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ON CERTAIN CONVEX SETS IN THE SPACE OF LOCALLY SCHLICHT FUNCTIONS

BY

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ABSTRACT. Let $H = H(*, [+])$ denote the real linear space of locally schlicht normalized functions in $|z| < 1$ as defined by Hornich. Let K and C respectively be the classes of convex functions and the close-to-convex functions. If $M \subset H$ there is a closed nonempty convex set in the $\alpha\beta$-plane such that for (α, β) in this set $\alpha^*f[+]\beta^*g \in C$ (in K) whenever $f, g \in M$. This planar convex set is explicitly given when M is the class K, the class C, and for other classes. Some consequences of these results are that K and C are convex sets in H and that the extreme points of C are not in K.

1. Introduction. Let H denote the class of locally schlicht analytic functions f in the open unit disk E, normalized by the conditions $f(0) = 0$, $f'(0) = 1$. Hornich [3] provided a linear space structure to H with the operations

$$f[+]g = \int_0^z f'(t)g'(t)\,dt, \qquad \alpha^*f = \int_0^z (f'(t))^\alpha\,dt,$$

where $f, g \in H$ and α is a real number.

This paper is primarily concerned with two convex subsets of H, namely, the convex schlicht functions K and the close-to-convex functions C [4]. More specifically, the main results are summarized in the following two theorems.

Theorem A. *Let $f, g \in K$. Define, for real α and β,*

$$G_{\alpha,\beta}(z) = \alpha^*f[+]\beta^*g = \int_0^z (f'(t))^\alpha(g'(t))^\beta\,dt. \tag{1}$$

Then (i) $G_{\alpha,\beta} \in K$ *if* $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta \leq 1$. (ii) $G_{\alpha,\beta} \in C$ *if* $-1/2 \leq \alpha, \beta \leq 3/2$, $-1/2 \leq \alpha + \beta \leq 3/2$. *In each case, the result is sharp.*

Sharpness here means that for each pair of real numbers α, β, not restricted as in the theorem, there exist functions $f, g \in K$ such that the corresponding $G_{\alpha\beta}$ is not in the stated subclass of H.

Theorem B. *Let $f, g \in C$. Then $G_{\alpha,\beta} \in C$ if $-1/3 \leq \alpha + \beta \leq 1$, $\alpha - 3\beta \leq 1$, $\beta - 3\alpha \leq 1$. This result is sharp.*

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In particular these theorems prove K and C are convex sets in H. Furthermore, a line segment joining two functions in K can always be extended from each end point into the convex set C. In particular, this implies that no extreme point of the convex set C in H is a convex function.

A number of authors ([5], [6], [7]) have considered the special case where only one function appears in the integrand (1). In fact, Theorem 1 and Theorem 2 reduce to theorems of Merkes and Wright [5] when $\beta = 0$. The one-dimensional results concern radial lines in H joining a point of K or C with the origin, $f(z) \equiv z$, in H. In addition to consideration of functions $G_{\alpha,0}$ earlier papers ([2], [6]) also discuss the close-to-convexity of integrals of the form

$$g_\alpha(z) = \int_0^z (f(t)/t)^\alpha \, dt$$

where f is in a given subclass of H. This suggests a study of functions $g \in H$ such that $g \neq 0$ in $0 < |z| < 1$. Some special cases of this problem are discussed in §5 of this paper.

2. Some lemmas. If $p(z) = 1 + a_1 z + a_2 z^2 + \cdots$ in E and if $\operatorname{Re} p(z) > 0$, $z \in E$, then, by the mean value theorem of harmonic functions, we have

$$0 \le \int_{\theta_1}^{\theta_2} \operatorname{Re}\{p(re^{i\theta})\}\, d\theta \le \int_0^{2\pi} \operatorname{Re}\{p(re^{i\theta})\}\, d\theta = 2\pi \tag{2}$$

for $0 \le \theta_1 < \theta_2 \le 2\pi$ and for all r, $0 \le r < 1$.

Lemma 1. *If $f \in K$, then for $0 \le r < 1$, $0 \le \theta_1 < \theta_2 \le 2\pi$ we have*

$$\frac{\theta_2 - \theta_1}{2} \le \int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\frac{re^{i\theta} f'(re^{i\theta})}{f(re^{i\theta})}\right\} d\theta \le \pi + \frac{\theta_2 - \theta_1}{2}, \tag{3}$$

and

$$0 \le \int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{1 + \frac{re^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})}\right\} d\theta \le 2\pi. \tag{4}$$

Proof. The inequalities (3) follow from (2) and the fact that $\operatorname{Re}\{zf'/f\} > 1/2$ for $f \in K$. The analytic definition of the class K and (2) imply (4).

Lemma 2. *If $f \in C$, then*

$$-\pi + \frac{\theta_2 - \theta_1}{2} \le \int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\frac{re^{i\theta} f'(re^{i\theta})}{f(re^{i\theta})}\right\} d\theta \le 2\pi + \frac{\theta_2 - \theta_1}{2} \tag{5}$$

and

$$-\pi \le \int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{1 + \frac{re^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})}\right\} d\theta \le 3\pi \tag{6}$$

where $0 \le \theta_1 < \theta_2 \le 2\pi$ and $0 \le r < 1$.

Proof. For $f \in C$ there is a function ϕ and a real number β, $|\beta| < \pi/2$, such that $e^{-i\beta}\phi \in K$ and $\operatorname{Re}\{f'(z)/\phi'(z)\} > 0$, $z \in E$. The last condition implies $\operatorname{Re}\{f(z)/\phi(z)\} > 0$, $z \in E$, as well [5]. It follows that

$$\left|\arg\left[\frac{f(re^{i\theta})}{re^{i\theta}}\right] - \arg\left[\frac{\phi(re^{i\theta})}{re^{i\theta}}\right]\right| < \frac{\pi}{2}.$$

Now this implies

$$\left|\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\frac{re^{i\theta}f'(re^{i\theta})}{f(re^{i\theta})}\right\}d\theta - \int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\frac{re^{i\theta}\phi'(re^{i\theta})}{\phi(re^{i\theta})}\right\}d\theta\right|$$

$$= \left|\arg\frac{f(re^{i\theta_2})}{re^{i\theta_2}} - \arg\frac{\phi(re^{i\theta_2})}{re^{i\theta_2}} - \arg\frac{f(re^{i\theta_1})}{re^{i\theta_1}} + \arg\frac{\phi(re^{i\theta_1})}{re^{i\theta_1}}\right|$$

$$\leq \pi/2 + \pi/2 = \pi.$$

Since $e^{-i\beta}\phi \in K$ the last inequality and (3) imply (5). The left-hand side of (6) is Kaplan's characterization of close-to-convex functions [4]. The right-hand inequality in (6) follows from (4) and the fact that

$$|\arg f'(re^{i\theta}) - \arg \phi'(re^{i\theta})| < \pi/2.$$

The next lemma provides a class of close-to-convex functions that serve as extremal functions for the sharpness arguments in this paper.

Lemma 3. *For real* α, *the function* $b_\alpha(z) = \int_0^z (1+t)^\alpha dt$ *is not univalent in* E *if* $\alpha \notin [-3, 1]$. *If* $-3 \leq \alpha \leq 1$, *then* b_α *is close-to-convex in* E. b_α *is convex if and only if* $-2 \leq \alpha \leq 0$.

Proof. $b_\alpha(z) = [(1+z)^{\alpha+1} - 1]/(1+\alpha)$ provided $\alpha \neq -1$. When $\alpha = -1$, this function is $\log(1+z)$ which is univalent and convex in E. If $\alpha \neq -1$, b_α is univalent in E if and only if $(1+z)^{\alpha+1}$ is univalent. The latter is the case if and only if $-3 \leq \alpha \leq 1$ [7]. For $-3 \leq \alpha \leq -1$, furthermore, let $\phi(z) = z/(1+z)$ and we have

$$\operatorname{Re}\frac{b_\alpha'(z)}{\phi'(z)} = \operatorname{Re}(1+z)^{\alpha+2} \geq 0, \quad z \in E.$$

For $-1 < \alpha \leq 1$, let $\phi = z$ and $\operatorname{Re}\{b_\alpha'(z)/\phi'(z)\} = \operatorname{Re}(1+z)^\alpha > 0$, $z \in E$. This proves b_α is close-to-convex for $-3 \leq \alpha \leq 1$. The convexity follows the fact, for real α,

$$\operatorname{Re}\left\{\frac{zb_\alpha''(z)}{b_\alpha'(z)} + 1\right\} = \operatorname{Re}\frac{1+(1+\alpha)z}{1+z} \geq 0$$

if and only if $-2 \leq \alpha \leq 0$.

3. Convexity results. The first question that we consider is a proof of the convexity of the sets K and C in the space H. The result for K is known [2].

Theorem 1. *Let f_1, f_2 be in K (in C) and let $0 \le \lambda \le 1$. Then $G_{\lambda,1-\lambda}$ given by* (1) *is in K (in C).*

Proof. If f_1, f_2 are in K, then from (1) we obtain

$$\operatorname{Re}\left\{\frac{zG''_{\lambda,1-\lambda}}{G'_{\lambda,1-\lambda}}+1\right\} = \lambda\operatorname{Re}\left\{\frac{zf''_1}{f'_1}+1\right\} + (1-\lambda)\operatorname{Re}\left\{\frac{zf''_2}{f'_2}+1\right\}$$

which is nonnegative since f_1 and f_2 are in K. It follows that $G_{\lambda,1-\lambda} \in K$. Now assume $f_j \in C$. Then there exist functions $\phi_j \in K$ and real numbers β_j, $|\beta_j| < \pi/2$, such that

$$\operatorname{Re}\{e^{i\beta_j} f'_j(z)/\phi'_j(z)\} > 0, \qquad z \in E \quad (j = 1, 2). \tag{7}$$

Define

$$\phi_{\lambda,1-\lambda}(z) = \int_0^z (\phi'_1(t))^\lambda (\phi'_2(t))^{1-\lambda}\, dt$$

and, by the first part of this theorem, $\phi_{\lambda,1-\lambda} \in K$. Furthermore,

$$e^{i\beta}\frac{G'_{\lambda,1-\lambda}}{\phi'_{\lambda,1-\lambda}} = \left(e^{i\beta_1}\frac{f'_1}{\phi'_1}\right)^\lambda \left(e^{i\beta_2}\frac{f'_2}{\phi'_2}\right)^{1-\lambda} \tag{8}$$

where $\beta = \lambda\beta_1 + (1-\lambda)\beta_2$. In view of (7), it follows that the real part of the left-hand side of (8) is nonnegative and, hence, that $G_{\lambda,1-\lambda}$ is close-to-convex [4].

Let f_1 and f_2 be two functions in H and assume the corresponding $G_{\alpha,\beta}$ function (1) is in C (in K) for $\alpha = \alpha_1$, $\beta = \beta_1$ and for $\alpha = \alpha_2$, $\beta = \beta_2$. Then $G_{\alpha,\beta}$ is in C (in K) for all points in the $\alpha\beta$-plane on the line segment joining (α_1, β_1) and (α_2, β_2). Indeed, the function

$$H_\lambda(z) = \int_0^z (G'_{\alpha_1,\beta_1}(t))^\lambda (G'_{\alpha_2,\beta_2}(t))^{1-\lambda}\, dt$$

$$= \int_0^z (f'_1(t))^{\lambda\alpha_1+(1-\lambda)\alpha_2} (f'_2(t))^{\lambda\beta_1+(1-\lambda)\beta_2}\, dt$$

is in C (in K) for $0 \le \lambda \le 1$ by Theorem 1. It follows that for each pair $f_1, f_2 \in H$ there is a convex set in the $\alpha\beta$-plane containing $(0, 0)$ and such that the corresponding $G_{\alpha,\beta}$ function of the pair is in C (in K). Furthermore, this set is closed in the plane since C (or K) is a compact subclass of H in the topology of uniform convergence. These remarks assist in the proof of the following useful result.

Lemma 4. *Let M be a nonempty subclass of H. Then the set of points in the*

$\alpha\beta$-plane such that $G_{\alpha,\beta}$ is in C (in K) for each pair f_1, f_2 in M is nonempty, closed, and convex.

Proof. For each pair f, $g \in M$ there is a maximal closed convex set in the $\alpha\beta$-plane such that the functions $G_{\alpha,\beta}$ in (1) are in C (in K). The intersection of these sets, taken over all pairs f, $g \in M$, is a closed, convex set containing the origin.

4. Proofs of the principal theorems. We begin with a proof of part (i) of Theorem A. Let f_1, f_2 be in K. Then from (1)

$$R_{\alpha,\beta}(z) \equiv \operatorname{Re}\left\{1+\frac{zG''_{\alpha,\beta}(z)}{G'_{\alpha,\beta}(z)}\right\}$$

$$(9) \qquad = (1-\alpha-\beta)+\alpha\operatorname{Re}\left\{1+\frac{zf_1''(z)}{f_1'(z)}\right\}+\beta\operatorname{Re}\left\{1+\frac{zf_2''(z)}{f_2'(z)}\right\}$$

which is clearly nonnegative if $\alpha \geq 0$, $\beta \geq 0$, and $\alpha+\beta \leq 1$. For the sharpness, let $f_1 = f_2 = z/(1+z)$. Then

$$G_{\alpha,\beta}(z) = \int_0^z \frac{dt}{(1+t)^{2(\alpha+\beta)}},$$

and, by Lemma 3, this function is in K if and only if $0 \leq \alpha+\beta \leq 1$. The fact that $\alpha \geq 0$ follows from consideration of $G_{\alpha,\beta}$ for $f_1 = z/(1+z)$ and $f_2 = z$. We obtain $\beta \geq 0$ by symmetry.

To prove (ii) of Theorem A, we first observe that the closed convex region $-1/2 \leq \alpha, \beta, \alpha+\beta \leq 3/2$ is the convex hull of the points $(3/2, 0)$, $(0, 3/2)$, $(-1/2, 0)$, $(0, -1/2)$, $(3/2, -1/2)$, and $(-1/2, 3/2)$. That the first four of these points are contained in the convex set D, that exists by Lemma 4 with $M = K$, is proved in [5]. For $\alpha = 3/2$, $\beta = -1/2$ we have from (9) that

$$R_{3/2,-1/2}(z) = \frac{3}{2}\operatorname{Re}\left\{1+\frac{zf_1''(z)}{f_1'(z)}\right\}-\frac{1}{2}\operatorname{Re}\left\{1+\frac{zf_2''(z)}{f_2'(z)}\right\}$$

$$\geq -\frac{1}{2}\operatorname{Re}\left\{1+\frac{zf_2''(z)}{f_2'(z)}\right\}.$$

Since $f_2 \in K$, this inequality and Lemma 1 imply

$$\int_{\theta_1}^{\theta_2} R_{3/2,-1/2}(re^{i\theta})\,d\theta \geq -\frac{1}{2}(2\pi) = -\pi,$$

from which we conclude that $G_{3/2,-1/2} \in C$ [4].

To verify the sharpness of the region in (ii) of Theorem A, consider the convex functions z and $z/(1+z)$. With $f_1 = z$, $f_2 = z/(1+z)$ we obtain $-1/2 \leq \beta \leq 3/2$ and $-1/2 \leq \alpha \leq 3/2$ follows by symmetry. The remaining restrictions on α and β are established by setting $f_1 = f_2 = z/(1+z)$.

We turn now to the proof of Theorem B. The convex set in the $\alpha\beta$-plane in this theorem is simply the closed convex hull of the points $(1, 0)$, $(0, 1)$, $(-1/3, 0)$ and $(0, -1/3)$. For each of these points it has been proved [5] that $G_{\alpha,\beta} \in C$ whenever $f_1, f_2 \in C$. Hence, by Lemma 4, $G_{\alpha,\beta} \in C$ for all α, β in this convex hull. The sharpness of this closed convex set remains to be established.

The function $s(z) = z(1 + \frac{1}{2}z)/(1+z)^2$ is close-to-convex relative to the convex function $\phi(z) = z/(1+z)$. Let $f_1(z) = f_2(z) = s(z)$ in (1). Then $G_{\alpha,\beta}(z) = \int_0^z (1+t)^{-3\alpha-3\beta}\,dt$. By Lemma 3, this function is univalent in E if and only if $-1/3 \leq \alpha + \beta \leq 1$ and is close-to-convex for α and β in this interval. Thus $G_{\alpha,\beta}$ is not univalent for the given pair of functions whenever $\alpha + \beta > 1$ or $\alpha + \beta < -1/3$. If we set $f_1(z) = z$ and $f_2(z) = s(z)$ then $G'_{\alpha,\beta} = (1+z)^{-3\beta}$ from which we conclude $G_{\alpha,\beta}$ is not univalent for $\beta > 1$ or $\beta < -1/3$. The restrictions on α are obtained by symmetry and the sharpness of the closed region in Theorem B is proved.

A curious result is obtained when f_1 is in C while f_2 is restricted to K.

Theorem C. *If $f_1 \in C$, $f_2 \in K$, then $G_{\alpha\beta} \in C$ when*

$$-1/3 \leq \alpha, \quad -1 \leq 3\alpha + 2\beta \leq 3, \quad -3 \leq \alpha - 2\beta \leq 1. \tag{10}$$

This result is sharp.

Proof. The convex set (10) is the closed convex hull of the points $(1, 0)$, $(-1/3, 0)$, $(0, -1/2)$, $(0, 3/2)$ and $(-1/3, 4/3)$. All, except the last, are known [5] to be points for which $G_{\alpha,\beta} \in C$ whenever $f_1 \in C$ and $f_2 \in K$. The proof that $(-1/3, 4/3)$ is also such a point parallels the proof of the analogous result for $(3/2, -1/2)$ in the justification of Theorem A. Sharpness of the first inequality follows by letting $f_1 = z/(1+z)$ and $f_2 = z$. The sharpness of the next two sets of inequalities in (10) are respectively established by setting $f_1(z) = z(1 + \frac{1}{2}z)/(1+z)^2$, $f_1(z) = z(1 + \frac{1}{2}z)$ and setting $f_2 = z/(1+z)$ in each case. The conclusion is obtained from these functions and Lemma 3.

5. Related results.

Theorem 2. *Let f_1, f_2 be in K. Define*

$$g_{\alpha,\beta}(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha} \left(\frac{f_2(t)}{t}\right)^{\beta} dt. \tag{11}$$

Then (i) *$g_{\alpha,\beta} \in K$ provided $\alpha \geq 0$, $\beta \geq 0$, and $\alpha + \beta \leq 2$ and* (ii) *$g_{\alpha,\beta} \in C$ provided $-1 \leq \alpha \leq 3$, $-1 \leq \beta \leq 3$, and $-1 \leq \alpha + \beta \leq 3$. These results are sharp.*

Proof. The convexity of the set D in the $\alpha\beta$-plane such that $g_{\alpha,\beta} \in K$ (in C) for all $f_1, f_2 \in K$ follows from Lemma 4 by setting $M = \{g \in H: zg' \in K\}$ (i) is a simple consequence of this since $(0, 0), (2, 0), (0, 2)$ are in D by [5]. The closed convex region in (ii) is the closed convex hull of the points $(-1, 0), (0, -1), (3, 0), (0, 3), (3, -1), (-1, 3)$. The first four are known [5] to be in the set D such that $g_{\alpha,\beta} \in C$ for all $f_1, f_2 \in K$. To prove $(3, -1)$ is also in D, we observe that

$$M_{3,-1}(z) \equiv \operatorname{Re}\left\{1 + \frac{zg''_{3,-1}(z)}{g'_{3,-1}(z)}\right\}$$

$$= -1 + 3\operatorname{Re}\left\{\frac{zf_1'(z)}{f_1(z)}\right\} - \operatorname{Re}\left\{\frac{zf_2'(z)}{f_2(z)}\right\}$$

$$\geq \frac{1}{2} - \operatorname{Re}\left\{\frac{zf_2'(z)}{f_2(z)}\right\}$$

since $\operatorname{Re}\{zf_1'/f_1\} \geq 1/2$ for $f_1 \in K$. Let $0 \leq \theta_1 < \theta_2 \leq 2\pi$ and integrate $M_{3,-1}(re^{i\theta})$ with respect to θ from θ_1 to θ_2 for a fixed r, $0 \leq r < 1$. By (3) in Lemma 1 and the above inequality, we have

$$\int_{\theta_1}^{\theta_2} M_{3,-1}(re^{i\theta})\, d\theta \geq \frac{\theta_2 - \theta_1}{2} - \left(\pi + \frac{\theta_2 - \theta_1}{2}\right) = -\pi.$$

This proves $g_{3,-1} \in C$ [4]. The fact that $g_{-1,3} \in C$ follows by symmetry. An application of Lemma 4 now completes the proof of (ii). Sharpness in (i) and in (ii) follows by considering the functions (11) obtained by the pairing $z/(1+z)$ with itself or with z for the choices of f_1 and f_2.

A similar argument to the proof of Theorem B can be used to prove the next result.

Theorem 3. *If $f_1, f_2 \in C$, then the function* (11) *is in C if (α, β) belongs to the closed convex region D given by $-1/2 \leq \alpha + \beta \leq 1$, $\alpha - 2\beta \leq 1$ and $\beta - 2\alpha \leq 1$. The result is sharp.*

The extremal functions for the first inequalities are

$$f_1(z) = f_2(z) = \frac{z(1 + \mu z)}{(1 + z)^2}, \qquad \mu = \cos \gamma e^{i\gamma}, \qquad 0 < \gamma < \pi$$

(see [5]). For the next set of inequalities in the definition of D the extremal functions are f_1 as given above and $f_2 = z/(1 + \mu z)^2$, $\mu = \cos \gamma e^{i\gamma}$, $0 < \gamma < \pi$.

Finally, the analog of Theorem C for functions of the form (11) can be proved by the methods in this paper.

Theorem 4. *If $f_1 \in C$ and $f_2 \in K$, then the function $g_{\alpha,\beta}$, defined by* (11), *be-*

longs to C if (α, β) satisfies all the following inequalities $-1/2 \leq \alpha \leq 1$, $-1 \leq 2\alpha + \beta \leq 3$, $-1 \leq \beta - \alpha \leq 3$. *The result is sharp.*

Sharpness is obtained from various pairings of the functions z, $z/(1+\mu z)$ in K and $z(1+\mu z)/(1+z)^2$ in C where $\mu = \cos \gamma e^{i\gamma}$, $0 \leq \gamma < \pi$.

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