LOCATION OF THE ZEROS OF POLYNOMIALS
WITH A PRESCRIBED NORM

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ABSTRACT. For monic polynomials \( f_n(z) \) of degree \( n \) with prescribed \( L^p \) norm (\( 1 \leq p \leq \infty \)) on the unit circle or supremum norm on the unit interval we determine bounded regions in the complex plane containing at least \( k \) (\( 1 \leq k \leq n \)) zeros. We deduce our results from some new inequalities which are similar to an inequality of Vicente Gonçalves and relate the zeros of a polynomial to its norm.

The location of some or all the zeros of a polynomial

\[
f_n(z) = \sum_{j=0}^{n} a_j z^j \quad (a_j \in \mathbb{C}, \ 0 \leq j \leq n)
\]

in terms of its coefficients has been extensively studied (see [3, Chapters VII—IX]). We may as well investigate the location of the zeros of \( f_n(z) \) in terms of a given norm. Such a problem is of interest in the theory of approximation [1, see §5]. Since multiplication by a nonzero constant does not change the zeros of \( f_n(z) \), norm alone cannot furnish any information regarding the location of any of the zeros. As a normalization we shall assume \( f_n(z) = \sum_{j=0}^{n} a_j z^j \) to be monic, i.e. the coefficient of \( z^n \) will be supposed to be 1. As typical norms we consider \( L^p \) norms on the unit circle and on the unit interval:

1. \( \|f_n\|_p = \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |f_n(e^{i\theta})|^p d\theta \right)^{1/p} \) (\( 1 \leq p \leq \infty \)), \( \|f_n\|_\infty = \max_{-\pi \leq \theta \leq \pi} |f_n(e^{i\theta})| \),
2. \( \|f_n\|_p = \left( \frac{1}{2} \int_{-1}^{1} |f_n(x)|^p dx \right)^{1/p} \) (\( 1 \leq p \leq \infty \)), \( \|f_n\|_\infty = \max_{-1 \leq x \leq 1} |f_n(x)| \).

We wish to determine the radius \( R(n, k, p, N) \) of the smallest disk centered at the origin containing at least \( k \) (\( 1 \leq k \leq n \)) zeros of every polynomial \( f_n(z) = \sum_{j=0}^{n} a_j z^j \) with prescribed norms.

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\[ z^n + \sum_{j=0}^{n-1} a_j z^j \] of degree \( n \) with \( \| f_n \|_2 = N \). In case \( N \) is given to be the supremum norm on the unit interval it turns out to be appropriate to find out the sum \( \rho(n, k, \infty, N) \) of the semiaxes of the ellipse with foci at \(-1, 1\) and containing at least \( k \) zeros of every polynomial \( f_n(z) = z^n + \sum_{j=0}^{n-1} a_j z^j \) with \( \| f_n \|_\infty = N \).

Quite a few results giving bounds for the zeros depending on the moduli of coefficients may be found in [3]. Since \( \| f_n \|_2 \) is expressible in terms of the coefficients some of these results may be used to determine estimates for \( R(n, k, 2, N) \). For example, the polynomial \( f_n(z) = z^n + \sum_{j=0}^{n-1} a_j z^j \) is known to have (see [3, (27, Formula 19)]) all its zeros in \( |z| < (1 + \sum_{j=1}^{n-1} |a_j|^2)^{1/2} \). Since \( (1 + \sum_{j=0}^{n-1} |a_j|^2)^{1/2} = \| f_n \|_2 \), this shows that \( R(n, n, 2, N) < N \). But \( R(n, n, 2, N) \) is easily seen to be equal to the positive root \( R(n, N) \) of the equation

\[ R^{2n} - (N^2 - 1) \sum_{\nu=0}^{n-1} R^{2\nu} = 0. \]

In fact, if \( \zeta \) is a zero of the polynomial \( f_n(z) = z^n + \sum_{j=0}^{n-1} a_j z^j \) with \( \| f_n \|_2 = N \), then

\[ |\zeta|^n = \left| \sum_{j=0}^{n-1} a_j \zeta^j \right| \leq \left\{ (N^2 - 1) \sum_{j=0}^{n-1} |\zeta|^{2j} \right\}^{1/2} \]

which shows that the largest positive root \( R(n, N) \) of (3) is a bound for the moduli of all the zeros. Given \( N \geq 1 \)

\[ f_n(z) = z^n - (N^2 - 1) \sum_{j=0}^{n-1} \frac{z^j}{R(n, N)^{n-j}} \]

is a polynomial of degree \( n \) with \( \| f_n \|_2 = N \) and having a zero on \( |z| = R(n, N) \). Substituting \( R^2 = N^2 - \alpha \) in (3) we get \( (N^2 - 1)/N^{2n} = \alpha(1 - \alpha/N^2)^n \). Hence for fixed \( n \) and large \( N \), \( \alpha = O(1/N^{2(n-1)}) \), i.e.

\[ R = N(1 - O(N^{-2n})). \]

If \( 1 \leq k < n \) an upper estimate for \( R(n, k, 2, N) \) can be deduced from the following result of Vicente Gonçalves ([10], also see [4] and [3, Exercise 4, p. 130]).

**Theorem A.** Consider the polynomial \( f_n(z) = z^n + \sum_{j=0}^{n-1} a_j z^j \) and let \( \zeta_1, \zeta_2, \ldots, \zeta_n \) denote the zeros of \( f_n(z) \) in an arbitrary order. Then for \( 1 \leq k \leq n \)

\[ |\zeta_1 \zeta_2 \cdots \zeta_{k-1}|^2 + |\zeta_k \zeta_{k+1} \cdots \zeta_n|^2 \leq \| f_n \|^2, \]

where for \( k = 1 \) the first term on the left-hand side is to be replaced by 1.
In particular

(6) \[ R(n, 1, 2, N) \leq (N^2 - 1)^{1/(2n)}. \]

The example \( f_n(z) = z^n + (N^2 - 1)^{1/2} \) (\( N \geq 1 \)) shows that, in fact,

(6*) \[ R(n, 1, 2, N) \equiv (N^2 - 1)^{1/(2n)}. \]

For \( p \neq 2 \) the known bounds for the moduli of the zeros in terms of the coefficients do not seem to be of much avail. But Jensen's formula gives (see [9, §3.61], and [7, §9, p. 21])

(7) \[ R(n, k, p, N) \leq N^{1/(n-k+1)} \quad (1 \leq k \leq n, 1 \leq p \leq \infty) \]

which for \( p = 2 \) is weaker than what is obtainable from (5). So, in order to improve on (7) we seek to extend (5) to values of \( p \) other than 2. In the case \( p = \infty \) this is done with the help of the following inequality due to Visser [11]:

(8) \[ |a_0| + |a_n| \leq \max_{|z|=1} \left| \sum_{j=0}^{n} a_j z^j \right|. \]

For \( 1 < p \leq 2 \) we use

(9) \[ (|a_0|^p + |a_n|^p)^{1/p} \leq \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{j=0}^{n} a_j e^{ij\theta} \right|^p d\theta \right)^{1/p} \]

which is a weaker form of the Hausdorff-Young inequality [13, p. 101].

From (8), (9) we deduce the following generalization of the inequality of Vicente Gonçalves (loc. cit.).

Theorem 1. Consider the polynomial \( f_n(z) = z^n + \sum_{j=0}^{n-1} a_j z^j \) and let \( \zeta_1, \zeta_2, \ldots, \zeta_n \) denote the zeros of \( f_n(z) \) in an arbitrary order. Then for \( 1 \leq k \leq n \)

(10) \[ (|\zeta_{1}^\ast \zeta_{2}^\ast \cdots \zeta_{k-1}^\ast|^q + |\zeta_k^\ast |^q)\cdots |\zeta_{n}^\ast |^q)^{1/q} \leq \|f_n\|_p \]

(\( p = \infty \) or \( 1 < p \leq 2, \quad p^{-1} + q^{-1} = 1 \))

where for \( k = 1 \) the term \( |\zeta_1^\ast \zeta_2^\ast \cdots \zeta_{k-1}^\ast|^q \) on the left-hand side is to be replaced by 1.

Proof. It is clear that if Theorem 1 holds for monic polynomials not vanishing at the origin then it also holds for those which have a simple or a multiple zero at the origin. So let \( \zeta_k^\ast, \zeta_{k+1}^\ast, \ldots, \zeta_n^\ast \) be different from zero and apply (8), (9) to
Since $|f_n(z)| = |g_n(z)|$ for $|z| = 1$ we get
\[
\left( \left| \frac{a_0}{\zeta_{k+1} \cdots \zeta_n} \right|^q + \left| \frac{z_0}{\zeta_{k+1} \cdots \zeta_n} \right|^q \right)^{1/q} \leq \|f_n\|_p
\]
which is equivalent to (10).

Remark 1. The example $f_n(z) = z^n + 1$ shows that (10) is false for $2 < p < \infty$. In fact, for $2 < p < \infty$
\[
\|z^n + 1\|_p < 2^{1/p} = (1 + 1^q)^{1/q}.
\]

Remark 2 (The case of equality in (10)). In (8) equality holds if and only if $a_j = 0$ for $j \neq 0, n$. The same is true of (9) if $p = 2$. If $1 < p < 2$ and $a_n \neq 0$ then by a result in [13] (see (2.25) on p. 105) there is strict inequality in (9) unless $a_j = 0$, $j = 0, 1, \cdots, n - 1$. Taking these facts into account and excluding the trivial case of $f_n(z) = z^n$ the proof of Theorem 1 shows that in (10) equality is not possible for $1 < p < 2$ and that for $p = \infty, p = 2$ equality holds if and only if $f_n(z)$ has the form
\[
\left( \zeta_0 \cdots \zeta_n \right) \prod_{k=1}^{n} \left( z - R \zeta_k \right)
\]
where $\zeta_k^{\frac{1}{n}}$ are the $n$th roots of unity in arbitrary order, $R$ is an arbitrary positive number and $\alpha$ an arbitrary real number.

Remark 3. It is seen from Jensen’s formula that inequality (10) may be extended to cover the case $p = 1$ by replacing the left-hand side by its limiting value (as $p \to 1$) max$(|\zeta_1 \zeta_2 \cdots \zeta_{k-1}|, |\zeta_k \zeta_{k+1} \cdots \zeta_n|)$.

Remark 4. If $f_n(z)$ is a polynomial of degree $n$ then by an inequality of Zygmund [12] we have
\[
\int_0^{2\pi} |f_n'(e^{i\theta})|^p d\theta \leq \gamma_p n^p \int_0^{2\pi} \left| \text{Re} f_n(e^{i\theta}) \right|^p d\theta \quad (p \geq 1)
\]
where
\[
(11) \quad \gamma_p = \sqrt{\pi} \Gamma(\frac{1}{2}p + 1)/\Gamma(\frac{1}{2}(p + 1))
\]
Applying Theorem 1 to $f_n(z)$ and noting that $\lim_{p \to \infty} \gamma_p^{1/p} = 1$ we obtain the following result on the location of critical points of $f_n(z)$. 

Corollary 1. Denote by $\eta_1, \eta_2, \cdots, \eta_{n-1}$ the critical points of a monic polynomial $f_n(z)$ of degree $n$. Then for $1 \leq k \leq n - 1$

$$
(|\eta_1 \eta_2 \cdots \eta_{k-1}|^q + |\eta_k \eta_{k+1} \cdots \eta_{n-1}|^q)^{1/q} \leq \gamma_1^{1/p} \|\text{Re } f_n\|_p
$$

$$(1 \leq p \leq 2, \frac{1}{p-1} + \frac{1}{q-1} = 1)$$

and

$$
|\eta_1 \eta_2 \cdots \eta_{k-1}| + |\eta_k \eta_{k+1} \cdots \eta_{n-1}| \leq \|\text{Re } f_n\|_\infty.
$$

For $k = 1$ the first term on the left hand side in both inequalities is to be replaced by 1.

The next result is an immediate consequence of Theorem 1.

Corollary 2. If $\zeta_1, \zeta_2, \cdots, \zeta_n$ are the zeros of $f_n(z) = z^n + \sum_{j=0}^{n-1} a_j z^j$ arranged in increasing order of moduli then we have

$$
|\zeta_1| \leq R(n, 1, p, N) \leq (Nq - 1)^{1/(2n)}
$$

$$(p = \infty \text{ or } 1 < p \leq 2, \frac{1}{p-1} + \frac{1}{q-1} = 1)$$

and for $2 \leq k \leq n$

$$
|\zeta_k| \leq |\zeta_k \zeta_{k+1} \cdots \zeta_n|^{1/(n-k+1)} \leq \left(\frac{1}{2}\|f_n\|_p^q + \frac{1}{2}(\|f_n\|_p^2 - 4|a_0|^2)^{1/2}\right)^{1/(n-k+1)}
$$

$$(p = \infty \text{ or } 1 < p \leq 2, \frac{1}{p-1} + \frac{1}{q-1} = 1).$$

Inequalities (12), (13) are best possible for $p = \infty$, 2. In particular

$$(12^*) R(n, 1, \infty, N) = (N - 1)^{1/n} \quad (N \geq 1)$$

which is attained for $f_n(z) = z^n + (N - 1)$.

Unfortunately the bound in (13) depends on $|a_0|$. But, in any case, it is at least as good as (7). It may be noted that for large $N$ and $p = 2$, there is not much room for improvement in (7). To see this let $f_{n-1}(z) = z^{n-1} + \sum_{j=0}^{n-2} a_j z^j$ be a monic polynomial of degree $n - 1$ with $\|f_{n-1}\|_p = N$. Then $g(z) = z f_{n-1}(z)$ has at least $k$ zeros in $|z| \leq R(n - 1, k - 1, p, N)$. Since $g(z)$ is a monic polynomial of degree $n$ with $\|g\|_p = N$ we have

$$R(n, k, p, N) \geq R(n - 1, k - 1, p, N).$$

This leads us to the conclusion that $R(n, k, p, N) \geq R(n - k + 1, 1, p, N)$, and by $(6^*)$, $(12^*)$ respectively, we get
\[ R(n, k, 2, N) \geq (N^2 - 1)^{1/(2(n-k+1))}, \quad R(n, k, \infty, N) \geq (N - 1)^{1/(n-k+1)} \]

showing that the bounds for \( R(n, k, 2, N), R(n, k, \infty, N) \) obtainable from (13) are not too bad for large \( N \).

With the help of Theorem 1 we obtain a slight improvement of (7) (it is only for sake of simplicity that we restrict ourselves to the case of supremum norm).

Let \( \zeta_1 \) be a zero of smallest modulus of \( f_n(z) = z^n + \sum_{j=0}^{n-1} a_j z^j \). Then by (10) \( f_n(z) \) has at least two zeros in

\[ |z| \leq \left( \left\| f_n \right\|_\infty - \left| \zeta_1 \right| \right)^{1/(n-1)}. \]

On the other hand, if \( f_{n-1}(z) = f_n(z)/(z - \zeta_1) \) then by (12), \( f_{n-1}(z) \) has at least one and \( f_n(z) \) at least two zeros in

\[ |z| \leq \left( \left\| f_n \right\|_\infty \left( 1 - \left| \zeta_1 \right| \right)^{-1} \right)^{1/(n-1)} \quad \left( \left| \zeta_1 \right| \neq 1 \right). \]

Hence, whatever \( \left| \zeta_1 \right| \) may be, \( f_n(z) \) has at least two zeros in

\[ |z| \leq \left\{ 2\left( \left\| f_n \right\|_\infty - 2 \right) + 4\sqrt{\left( \left\| f_n \right\|_\infty + 2 \right)^2 - 4} \right\}^{1/(n-1)}, \]

i.e. we have

**Corollary 3.** \( R(n, 2, \infty, N) \leq \left\{ 2(N - 2) + 4\sqrt{(N + 2)^2 - 4} \right\}^{1/(n-1)}. \)

Thus, \( R(n, 2, \infty, N) \) has an upper bound independent of \( n \) which we denote by \( r_2(N) \)—the subscript 2 refers to 2 zeros. Now suppose that an upper bound \( r_k(N) \) (independent of \( n \)) for \( R(n, k, \infty, N)^{1/(n-k+1)} \) has been found. Then \( f_n(z)/(z - \zeta_1) \) has at least \( k \) and \( f_n(z) \) at least \( k + 1 \) zeros in

\[ D_1(\zeta_1) = \{ z : |z| \leq (r_k(N)/\left| 1 - |\zeta_1| \right|)^{1/(n-k)} \}, \quad \left( |\zeta_1| \neq 1 \right). \]

On the other hand we may conclude from (10) that \( f_n(z) \) has at least \( k + 1 \) zeros in

\[ D_2(\zeta_1) = \{ z : |z| \leq (N - \left| \zeta_1 \right|)^{1/(n-k)} \}. \]

Comparing the radii of \( D_1(\zeta_1) \) and \( D_2(\zeta_1) \) we see that \( r_k(N) \) may be taken to be equal to \( N - (\lambda_k(N))^k \) where \( \lambda_k(N) \) is the smallest positive root of the equation \( r_k(N/(1 - \lambda) = N - \lambda^k \). Thus

\[ R(n, k + 1, \infty, N) \leq (N - (\lambda_k(N))^k)^{1/(n-k)} \]

which is an improvement on (7).

As pointed out in Remark 1 inequality (10) does not hold for \( 2 < p < \infty \).
Since \( ||f_n||_p \) is a nondecreasing function of \( p \) we obtain from Theorem A
\[
(|z_{k-1}^2| + |z_{k+1}|^2)^{1/2} \leq ||f_n||_p \quad (2 \leq p < \infty)
\]
and in particular
\[
R(n, 1, p, N) \leq (N^2 - 1)^{1/2n} \quad (2 \leq p < \infty).
\]

Another result like inequality (10) but valid for \( 1 \leq p < \infty \) is the following.

Theorem 2. In the notations of Theorem 1 we have for \( 1 \leq p < \infty \) and \( 1 \leq k \leq n \)
\[
|z_{1}^2z_{2} \cdots z_{k-1}' + |z_{k}z_{k+1}' \cdots z_{n}'| \leq \gamma_p \|f_n\|_p
\]
where \( \gamma_p \) is given by (11). For \( k = 1 \) the first term on the left-hand side of (15) is to be replaced by 1.

This result can be deduced from the following lemma (see [5, Theorem 2]) in the same way as Theorem 1 was deduced from (8), (9).

Lemma 1. If \( f(z) = \sum_{j=0}^{n} a_j z^j \) is a polynomial of degree \( n \) and \( a_u, a_v \) \( (u < v) \) are two coefficients such that for no other coefficients \( a_w \neq 0 \) do we have \( w \equiv u \mod (v - u) \), then for every \( p \geq 1 \), \( |a_u| + |a_v| \leq \gamma_p \|f_n\|_p \) where \( \gamma_p \) is given by (11).

From (15) it follows that
\[
R(n, 1, p, N) \leq (\gamma_p \|f_n\|_p N - 1)^{1/n} \quad (1 \leq p < \infty).
\]
The limiting case as \( p \to \infty \) of (16) agrees with (12*). The bound in (16) is attained for \( f_n(z) = z^n + e^{i\alpha} \), \( \alpha \) real.

Comparing (16) with (12) for \( 1 \leq p < 2 \) and with (14) for \( 2 \leq p < \infty \) it is seen, that in both cases the bound for \( R(n, 1, p, N) \) given by (16) is better or worse than the other one depending on the value of \( N \).

We now turn to the study of the location of zeros of a monic polynomial \( f_n(z) \) in terms of \( \Pi_p(f_n) \). As \( \Pi_2(f_n) \) may be expressed in terms of the moduli of the coefficients in the Legendre-development of \( f_n(z) \) regions containing at least \( k \) \( (1 \leq k \leq n) \) zeros of \( f_n(z) \) may be obtained from the following (specialized versions of) known results.

Theorem B [8]. For \( f_n(z) = \Pi_{\nu=1}^{n} (z - \zeta_{\nu}) \) we have
\[
\sum_{\nu=1}^{n} \left( \frac{d_n(\zeta_{\nu}) d_n(\zeta_{\nu-1}) \cdots d_n(\zeta_{\nu})}{\lambda_{\nu-1}} \right)^2 \leq (\Pi_2(f_n))^2 - \lambda_n^{-2}
\]
where
\[ \lambda_0 = 1, \quad \lambda_\nu = \frac{1}{2^\nu} \left( \frac{2\nu}{\nu} \right)^{2\nu + 1} \quad (1 \leq \nu \leq n) \]

and \( d_n(z) \) denotes the distance of \( z \) from the span of the zeros of the \( n \)th Legendre polynomial.

**Theorem C [2].** In the notations of Theorem B we have

\[ \frac{1}{n} \sum_{\nu=1}^{n} d_n(\zeta_\nu) \leq (\|f_n\|_\infty^2 - \lambda_n^{-2})^{1/2}. \]

For the purpose of determining the location of \( k \) \((1 \leq k \leq n)\) zeros of \( f_n(z) = z^n + \sum_{\nu=0}^{n-1} a_\nu z^\nu \) in terms of \( \|f_n\|_\infty \) we prove the following inequality which is somewhat similar to (10).

**Theorem 3.** Let \( f_n(z) = \prod_{\nu=1}^{n} (z - \zeta_\nu) \) be a real polynomial of degree \( n \) which does not change sign on the unit interval. If \( R_\nu \) denotes the sum of the semi-axes of the ellipse \( \mathcal{E}(R_\nu) \) with foci at \( +1, -1 \) and passing through the point \( \zeta_\nu \) \((\nu = 1, 2, \ldots, n)\) then for \( 1 \leq k \leq n \)

\[ \frac{R_1 R_2 \cdots R_{k-1}}{R_k} + \frac{R_k R_{k+1} \cdots R_n}{R_1 R_2 \cdots R_{k-1}} \leq 2(2^n - \|f_n\|_\infty^2 - 1). \]

For \( k = 1 \) the product \( R_1 R_2 \cdots R_{k-1} \) is to be replaced by 1.

**Proof.** Under the hypothesis \( |f_n(\cos \theta)| \) is a nonnegative trigonometric polynomial of degree \( n \). By a well-known theorem of Fejér and Riesz (see [6, p. 117]) there exists a polynomial \( F_n(z) = A_n \prod_{\nu=1}^{n} (z - \zeta_\nu) \) with \( |Z_\nu| \geq 1 \) and

\[ Z_1^{-1} + Z_\nu = 2\zeta_\nu \quad (\nu = 1, 2, \ldots, n) \]

such that

\[ |f_n(\cos \theta)| = |F_n(e^{i\theta})|^2 \quad (\theta \text{ real}). \]

Replacing \( \cos \theta \) by \( \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \) and equating the coefficients of \( e^{in\theta} \) on the two sides of (18) we get \( 1/2^n = |A_n|^2 \prod_{\nu=1}^{n} |Z_\nu| \). Hence, by Theorem 1 we have

\[ |Z_1 Z_2 \cdots Z_{k-1}| + |Z_k Z_{k+1} \cdots Z_n| \leq \left( 2^n \prod_{\nu=1}^{n} |Z_\nu| \right)^{1/2} \|F_n\|_\infty \]

which is equivalent to

\[ \frac{|Z_1 Z_2 \cdots Z_{k-1}|}{|Z_k Z_{k+1} \cdots Z_n|} + \frac{|Z_k Z_{k+1} \cdots Z_n|}{|Z_1 Z_2 \cdots Z_{k-1}|} \leq 2(2^n - \|f_n\|_\infty^2 - 1). \]

Inequality (17) follows from this on noting that \( \zeta_\nu \) lies on the ellipse \( \mathcal{E}(|Z_\nu|) \) \((\nu = 1, 2, \ldots, n)\).

**Remark 5 (The case of equality in (17)).** Taking into account the case of
equality in (10) (as discussed in Remark 2) and the identity (18) we easily see that for \( k = 1 \) and given \( N = \|f_n\| \geq 1/2^{n-2} \) equality holds in (17) for

\[
(19) \quad f_n(z) = \frac{1}{2^{n-1}} \left( T_n(z) \pm (2^{n-1}N - 1) \right)
\]

where \( T_n(x) = \cos n \arccos x \) is the \( n \)-th Chebyshev polynomial. If \( 2 \leq k \leq n \) equality holds only for \( f_n(z) = 2^{-(n-1)}(T_n(z) \pm 1) \).

We may apply Theorem 3 to the polynomial \( f_n(z)/n(z) \) of degree \( 2n \) to obtain the following

**Corollary 4.** If \( f(z) = \Pi_{\nu=1}^n (z - \zeta_{\nu}) \) is a polynomial of degree \( n \) then with \( R_{\nu} (\nu = 1, 2, \cdots, n) \) as defined in Theorem 3 we have for \( 1 \leq k \leq n \)

\[
\frac{R_1 R_2 \cdots R_{k-1}}{R_k R_{k+1} \cdots R_n} + \frac{R_k R_{k+1} \cdots R_n}{R_1 R_2 \cdots R_{k-1}} \leq 2^n \|f_n\|.
\]

For \( k = 1 \) the product \( R_1 R_2 \cdots R_{k-1} \) is to be replaced by 1. Equality holds for \( f_n(z) = 2^{-(n-1)}T_n(z) \) where \( T_n(z) \) is the \( n \)-th Chebyshev polynomial.

From Theorem 3 (in conjunction with Remark 5) and Corollary 4 we may deduce the following results.

**Corollary 5.** If \( \rho^*(n, k, \infty, N) \) denotes the sum of the semiaxes of the ellipse with foci at \( +1, -1 \) and containing at least \( k \) zeros of every real monic polynomial \( f_n(z) \) with \( \|f_n\| = N \) then

\[
(20) \quad \rho^*(n, 1, \infty, N) = \begin{cases} 
1 & \text{for } 2^{-(n-1)} \leq N \leq 2^{-(n-2)}, \\
\left(\sqrt{2^{n-2}N + 2^{n-2}N - 1}\right)^{2/n} & \text{for } N \geq 2^{-(n-2)}.
\end{cases}
\]

where the polynomials \( f_n(z) \) given in (19) are extremal.

**Corollary 6.** Let \( f_n(z) = \Pi_{\nu=1}^n (z - \zeta_{\nu}) \) be a real polynomial which does not change sign in \((-1, 1)\). If \( \|f_n\| = N \) then \( f_n(z) \) has at least \( k \) \( 1 \leq k \leq n \) zeros in

\[
\delta((\sqrt{2^{n-2}N + 2^{n-2}N - 1})^{2/(n-k+1)}) \quad (N \geq 2^{-(n-2)}).
\]

Proof. Let the zeros of \( f_n(z) \) be arranged in such a way that the corresponding numbers \( R_{\nu} \) are nondecreasing in magnitude and put

\[
S = R_n R_{n-1} \cdots R_k \frac{R_{k-1}}{R_{k-2}} \frac{R_{k-2}}{R_{k-3}} \cdots \omega,
\]

where \( \omega \) is equal to 1 for \( k = 1 \) and equal to \( R_2/R_1 \) or \( R_1 \) according as \( k \neq 1 \) is odd or even respectively. Since (17) holds for every arrangement of the numbers \( R_{\nu} \) we get \( S + S^{-1} \leq 2(2^{n-1}N - 1) \). From this Corollary 6 follows on noting
that \( R_{n-k+1} \leq R_n R_{n-1} \cdots R_k \leq S \). We observe that the monic polynomials 
\( f_n(z) \) having no sign change in \((-1, 1)\) and deviating least from zero on the unit
interval are \( z^{n-1}(T_n(z) \pm 1) \) with deviation \( N = 2^{-(n-2)} \).

In the same way we can deduce from Corollary 4 the following result.

Corollary 7. If \( \rho(n, k, \alpha, N) \) is as defined in the beginning of this paper we have

\[
\rho(n, k, \alpha, N) \leq \left( 2^{n-1}N \pm \sqrt{(2^{n-1}N)^2 - 1} \right)^1/(n-k+1) \quad (N \geq 2^{-(n-1)}).
\]

The Corollaries 5–7 add to the information available to us from the work of
S. N. Bernštejn [1, §5].

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