UNITARY MEASURES ON LCA GROUPS

BY

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ABSTRACT. A unitary measure on a locally compact Abelian (LCA) group \( G \) is a complex measure whose Fourier transform is of absolute value 1 everywhere. The problem of finding all such measures is known to be closely related to that of finding all invertible measures on \( G \). In this paper, we find all unitary measures when \( G \) is the circle or a discrete group. If \( G \) is a torsion-free discrete group, the characterization generalizes a theorem of Bohr.

1. Introduction. Let \( G \) be a locally compact group, and let \( \mu \) be a finite (complex-valued) regular Borel measure on \( G \). Define \( \mu^* \) by \( \mu^*(E) = \mu(E^{-1}) \) for all Borel sets \( E \). We call \( \mu \) unitary if \( \mu^* = \mu \) (i.e., \( \mu^* = \mu \)). If \( G \) is Abelian, this condition is equivalent to saying that \( |\hat{\mu}(\xi)| = 1 \) for every \( \xi \in \Gamma \), the dual group of \( G \).

We investigate the problem of finding all unitary measures on a locally compact Abelian group. The key tools are results of J. Taylor [7] and the Arens-Royden theorem (one form of which is given as Proposition 4.1 of [7]). We obtain complete results when \( G \) is discrete (Theorem 5); in the particular case where \( G \) is also torsion-free, the answer has a nicer form (Theorem 4), and this result generalizes a theorem of Bohr [1]. We also obtain all unitary measures on the circle group \( T \); this was the original aim of the paper. In view of Corollaries 4.6 and 4.7 of [7], these results give characterizations of the measures in \( \mathbb{M}(G)^* \), the multiplicative group of invertible measures on \( G \), when \( G \) is one of the above groups.

Before getting down to serious work, we make a few preliminary remarks. If \( \mu \) is unitary, write \( \mu = \mu_d + \mu_c \), where \( \mu_d \) is discrete and \( \mu_c \) is continuous. Then \( \mu^* \) is also unitary, and so \( \mu = \mu_d + (\delta_e + \mu_c \ast \mu_d) \). Thus classifying the unitary measures amounts to classifying the discrete ones and those which are (continuous + \( \delta_e \)). We call the latter continuous unitary measures.

Next, if \( \nu \) is a measure on \( \nu \) satisfying \( \nu^* = -\nu \) (for Abelian \( G \), this \( \Leftrightarrow \) \( \hat{\nu} \) is purely imaginary), then \( \exp \nu = \delta_e + \nu + (\nu \ast \nu)/2! + \cdots \) is unitary. (Conversely, if \( \exp \nu \) is unitary, then \( \nu^* = -\nu \).) Unitary measures of this form are precisely the ones lying in the connected component of the identity of \( \mathbb{M}(G)^* \). We gener-
ally regard these measures as trivial and look for others. Corollaries 4.6 and 4.7 of [7] show that if \( \mu \in \mathbb{M}(G)^{\times} \), then \( \exists \) a measure \( \nu_0 \in \mathbb{M}(G) \) such that \( (\exp \nu_0)^{-1}(y) = |\mu(y)| \), \( \forall y \in \Gamma \); it is this result that makes the problem of classifying unitary measures equivalent to the problem of classifying elements of \( \mathbb{M}(G)^{\times} \).

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2. Continuous unitary measures. The main purpose of this section is to prove the following theorem for \( T \).

**Theorem 1.** If \( \mu \) is a continuous unitary measure on \( T \), \( \mu = \exp \nu \) for some continuous \( \nu \).

**Proof.** Theorem 3 of [7] says that there are measures \( \mu_1, \ldots, \mu_n, \nu_0 \in \mathbb{M}(T) \), topologies \( T_1, \ldots, T_n \) on \( T \) (all at least as fine as the usual topology, and all making \( T \) into a locally compact group), and complex numbers \( \lambda_1, \ldots, \lambda_n \) such that \( \mu_j - \lambda_j \delta_e \) is absolutely continuous with respect to Haar measure on \( (T, T_j) \) and \( \mu = \mu_1 \ast \cdots \ast \mu_n \ast \exp(\nu_0) \). But the only topologies on \( T \) satisfying the condition are the usual one and the discrete topology. Hence \( \mu = \mu_1 \ast \mu_2 \ast \exp(\nu_0) \), where \( \mu_1 \) is discrete and \( \mu_2 - \lambda \delta_e \) is absolutely continuous. Write \( \nu_0 = \nu_1 + \nu_2 \), where \( \nu_1 \) is discrete and \( \nu_2 \) is continuous; then \( \mu = (\mu_1 \ast \exp \nu_1) \ast (\mu_2 \ast \exp \nu_2) \); the first expression in parentheses is a discrete measure, and the second is (except for a multiple of \( \delta_e \)) continuous. As \( \mu - \delta_e \) is continuous, the first term must be a multiple of \( \delta_e \).

We have reduced the theorem to the following: If \( \mu \) is unitary and \( \mu - \delta_e \) is absolutely continuous, then \( \mu = \exp \nu \) for some continuous \( \nu \). In fact, one can even pick \( \nu \) absolutely continuous, as we shall see. We work in \( L^1(T) \otimes C\delta_e \); it suffices to show that \( \mu \) is in the connected component of \( \delta_e \) in \( \mathbb{M}(T)^{\times} \). If \( \hat{\mu}(m) \neq -1 \), \( \forall m \), then \( \delta_e + i(\mu - \delta_e) \) never has zero transform, and hence is invertible for all \( t \). In general, \( \lim_{k \to \infty} \hat{\mu}_0(k) = 0 \); hence \( \hat{\mu} \) is \( -1 \) on a finite set. Suppose \( \hat{\mu}(k) = -1 \) for \( k = m_1, \ldots, m_n \); let \( \sigma \) be the measure given by \( \sigma = \int (z^{m_1+\cdots+m_l}) \, dz \) (\( dz \) = Haar measure). Then \( (\mu + it\sigma) \neq (\mu(k) \delta_e) \) except for \( k = m_1, \ldots, m_n \); at those points, \( (\mu + it\sigma) = -1 + it \). Hence \( \mu \) and \( \mu + \sigma \) are connected by a line of invertible measures and, as in the first part, \( \mu + \sigma \) is in the connected component of \( \delta_e \). That proves the theorem.

The last part of the proof gives the following result.

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(1) This follows from structure theory. Another proof: If \( G = (T, \mathcal{J}) \), then the identity map of \( G \to T \) gives (by duality) a dense map of \( \mathbb{Z} \) into \( \hat{G} \). Hence \( \hat{G} \) is monothetic. All monothetic groups are \( \mathbb{Z} \) or compact [5, Theorem 2.3.2], hence \( G \) is \( T \) or discrete.
Corollary 1. Suppose that $G$ is compact. Then if $\mu = \delta_e + \mu_0$ is unitary and $\mu_0$ is absolutely continuous, $\mu = \exp \nu$ for some absolutely continuous $\nu$.

The argument is the same as above. Suppose $\hat{\mu} = -1$ at $y_1, \ldots, y_n$; one lets 
\[ \text{d} \nu(x) = i(y_1(x) + \cdots + y_n(x)) \text{d}x \]
and reasons the same way. (The set where $\hat{\mu} = -1$ is finite because $\hat{\mu}_0$ is 0 at $\infty$.)

The argument of the first half of the theorem reduces the problem for general Abelian $G$ to finding absolutely continuous unitary measures on various other groups (viz., $G$ with finer topologies). But that still leaves a good deal of work in most cases.

A similar attack does, however, work in at least one other case.

Theorem 2. If $\mu$ is a unitary measure on a discrete torsion group $G$, then $\mu = \exp \nu$ for some $\nu \in M(G)$.

Proof. Since $|\hat{\mu}| = 1$ and $\Gamma$ is totally disconnected, $\hat{\mu}(\gamma) = e^{a(\gamma)}$ for some continuous function $a$. Now the Arens-Royden theorem says that $\mu = \exp(\nu)$ for some $\nu \in M(G)$.

3. Delta measures; unitary measures on torsion-free discrete groups. Theorem 2 implies that if $z \in T$ is of finite order, then $\delta_z$, the point mass at $z$ with mass 1, is of the form $\exp(\nu)$. Here is a converse.

Theorem 3. If $z \in T$ has infinite order, then $\delta_z \neq \exp(\nu)$ for any measure $\nu$.

Proof. Let $z = e^{i\theta}$. Then $\delta_z(n) = e^{-in\theta}$; if, therefore, $\exp(\nu) = \delta_z$, then $\hat{\nu}(n) = -in\theta + 2nk$, $k \in \mathbb{Z}$. Let $K = \|\nu\|$, so that $K \geq |2nk - n\theta|$, and let $p$ be an integer $> K$. Let $\nu = e^{-i\theta/p}$. Then $\exp(\nu/p) * \delta_z$ has a Fourier-Stieltjes transform whose range consists of $p$th roots of unity. Let the roots be $\omega_1, \ldots, \omega_p$; then the set $S_j$ on which the transform is $\omega_j$ is, according to results on idempotent measures (see, e.g., [5, p. 61 ff.]), a union of arithmetic progressions (with finitely many exceptions). The idea in what follows is that the irrationality of $\theta/p$ makes it impossible for the $S_j$ to be so orderly.

Let $N$ be large enough so that the variations in the progressions have been ironed out by then; pick $S_j$ so that it contains an infinite arithmetic progression, with common difference $r$, say. Since $|2nk - n\theta| < K$, we have $(n\theta - K)/2\pi \leq k_n \leq (n\theta + K)/2\pi$.

Replace $n$ by $n + mr$; we get

\[
\frac{(n + mr)\theta - K}{2\pi} \leq k_{n+mr} \leq \frac{(n + mr)\theta + K}{2\pi}, \quad \text{or} \quad \frac{mr\theta - 2K}{2\pi} \leq k_{n+mr} - k_n \leq \frac{mr\theta + 2K}{2\pi}.
\]

But $k_n - k_{n+mr}$ is a multiple of $p$; thus there is a multiple of $p$ between $(mr\theta - 2K)/2\pi$ and $(mr\theta + 2K)/2\pi$, $\forall m$. That means that for all $m$, there is an
integer between \((\frac{m\theta - 2K}{2n})/2n\) and \((\frac{m\theta + 2K}{2n})/2n\), or that \(-\frac{m\theta}{2n}\) is congruent (mod 1) to a number between \(-\frac{2K}{2n} > -\frac{1}{n}\) and \(\frac{2K}{2n} < \frac{1}{n}\). As \(\frac{m\theta}{2n}\) is irrational, this is impossible; in fact, the numbers \(-\frac{m\theta}{2n}\) are dense.

**Corollary 2.** If \(G\) is any locally compact group and \(x \in G\) has infinite order, then \(\delta_x\) is not an exponential.

**Proof.** We may as well assume that \(G\) is discrete, since if \(\delta_x = \exp \nu\) and \(\nu = \nu_1 + \nu_2\), with \(\nu_1\) discrete and \(\nu_2\) continuous, then \(\delta_x = \exp \nu_1\) also. As \(T\) is divisible, we can extend the map \(\alpha : nx \mapsto e^{ix}\) to a homomorphism (also called \(\alpha\)) of \(G\) into \(T\). If \(\delta_x = \exp(\nu)\), then it is easily checked that \(\delta_{\alpha(x)} = \exp(\alpha \nu)\), where \(\alpha \nu(E) = \nu(\alpha^{-1}E)\). But Theorem 3 makes this impossible.

Theorem 3 makes it possible to find all the unitary measures on any torsion-free discrete group.

**Theorem 4.** Let \(G\) be discrete and torsion-free. Then every unitary measure \(\mu\) on \(G\) is of the form \(\delta_x \ast \exp \nu\), for some \(x \in G\) and some measure \(\nu\) on \(G\) with \(\nu_- = -\nu\); moreover, \(x\) is uniquely determined by \(\mu\).

**Proof.** It suffices to show that any function \(f : \Gamma \to \mathbb{C}\) is homotopic to a character \(X_x : \gamma \mapsto (x, \gamma), x \in G\). For then we can choose \(x \in G\) such that \(\mu \ast X_x = \delta_x\), homotopic to the trivial map. Since \(X_x = \delta_{-x}\), we can use Arens-Royden to show that \(\mu \ast \delta_{-x} = \exp(\nu)\) for some \(\nu\), and the theorem follows. The uniqueness of \(x\) follows from Theorem 3.

Now we prove the homotopy result. Since \(\Gamma\) is compact, we can use Stone-Weierstrass to approximate \(f\) by a finite linear combination of characters, \(f \approx \sum_{j=1}^n a_j \chi_{x_j} = g\), say, so that \(\|f - g\|_\infty < \frac{1}{2} \inf_{\gamma \in \Gamma} |f(\gamma)|\). Then \(f\) and \(g\) are homotopic. Let \(\Gamma_0\) be the intersection of the kernels of the \(\chi_{x_j}\). Then \(g\) is constant on \(\Gamma_0\)-cosets, and therefore we can define \(\tilde{g}\) on \(\Gamma/\Gamma_0\) by \(\tilde{g}(x\Gamma_0) = g(x)\). \((\Gamma/\Gamma_0)^\wedge\) is the group generated by the \(x_j\); therefore \(\Gamma/\Gamma_0\) is isomorphic to a torus. But it is well known (see, e.g., [3, Theorem II.7.1]) that the characters of \(T_m\) represent the homotopy classes of maps on \(T_m\); hence \(\tilde{g}\) is homotopic to a character \(\tilde{X}\) of \(\Gamma/\Gamma_0\). Pull \(\tilde{X}\) back to \(\Gamma\), getting \(\tilde{X}\); then \(g\) and \(\tilde{X}\) are homotopic, as desired.

A corollary of the proof is

**Corollary 3.** If \(\Gamma\) is a connected compact group, then \(H^1(\Gamma, \mathbb{Z}) \cong G\). (The cohomology is Čech cohomology.)

**Proof.** From [4], \(H^1(\Gamma, \mathbb{Z}) \cong \text{group of homotopy classes of maps from } \Gamma \text{ to } T\). Since \(\Gamma\) is connected, \(G\) is torsion-free [5, Theorem 2.5.6]; the last part of the above proof does the rest. (The result is dual to one of Steenrod's: \(H_1(G, T) \cong G\). See [6, Theorem 15]).
4. Unitary measures on arbitrary discrete groups and $T$. We have still not solved the problem of finding the unitary measures on $T_d$, since $T_d$ has torsion elements. The following example shows that we can actually find other unitary measures besides $\delta$-measures $\ast$ exponentials.

Let $z_0 \in T$ have infinite order, and let $\mu = \frac{1}{2}(\delta_1 + \delta_{-1} - \delta_{-z_0})$. Then if $n$ is even, $\hat{\mu}(n) = \frac{1}{2}(1 + z_0^{-n} + (-1)^n - (-z_0)^{-n}) = 1$, while if $n$ is odd, $\hat{\mu}(n) = z_0^{-n}$. Suppose now that $\mu = \delta_{z_1} \ast \exp(\nu)$ for some $z \in T$ and some measure $\nu$. Then $\mu \ast \delta_{z_1} = \exp(\nu)$. Let $\mu_0 = \frac{1}{2}(\mu \ast \delta_{z_1}^{-1}) \ast (\delta_1 + \delta_{-1})$, $\mu_1 = \frac{1}{2}(\mu \ast \delta_{z_1} \ast (\delta_1 - \delta_{-1})$. Then

$$\mu_0 + \mu_1 = \mu \ast \delta_{z_1}^{-1}, \quad \text{and} \quad \hat{\mu}_0(n) = \begin{cases} (\mu \ast \delta_{z_1}^{-1})^\wedge(n), & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

Let $a : T \to T$ take $z \mapsto z^2$, and let $\mu^* = \mu \circ a$. Then $\hat{\mu}_0^*(n) = \hat{\mu}_0(2n) = z_1^{2n}$; also, $\hat{\mu}_0(2n) = \exp(\nu)(2n) = \exp(\nu)(n)$. Hence $\mu^* \ast \delta_{z_1}^{-2} = \mu_0^* = \exp(\nu^*)$. It follows that $z_1$ has finite order in $T$.

On the other hand,

$$\hat{\mu}_1(n) = \begin{cases} (\mu \ast \delta_{z_1}^{-1})^\wedge(n) = (z_1z_0^{-1})^n, & n \text{ odd}, \\ 0, & n \text{ even}, \end{cases}$$

and $z_2 = z_1z_0^{-1}$ has infinite order. Also, $\hat{\mu}_1(n) = \exp(\nu)(n)$ whenever $n$ is odd.

We can now use the same reasoning as in the proof of Theorem 3 to get a contradiction. Let $z_2 = e^{i\theta}$, and define $K$, $p$, and $\gamma$ as in Theorem 3. Then $\exp(\nu/p) \ast \delta_{\gamma} \ast \frac{1}{2}(\delta_1 - \delta_{-1})$ has a Fourier-Stieltjes transform whose range consists of $p$th roots of unity and 0; the value is 0 on $2\mathbb{Z}$. Define the $S_j$ as in Theorem 3, and the rest of the argument in Theorem 3 goes through. It follows that $\mu$ is not a point measure convolved with an exponential.

What this argument says is that Theorem 4 is false for $G = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. ($G$ is embedded in $T$ as $z_0 \mathbb{Z} \oplus \{-1, 1\}$.) Then $\Gamma \cong T \oplus \mathbb{Z}/2\mathbb{Z}$; $\hat{\mu}$ is 1 on one circle and $z$ on the other. This construction generalizes.

Let $G$ be any discrete group, and let $G_1$ be a finite subgroup (of order $n$, say). Then $\Gamma_1 = G_1^\perp$ is of index $n$; let $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ be the cosets. Let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be the idempotent measures whose Fourier-Stieltjes transforms are the characteristic functions of $\Gamma_1, \ldots, \Gamma_n$ respectively. Let $x_1, \ldots, x_n$ be elements of $G$, and let $\nu$ be a measure on $G$ with $\nu^* = -\nu$. Then $\mu = (\Sigma_j \delta_{x_j} \ast \sigma_j) \ast \exp(\nu)$ is unitary.

**Theorem 5.** Every unitary measure on a discrete group $G$ arises in this way.
Proof. Let $\mu_1$ be a unitary measure. As in Theorem 3, it suffices to show that $\hat{\mu}_1$ and $\hat{\mu}$ are homotopic for some $\mu = (\sum_{j=1}^{n} \delta_{x_j} \ast \sigma_j) \ast \exp(\nu)$. Again, as in Theorem 3, $\hat{\mu}_1$ is homotopic to a linear combination of finitely many characters: $\hat{\mu}_1 \sim \sum_{j=1}^{n} a_j \chi_{y_j} = f$, say. Let $\Gamma_0$ be the common kernel of $\chi_{y_1}, \ldots, \chi_{y_m}$; $f$ gives $f$ on $\Gamma/\Gamma_0$. We may assume from now on that $\Gamma_0 = \{1\}$, since from now on everything will be constant on $\Gamma_0$-cosets. Note that $f$ is the transform of a measure; thus $f^{-1} \hat{\mu}_1 = (\exp(\nu_0))$. 

Given our assumption, $G$ is generated by $y_1, \ldots, y_m$; hence $G \cong \mathbb{Z}^k \oplus G_1$, where $G_1$ is finite of order $n$. Thus $\Gamma_1, \ldots, \Gamma_m$ are $k$-tori. Hence there are elements $x_1, \ldots, x_n$ such that $\chi_{x_j}$ and $\exp(\nu_0)$ are homotopic on $\Gamma_j$. It follows that if $\mu_0 = \sum_{j=1}^{n} \delta_{x_j} \ast \sigma_j$, then $\mu_0^{-1} f$ is homotopic to the trivial map on each component. Hence $\mu_0^{-1} f = (\exp(\nu_1))$, and the theorem follows.

In the case of $\mathbb{T}_d$, $G_1$ is necessarily cyclic. A more careful analysis along the lines of the example shows that if one picks $m$ as small as possible, then each $x_j$ is determined modulo the torsion group of $\mathbb{T}_d$.

Theorems 1 and 5 together determine all the unitary measures on $\mathbb{T}$. As noted earlier, they also determine all the connected components of $\mathbb{M}(\mathbb{T})^\times$. We state the result here for completeness.

Corollary 4. Let $\mu$ be an invertible measure on $\mathbb{T}$. Then there are an integer $m$, elements $z_1, \ldots, z_m \in \mathbb{T}$, and a measure $\nu$ on $\mathbb{T}$ such that $\mu = \exp(\nu) \ast (\sum_{j=1}^{m} \delta_{x_j} \ast \sigma_{m,j})$, where $\sigma_{m,j}$ is the idempotent measure on $\mathbb{T}$ whose Fourier-Stieltjes transform is 1 on $m\mathbb{Z} + j$ and 0 elsewhere. If $\mu$ is unitary, $\mu$ can be expressed in the same form, but with $\nu \sim -\nu$.

BIBLIOGRAPHY