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LOCATION OF THE ZEROS OF POLYNOMIALS WITH A PRESCRIBED NORM

BY

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ABSTRACT. For monic polynomials $f_n(z)$ of degree n with prescribed L^p norm $(1 \leq p \leq \infty)$ on the unit circle or supremum norm on the unit interval we determine bounded regions in the complex plane containing at least k $(1 \leq k \leq n)$ zeros. We deduce our results from some new inequalities which are similar to an inequality of Vicente Gonçalves and relate the zeros of a polynomial to its norm.

The location of some or all the zeros of a polynomial

$$f_n(z) = \sum_{j=0}^{n} a_j z^j \qquad (a_j \in \mathbf{C},\ 0 \leq j \leq n)$$

in terms of its coefficients has been extensively studied (see [3, Chapters VII–IX]). We may as well investigate the location of the zeros of $f_n(z)$ in terms of a given norm. Such a problem is of interest in the theory of approximation [1, see §5]. Since multiplication by a nonzero constant does not change the zeros of $f_n(z)$, norm alone cannot furnish any information regarding the location of any of the zeros. As a normalization we shall assume $f_n(z) = \sum_{j=0}^{n} a_j z^j$ to be monic, i.e. the coefficient of z^n will be supposed to be 1. As typical norms we consider L^p norms on the unit circle and on the unit interval:

$$\|f_n\|_p = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f_n(e^{i\theta})|^p \, d\theta\right)^{1/p} \quad (1 \leq p < \infty), \quad \|f_n\|_\infty = \max_{-\pi \leq \theta < \pi} |f_n(e^{i\theta})|, \tag{1}$$

$$\mathfrak{M}_p(f_n) = \left(\frac{1}{2} \int_{-1}^{1} |f_n(x)|^p \, dx\right)^{1/p} \qquad (1 \leq p < \infty), \quad \mathfrak{M}_\infty(f_n) = \max_{-1 \leq x \leq 1} |f_n(x)|. \tag{2}$$

We wish to determine the radius $R(n, k, p, N)$ of the smallest disk centered at the origin containing at least k $(1 \leq k \leq n)$ zeros of every polynomial $f_n(z) =$

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$z^n + \sum_{j=0}^{n-1} a_j z^j$ of degree n with $\|f_n\|_p = N$. In case N is given to be the supremum norm on the unit interval it turns out to be appropriate to find out the sum $\rho(n, k, \infty, N)$ of the semiaxes of the ellipse with foci at $-1, 1$ and containing at least k zeros of every polynomial $f_n(z) = z^n + \sum_{j=0}^{n-1} a_j z^j$ with $\mathfrak{M}_\infty(f_n) = N$.

Quite a few results giving bounds for the zeros depending on the moduli of coefficients may be found in [3]. Since $\|f_n\|_2$ is expressible in terms of the coefficients some of these results may be used to determine estimates for $R(n, k, 2, N)$. For example, the polynomial $f_n(z) = z^n + \sum_{j=0}^{n-1} a_j z^j$ is known to have (see [3, (27, Formula 19)]) all its zeros in $|z| < (1 + \sum_{j=1}^{n-1} |a_j|^2)^{1/2}$. Since $(1 + \sum_{j=0}^{n-1} |a_j|^2)^{1/2} \equiv \|f_n\|_2$ this shows that $R(n, n, 2, N) < N$. But $R(n, n, 2, N)$ *is* easily seen to be *equal to the positive root* $R(n, N)$ *of the equation*

$$R^{2n} - (N^2 - 1) \sum_{\nu=0}^{n-1} R^{2\nu} = 0. \tag{3}$$

In fact, if ζ is a zero of the polynomial $f_n(z) = z^n + \sum_{j=0}^{n-1} a_j z^j$ with $\|f_n\|_2 = N$, then

$$|\zeta|^n = \left| - \sum_{j=0}^{n-1} a_j \zeta^j \right| \le \left\{ (N^2 - 1) \sum_{j=0}^{n-1} |\zeta|^{2j} \right\}^{1/2}$$

which shows that the largest positive root $R(n, N)$ of (3) is a bound for the moduli of all the zeros. Given $N \ge 1$

$$f_n(z) = z^n - (N^2 - 1) \sum_{j=0}^{n-1} \frac{z^j}{\{R(n, N)\}^{n-j}}$$

is a polynomial of degree n with $\|f_n\|_2 = N$ and having a zero on $|z| = R(n, N)$. Substituting $R^2 = N^2 - \alpha$ in (3) we get $(N^2 - 1)/N^{2n} = \alpha(1 - \alpha/N^2)^n$. Hence for fixed n and large N, $\alpha = O(1/N^{2(n-1)})$, i.e.

$$R = N(1 - O(N^{-2n})). \tag{4}$$

If $1 \le k < n$ an upper estimate for $R(n, k, 2, N)$ can be deduced from the following result of Vicente Gonçalves ([10], also see [4] and [3, Exercise 4, p. 130]).

Theorem A. *Consider the polynomial* $f_n(z) = z^n + \sum_{j=0}^{n-1} a_j z^j$ *and let* $\zeta_1, \zeta_2, \cdots, \zeta_n$ *denote the zeros of* $f_n(z)$ *in an arbitrary order. Then for* $1 \le k \le n$

$$|\zeta_1 \zeta_2 \cdots \zeta_{k-1}|^2 + |\zeta_k \zeta_{k+1} \cdots \zeta_n|^2 \le \|f_n\|_2^2, \tag{5}$$

where for $k = 1$ *the first term on the left-hand side is to be replaced by* 1.

In particular

$$R(n,\,1,\,2,\,N) \leq (N^2-1)^{1/(2n)}. \tag{6}$$

The example $f_n(z) = z^n + (N^2 - 1)^{1/2}$ $(N \geq 1)$ shows that, in fact,

$$R(n,\,1,\,2,\,N) \equiv (N^2-1)^{1/(2n)}. \tag{6*}$$

For $p \neq 2$ the known bounds for the moduli of the zeros in terms of the coefficients do not seem to be of much avail. But Jensen's formula gives (see [9, §3.61], and [7, §9, p. 21])

$$R(n,\,k,\,p,\,N) \leq N^{1/(n-k+1)} \qquad (1 \leq k \leq n,\ 1 \leq p \leq \infty) \tag{7}$$

which for $p = 2$ is weaker than what is obtainable from (5). So, in order to improve on (7) we seek to extend (5) to values of p other than 2. In the case $p = \infty$ this is done with the help of the following inequality due to Visser [11]:

$$|a_0| + |a_n| \leq \max_{|z|=1} \left| \sum_{j=0}^{n} a_j z^j \right|. \tag{8}$$

For $1 < p \leq 2$ we use

$$(|a_0|^q + |a_n|^q)^{1/q} \leq \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{j=0}^{n} a_j e^{ij\theta} \right|^p d\theta \right)^{1/p} \qquad (1 < p \leq 2,\ p^{-1} + q^{-1} = 1) \tag{9}$$

which is a weaker form of the Hausdorff-Young inequality [13, p. 101].

From (8), (9) we deduce the following generalization of the inequality of Vicente Gonçalves (loc. cit.).

Theorem 1. *Consider the polynomial* $f_n(z) = z^n + \Sigma_{j=0}^{n-1} a_j z^j$ *and let* $\zeta_1, \zeta_2, \cdots, \zeta_n$ *denote the zeros of* $f_n(z)$ *in an arbitrary order. Then for* $1 \leq k \leq n$

$$(|\zeta_1\zeta_2 \cdots \zeta_{k-1}|^q + |\zeta_k\zeta_{k+1} \cdots \zeta_n|^q)^{1/q} \leq \|f_n\|_p \qquad (p = \infty \text{ or } 1 < p \leq 2,\ p^{-1} + q^{-1} = 1) \tag{10}$$

where for $k = 1$ *the term* $|\zeta_1\zeta_2 \cdots \zeta_{k-1}|^q$ *on the left-hand side is to be replaced by* 1.

Proof. It is clear that if Theorem 1 holds for monic polynomials not vanishing at the origin then it also holds for those which have a simple or a multiple zero at the origin. So let $\zeta_k, \zeta_{k+1}, \cdots, \zeta_n$ be different from zero and apply (8), (9) to

$$g_n(z) = f_n(z) \prod_{j=k}^{n} \left(\frac{\overline{\zeta}_j z - 1}{z - \zeta_j} \right).$$

Since $|f_n(z)| = |g_n(z)|$ for $|z| = 1$ we get

$$\left(\left| \frac{a_0}{\zeta_k \zeta_{k+1} \cdots \zeta_n} \right|^q + |\overline{\zeta}_k \overline{\zeta}_{k+1} \cdots \overline{\zeta}_n|^q \right)^{1/q} \leq \|f_n\|_p$$

$$(p = \infty,\ 1 < p \leq 2,\ p^{-1} + q^{-1} = 1)$$

which is equivalent to (10).

Remark 1. The example $f_n(z) = z^n + 1$ shows that (10) is false for $2 < p < \infty$. In fact, for $2 < p < \infty$

$$\|\tfrac{1}{2}(z^n + 1)\|_p^p < \|\tfrac{1}{2}(z^n + 1)\|_2^2 = \tfrac{1}{2}$$

so that $\|z^n + 1\|_p < 2(\tfrac{1}{2})^{1/p} = 2^{1/q} = (1^q + 1^q)^{1/q}$.

Remark 2 (*The case of equality in* (10)). In (8) equality holds if and only if $a_j = 0$ for $j \neq 0, n$. The same is true of (9) if $p = 2$. If $1 < p < 2$ and $a_n \neq 0$ then by a result in [13] (see (2.25) on p. 105) there is strict inequality in (9) unless $a_j = 0$, $j = 0, 1, \cdots, n-1$. Taking these facts into account and excluding the trivial case of $f_n(z) = z^n$ the proof of Theorem 1 shows that in (10) equality is not possible for $1 < p < 2$ and that for $p = \infty$, $p = 2$ equality holds if and only if $f_n(z)$ has the form

$$\prod_{\nu=1}^{k-1} (z - R^{-1} e^{i(\alpha + \phi_\nu)}) \prod_{\nu=k}^{n} (z - R e^{i(\alpha + \phi_\nu)})$$

where $\{e^{i\phi_\nu}\}_{\nu=1}^n$ are the nth roots of unity in arbitrary order, R is an arbitrary positive number and α an arbitrary real number.

Remark 3. It is seen from Jensen's formula that inequality (10) may be extended to cover the case $p = 1$ by replacing the left-hand side by its limiting value (as $p \to 1$) $\max(|\zeta_1 \zeta_2 \cdots \zeta_{k-1}|, |\zeta_k \zeta_{k+1} \cdots \zeta_n|)$.

Remark 4. If $f_n(z)$ is a polynomial of degree n then by an inequality of Zygmund [12] we have

$$\int_0^{2\pi} |f_n'(e^{i\theta})|^p \, d\theta \leq \gamma_p n^p \int_0^{2\pi} |\operatorname{Re} f_n(e^{i\theta})|^p \, d\theta \qquad (p \geq 1)$$

where

$$\gamma_p = \sqrt{\pi}\,\Gamma(\tfrac{1}{2}p + 1)/\Gamma(\tfrac{1}{2}(p + 1)). \tag{11}$$

Applying Theorem 1 to $f_n'(z)$ and noting that $\lim_{p \to \infty} \gamma_p^{1/p} = 1$ we obtain the following result on the location of critical points of $f_n(z)$.

Corollary 1. *Denote by* $\eta_1, \eta_2, \cdots, \eta_{n-1}$ *the critical points of a monic polynomial* $f_n(z)$ *of degree* n*. Then for* $1 \le k \le n-1$

$$(|\eta_1\eta_2 \cdots \eta_{k-1}|^q + |\eta_k\eta_{k+1} \cdots \eta_{n-1}|^q)^{1/q} \le \gamma_p^{1/p}\|\mathrm{Re}\, f_n\|_p \qquad (1 \le p \le 2,\ p^{-1} + q^{-1} = 1)$$

and

$$|\eta_1\eta_2 \cdots \eta_{k-1}| + |\eta_k\eta_{k+1} \cdots \eta_{n-1}| \le \|\mathrm{Re}\, f_n\|_\infty.$$

For $k = 1$ *the first term on the left hand side in both inequalities is to be replaced by* 1.

The next result is an immediate consequence of Theorem 1.

Corollary 2. *If* $\zeta_1, \zeta_2, \cdots, \zeta_n$ *are the zeros of* $f_n(z) = z^n + \sum_{j=0}^{n-1} a_j z^j$ *arranged in increasing order of moduli then we have*

$$|\zeta_1| \le R(n, 1, p, N) \le (N^q - 1)^{1/(qn)} \qquad (p = \infty \text{ or } 1 < p \le 2,\ p^{-1} + q^{-1} = 1) \tag{12}$$

and for $2 \le k \le n$

$$\begin{aligned} |\zeta_k| &\le |\zeta_k\zeta_{k+1} \cdots \zeta_n|^{1/(n-k+1)} \\ &\le (\tfrac{1}{2}\|f_n\|_p^q + \tfrac{1}{2}(\|f_n\|_p^{2q} - 4|a_0|^2)^{1/2})^{1/\{q(n-k+1)\}} \end{aligned} \qquad (p = \infty \text{ or } 1 < p \le 2,\ p^{-1} + q^{-1} = 1). \tag{13}$$

Inequalities (12), (13) *are best possible for* $p = \infty, 2$*. In particular*

$$R(n, 1, \infty, N) \equiv (N-1)^{1/n} \qquad (N \ge 1) \tag{12*}$$

which is attained for $f_n(z) = z^n + (N-1)$.

Unfortunately the bound in (13) depends on $|a_0|$. But, in any case, it is at least as good as (7). It may be noted that for large N and $p = 2, \infty$ there is not much room for improvement in (7). To see this let $f_{n-1}(z) = z^{n-1} + \sum_{j=0}^{n-2} a_j z^j$ be a monic polynomial of degree $n-1$ with $\|f_{n-1}\|_p = N$. Then $g(z) = z f_{n-1}(z)$ has at least k zeros in $|z| \le R(n-1, k-1, p, N)$. Since $g(z)$ is a monic polynomial of degree n with $\|g\|_p = N$ we have

$$R(n, k, p, N) \ge R(n-1, k-1, p, N).$$

This leads us to the conclusion that $R(n, k, p, N) \ge R(n-k+1, 1, p, N)$, and by (6*), (12*) respectively, we get

$$R(n,\ k,\ 2,\ N) \geq (N^2-1)^{1/\{2(n-k+1)\}}, \qquad R(n,\ k,\ \infty,\ N) \geq (N-1)^{1/(n-k+1)}$$

showing that the bounds for $R(n,\ k,\ 2,\ N)$, $R(n,\ k,\ \infty,\ N)$ obtainable from (13) are not too bad for large N.

With the help of Theorem 1 we obtain a slight improvement of (7) (it is only for sake of simplicity that we restrict ourselves to the case of supremum norm).

Let ζ_1 be a zero of smallest modulus of $f_n(z) = z^n + \sum_{j=0}^{n-1} a_j z^j$. Then by (10) $f_n(z)$ has at least two zeros in

$$|z| \leq (\|f_n\|_\infty - |\zeta_1|)^{1/(n-1)}.$$

On the other hand, if $f_{n-1}(z) = f_n(z)/(z-\zeta_1)$ then by (12), $f_{n-1}(z)$ has at least one and $f_n(z)$ at least two zeros in

$$|z| \leq \left(\frac{\|f_n\|_\infty}{|1-|\zeta_1||} - 1\right)^{1/(n-1)} \qquad (|\zeta_1| \neq 1).$$

Hence, whatever $|\zeta_1|$ may be, $f_n(z)$ has at least two zeros in

$$|z| \leq \left\{\tfrac{1}{2}(\|f_n\|_\infty - 2) + \tfrac{1}{2}\sqrt{(\|f_n\|_\infty + 2)^2 - 4}\right\}^{1/(n-1)},$$

i.e. we have

Corollary 3. $R(n,\ 2,\ \infty,\ N) \leq \{\tfrac{1}{2}(N-2) + \tfrac{1}{2}\sqrt{(N+2)^2-4}\}^{1/(n-1)}$.

Thus, $(R(n,\ 2,\ \infty,\ N))^{n-1}$ has an upper bound independent of n which we denote by $r_2(N)$—the subscript 2 refers to 2 zeros. Now suppose that an upper bound $r_k(N)$ (independent of n) for $(R(n,\ k,\ \infty,\ N)^{n-k+1}$ has been found. Then $f_n(z)/(z-\zeta_1)$ has at least k and $f_n(z)$ at least $k+1$ zeros in

$$D_1(\zeta_1) = \{z\colon\ |z| \leq (r_k(N/|1-|\zeta_1||))^{1/(n-k)}\}, \qquad (|\zeta_1| \neq 1).$$

On the other hand we may conclude from (10) that $f_n(z)$ has at least $k+1$ zeros in

$$D_2(\zeta_1) = \{z\colon\ |z| \leq (N - |\zeta_1|^k)^{1/(n-k)}\}.$$

Comparing the radii of $D_1(\zeta_1)$ and $D_2(\zeta_1)$ we see that $r_{k+1}(N)$ may be taken to be equal to $N - (\lambda_k(N))^k$ where $\lambda_k(N)$ is the smallest positive root of the equation $r_k(N/(1-\lambda)) = N - \lambda^k$. Thus

$$R(n,\ k+1,\ \infty,\ N) \leq (N - (\lambda_k(N))^k)^{1/(n-k)}$$

which is an improvement on (7).

As pointed out in Remark 1 inequality (10) does not hold for $2 < p < \infty$.

Since $\|f_n\|_p$ is a nondecreasing function of p we obtain from Theorem A

$$(|\zeta_1\zeta_2\cdots\zeta_{k-1}|^2+|\zeta_k\zeta_{k+1}\cdots\zeta_n|^2)^{1/2}\le\|f_n\|_p \qquad (2\le p<\infty)$$

and in particular

$$R(n,1,p,N)\le(N^2-1)^{1/2n} \qquad (2\le p<\infty). \tag{14}$$

Another result like inequality (10) but valid for $1\le p<\infty$ is the following.

Theorem 2. *In the notations of Theorem* 1 *we have for* $1\le p<\infty$ *and* $1\le k \le n$

$$|\zeta_1\zeta_2\cdots\zeta_{k-1}|+|\zeta_k\zeta_{k+1}\cdots\zeta_n|\le\gamma_p^{1/p}\|f_n\|_p \tag{15}$$

where γ_p *is given by* (11). *For* $k=1$ *the first term on the left-hand side of* (15) *is to be replaced by* 1.

This result can be deduced from the following lemma (see [5, Theorem 2]) in the same way as Theorem 1 was deduced from (8), (9).

Lemma 1. *If* $f(z)=\sum_{j=0}^n a_jz^j$ *is a polynomial of degree* n *and* a_u, a_v $(u<v)$ *are two coefficients such that for no other coefficients* $a_w\neq 0$ *do we have* $w\equiv u \bmod (v-u)$, *then for every* $p\ge 1$, $|a_u|+|a_v|\le\gamma_p^{1/p}\|f\|_p$ *where* γ_p *is given by* (11).

From (15) it follows that

$$R(n,1,p,N)\le(\gamma_p^{1/p}N-1)^{1/n} \qquad (1\le p<\infty). \tag{16}$$

The limiting case as $p\to\infty$ of (16) agrees with (12*). The bound in (16) is attained for $f_n(z)=z^n+e^{i\alpha}$, α real.

Comparing (16) with (12) for $1<p<2$ and with (14) for $2<p<\infty$ it is seen, that in both cases the bound for $R(n,1,p,N)$ given by (16) is better or worse than the other one depending on the value of N.

We now turn to the study of the location of zeros of a monic polynomial $f_n(z)$ in terms of $\mathfrak{M}_p(f_n)$. As $\mathfrak{M}_2(f_n)$ may be expressed in terms of the moduli of the coefficients in the Legendre-development of $f_n(z)$ regions containing at least k $(1\le k\le n)$ zeros of $f_n(z)$ may be obtained from the following (specialized versions of) known results.

Theorem B [8]. *For* $f_n(z)=\prod_{\nu=1}^n(z-\zeta_\nu)$ *we have*

$$\sum_{\nu=1}^n\left(\frac{d_n(\zeta_n)d_n(\zeta_{n-1})\cdots d_n(\zeta_\nu)}{\lambda_{\nu-1}}\right)^2\le(\mathfrak{M}_2(f_n))^2-\lambda_n^{-2}$$

where

$$\lambda_0 = 1, \qquad \lambda_\nu = \frac{1}{2^\nu}\binom{2\nu}{\nu}\sqrt{2\nu+1} \qquad (1 \le \nu \le n)$$

and $d_n(z)$ *denotes the distance of* z *from the span of the zeros of the nth Legendre polynomial.*

Theorem C [2]. *In the notations of Theorem* B *we have*

$$\frac{1}{\lambda_{n-1}} \sum_{\nu=1}^{n} d_n(\zeta_\nu) \le ((\mathfrak{M}_2(f_n))^2 - \lambda_n^{-2})^{1/2}.$$

For the purpose of determining the location of k $(1 \le k \le n)$ zeros of $f_n(z) = z^n + \sum_{\nu=0}^{n-1} a_\nu z^\nu$ in terms of $\mathfrak{M}_\infty(f_n)$ we prove the following inequality which is somewhat similar to (10).

Theorem 3. *Let* $f_n(z) = \prod_{\nu=1}^{n} (z - \zeta_\nu)$ *be a real polynomial of degree* n *which does not change sign on the unit interval. If* R_ν *denotes the sum of the semi-axes of the ellipse* $\mathfrak{E}(R_\nu)$ *with foci at* $+1, -1$ *and passing through the point* ζ_ν $(\nu = 1, 2, \cdots, n)$ *then for* $1 \le k \le n$

$$\frac{R_1 R_2 \cdots R_{k-1}}{R_k R_{k+1} \cdots R_n} + \frac{R_k R_{k+1} \cdots R_n}{R_1 R_2 \cdots R_{k-1}} \le 2(2^{n-1}\mathfrak{M}_\infty(f_n) - 1). \tag{17}$$

For $k = 1$ *the product* $R_1 R_2 \cdots R_{k-1}$ *is to be replaced by* 1.

Proof. Under the hypothesis $|f_n(\cos\theta)|$ is a nonnegative trigonometric polynomial of degree n. By a well-known theorem of Fejér and Riesz (see [6, p. 117]) there exists a polynomial $F_n(z) = A_n \prod_{\nu=1}^{n} (z - Z_\nu)$ with $|Z_\nu| \ge 1$ and $Z_\nu^{-1} + Z_\nu = 2\zeta_\nu$ $(\nu = 1, 2, \cdots, n)$ such that

$$|f_n(\cos\theta)| = |F_n(e^{i\theta})|^2 \quad (\theta \text{ real}). \tag{18}$$

Replacing $\cos\theta$ by $\frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and equating the coefficients of $e^{in\theta}$ on the two sides of (18) we get $1/2^n = |A_n|^2 \prod_{\nu=1}^{n} |Z_\nu|$. Hence, by Theorem 1 we have

$$|Z_1 Z_2 \cdots Z_{k-1}| + |Z_k Z_{k+1} \cdots Z_n| \le \left(2^n \prod_{\nu=1}^{n} |Z_\nu|\right)^{1/2} \|F_n\|_\infty$$

which is equivalent to

$$\left|\frac{Z_1 Z_2 \cdots Z_{k-1}}{Z_k Z_{k+1} \cdots Z_n}\right| + \left|\frac{Z_k Z_{k+1} \cdots Z_n}{Z_1 Z_2 \cdots Z_{k-1}}\right| \le 2(2^{n-1}\mathfrak{M}_\infty(f_n) - 1).$$

Inequality (17) follows from this on noting that ζ_ν lies on the ellipse $\mathfrak{E}(|Z_\nu|)$ $(\nu = 1, 2, \cdots, n)$.

Remark 5 (*The case of equality in* (17)). Taking into account the case of equality in (10) (as discussed in Remark 2) and the identity (18) we easily see that for $k = 1$ and given $N = \mathfrak{M}_\infty(f_n) \geq 1/2^{n-2}$ equality holds in (17) for

$$f_n(z) = \frac{1}{2^{n-1}}(T_n(z) \pm (2^{n-1}N - 1)) \tag{19}$$

where $T_n(x) \doteq \cos n \arccos x$ is the nth Chebyshev polynomial. If $2 \leq k \leq n$ equality holds only for $f_n(z) = 2^{-(n-1)}(T_n(z) \pm 1)$.

We may apply Theorem 3 to the polynomial $f_n(z)\overline{f_n(\bar{z})}$ of degree $2n$ to obtain the following

Corollary 4. *If $f(z) = \Pi_{\nu=1}^n (z - \zeta_\nu)$ is a polynomial of degree n then with R_ν $(\nu = 1, 2, \cdots, n)$ as defined in Theorem 3 we have for $1 \leq k \leq n$*

$$\frac{R_1R_2 \cdots R_{k-1}}{R_kR_{k+1} \cdots R_n} + \frac{R_kR_{k+1} \cdots R_n}{R_1R_2 \cdots R_{k-1}} \leq 2^n\mathfrak{M}_\infty(f_n).$$

For $k = 1$ the product $R_1R_2 \cdots R_{k-1}$ is to be replaced by 1. Equality holds for $f_n(z) = 2^{-(n-1)}T_n(z)$ where $T_n(z)$ is the nth Chebyshev polynomial.

From Theorem 3 (in conjunction with Remark 5) and Corollary 4 we may deduce the following results.

Corollary 5. *If $\rho^*(n, k, \infty, N)$ denotes the sum of the semiaxes of the ellipse with foci at $+1, -1$ and containing at least k zeros of every real monic polynomial $f_n(z)$ with $\mathfrak{M}_\infty(f_n) = N$ then*

$$\rho^*(n, 1, \infty, N) \equiv \begin{cases} 1 & \text{for } 2^{-(n-1)} \leq N \leq 2^{-(n-2)}, \\ (\sqrt{2^{n-2}N} + \sqrt{2^{n-2}N - 1})^{2/n} & \text{for } N \geq 2^{-(n-2)}, \end{cases} \tag{20}$$

where the polynomials $f_n(z)$ given in (19) are extremal.

Corollary 6. *Let $f_n(z) = \Pi_{\nu=1}^n (z - \zeta_\nu)$ be a real polynomial which does not change sign in $(-1, 1)$. If $\mathfrak{M}_\infty(f_n) = N$ then $f_n(z)$ has at least k $(1 \leq k \leq n)$ zeros in*

$$\mathfrak{E}((\sqrt{2^{n-2}N} + \sqrt{2^{n-2}N - 1})^{2/(n-k+1)}) \qquad (N \geq 2^{-(n-2)}).$$

Proof. Let the zeros of $f_n(z)$ be arranged in such a way that the corresponding numbers R_ν are nondecreasing in magnitude and put

$$S = R_nR_{n-1} \cdots R_k \cdot \frac{R_{k-1}}{R_{k-2}} \cdot \frac{R_{k-3}}{R_{k-4}} \cdots \omega,$$

where ω is equal to 1 for $k = 1$ and equal to R_2/R_1 or R_1 according as $k \neq 1$ is odd or even respectively. Since (17) holds for every arrangement of the numbers R_ν we get $S + S^{-1} \leq 2(2^{n-1}N - 1)$. From this Corollary 6 follows on noting that $R_k^{n-k+1} \leq R_n R_{n-1} \cdots R_k \leq S$. We observe that the monic polynomials $f_n(z)$ having no sign change in $(-1, 1)$ and deviating least from zero on the unit interval are $2^{-(n-1)}(T_n(z) \pm 1)$ with deviation $N = 2^{-(n-2)}$.

In the same way we can deduce from Corollary 4 the following result.

Corollary 7. *If $\rho(n, k, \infty, N)$ is as defined in the beginning of this paper we have*

$$(21)\quad \rho(n, k, \infty, N) \leq \left(2^{n-1}N + \sqrt{(2^{n-1}N)^2 - 1}\right)^{1/(n-k+1)} \qquad (N \geq 2^{-(n-1)}).$$

The Corollaries 5–7 add to the information available to us from the work of S. N. Bernšteĭn [1, §5].

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