)$ be the ν_bth expression $(c_1, c_2, \cdots, c_{k-b})$ of k by the sum of $k-b$ natural numbers; $k = c_1 + c_2 + \cdots + c_{k-b}$. Let

$$A_{k-b,\nu_b} = \sum_{p_i<x}{}' \frac{|a(p_1)|^{2c_1}|a(p_2)|^{2c_2}\cdots|a(p_{k-b})|^{2c_{k-b}}}{p_1^{c_1}\cdots p_{k-b}^{c_{k-b}}}$$

for $\nu_b = 1, 2, \cdots, \sigma_k(k-b)$, where in $\sum'_{p_i<x}$ we sum over all $p_1, \cdots, p_{k-b}$ with $p_i < x$ for $i = 1, \cdots, k-b$ whenever $p_1, \cdots, p_{k-b}$ are all different, and $(c_1, c_2, \cdots, c_{k-b}) = \Phi(k-b, \nu_b)$. Also let

$$\bar{A}_{k-b,\nu_b} = F_{c_1}(x)F_{c_2}(x)\cdots F_{c_{k-b}}(x)$$

$$= \left(\sum_{p<x} \frac{|a(p)|^{2c_1}}{p^{c_1}}\right)\cdots\left(\sum_{p<x} \frac{|a(p)|^{2c_{k-b}}}{p^{c_{k-b}}}\right).$$

Now

$$\sum_{p_i<x}^{*} \frac{a(p_1)a(p_2)\cdots a(p_k)\overline{a(p_{k+1})}\cdots\overline{a(p_{2k})}}{\sqrt{p_1p_2\cdots p_{2k}}} = \sum_{p_i<x}^{*} \frac{|a(p_1)|^2\cdots|a(p_k)|^2}{p_1\cdots p_k}$$

(1)

$$= \sum_{b=0}^{k-1} \sum_{\nu_b=1}^{\sigma_k(k-b)} (k-b)!A_{k-b,\nu_b}.$$

We remark that $\overline{A}_{k-b,\nu_b} = A_{k-b,\nu_b} + \sum_{b'=b+1}^{k-1} \sum_{\nu_{b'}=1}^{\sigma_k(k-b')} A_{k-b',\nu_{b'}}$. Substituting this into the above, we get

$$(1) = k!F_1(x)^k + \sum_{b=1}^{k-1} (b-k)\cdot(k-b)! \sum_{\nu_b=1}^{\sigma_k(k-b)} \overline{A}_{k-b,\nu_b}$$

since $\overline{A}_{k,\nu_0} = (\Sigma_{p<x}|a(p)|^2/p)^k = F_1(x)^k$. Now by the assumption if $F_1(x)\to\infty$ as $x\to\infty$

(2) $$\overline{A}_{k-b,\nu_b} \ll F_1(x)^{ov(k-2)} \quad \text{for } 1\le b\le k-1.$$

Hence it is enough to show that

(3). $$(b-k)\cdot(k-b)! \ll \sigma_k(k-b) \ll (k-1)!$$

for $b=1,2,\cdots,k-1$. Moreover it is enough to estimate this for sufficiently big k. We define $\sigma_k'(n)$ by

$$\frac{x^n}{(1-x)^n} = \sum_{k=n}^{\infty} \sigma_k'(n)x^n;$$

then $\sigma_k(n)\le\sigma_k'(n)$. Since

$$\frac{x^n}{(1-x)^n} = \sum_{k\ge n} \frac{(k-1)!}{(n-1)!(k-n)!}x^k,$$

$\sigma_k(n)\le\sigma_k'(n) = (k-1)!/(n-1)!(k-n)!$. Now (3) is clear for $b\ll 1$ or $k-b\ll 1$. In other cases since $(k-b)^2\le b!$ for sufficiently big k,

$$|(b-k)\cdot(k-b)!\sigma_k(k-b)| \le \frac{(k-1)!}{(k-b-1)!(k-k+b)!}(k-b)!\cdot(k-b)$$

$$\le (k-1)!(k-b)^2/b! \le (k-1)!.$$

Hence by (2) and (3) $(1) = k!F_1(x)^k + O(k!F_1(x)^{ov(k-2)})$. If $F_1(x)\ll 1$, $\overline{A}_{k-b,\nu_b} \ll 1$ for $0\le b\le k-1$. Hence we get our conclusion. Q.E.D.

Lemma 3. *Assume that* $F_\alpha(x) \ll 1$ *for* $\alpha \geq 2$ *and* $F_{1/2}(x) \ll x^c$ *with some positive* c. *Then for* $x = T^{(a_2+\frac{1}{2})/2k(c+1)}$

$$\int_T^{T+H} \left| \operatorname{Im} \sum_{p<x} \frac{a(p)}{p^{1/2+it}} \right|^{2k} dt$$

$$= \begin{cases} \dfrac{2k!}{2^{2k}k!} HF_1(x)^k + O\left(\dfrac{2k!}{2^{2k}k!} HF_1(x)^{ov(k-2)}\right) & \text{if } F_1(x) \to \infty \text{ as } x \to \infty, \\ O(k^k H) & \text{if } F_1(x) \ll 1, \\ O(k^k A^{2k} H) & \text{always,} \end{cases}$$

where A *satisfies* $|a(p)| \leq A \log p/\log x$ *for any* $p \leq x$.

Proof (*First case*). We write

$$\tau = \sum_{p<x} \frac{a(p)}{p^{1/2+it}},$$

and

$$\operatorname{Im} \sum_{p<x} \frac{a(p)}{p^{1/2+it}} = \frac{\tau - \bar{\tau}}{2i}$$

and

$$\left| \operatorname{Im} \sum_{p<x} \frac{a(p)}{p^{1/2+it}} \right|^{2k} = \frac{1}{2^{2k}} \sum_{b=0}^{2k} (-1)^{k-b} \binom{2k}{b} \cdot \int_T^{T+H} \tau^b \bar{\tau}^{2k-b}\, dt$$

$$= \frac{1}{2^{2k}} \sum_{b=0}^{2k} (-1)^{k-b} \binom{2k}{b} \sum_{p<x} \frac{a(p_1) \cdots a(p_b)\overline{a(p_{b+1})} \cdots \overline{a(p_{2k})}}{\sqrt{p_1 p_2 \cdots p_{2k}}} \int_T^{T+H} \left(\frac{p_{b+1} \cdots p_{2k}}{p_1 \cdots p_b} \right)^{it} dt.$$

If $p_{b+1} \cdots p_{2k} \neq p_1 \cdots p_b$,

$$\int_T^{T+H} \left(\frac{p_{b+1} \cdots p_{2k}}{p_1 \cdots p_b} \right)^{it} dt = O\left(1 \middle/ \left| \log \frac{p_{b+1} \cdots p_{2k}}{p_1 \cdots p_b} \right| \right) = O(x^{2k}).$$

Hence for $b \neq k$,

$$\int_T^{T+H} \tau^b \bar{\tau}^{2k-b}\, dt = O(x^{2k} F_{1/2}(x)^{2k}) = O(x^{2k(1+c)}) = O(H).$$

If $b = k$, we get by Lemma 1

$$\int_T^{T+H} |\tau|^{2k}\, dt = H \sum_{p_i<x}^{*} \frac{a(p_1) \cdots a(p_k)\overline{a(p_{k+1})} \cdots \overline{a(p_{2k})}}{p_1 \cdots p_k} + O(H)$$

$$= k! HF_1(x)^k + O(Hk! F(x)^{ov(k-2)}),$$

where the meaning of $\Sigma^*_{p_i<x}$ is the same as in Lemma 1. Hence

$$\int_T^{T+H}\left|\operatorname{Im}\sum_{p<x}\frac{a(p)}{p^{1/2+it}}\right|^{2k}dt=\frac{1}{2^{2k}}\binom{2k}{k}Hk!F_1(x)^k$$
$$+O\left(\frac{1}{2^{2k}}\binom{2k}{k}Hk!F_1(x)^{o\nu(k-2)}\right)$$
$$+O\left(\frac{1}{2^{2k}}\sum_{b=0;b\neq k}^{2k}\binom{2k}{b}H\right)$$
$$=\frac{2k!}{2^{2k}k!}HF_1(x)+O\left(\frac{2k!}{2^{2k}k!}HF_1(x)^{o\nu(k-2)}\right).$$

(*Other cases*). For $b\neq k$, $\int_T^{T+H} r^b\bar{r}^{2k-b}\,dt=O(H)$ as above. For $b=k$,

$$\int_T^{T+H}|r|^{2k}\,dt=H\sum_{p_i<x}{}^*\frac{a(p_1)\cdots a(p_k)\overline{a(p_{k+1})}\cdots\overline{a(p_{2k})}}{p_1\cdots p_k}+O(H)=O(k!H)$$

by Lemma 1. Hence we get the second part. The third part comes from Lemma 4 below. Q.E.D.

For $x\geq 2$ we define (see p. 47 of [3]) a number $\sigma_\chi(t)$ by

$$\sigma_\chi(t)\equiv\sigma_\chi(x,\ t)=\tfrac12+2\operatorname*{Max}_{\rho}(\beta-\tfrac12,\ 2/\log x),$$

where $\rho=\beta+i\gamma$ runs through the zeros of $L(s,\chi)$ for which $|t-\gamma|\leq x^{3(\beta-1/2)}/\log x$. Now modifying A. Selberg's arguments (i.e., Lemma 12 on p. 121 of [2] and Lemma 13 on p. 123 of [2]) we get the following two lemmas. We omit these proofs. We use Lemma 1 to prove Lemma 4. The condition on b in the Main Theorem comes from Lemma 4.

Lemma 4. *Under the same hypothesis as Lemma* 1, *for* ξ *satisfying* $1\leq\xi\leq x^{8k}$ *and* $x^3\xi^2\leq(H/\sqrt{T})^{1/4}$, *for* ν *in* $0\leq\nu\leq 8k$, *and for* $b=b(T)\leq H-(H/\sqrt{T})^{1/8}$,

$$\int_T^{T+H}(\sigma_\chi(t+b)-\tfrac12)^\nu\xi^{\sigma_\chi(t+b)-1/2}\,dt\ll\frac{HA_1^k\nu!}{(\log x)^\nu},$$

where A_1 *is some positive absolute constant.*

Lemma 5. *Let* $H>1$, $1<y\leq H^{1/k}$ *and* $|a(p)|<A\log p/\log y$ *for* $p<y$ *and* $|a'(p)|\leq B$ *for* $p<y$. *Then*

$$\int_0^H\left|\sum_{p<y}\frac{a(p)}{p^{1/2+it}}\right|^{2k}dt=O((A^2k)^kA_1^kH)$$

and

$$\int_0^H \left| \sum_{p<y} \frac{a'(p)}{p^{1+2it}} \right|^{2k} dt = O((B^2k)^k A_1^k H),$$

where A_1's are some positive absolute constants.

3. **Proof of theorem.** We need the following expression for $x \geq 2$ due to A. Selberg (cf. [3]):

$$S(t, \chi) - \frac{1}{\pi} \operatorname{Im} \sum_{p<x^3} \frac{\chi(p)}{p^{1/2+it}} = \sum_{i=1}^{5} O(B_i(t)),$$

where

$$B_1(t) = \left| \sum_{p<x^3} \frac{\Lambda(p) - \Lambda_x(p)}{p^{1/2+it} \log p} \cdot \chi(p) \right|, \qquad B_2(t) = (\sigma_\chi(t) - ½) \log x,$$

$$B_3(t) = (\sigma_\chi(t) - ½) \log(q(1 + |t|)), \qquad B_4(t) = \left| \sum_{p<x^{3/2}} \frac{\Lambda_x(p^2)\chi(p^2)}{p^{1+2it} \log p} \right|,$$

$$B_5(t) = (\sigma_\chi(t) - ½) x^{\sigma_\chi(t)-1/2} \int_{1/2}^{\infty} x^{1/2-\sigma} \left| \sum_{p<x^3} \frac{\Lambda_x(p) \log(xp)}{p^{\sigma+it}} \chi(p) \right| d\sigma$$

and

$$\Lambda_x(n) = \Lambda(n) \qquad \text{if } 1 \leq n \leq x,$$

$$= \Lambda(n) \frac{(\log x^3/n)^2 - 2(\log x^2/n)^2}{2(\log x)^2} \qquad \text{if } x \leq n \leq x^2,$$

$$= \Lambda(n) \frac{(\log x^3/n)^2}{2(\log x)^2} \qquad \text{if } x^2 \leq n \leq x^3.$$

For simplicity we write

$$\|f\| = \left(\int_T^{T+H} |f(t)|^{2k} dt \right)^{1/2k}.$$

We use the above expression for $x = T^{a_2/60k}$.

Now

$$A_{x^3}(t) \overset{\text{def}}{=} S(t+h, \chi) - S(t, \chi) - f_{x^3}(t) = \sum_{i=1}^{5} O(B_i(t+h)) + \sum_{i=1}^{5} O(B_i(t)),$$

where $f_{x^3}(t) = \pi^{-1} \operatorname{Im} \sum_{p<x^3} a(p)/p^{(1/2+it)}$ with $a(p) = \chi(p)(e^{-ih \log p} - 1)$. By Lemma 5 $\|B_i(t+h)\| \ll k^{1/2} H^{1/2k}$ and $\|B_i(t)\| \ll k^{1/2} H^{1/2k}$ for $i = 1$ and 4. By Lemma 4 with $\xi = 1$ and by the assumption on q, $\|B_i(t+h)\| \ll k^{1/2} H^{1/2k}$ and $\|B_i(t)\| \ll k^2 H^{1/2k}$ for $i = 2$ and 3. By Lemmas 4 and 5 as on p. 126 of [2],

$$\|B_5(t+h)\| \ll k^{3/2} H^{1/2k} \quad \text{and} \quad \|B_5(t)\| \ll k^{3/2} H^{1/2k}.$$

Hence for $x = T^{a_2/60k}$, $\|A_{x^3}(t)\| \ll k^2 H^{1/2k}$. For $T^{a_2/20k} \le x \le H^{1/k}$,

$$\|A_x(t)\| \le \|A_{T^{a_2/20k}}(t)\| + \left\| \sum_{T^{a_2/20k} \le p \le x} \frac{a(p)}{p^{1/2+it}} \right\|$$

$$\ll k^2 H^{1/2k} + k^{1/2} H^{1/2k} \ll k^2 H^{1/2k} \quad \text{by Lemma 5.}$$

Hereafter we take $x = T^{(a_2+1/2)/2k(c+1)}$, where c is some nonnegative constant satisfying $F_{1/2}(x) = \sum_{p<x} a(p)/\sqrt{p} \ll x^c$.

Now in using Lemma 3 in our case

$$F_1(x) = \sum_{p<x} \frac{|e^{-ih \log p} - 1|^2}{p} = 2 \sum_{p<x} \frac{1 - \cos(h \log p)}{p}$$

$$= 2 \sum_{p<x} \frac{1 - \cos(h \log p)}{\log p} \frac{\log p}{p} = 2 \int_1^x \frac{1 - \cos(h \log \xi)}{\log \xi} d(\log \xi + O(1))$$

since $\sum_{p<\xi} \log p/p = \log \xi + O(1)$. Hence

$$F_1(x) = 2 \int_0^{h \log x} \frac{1 - \cos u}{u} du + O(1) = 2 \log(h \log x) + O(1).$$

Hence if $h \log T \to \infty$ as $T \to \infty$, then $F_1(x) \to \infty$ and by Lemma 3

$$\|f_x(t)\|^{2k} = c(k) H F_1(x)^k + O(c(k) H F_1(x)^{0 \vee (k-2)}).$$

Hence

$$\|S(t+h, \chi) - S(t, \chi)\|^{2k} = \|A_x(t) + f_x(t)\|^{2k}$$

$$= c(k) H (2 \log(h \log T))^k + O((Ak)^{4k} H (\log(h \log T))^{k-1/2})$$

if $h \log T \to \infty$ as $T \to \infty$.

If $h \log T \ll 1$, using Lemma 3 again we get similarly

$$\|S(t+h, \chi) - S(t, \chi)\|^{2k} \ll (Ak)^{4k} H.$$

Hence we get our conclusion. Q.E.D.

4. Concluding remarks. We can get mean estimates of $\sum_{m=0}^{L}\sum_\chi b(m)g(\chi)S(t+hm,\chi)$ similarly as we did in §3, where $b(0),\cdots,b(L)$ are real numbers and $g(\chi)$ is a complex number for a character $\chi \bmod q$. Instead of describing a general theorem to this, we will give only some examples of this. We always assume that $q \le T^{1/4-a_1}$ and $T^{1/2+a_2} \le H \le T$ for a fixed a_1 and a_2 in $0 < a_i \le \frac{1}{2}$ for $i = 1, 2$.

(i)
$$\int_T^{T+H}\left(\sum_\chi{}' S(t,\chi)\right)^{2k} dt = c(k)H(\phi(q)-1)^{2k}(\log\log T)^k + O_k(H(\phi(q)-1)^{2k-1}(\log\log T)^{k-1/2}),$$

where in $\sum_\chi' \chi$ runs over all nonprincipal characters mod q.

(ii)
$$\int_T^{T+H}\left(\sum_\chi{}' S(t+h,\chi) - \sum_\chi{}' S(t,\chi)\right)^{2k} dt$$
$$= c(k)(\phi(q)-1)^{2k}H(2\log(2+h\log T))^k \cdot (1 + O((Ak)^{3k}(\phi(q)-1)^{-1}(\log(2+h\log T))^{-1/2}))$$

uniformly for h satisfying $0 < h \le (H - (H/\sqrt{T})^{1/8})$, where A is some positive absolute constant.

(iii)
$$\int_T^{T+H}\left(\frac{S(t,\chi)+S(t+h,\chi)}{2}\right)^{2k} dt = c_k H(\log\log T)^k + O_k(H(\log\log T)^{k-1/2})$$

if $h \log T \to 0$ as $T \to \infty$.

(iv) For any Δ satisfying $\Delta \log T \to 0$ as $T \to \infty$, there exists a function $\Phi(T)$ satisfying $\Phi(T) \to \infty$ as $T \to \infty$ such that

$$\int_T^{T+H}\left(\frac{1}{\Delta}\int_t^{t+\Delta} S(u,\chi)\,du\right)^{2k} dt = c_k H(\log\log T)^k + O_k(H(\log\log T)^k/\Phi(T)).$$

Also

$$\int_T^{T+H}\left(\frac{1}{\Delta(\phi(q)-1)}\sum_\chi{}'\int_t^{t+\Delta} S(u,\chi)\,du\right)^{2k} dt$$
$$= c_k H(\log\log T)^k + O_k(H(\log\log T)^k/\Phi(T)).$$

(v)
$$\int_T^{T+H} S(t,\chi)^{2k}\,dt = c(k)H(\log\log T)^k + O_k(H(\log\log T)^{k-1/2}).$$

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