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DENSITY OF PARTS OF ALGEBRAS ON THE PLANE

BY

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ABSTRACT. We study the Gleason parts of a uniform algebra A on a compact subset of the plane, where it is assumed that for each point $x \in \mathbf{C}$ the functions in A which are analytic in a neighborhood of x are uniformly dense in A. We prove that a part neighborhood N of a nonpeak point x for A satisfies a density condition of Wiener type at x: $\sum_{n=1}^{+\infty} 2^n C(A_n(x)\setminus N) < +\infty$, and if A admits a pth order bounded point derivation at x, then N satisfies a stronger density condition: $\sum_{n=1}^{+\infty} 2^{(p+1)n} C(A_n(x)\setminus N) < +\infty$. Here C is Newtonian capacity and $A_n(x)$ *is* $\{z \in \mathbf{C}: 2^{-n-1} \le |z-x| \le 2^{-n}\}$. These results strengthen and extend Browder's metric density theorem. The relation with potential theory is examined, and analogous results for the algebra $H^\infty(U)$ are obtained as corollaries.

Let $X \subset \mathbf{C}$ be compact. Browder's metric density theorem [3, p. 177] states that *any part neighborhood (with respect to $R(X)$) of a nonpeak point x for $R(X)$ has full area density at x*. We extend this theorem in three ways. We replace $R(X)$ by any of a large class of algebras (those satisfying conditions (1)–(4) of §1); we strengthen the conclusion, using Newtonian capacity (Theorem 1); and we show that, by strengthening the hypothesis to allow the existence of a bounded point derivation at x, the conclusion may be further improved (Theorem 2). One consequence (Corollary 2(d)) is that if a pth order bounded point derivation exists at x, then plane area $\mathfrak{L}^2$, restricted to the complement of any given part neighborhood of x, has zero $(2p+2)$-dimensional density at x.

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1. Consider a compact set $X \subset \mathbf{C}$ and an algebra A of continuous complex-valued functions defined on $\mathbf{C}$ such that

(1) A contains the constants;

(2) A separates points on X;

(3) the restriction, B, of A to X is closed with respect to $\|\cdot\|_X$, the uniform norm on X;

(4) for each $x \in \mathbf{C}$ the set $A_x = A \cap \{f: f \text{ is analytic on a neighborhood of } x\}$ is dense in A with respect to $\|\cdot\|_X$.

The first three conditions state that B is a *function algebra* on X [3], while the fourth first occurred in the work of Arens [1] on maximal ideals. Many interesting spaces have these properties.

The *Gleason metric* of A on $\mathbf{C}$ is defined by

$$d(x, y) = \sup\{|f(x) - f(y)|: f \in A, \|f\|_{\mathbf{C}} \le 1\},$$

whenever $x, y \in \mathbf{C}$. A *Gleason part* of A is an equivalence class under the relation $x \sim y \Leftrightarrow d(x, y) < 2$. Parts are discussed in [2], [3], [6], [8], [9], [11], [12], [13], [15], [18], [25], [26], [29]. A point $x \in X$ is a *peak point* for B if there is a function in B whose modulus takes its maximum value at x, and only at x. Peak points form trivial one-point parts [9]. This paper is about the structure of nontrivial parts.

We need some notation.

For $x \in \mathbf{C}$ and $0 < a \in \mathbf{R}$, set $P(x, a) = \{y \in \mathbf{C}: d(x, y) \le a\}$, $G(x, a) = \{y \in \mathbf{C}: d(x, y) < a\}$. For a (complex Radon [7, p. 62]) measure μ on $\mathbf{C}$ the *Cauchy transform* $\hat{\mu}$ and the *Newtonian potential* $\tilde{\mu}$ are the $L_1^{\text{loc}}(\mathbf{C}, \mathfrak{L}^2)$ functions defined by

$$\hat{\mu}(z) = \int \frac{d\mu\zeta}{\zeta - z}, \qquad \tilde{\mu}(z) = \int \frac{d|\mu|\zeta}{|\zeta - z|}$$

for $z \in \mathbf{C}$. Here $|\mu|$ denotes the total variation measure of μ. The *Newtonian capacity* of a set $E \subset \mathbf{C}$ is

$$C(E) = \sup\{\|\mu\|: \mu \text{ is a measure, } \operatorname{spt}\mu \subset E, \tilde{\mu} \le 1\},$$

where $\operatorname{spt}\mu$ refers to the compact *support* of μ, and $\|\mu\|$ is its *total variation norm*, i.e. $\|\mu\| = |\mu|(\mathbf{C})$. For $x \in \mathbf{C}$, $n \in \mathbf{Z}$ and $r \in \mathbf{R}$ we set

$$A_n(x) = \{z \in \mathbf{C}: 2^{-n-1} \le |z - x| \le 2^{-n}\}, \qquad B(x, r) = \{z \in \mathbf{C}: |z - x| \le r\}.$$

2. The main results of this paper are as follows.

Theorem 1. *Suppose $x \in X$ is not a peak point for B, and $a > 0$. Then*

$$\sum_{n=1}^{+\infty} 2^n C(A_n(x)\setminus P(x, a)) < +\infty.$$

Corollary 1. *Suppose $x \in X$ is not a peak point for B, and $a > 0$. Then the following statements are true.*

(a) $$\lim_{r\to 0} \frac{C(B(x, r)\setminus P(x, a))}{r} = 0.$$

(b) $$\lim_{r\to 0} \frac{M_b(B(x, r)\setminus P(x, a))}{rk(r)} = 0,$$

whenever b is a measure function [4] *such that $\int_0 db(t)/t < +\infty$, and $k(r) = \int_0^r db(t)/t$.*

(c) $$\lim_{r\to 0} \frac{M^\beta(B(x, r)\setminus P(x, a))}{r^\beta} = 0 \quad \text{whenever } \beta > 1.$$

(d) $\mathbf{C}\setminus P(x, a)$ *has zero area density at x.*

Here M_b denotes the *Hausdorff content* corresponding to the measure function b: if $E \subset \mathbf{C}$, then

$$M_b(E) = \inf\left\{\sum_{S\in\mathcal{S}} b(\operatorname{diam} S)\right\},$$

the infimum being taken over all countable coverings $\mathcal{S}$ of E by closed discs; $M^\beta = M_b$ for $b(r) = r^\beta$.

B is said to admit a pth order *bounded point derivation* at $x \in X$ if the linear functional $f \longrightarrow f^{(p)}(x)$ on A_x extends to a continuous linear functional on the Banach space $(B, \|\cdot\|_X)$. The notion is discussed in [16], [22], [23], [27].

Theorem 2. *If B admits a pth order bounded point derivation at $x \in X$, and $a > 0$, then*

$$\sum_{n=1}^{+\infty} 2^{(p+1)n} C(A_n(x)\setminus P(x, a)) < +\infty.$$

Corollary 2. *If B admits a pth order bounded point derivation at $x \in X$, and $a > 0$, then*

(a) $$\lim_{r\to 0} \frac{C(B(x, r)\setminus P(x, a))}{r^{p+1}} = 0,$$

(b) $$\lim_{r\to 0} \frac{M_h(B(x, r)\setminus P(x, a))}{r^{p+1}k(r)} = 0$$

whenever h, k are as in Corollary 1(b),

(c) $$\lim_{r\to 0} \frac{M^{\beta}(B(x, r)\setminus P(x, a))}{r^{\beta(p+1)}} = 0 \textit{ whenever } \beta > 1,$$

(d) $$\lim_{r\to 0} \frac{\mathfrak{L}^2(B(x, r)\setminus P(x, a))}{r^{2p+2}} = 0.$$

All these results are new, with the exception of Corollary 1(d), which is Browder's metric density theorem, in case $A = R(X)$. Another case of 1(d) was noted by Gamelin and Garnett [11]. The results are somewhat analogous to theorems of [16], [28], [20], [21, Theorem 5], [5], [24], although quite independent of all of these.

3. Our two main tools are a lemma of Browder's and the Wiener series machinery, as perfected by Mel′nikov and Curtis. So our methods are a blend of the dual space methods of the disciples of F. Riesz and the constructive techniques of the Russian school.

If L is a linear functional on A, a *representing measure* for L is a (complex!) measure μ such that $Lf = \int f\,d\mu$ whenever $f \in A$. A *representing measure for a point* $y \in X$ is a representing measure for the functional $f \mapsto f(y)$. By the Hahn-Banach and Riesz representation theorems every continuous linear functional on the Banach space B has at least one representing measure supported on X. A point $y \in X$ is a nonpeak point for B if and only if y has a representing measure with no mass at y [9, p. 54].

Browder's lemma. *Suppose μ is a representing measure for a point $x \in X$, $a > 0$, and $\delta = a(a + 1 + \|\mu\|)^{-1}$. Then $d(x, y) < a$ whenever $|x-y|\tilde{\mu}(y) < \delta$.*

This is proved in case $A = R(X)$ in [3, p. 176]. The same proof works for general A satisfying conditions (1)–(4) of §1, a fact which was known to Browder.

Lemma 2. *Let $x \in \mathbf{C}$, μ be a measure with no mass at x, $\delta > 0$, $E = \{y \in \mathbf{C}: |x-y|\tilde{\mu}(y) \geq \delta\}$. Then*

$$\sum_{n=1}^{+\infty} 2^n C(A_n(x) \cap E) < +\infty.$$

Proof. Abbreviate $A_n(x) \cap E = E_n$. Let x, μ, δ and E be as in the hypotheses, and suppose $\sum_{n=1}^{+\infty} 2^n C(E_n) = +\infty$. We seek a contradiction.

Fix $1 \leq N \in \mathbf{Z}$. Since $C(E_n) \leq C(B(x, 2^{-n})) \leq 2^{-n}$ we may choose $M \in \mathbf{Z}$,

$M > N$ such that $2 \le \Sigma_{n=N}^{M} 2^n C(E_n) \le 4$. For each $n \in \mathbf{Z}$, with $N \le n \le M$, choose a positive measure ν_n with support in E_n, such that $\tilde{\nu}_n \le 1$ and $\frac{1}{2}C(E_n) \le \|\nu_n\| < C(E_n)$, and set $\alpha_N = \Sigma_{n=N}^{M} 2^n \nu_n$. Then α_N is a positive measure with support in $E \cap B(x, 2^{-N})$, and $1 \le \|\alpha_N\| \le 4$.

From the definition of E and Fubini's theorem we deduce

$$\delta \le \int |y - x| \tilde{\mu}(y)\, d\alpha_N(y) = \int G_N(z) d|\mu|(z),$$

where

$$G_N(z) = \int \frac{|y - x|}{|y - z|}\, d\alpha_N(y).$$

We claim there is a sequence $\{N_i\}_1^\infty \subset \mathbf{Z}$ such that G_{N_i} tends pointwise boundedly to some positive multiple of $\chi_{\{x\}}$, the characteristic function of $\{x\}$.

Fix $z \neq x$. Then, provided N is large enough to ensure that $2^{-N} < \frac{1}{2}|z-x|$, we have $1/|z-y| < 2/|z-x|$ for $y \in B(x, 2^{-N})$, hence

$$|G_N(z)| \le \frac{2}{|z - x|} \int |y - x|\, d\alpha_N(y) \le \frac{2^{1-N}\|\alpha_N\|}{|z - x|} \le \frac{2^{3-N}}{|z - x|} \to 0 \quad \text{as } N \to +\infty.$$

Also $G_N(x) = \|\alpha_N\|$, so $1 \le G_N(x) \le 4$ for every N, hence there is a sequence $\{N_i\}$ and a number $\beta \in [1, 4]$ such that G_{N_i} tends pointwise to $\beta\chi_{\{x\}}$.

Next, $|G_N(x)| \le 4$; also, each G_N is the potential of measure with support inside $B(x, 2^{-1})$, so it suffices to show that G_N is bounded on $B(x, 2^{-1})$. Accordingly, fix $z \in \mathbf{C}$ with $0 < |z-x| \le \frac{1}{2}$. Choose p such that $z \in A_p(x)$. For $n \neq p - 1$, p or $p + 1$, and $y \in A_n(x)$ we have $|y-z| \ge 2^{-n-2}$, $|y-x| \le 2^{-n}$,

$$2^n \int \frac{|y - x|}{|z - y|}\, d\nu_n(y) \le 4 \cdot 2^n \|\nu_n\|.$$

For $p - 1 \le n \le p + 1$, we have

$$2^n \int \frac{|y - x|}{|z - y|}\, d\nu_n(y) \le \int \frac{d\nu_n(y)}{|z - y|} = \tilde{\nu}_n(z) \le 1.$$

Hence

$$|G_N(z)| = \sum_{n=N}^{M} 2^n \int \frac{|y - x|}{|y - z|}\, d\nu_n(y) = \sum_{N}^{p-2} + \sum_{p-1}^{p+1} + \sum_{p+2}^{M}$$

$$\le 4 \sum_{N}^{p-2} 2^n \|\nu_n\| + 3 + 4 \sum_{p+2}^{M} 2^n \|\nu_n\| \le 3 + 4 \sum_{N}^{M} 2^n \|\nu_n\| \le 19.$$

So we have shown that G_N tends pointwise boundedly to $\beta\chi_{\{x\}}$. Hence

$$0 < \delta \le \int G_N(z) d|\mu|(z) \to \int \beta \chi_{\{x\}} d|\mu| = \beta |\mu\{x\}| = 0,$$

a contradiction.

Proof of Theorem 1. Suppose $x \in X$ is not a peak point for B and $a > 0$. Choose a representing measure for x with no mass at x, and set $\delta = a(a + 1 + \|\mu\|)^{-1}$, $E = \{y \in \mathbf{C}: |x-y|\tilde{\mu}(y) \ge \delta\}$. By Browder's lemma, $\mathbf{C} \backslash E \subset P(x, a)$, and by Lemma 2 and the monotonicity of C,

$$\sum_{n=1}^{+\infty} 2^n C(A_n(x) \backslash P(x, a)) \le \sum_{n=1}^{+\infty} 2^n C(A_n(x) \cap E) < +\infty.$$

Lemma 3. *Let τ be any measure, $0 < p \in \mathbf{Z}$, $0 < \eta \in \mathbf{R}$, $x \in \mathbf{C}$,*

$$E = \left\{ y \in \mathbf{C}: |y - x| \int \frac{|z-x|^p}{|z-y|} d|\tau|(z) \ge \eta \right\},$$
$$E_n = A_n(x) \cap E.$$

Then $\sum_{n=1}^{+\infty} 2^{(p+1)n} C(E_n) < +\infty$.

We omit the proof, since it follows the same lines as that of Lemma 2.

Proof of Theorem 2. Let $x \in X$, p be a positive integer, and suppose B admits a pth order bounded point derivation at x. Choose a representing measure τ for the derivation. Then $\mu = (z-x)^p \tau / p!$ represents x, hence by Browder's lemma, if $a > 0$, $\delta = p! a(a + 1 + \|\mu\|)^{-1}$, then

$$\mathbf{C} \backslash P(x, a) \subset \{y \in \mathbf{C}: |x-y|\tilde{\tau}(y) \ge \delta\} = \left\{ y \in \mathbf{C}: |y - x| \int \frac{|z-x|^p}{|z-y|} d|\tau|(z) \ge \delta p! \right\};$$

and so, by Lemma 3 and the monotonicity of C,

$$\sum_{n=1}^{+\infty} 2^{(p+1)n} C(A_n(x) \backslash P(x, a)) < +\infty.$$

Proof of Corollary 1(a). The conclusion of Theorem 1 implies the conclusion of the corollary, as follows. Let us abbreviate $P(x, a) = P$, and assume

$$\limsup_{r \to 0} \frac{C(B(x, r) \backslash P)}{r} > \kappa > 0.$$

Choose a sequence of positive numbers r_n, tending to 0, such that $C(B(x, r_n) \backslash P) \ge \kappa r_n$, and for each n choose an integer N_n such that $2^{-N_n - 1} \le r_n \le 2^{-N_n}$. Then $N_n \to +\infty$ as $n \to +\infty$, and

$$\sum_{m=N_n}^{+\infty} 2^m C(A_n(x)\setminus P) \geq 2^{N_n} \sum_{m=N_n}^{+\infty} C(A_m(x)\setminus P) \geq 2^{N_n} C(B(x, 2^{-N_n})\setminus P)$$

$$\geq C(B(x, r_n)\setminus P)/2r_n \geq \tfrac{1}{2}\kappa,$$

hence $\Sigma_{m=1}^{+\infty} 2^m C(A_m(x)\setminus P) = +\infty$. Here we used the fact that C is subadditive [17].

Proof of Corollary 1(b). If $E \subset \mathbf{C}$, h is a measure function, $\int_0 dh(t)/t < +\infty$, and $k(r) = \int dh(t)/t$, then $C(E) \geq M_h(E)/k(\operatorname{diam} E)$ (for a proof see [14]) and hence the assertion of 1(b) follows from 1(a).

Proof of Corollary 1(c). If $\beta > 1$, and $E \subset \mathbf{C}$, then $C(E) \geq \kappa\, M^{\beta}(E)^{1/\beta}$, where κ depends only on β [14].

Proof of Corollary 1(d). If $E \subset \mathbf{C}$, then $C(E) \geq (4\pi)^{-1/2}\mathfrak{L}^2(E)^{1/2}$. This is proved, in essence, in [3, p. 150]. Thus whenever

$$\lim_{r\to 0} \frac{C(B(x, r)\setminus P(x, a))}{r} = 0,$$

it follows that

$$\lim_{r\to 0} \frac{\mathfrak{L}^2(B(x, r)\setminus P(x, a))}{r^2} = 0,$$

so that 1(d) follows from 1(c).

Corollary 2 follows from Theorem 2 in the same way as Corollary 1 follows from Theorem 1. We omit the details.

4. Example 1. Consider a *Zalcman L-set* [30] X, a compact set obtained by deleting from the closed unit disc a sequence of disjoint open balls with centers on $(0, 1)$, accumulating only at 0. If the balls are chosen sparse enough, then 0 is not a peak point for $R(X) = A(X)$ [30]. More precisely, if their centers are at $a_1, a_2, a_3, \cdots$ and their radii are $r_1, r_2, r_3, \cdots$, respectively, then 0 is not a peak point provided $\Sigma_{n=1}^{+\infty} r_n/a_n < +\infty$.

Assume 0 is not a peak point, and fix $0 < a < 2$, and apply Theorem 1 with $A = R(X)$. Then $P(0, a)$ is roughly heart-shaped (its complement is not connected), and $P(0, a) \cap \partial X = \{0\}$, so since the Gleason metric is continuous on $\mathring{X}$ [23], it follows that the set $U = G(0, a)\setminus\{0\}$ is open. Also $G(0, a) \supset P(0, a/2)$, so by Theorem 1, $\Sigma_{n=1}^{+\infty} 2^n C(A_n(0)\setminus U) < +\infty$. Now $E = \mathbf{C}\setminus U$ is closed and has a fragmented "spike" at 0. Look at the plane as embedded in $\mathbf{R}^3$ and form an open set $V \in \mathbf{R}^3$ by deleting the "flat spike" E from the open unit ball in $\mathbf{R}^3$. Then, denoting $A_n^3(0) = \{x \in \mathbf{R}^3 \colon 2^{-n-1} \leq |x| \leq 2^{-n}\}$ we have $\Sigma_{n=1}^{+\infty} 2^n C(A_n^3(0)\setminus V) < +\infty$. Now this is just *Wiener's criterion* [28] that 0 be a nonregular point for the

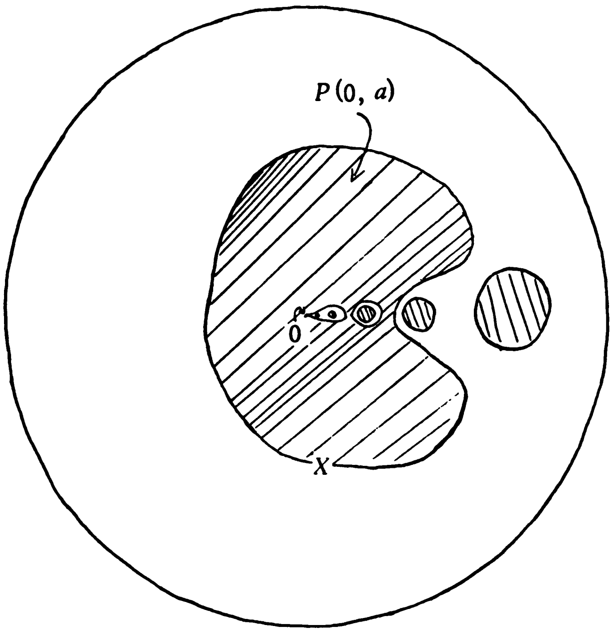

Figure 1

solution of the Dirichlet problem in V, i.e. 0 is not a peak point for the space $D(V)$ of functions harmonic in V and continuous on $\overline{V}$, and the Dirichlet problem cannot be solved in V.

For general X, $G(x, a)\setminus\{a\}$ is not open, so this discussion does not apply.

Example 2. There is a connection between Lemma 2 and the *fine topology* of potential theory [17]. The fine topology on $\mathbf{R}^3$ is the smallest topology on $\mathbf{R}^3$ in which all superharmonic functions are continuous $\overline{\mathbf{R}}$-valued functions. Let us adopt the notation of Lemma 2, and think of the plane as embedded in $\mathbf{R}^3$. The functions $\tilde{\mu}(y)$ and $|x-y|^{-1}$ make sense for $y \in \mathbf{R}^3$, and are superharmonic, so that $N = \{y \in \mathbf{R}^3 : \tilde{\mu}(y) < \delta|x-y|^{-1}\}$ is a fine-open set, and N is a fine neighborhood of x if and only if $\tilde{\mu}(x) < +\infty$. A theorem on the fine topology [17, p. 220] states that if N is a fine neighborhood of x, then $\sum_{n=1}^{+\infty} 2^n C(A_n^3(x)\setminus N) < +\infty$.

Thus, if we add to the hypotheses of Lemma 2 the assumption $\tilde{\mu}(x) < +\infty$, then the conclusion of the lemma follows from this theorem, for then $\mathbf{R}^3\setminus V \supset \mathbf{C}\setminus V = E$, and

$$\sum_{n=1}^{+\infty} 2^n C(A_n(x) \cap E) \le \sum_{n=1}^{+\infty} 2^n C(A_n^3(x) \cap E) \le \sum_{n=1}^{+\infty} 2^n C(A_n^3(x)\setminus V) < +\infty.$$

However, if $\tilde{\mu}(x) < +\infty$, then $\tau = (z-x)^{-1}\mu$ is a Radon measure, and for $y \in \mathbf{C}$,

$$|y - x|\tilde{\tau}(y) = |y - x| \int \frac{|z - x|}{|z - y|} \, d|\tau|(z),$$

so Lemma 3 applies (with $p = 1$), and $\sum_{n=1}^{+\infty} 4^n C(A_n(x) \cap E) < +\infty$. Thus our methods yield an even stronger result under the new hypothesis.

In terms of the algebra A this shows that the fine topology merely allows us to deduce *the conclusion of Theorem* 1 *under the hypothesis of Theorem* 2 (with $p = 1$). In fact, as Wilken first observed, A admits a (first order) bounded point derivation at x if and only if x has a representing measure μ with $\tilde{\mu}(x) < +\infty$.

Example 3. It may be asked whether Newtonian capacity C may be replaced by analytic capacity γ in Theorem 1 or Theorem 2. The answer is no. For a compact set $K \subset \mathbf{C}$,

$$\gamma(K) = \sup\{|f'(\infty)|: f \text{ is analytic off } K, \; \|f\|_{\mathbf{C}\setminus K} \leq 1\}.$$

Mel'nikov's theorem [30] states that a point a in a compact set $Y \subset \mathbf{C}$ is a peak point for $R(Y)$ if and only if $\sum_{n=1}^{+\infty} 2^n \gamma(A_n(x) \setminus Y) < +\infty$. If C could be replaced by γ in the conclusion of Theorem 1, then the theorem would imply that a nonpeak point for $R(X)$ is a nonpeak point for $R(P(x, a))$. This cannot be so, for Gamelin and Garnett [12] have constructed an example of a set X such that 0 is not a peak point for $R(X)$, but $P(0, 19^{-1})$ meets none of the circles $\{z \in \mathbf{C}: |z| = 2^{-n}\}$, $n = 1, 2, 3, \cdots$, hence $P(0, 19^{-1})$ consists of a collection of closed disjoint bands surrounding 0, hence by Bishop's $1/4 - 3/4$ criterion [9], 0 is a peak point for $R(P(0, 19^{-1}))$.

Fix $1 < p \in \mathbf{Z}$. By use of Hallstrom's capacity criterion for the existence of a bounded point derivation [16], the Gamelin-Garnett example may be modified so as to ensure that $R(X)$ admits a pth order bounded point derivation at 0, while $P(0, 19^{-1})$ remains disconnected at 0, so that by Mel'nikov

$$\sum_{n=1}^{+\infty} 2^n \gamma(A_n(x) \setminus P(0, 19^{-1})) = +\infty,$$

and a fortiori,

$$\sum_{n=1}^{+\infty} 2^{(p+1)n} \gamma(A_n(x) \setminus P(0, 19^{-1})) = +\infty.$$

Thus C cannot be replaced by γ in Theorem 2. Similar reasoning shows that C cannot be replaced by α, the *continuous analytic capacity* [9].

Example 4. Easy examples show that Corollary 1(a) is stronger than 1(b), 1(b) is stronger than 1(c), etc., and similarly for Corollary 2.

We give an example which may provide some geometric "feel" for Theorem 1.

Fix $x \in \mathbf{C}$, and take any sequence of nonnegative numbers $r_n \leq 1/4$ with $\sum_{n=1}^{+\infty} r_n = +\infty$. Let $D_n \subset A_n(x)$ be any open disc with radius $r_n/2^n$ and let $D = \sum_{n=1}^{+\infty} D_n$. Then $C(A_n \cap D) = C(D_n) \geq r_n 2^{-n-1}$, so $\sum_{n=1}^{+\infty} 2^n C(A_n \cap D) \geq \frac{1}{2}\sum_{n=1}^{+\infty} r_n = +\infty$. Hence if X is a compact set, and X is not a peak point for A, then D meets $P(x, a)$ for every $a > 0$, and in particular D contains nonpeak points.

If we choose $r_n = (3+n)^{-1}$, then D has zero area density at x. More than that, $\lim_{r \to 0}[C(B(x, r) \cap D)/r] = 0$, as is easily seen, so the conclusion we have drawn cannot be deduced from Corollary 1.

In case A admits a pth order bounded point derivation at x, we may modify the construction of D by taking radius $(D_n) = r_n/2^{(p+1)n}$, and still conclude that D meets $P(x, a)$.

5. By means of a trick due to Gamelin and Garnett, these results may be extended to cover the algebra $H^\infty(U)$, for a bounded open set $U \subset \mathbf{C}$, a Banach algebra in L^∞-norm. Let $\mathfrak{M}$ denote the space of multiplicative linear functionals on $H^\infty(U)$ with its usual compact Hausdorff topology [19], and let $\mathfrak{M}_x$ denote the fibre in $\mathfrak{M}$ over a point $x \in \overline{U}$, with respect to the natural projection $\phi \mapsto \phi(z)$ of $\mathfrak{M} \to \overline{U}$ [12]. If $x \in U$, then $\mathfrak{M}_x = \{\phi_x\}$ consists of just one point, but for $x \in \partial U$, $\mathfrak{M}_x$ may be very large. $\mathfrak{M}$ may be regarded as an extension of U, and all the functions of $H^\infty(U)$ as continuous functions on $\mathfrak{M}$. The *part metric* of $H^\infty(U)$ on $\mathfrak{M}$ is defined by

$$\partial(\phi, \psi) = \sup\{|f(\phi) - f(\psi)|: f \in H^\infty(U), \|f\|_U \leq 1\}$$

whenever $\phi, \psi \in \mathfrak{M}$, and a part of $H^\infty(U)$ is an equivalence class under the relation on $\mathfrak{M}$ given by $\phi \sim \psi \Leftrightarrow \partial(\phi, \psi) < 2$. We set

$$\Pi(\phi, a) = \{z \in U: \partial(\phi, \phi_z) \leq a\}$$

for $\phi \in \mathfrak{M}$ and $a > 0$. $\Pi(\phi, a)$ is the intersection with U of a part neighborhood. A fibre $\mathfrak{M}_x$ is a *peak fibre* if there is a function $f \in H^\infty(U)$ such that $f = 1$ on $\mathfrak{M}_x$ and $|f| < 1$ on $\mathfrak{M} \setminus \mathfrak{M}_x$. Gamelin and Garnett [12] studied the conditions under which $\mathfrak{M}_x$ is a peak fibre. They showed that, while any representing measure on $\mathfrak{M}$ for a point of a peak fibre $\mathfrak{M}_x$ must have mass on $\mathfrak{M}_x$, a nonpeak fibre $\mathfrak{M}_x$ always contains a unique homomorphism, ϕ_x, distinguished by the property of having a representing measure with no mass on $\mathfrak{M}_x$. They gave a capacity condition on the behavior of U near x which is necessary and sufficient for $\mathfrak{M}_x$ to be a peak fibre, and in the course of the proof (p. 459) they showed that *if* $\mathfrak{M}_x$ *is not a peak fibre for* $H^\infty(U)$, *then there exists a compact set* $X \subset U \cup \{x\}$ *such that* x *is not a peak point for* $R(X)$. They also observed (pp. 457–458) that *each function* $f \in H^\infty(U)$

may be approximated pointwise on $\mathfrak{M}\setminus\mathfrak{M}_x$ *by a sequence of functions* $f_n \in H^\infty(U)$, *each of which extends analytically to a neighborhood of x and satisfies* $\|f_n\|_U \leq 17\|f\|_U$. This implies that, denoting by d the Gleason metric of the above $R(X)$, we have $\partial(\phi_y, \phi_x) \leq 17d(y, x)$, hence $P(x, 17^{-1}a) \subset \Pi(\phi_x, a) \cup \{x\}$. If we now apply Theorem 1 we obtain a part density theorem for $H^\infty(U)$. This improves on [12, 3.5].

Theorem 5. *If* $\mathfrak{M}_x$ *is not a peak fibre for* $H^\infty(U)$, *and* $a > 0$, *then*

$$\sum_{n=1}^{+\infty} 2^n C\left(A_n(x)\setminus\Pi(\phi_x, a)\right) < +\infty.$$

In [23] the notion of a *regular pth order bounded point derivation* on $H^\infty(U)$ at a point $\phi \in \mathfrak{M}$ is introduced, and it is shown that such derivations exist only at distinguished homomorphisms. By the methods of the proof of Theorem 5 of that paper one may show that *if* $H^\infty(U)$ *admits a regular pth order bounded point derivation at* ϕ_x *then there is a compact set* $X \subset U \cup \{x\}$ *such that* $R(X)$ *admits a pth order bounded point derivation at x*. Hence Theorem 2 can be applied, and the following theorem results.

Theorem 6. *If* $H^\infty(U)$ *admits a pth order regular bounded point derivation at* ϕ_x, *and* $a > 0$, *then*

$$\sum_{n=1}^{+\infty} 2^{(p+1)n} C\left(A_n(x)\setminus\Pi(\phi_x, a)\right) < +\infty.$$

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