TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 197, 1974

PERIODIC SOLUTIONS OF $x'' + g(x) + \mu h(x) = 0$

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ABSTRACT. Necessary and sufficient conditions for $x'' + f(x) = 0$ to admit at least one nontrivial periodic solution are given. The results are applied to $x'' + g(x) + \mu h(x) = 0$, $x(0) = A$, $x'(0) = 0$ in order to characterize those regions of the (μ, A)-plane for which nontrivial periodic solutions exist. A converse theorem is given, together with some illustrative examples.

1. Introduction. The problem of determining the existence of nontrivial periodic solutions to the equation

$$x'' + f(x) = \epsilon p(t) \tag{1}$$

has received considerable attention. In order to investigate this problem, it is necessary to consider the problem of determining periodic solutions of the corresponding unforced equation

$$x'' + f(x) = 0. \tag{2}$$

The existence of periodic solutions of (2) has been studied, for example, by Loud [7], Opial [12], Cesari [2] and Utz [16].

Recently, Maekawa [9] has considered the construction of periodic solutions of the equation

$$x'' + x + \mu x^2 = \epsilon \cos wt, \tag{3}$$

with the initial condition $x(0) = A$ (> 0), $x'(0) = 0$.

In the case that $\epsilon = 0$, he determined that periodic solutions exist if $0 < \mu A < \frac{1}{2}$.

In the following section, we shall obtain necessary and sufficient conditions for the existence of periodic solutions of equation (2) under rather general

Presented to the Society, January 17, 1972; received by the editors May 11, 1973 and, in revised form, September 24, 1973.

AMS (MOS) subject classifications (1970). Primary 34C25.

Key words and phrases. Ordinary differential equation, periodic solution, boundary curves, admissible sets.

(1) The research for this paper was partially supported by the National Research Council of Canada, Grant No. NRC-A-8130.

(2) The research for this paper was partially supported by the National Research Council of Canada, Grant No. NRC-A-4823.

conditions on f. In §3 we shall apply these results to characterize those points of the (μ, A)-plane for which the initial value problem

$$x'' + g(x) + \mu b(x) = 0, \tag{4}$$

$x(0) = A$, $x'(0) = 0$, $\mu > 0$ has at least one nontrivial periodic solution.

In §4, we state and prove a converse theorem, and in §5 several examples illustrating our results will be given.

Throughout this paper, the following notation will be used:

$$F(y) \equiv \int_0^y f(u)\,du, \qquad G(y) \equiv \int_0^y g(u)\,du, \qquad H(y) \equiv \int_0^y h(u)\,du.$$

$$\phi(\mu, y) \equiv G(y) + \mu H(y). \tag{5}$$

Further, in what follows, by periodic solution we shall always mean a nontrivial (i.e. nonconstant) periodic solution.

2. **Periodic solutions of $x'' + f(x) = 0$.** The following theorem generalizes known results (see, for example, [2]).

Theorem 1. *Let $f(x)$ be locally integrable on $(-\infty, \infty)$. Then a necessary and sufficient condition for there to be a periodic solution of* (2) *is that there exist real numbers α, β with $\alpha < \beta$ such that $F(\alpha) = F(\beta) > F(x)$ for $\alpha < x < \beta$.*

Proof of necessity. Assume that $z(t)$ is a periodic solution of (2) with period ω. The energy equation satisfied by $z(t)$ is

$$\tfrac{1}{2}z'^2(t) + F(z(t)) = \text{const} = E, \text{ say.} \tag{6}$$

Let $m = \min_{0 \le t \le \omega} z(t)$, $M = \max_{0 \le t \le \omega} z(t)$. Then $m < M$. Let $t_1 = \inf\{t\colon t \ge 0$ and $z(t) = m\}$ and let $t_2 = \inf\{t\colon t \ge t_1$ and $z(t) = M\}$. We have $F(m) = F(M) = E \ge F(x)$ for $m < x < M$. Choose t_0 with $t_1 < t_0 < t_2$ such that $z'(t_0) > 0$. Let (t_3, t_4) be the maximal interval I about t_0 for which $z'(t) > 0$. Then $t_1 \le t_3 < t_0 < t_4 \le t_2$ and $z'(t_3) = z'(t_4) = 0$. Defining α to be $z(t_3)$, $\beta = z(t_4)$, the necessity follows.

Before proving the sufficiency, we require the following lemma, which is related to Sard's theorem [10], [11], [14], but does not appear to exist in the literature in the form in which we need it.

Lemma 1. *Let $F(x) \in AC[a, b]$. Let $C = \{y$: there exists $x \in [a, b]$ such that $y = F(x)$, $F'(x) = 0\}$ (the critical range of F). Then C has measure zero.*

Proof. Define $T = \{x\colon x \in [a, b]$ and $F'(x) = 0\}$. Let $\epsilon > 0$ be given. Since $F(x) \in AC[a, b]$, there exists $\delta > 0$ such that $\int_M |F'(x)|\,dx < \epsilon$ for any measurable set M with meas $M < \delta$. Let $I_1, I_2, \cdots, I_n$ be interior disjoint intervals such

that $T \subset I = \bigcup_{j=1}^{n} I_j$ and meas $T > \sum_{j=1}^{n}$ (meas I_j) $- \delta$. Then $\int_I |F'(x)|\,dx = \int_{I \setminus T} |F'(x)|\,dx < \epsilon$ ($\setminus$ denotes complement). But

$$\int_I |F'(x)|\,dx \geq \sum_{j=1}^{n} \left(\max_{x \in I_j} F(x) - \min_{x \in I_j} F(x) \right) = \sum_{j=1}^{n} \text{meas } D_j,$$

where $D_j = [\min_{x \in I_j} F(x), \max_{x \in I_j} F(x)]$. However $y \in C$ implies that $y = F(x)$ for some x belonging to some I_j, and hence $y \in D_j$ for some j. Thus $\bigcup_{j=1}^{n} D_j \supset C$, and meas $C < \epsilon$. Since $\epsilon > 0$ is arbitrary, the lemma follows.

Proof of sufficiency. Let $\alpha < \gamma < \beta$ such that $\Gamma = F(\gamma) = \min_{[\alpha,\beta]} F(x)$. Let $E_0 = F(\alpha) = F(\beta)$ and for $\Gamma < y \leq E_0$, define $\xi(y)$ to be $\sup\{x\colon \alpha \leq x \leq \gamma$ and $F(x) = y\}$ and $\eta(y)$ to be $\inf\{x\colon \gamma \leq x \leq \beta$ and $F(x) = y\}$. Then $F(\xi(y)) = F(\eta(y)) > F(x)$ whenever $\xi(y) < x < \eta(y)$. By Lemma 1, there is a y^* with $\Gamma < y^* \leq E_0$ such that for every x for which $F(x) = y^*$, we have $F'(x) = f(x) \neq 0$. Define α^* to be $\xi(y^*)$ and β^* to be $\eta(y^*)$. Then, for $\alpha^* < x < \beta^*$, we have $F(\alpha^*) = F(\beta^*) > F(x)$, and, moreover,

$$F'(\alpha^*)F'(\beta^*) = f(\alpha^*)f(\beta^*) \neq 0. \tag{7}$$

Furthermore it is easily seen that $f(\alpha^*) < 0$, while $f(\beta^*) > 0$. We consider the initial value problem

$$x'' + f(x) = 0, \qquad x(0) = \alpha^*, \qquad x'(0) = 0. \tag{8}$$

Define $\phi(t)$ on its maximal domain $D \subset [0, \infty)$ by $\int_{\alpha^*}^{\phi(t)} du/\sqrt{E_0 - F(u)} = t$. (7) implies that there exists $\delta > 0$, $c > 0$ such that for $\alpha^* \leq u \leq \alpha^* + \delta$, we have $E_0 - F(u) \geq c(u - \alpha^*)$, and for $\beta^* - \delta \leq u \leq \beta^*$, we have $E_0 - F(u) \geq c(\beta^* - u)$. For $\alpha^* + \delta \leq u \leq \beta^* - \delta$, we have $E_0 - F(u) > c\delta$. It follows that

$$\int_{\alpha^*}^{\beta^*} \frac{du}{\sqrt{E_0 - F(u)}} \leq 4\,\frac{8}{c} + \frac{\beta^* - \alpha^*}{\sqrt{c\delta}}$$

which implies that $\phi(t)$ obtains the value β^* for $t = \frac{1}{2}T$, where $T > 0$. Defining $\phi(t)$ to be $\phi(T - t)$, $\frac{1}{2}T < t \leq T$, and then extending it periodically we obtain a periodic solution of (2). This completes the proof of the theorem.

Remark. Had we assumed the existence of solutions of IVP's for (2), the existence of a periodic solution would have followed from (7) and classical considerations. The following corollary will prove useful in the sequel.

Corollary 1. *Let $f(x)$ be continuous and assume uniqueness of IVP's for (2). Let $f(\alpha) < 0$ ($f(\alpha) > 0$). Then a necessary and sufficient condition for the solution of $x'' + f(x) = 0$, $x(0) = \alpha$, $x'(0) = 0$ to be periodic is that there exist $\beta > \alpha$ ($\beta < \alpha$) such that $F(\alpha) = F(\beta) > F(y)$ for $\alpha < y < \beta$ ($\beta < y < \alpha$).*

3. **Admissible regions for equation (4).** Now we shall concern ourselves with those regions of the $\mu - A$ plane, $\mu > 0$, for which equation (4) has a periodic solution.

Definition 1. The pair (μ, A) is said to be admissible ($(\mu, A) \in \mathfrak{A}$) if equation (4) has a periodic solution.

In the rest of this paper we shall assume the following: (i) $g(x)$, $h(x)$ are continuous for all x; (ii) solutions of equation (4) are unique.

Theorem 2. *$\mathfrak{A}$ is open.*

Proof. Let $(\mu_0, A_0) \in \mathfrak{A}$. By Corollary 1, there exists B_0 such that $\phi(\mu_0, A_0) = \phi(\mu_0, B_0) > \phi(\mu_0, y)$, for y between A_0 and B_0. Assume, without loss of generality, that $B_0 < A_0$. Then $\partial\phi(\mu_0, B_0)/\partial y < 0$ and $\partial\phi(\mu_0, A_0)/\partial y > 0$. Hence there exist C_1, C_2, δ_1, $\delta_2 > 0$ such that if $|\mu - \mu_0| < \delta_1$, $|B - B_0| < \delta_1$, then $\partial\phi(\mu, B)/\partial y < -C_1 < 0$, and $\partial\phi(\mu, y)/\partial\mu < C_2$, provided $B_0 - \delta_2 \leq y \leq (A_0 + B_0)/2$. Hence, if $|\mu - \mu_0| < \delta_1$, $|B - B_0| < \delta_2 \leq \delta_1$, and $B_0 - \frac{1}{2}\delta_2 \leq y \leq \frac{1}{2}(A_0 + B_0)$, then $\phi(\mu, B) > \phi(\mu, y)$, for $B < y$. A similar argument shows that there exists a $\delta_3 > 0$ such that if $|\mu - \mu_0| < \delta_3$, $|A - A_0| < \delta_3$ and $\frac{1}{2}(A_0 + B_0) \leq y < A$, then $\phi(\mu, A) > \phi(\mu, y)$. There exists $\delta_4 > 0$ such that $\delta_4 \leq \min\{\delta_2, \delta_3, C_1\delta_2/4C_2\}$ for which $|\mu - \mu_0| < \delta_4$, $|A - A_0| < \delta_4$ implies that $|\phi(\mu, A) - \phi(\mu_0, A_0)| < C_1\delta_2/4$. For such μ, we have that $\phi(\mu, B_0 - \delta_2/2) - \phi(\mu_0, B_0) > \frac{1}{2}C_1\delta_2 - C_2\delta_4 > \frac{1}{4}C_1\delta_2$, since in this range $\partial\phi(\mu, B)/\partial y < -C_1$ and $|\partial\phi(\mu, y)/\partial\mu| < C_2$. Similarly $\phi(\mu_0, B_0) - \phi(\mu, B_0 + \frac{1}{2}\delta_2) > \frac{1}{4}C_1\delta_2$. Hence there exists B with $|B - B_0| < \frac{1}{2}\delta_2$ such that $\phi(\mu, A) = \phi(\mu, B)$. Further $\partial\phi(\mu, B)/\partial y < 0$, since $|B - B_0| < \delta_2$ and $|\mu - \mu_0| < \delta_1$. The theorem is now proved.

Definition 2. Let $(\mu_0, A_0) \in \partial\mathfrak{A}$ such that $\mu_0 > 0$. Then we shall say that (μ_0, A_0) is of type I if $\partial\phi(\mu_0, A_0)/\partial y = 0$; is of type II if it is in $\mathrm{cl}\{(\mu, A) \in \partial\mathfrak{A}$: there exists $B \neq A$ such that $\phi(\mu, B) = \phi(\mu, A)$ and $\partial\phi(\mu, B)/\partial y = 0$ and (μ, A) is not of type I$\}$; and is of type III if it is in $\mathrm{cl}\{(\mu, A) \in \partial\mathfrak{A}$: $\phi(\mu, A) > \phi(\mu, y)$, either for all $y > A$ or for all $y < A$, and (μ, A) is not of type I or II$\}$.

Theorem 3. *Let $(\mu_0, A_0) \in \partial\mathfrak{A}$. Then one of the following is true: $\mu = 0$, or (μ, A) is of one of the types* I, II *or* III.

Proof. Suppose that none of the first three alternatives holds. Since $(\mu, A) \in \partial\mathfrak{A}$, there is a sequence $(\mu_n, A_n) \in \mathfrak{A}$ with $(\mu_n, A_n) \to (\mu, A)$. By the corollary to Theorem 1, there exist $B_n \neq A_n$ such $\phi(\mu_n, A_n) = \phi(\mu_n, B_n) > \phi(\mu_n, y)$ for $A_n < y < B_n$ (without loss of generality, we may assume that $A_n < B_n$). If a subsequence B_{n_k} converges to B, then $A \leq B$. By continuity, $\phi(\mu, A) = \phi(\mu, B) \geq \phi(\mu, y)$ for $A \leq y \leq B$. Since $\partial\phi(\mu, A)/\partial y$ and $\partial\phi(\mu, B)/\partial y$ have opposite sign (otherwise (μ, A) would be of type I or II) we have $A < B$; if there exists C with $A < C < B$ and $\phi(\mu, A) = \phi(\mu, C)$, then we must have $\partial\phi(\mu, C)/\partial y = 0$ and again

(μ, A) is of type II. Hence $\phi(\mu, A) > \phi(\mu, y)$ for $A < y < B$. But this implies that $(\mu, A) \in \mathfrak{A}$ which is a contradiction. It follows, therefore, that the sequence $B_n \to +\infty$, and so $\phi(\mu, A) > \phi(\mu, y)$ for $y > A$ (equality is again excluded as (μ, A) would be of type II). Thus (μ, A) is of type III, and the theorem is proved.

It should be noted that a point of ∂A may be simultaneously more than one of the types I, II or III.

We now proceed to investigate the nature of the boundary of $\mathfrak{A}$. In the next several theorems we show that under suitable hypotheses, boundary points exclusively of a given type (I, II or III) are interior to a continuous arc Γ of such points. Suppose that $xg(x) > 0$ for $x \neq 0$. The existence of periodic solutions in the case $\mu = 0$ has been much discussed in the literature (see, for example, [2] – [4], [6], [7], [9], [12]). Clearly nothing additional is obtained when $\mu > 0$, $xh(x) > 0$ for $x \neq 0$. Thus the case $h(x) > 0$ (or $h(x) < 0$) for all x will be of some interest. For this case we show that the above mentioned arc is *strictly decreasing*, where by strictly decreasing we mean $\Gamma = \{(\mu(s), A(s)): s_0 \leq s < s_1\}$ with $\mu(s)$ monotone increasing and $A(s)$ strictly decreasing.

Theorem 4. (a) *Let (μ_0, A_0) be a point of type* I *exclusively such that $h(A_0) \neq 0$. Then it is relatively interior to a continuous arc of such points.*

(b) *Let (μ_0, A_0) be a point of type* II *exclusively, such that $g(x)$, $h(x)$ are continuously differentiable in some neighbourhood of B_0 (see Definition* 2) *and such that*

$$h(B_0)[H(B_0) - H(A_0)] \frac{d}{dB}\left(\frac{g(B)}{h(B)}\right)\bigg|_{B=B_0} \neq 0.$$

Then (μ_0, A_0) is relatively interior to a continuous arc of points of type II.

(c) *Let (μ_0, A_0) be of type* III *exclusively, and suppose that $\phi(\mu_0, A_0) > \phi(\mu_0, y)$ for all $y > A_0$. Assume that $\sup_{0 \leq y < \infty} H(y) = \hat{H}$, and $\sup_{0 \leq y < \infty} G(y) = \hat{G}$ are finite, and that $\sup_{0 \leq x < \infty} \phi(\mu, x) = \hat{G} + \mu\hat{H}$ in some neighbourhood of μ_0. Then (μ_0, A_0) is relatively interior to a continuous arc of points of type* III. *A corresponding result holds if $\phi(\mu_0, A_0) > \phi(\mu_0, y)$ for all $y < A_0$.*

Proof. (a) (μ_0, A_0) is of type I implies that $g(A_0) + \mu_0 h(A_0) = 0$. Since $h(A_0) \neq 0$, we have, by the continuity of h, that there is a neighbourhood of A_0 in which $h(A) \neq 0$. In this neighbourhood define $\mu(A)$ to be $-g(A)/h(A)$. This defines a continuous arc Γ containing (μ_0, A_0) in its relative interior. We note that $\Gamma \cap \mathfrak{A} = \emptyset$. Let $N_{\epsilon,\delta}$ be the rectangular neighbourhood $\{(\mu, A): |\mu - \mu_0| < \epsilon, |A - A_0| < \delta\}$. Since $(\mu_0, A_0) \in \partial\mathfrak{A}$, $N_{\epsilon,\delta} \cap \mathfrak{A} \neq \emptyset$; also $N_{\epsilon,\delta} \cap \mathfrak{A}^c$ contains points other than (μ_0, A_0). It follows that $N_{\epsilon,\delta} \cap \partial\mathfrak{A}$ contains points other than (μ_0, A_0). Since (μ_0, A_0) is a point exclusively of type I, it follows from considerations of continuity that we may choose δ, ϵ so small that $\partial\mathfrak{A} \cap N_{\epsilon,\delta}$ con-

sists of points exclusively of type I. Since all points of type I satisfy $g(A) + \mu h(A) = 0$, it follows that $\partial\mathfrak{A} \cap N_{\epsilon,\delta} \subset \Gamma$ (here we use $h(A) \neq 0$). Since Γ is in fact the graph of a continuous function, it is clear that we may, in addition, choose δ, ϵ so small that $N_{\epsilon,\delta} \cap \Gamma$ is a continuous arc (the graph of a continuous function) containing (μ_0, A_0) in its relative interior. If $(\mu_1, A_1) \in N_{\epsilon,\delta} \cap \Gamma$, $(\mu_2, A_2) \in N_{\epsilon,\delta} \cap \mathfrak{A}$, we may join (μ_1, A_1), (μ_2, A_2) by a continuous arc intersecting Γ only at (μ_1, A_1). Since this arc must intersect $\partial\mathfrak{A}$, it follows that $(\mu_1, A_1) \in \partial\mathfrak{A}$. Thus $\partial\mathfrak{A} \cap N_{\epsilon,\delta} = N_{\epsilon,\delta} \cap \Gamma$ and the proof of (a) is complete.

(b) We have $g(B_0) + \mu_0 h(B_0) = 0$ and $\phi(\mu_0, A_0) = \phi(\mu_0, B_0)$. Define $J(\mu, A, B)$ to be $\phi(\mu, B) - \phi(\mu, A)$. Since $h(B_0) \neq 0$, there exists $\delta > 0$ such that $h(B) \neq 0$ for $|B - B_0| < \delta$. For B in this interval, define $\mu(B)$ to be $-g(B)/h(B)$. Note that $\mu(B_0) = \mu_0$. Consider now $J(\mu(B), A, B)$.

$$J(\mu(B_0), A_0, B_0) = \phi(\mu_0, B_0) - \phi(\mu_0, A_0) = 0,$$

$$\frac{\partial J}{\partial B}(\mu(B), A, B) = \frac{\partial J}{\partial \mu}(\mu, A, B)\frac{d\mu(B)}{dB} + \frac{\partial J}{\partial B}(\mu, A, B)$$

$$= [H(B) - H(A)]\frac{d}{dB}\left(\frac{g(B)}{h(B)}\right) + g(B) + \mu(B)h(B).$$

Thus

$$\frac{\partial J}{\partial B}(\mu(B_0), A_0, B_0) = [H(B_0) - H(A_0)]\frac{d}{dB}\left(\frac{g(B_0)}{h(B_0)}\right) \neq 0,$$

by hypothesis. Hence, by the implicit function theorem, we can solve $J(\mu(B), A, B) = 0$ for $B = B(A)$ as a continuous function of A in a neighbourhood of A_0. Define $\mu(A)$ to be $\mu = (-g(B(A))/h(B(A)))$. Clearly, this defines a continuous arc of points containing (μ_0, A_0) in its relative interior. Furthermore, this is a unique such arc. The remainder of the proof follows as in part (a).

(c) First we show that our hypotheses imply that $\phi(\mu_0, A_0) = \sup_{A_0 \le y < \infty} \phi(\mu_0, y)$. For suppose $\phi(\mu_0, A_0) > \sup_{0 \le y < \infty} \phi(\mu_0, y)$. Define $\psi(\mu, A)$ to be $\phi(\mu, A) - \hat{G} - \mu\hat{H}$. Then $\psi(\mu_0, A_0) > 0$. Thus there exists a neighbourhood of (μ_0, A_0) in which $\psi(\mu, A) > 0$, i.e. $\phi(\mu, A) > \sup_{A \le y < \infty} \phi(\mu, y)$ in this neighbourhood. Hence there are no points of this neighbourhood in the set $\mathfrak{A}$, contradicting the fact that $(\mu_0, A_0) \in \partial\mathfrak{A}$.

It follows that $\phi(\mu_0, A_0) = \sup_{A_0 \le y < \infty} \phi(\mu_0, y)$, as stated.

Since $H(A_0) \neq \hat{H}$, choose $\delta > 0$ such that for $|A - A_0| < \delta$, we have $H(A) \neq \hat{H}$. Define $\mu(A)$ by $\mu(A) = -(G(A) - \hat{G})/(H(A) - \hat{H})$. Then $\phi(\mu, A) = G(A) + \mu H(A) = \hat{G} + \mu\hat{H} = \sup_{A \le y < \infty} \phi(\mu, y)$. Hence we have an arc of points containing (μ_0, A_0) in its relative interior. The remainder of the proof follows as in parts (a) and (b).

Remark 1. Some of the conditions required in the hypotheses of Theorem 4

are somewhat technical in nature, but they appear to be necessary for the method of proof adopted; for example in (a) if A_0 is an isolated zero of h, we may still define a continuous arc Γ by

$$\Gamma = \{(\mu, A) : \mu = -g(A)/h(A),\ A \neq A_0\} \cup \{(\mu_0, A_0)\};$$

however this may not be wholly contained in $\partial\mathcal{P}$.

Remark 2. The theorem may no longer hold if the assumption that (μ_0, A_0) is *exclusively* of type I, II or III is removed.

The case where $\mu = 0$, $xg(x) > 0$ for $x \neq 0$, and $A > 0$, has been extensively discussed. In the case that $\mu > 0$, $h(x) > 0$ when $x \neq 0$ is also of interest as was earlier remarked. In this case $(A > 0)$ (μ, A) cannot be a boundary point of type I. It is the purpose of the following theorem to obtain further information about points of type II.

Theorem 5. *Assume that $xg(x) > 0$ and $h(x) > 0$ for all $x \neq 0$. Let (μ_0, A_0) be a point of type* II *but not of type* III, *with $\mu_0 A_0 > 0$. Then there exists a continuous strictly monotone decreasing arc $\Gamma = \{(\mu, A(\mu)) : \mu_0 \leq \mu < \mu^*\}$ of such points, with $\lim_{u \to \mu^*} A(\mu) = 0$ if the maximal interval $[\mu, \mu^*)$ of definition of the arc is finite.*

Proof. Since (μ_0, A_0) is not of type III, there exists B_0 (< 0, by virtue of the hypotheses of the theorem) such that $\phi(\mu_0, B_0) = \phi(\mu_0, A_0)$. Choosing B_0 to be the largest value of B for which $\phi(\mu_0, B) = \phi(\mu_0, A_0)$, it follows that $\partial\phi(\mu_0, B_0)/\partial y = 0$, and $\phi(\mu_0, y) < \phi(\mu_0, A_0) = \phi(\mu_0, B_0)$ for $B_0 < y < A_0$.

Now $\phi(\mu, B_0) > 0$ for $\mu_0 \leq \mu < \mu^*$, say, and $\partial\phi(\mu, B_0)/\partial y = g(B_0) + \mu_0 h(B_0) + (\mu - \mu_0)h(B_0) > 0$ for $\mu > \mu_0$. Thus defining $B(\mu)$ to be (for $\mu_0 \leq \mu < \mu^*$) $\sup\{y : B_0 < y < 0 \text{ and } \phi(\mu, y) = \sup_{B_0 < y < 0} \phi(\mu, B)\}$, we have $B_0 < B(\mu) < 0$ and $\partial\phi(\mu, B(\mu))/\partial y = 0$. Clearly $B(\mu_0) = B_0$.

We shall show that $B(\mu)$ is nondecreasing. Let $\mu_0 \leq \mu_1 < \mu_2 < \mu^*$ and let $B(\mu_i) = B_i$. Suppose $B_1 > B_2$. Then

$$\phi(\mu_2, B_2) = G(B_2) + \mu_2 H(B_2) > G(B_1) + \mu_2 H(B_1) = \phi(\mu_2, B_1),$$

i.e. $G(B_2) - G(B_1) > \mu_2[H(B_1) - H(B_2)]$. However,

$$g(B_2) + \mu_1 h(B_2) < g(B_2) + \mu_2 h(B_2) = 0$$

and so $\phi(\mu_1, B_2) < \sup_{B_0 \leq y \leq 0} \phi(\mu_1, y) = \phi(\mu_1, B_1)$. Therefore

$$0 < G(B_2) - G(B_1) < \mu_1[H(B_1) - H(B_2)] < \mu_2[H(B_1) - H(B_2)]$$

giving a contradiction. Thus $B_1 \leq B_2$, and $B(\mu)$ is nondecreasing.

Next we show that $\phi(\mu, B(\mu))$ is continuous in μ. Let $\mu_i \uparrow \mu$ with $B(\mu_i) \uparrow$

$B(\mu) - \epsilon$, where $0 \le \epsilon < \infty$. Then $\phi(\mu_i, B(\mu_i)) \to \phi(\mu, B(\mu) - \epsilon)$ and by continuity, we have $\phi(\mu, B(\mu) - \epsilon) \ge \phi(\mu, y)$ for $B(\mu) - \epsilon \le y \le 0$. It follows from the definition of $B(\mu)$ that ϵ must be 0, and so $\phi(\mu, B(\mu))$ is continuous to the left. Now let $\mu_i \downarrow \mu$ with $B(\mu_i) \downarrow B(\mu) + \epsilon$ $(0 \le \epsilon < \infty)$. We have $\phi(\mu_i, B(\mu_i)) \to \phi(\mu, B(\mu) + \epsilon)$. If $\epsilon = 0$, there is nothing to prove; otherwise we have $\phi(\mu, B(\mu)) > \phi(\mu, B(\mu) + \epsilon)$, on account of the definition of $B(\mu)$. It follows that for i sufficiently large, we have $\phi(\mu_i, B(\mu_i)) < \phi(\mu_i, B(\mu))$. Since $B_0 \le B(\mu) < B(\mu_i) < 0$, this clearly contradicts the definition of $B(\mu_i)$. It now follows that $B(\mu)$ is also continuous to the right and hence continuous. For $\mu_0 < \mu < \mu^*$, we have

$$\begin{aligned}\phi(\mu, A_0) &> \phi(\mu_0, A_0) = \phi(\mu_0, B_0) = G(B_0) + \mu_0 H(B_0)\\ &= [G(B(\mu)) + \mu H(B(\mu))] + [(G(B_0) + \mu_0 H(B_0)) - (G(B(\mu)) + \mu_0 H(B(\mu)))]\\ &\quad + (\mu_0 - \mu)H(B(\mu))\\ &> G(B(\mu)) + \mu H(B(\mu)) > \phi(\mu, 0) = 0,\end{aligned}$$

since $(\mu_0 - \mu)$ and $H(B(\mu))$ are negative, and the expression in the second square brackets is positive on account of the definition of B_0.

It follows that the equation $\phi(\mu, A) = \phi(\mu, B(\mu))$ has at least one solution A with $0 < A < A_0$; in fact, exactly one, since the hypotheses of the theorem imply that $\phi(\mu, A)$ is strictly increasing for $A > 0$. We shall denote this solution by $A(\mu)$.

Let $\mu_0 < \mu_1 < \mu^*$ and let $A_1 = A(\mu_1)$, $B_1 = B(\mu_1)$. Define η_1, η_2 by

$$\eta_1 = \min_{A_1/2 \le y \le 3A_1/2} g(y), \qquad \eta_2 = \max_{A_1/2 \le y \le 3A_1/2} h(y).$$

Let $\epsilon > 0$ and choose $\delta > 0$ so that $\delta < \epsilon\eta_1/2\eta_2$, and $\phi(\mu_1, B_1) - \phi(\mu, B(\mu)) < \frac{1}{2}\epsilon$ whenever $|\mu - \mu_1| < \delta$. Then if $|A - A_1| > \epsilon$ and $|\mu - \mu_1| < \delta$, we have

$$|\phi(\mu_1, A_1) - \phi(\mu, A(\mu))| \ge \epsilon\eta_1 - \delta\eta_2 > \tfrac{1}{2}\epsilon;$$

however $|\phi(\mu_1, B_1) - \phi(\mu, B(\mu))| < \frac{1}{2}\epsilon$ which is a contradiction since $\phi(\mu, A(\mu)) = \phi(\mu, B(\mu))$, and $\phi(\mu_1, A_1) = \phi(\mu_1, B_1)$. It follows that $A(\mu)$ is continuous for $\mu_0 < \mu < \mu^*$. Now let $\mu_0 \le \mu_1 < \mu_2 < \mu^*$, and let $A_i = A(\mu_i)$, $B_i = B(\mu_i)$, $i = 1, 2$. We have

$$\phi(\mu_2, B_2) = G(B_2) + \mu_2 H(B_2) < G(B_2) + \mu_1 H(B_2) \le G(B_1) + \mu_1 H(B_1) = \phi(\mu_1, B_1).$$

It follows that $\phi(\mu_2, A_2) < \phi(\mu_1, A_1)$ and so $A_2 < A_1$. Thus $A(\mu)$ is strictly decreasing in μ and $A(\mu)$ is defined in $[\mu_0, \mu^*)$. An examination of the definition of μ^* reveals that we require only that there exists for each $\mu \in [\mu_0, \mu^*]$, a B with $B_0 < B < 0$ such that $\phi(\mu, B(\mu)) > 0$. Defining μ^*, therefore, so that this

interval is maximal, it follows that either $\mu^* = +\infty$, or $\mu^* < \infty$ and $\phi(\mu^*, B) \leq 0$ for $B_0 \leq B \leq 0$, and by continuity of ϕ and $A(\mu)$, we must have $\phi(\mu, A(\mu)) \to 0$ as $\mu \to \mu^*$. The hypotheses of the theorem will then imply that $\lim_{\mu\to\mu^*} A(\mu) = 0$. This completes the proof of the theorem.

Corollary 2. *The energy function $\phi(\mu, A)$ is decreasing along the arcs of type* II *defined in the above theorem.*

Proof. $\phi(\mu, A(\mu)) = \phi(\mu, B(\mu))$. So

$$\partial\phi(\mu, A(\mu))/\partial\mu = \partial\phi(\mu, B(\mu))/\partial\mu = [g(B(\mu)) + \mu h(B(\mu))]B'(\mu) + H(B(\mu))$$

for almost all μ, since $B(\mu)$ is monotone.

$$g(B(\mu)) + \mu h(B(\mu)) = 0 \quad \text{and so} \quad \partial\phi(\mu, A(\mu))/\partial\mu = H(B(\mu)) < 0.$$

4. A converse theorem. We now consider a converse problem, namely, given a decreasing continuous function $A(\mu)$, can functions $g(x)$, $h(x)$, $xg(x) > 0$, $h(x) > 0$ for $x \neq 0$ be found such that $A(\mu)$ is the boundary of the admissible set in the first quadrant, of $x'' + g(x) + \mu h(x) = 0$?

With some additional restrictions on $A(\mu)$, the next theorem shows that the answer is yes.

Theorem 6. *Let $A(\mu)$ be a positive, continuously differentiable function which is strictly decreasing for $\mu > 0$ such that $\lim_{\mu\to 0} A(\mu) = +\infty$ and $\lim_{\mu\to+\infty} A(\mu) = 0$. Then there exist $g(x)$, $h(x)$ with $g(-x) = -g(x)$ and $h(-x) = h(x) > 0$ for $x \neq 0$ such that*

$$x'' + g(x) + \mu h(x) = 0 \tag{9}$$

has $(\mu, A(\mu))$ as its only boundary points of $\mathfrak{A}$ in the interior of the first quadrant.

Proof. We wish to construct $g(x)$ and $h(x)$ with the above properties, such that $G(A) + \mu H(A) = G(B) + \mu H(B)$ and $g(B) + \mu h(B) = 0$, where $B = B(\mu) < 0$. Let there be a function $q(x)$, and constants $C_1 \neq 0$, C_2, such that

$$G(x) = C_1 q(x) g(x), \qquad H(x) = C_2 q(x) h(x). \tag{10}$$

Then $\mu H(B) = \mu C_2 q(B) h(B) = -C_2 q(B) g(B) = -C_2 G(B)/C_1$. We shall arrange that

$$\mu H(A) = C_3 G(A), \quad \text{for some constant } C_3. \tag{11}$$

From (10) and the definition of $G(x)$, $G(x) = C_1 q(x) G'(x)$. Solving gives

$$G(x) = G(x_0) \exp\left(C_1^{-1} \int_{x_0}^{x} q(s)^{-1}\, ds\right). \tag{12}$$

Similarly

$$H(x) = H(x_0) \exp\left(C_2^{-1} \int_{x_0}^{x} q(s)^{-1}\, ds\right), \tag{13}$$

and hence

$$H(x) = C_0 G^{\alpha}(x), \quad \text{where } C_0 = H(x_0)/G^{\alpha}(x_0),\ \alpha = C_1/C_2. \tag{14}$$

In order for (11) to be valid we want $G(A) = C_3^{-1}\mu H(A) = \mu C_3^{-1} C_0 G^{\alpha}(A)$. Hence $G^{\alpha-1}(A) = C_3 C_0^{-1}\mu^{-1}$, or

$$G(A) = (C_3 C_0^{-1}\mu^{-1})^{1/(\alpha-1)}. \tag{15}$$

Define

$$\psi(x) = \frac{1}{G(x_0)}\left(\frac{C_3}{C_0 A^{-1}(x)}\right)^{1/(\alpha-1)}, \tag{16}$$

where $A^{-1}(x)$ stands for the inverse function. Then

$$\psi(A(\mu)) = \frac{1}{G(x_0)}\left(\frac{C_3}{C_0\mu}\right)^{1/(\alpha-1)} = \frac{G(A)}{G(x_0)}$$

by (15). Hence, letting $G_0 = G(x_0)$, we define

$$G(x) = G_0\psi(x) = \left(\frac{C_3}{C_0 A^{-1}(x)}\right)^{1/(\alpha-1)}, \qquad x \geq 0, \tag{17}$$

and then by (14)

$$H(x) = C_0\left(\frac{C_3}{C_0 A^{-1}(x)}\right)^{\alpha/(\alpha-1)}, \qquad x \geq 0. \tag{18}$$

We extend the definition to all x by defining $G(-x) = G(x)$ and $H(-x) = -H(x)$. We observe that

$$q(x) = -\frac{C_0}{C_3}\,\frac{(\alpha-1)}{\alpha}\left(\frac{C_3}{C_0 A^{-1}(x)}\right)^{(\alpha-1)/\alpha}\frac{(A^{-1}(x))^2}{(A^{-1}(x))'}. \tag{19}$$

By the hypotheses on $A(x)$, $g(x)$ and $h(x)$ are well defined and continuous, and are such that $xg(x) > 0$, $x \neq 0$ and $h(x) > 0$, $x \neq 0$.

It remains to ensure that

$$G(A) + \mu H(A) = G(B) + \mu H(B) \quad \text{and} \quad g(B) + \mu h(B) = 0. \tag{20}$$

The first of the equations in (20) requires that

$$(1 + C_3)(C_3/C_0\mu)^{1/(\alpha-1)} = (1 - C_2/C_1)(-C_2/C_0C_1\mu)^{1/(\alpha-1)}, \tag{21}$$

the last factor on the right being obtained by combining $\mu H(B) = -C_2G(B)/C_1$ with (14). Equation (21) reduces after simplification to

$$C_3^{1/(\alpha-1)}(1 + C_3) = (1 - 1/\alpha)(1/\alpha)^{1/(\alpha-1)}, \tag{22}$$

which is a constraint on C_1, C_2, C_3. C_0 may be chosen arbitrarily. The second of equations (20) is satisfied automatically.

Since, by the constructed properties of $g(x)$ and $h(x)$, for μ, $A > 0$ there can occur boundary points only of type II, the theorem is proved.

5. **Examples.** We first give an example to show that the boundaries of $\mathfrak{A}$ can be very complicated.

Example 1. Let $\mathcal{C}$ be the obvious extension to the real line of the Cantor set (removing middle thirds) on $[0, 1]$. Let

$$g(x) = x, \qquad h(x) = \begin{cases} 0, & x \geq 0, \\ (\rho(x, \mathcal{C}) - 1)x, & x < 0, \end{cases} \tag{23}$$

where $\rho(x, \mathcal{C}) = \inf_{y \in \mathcal{C}} |x - y|$. Note $0 \leq \rho(x, \mathcal{C}) \leq 1/6$. Then

$$g(x) + \mu h(x) = \begin{cases} x, & x \geq 0, \\ x + \mu x(\rho - 1), & x < 0. \end{cases} \tag{24}$$

If $\mu < 1$, $\phi(\mu, x) > 0$, $x \neq 0$ and $\lim_{x \to -\infty} \phi(\mu, x) = +\infty$, and $g(x) + \mu h(x) \neq 0$ for $x \neq 0$ and hence all solutions are periodic.

For $\mu = 1$, $g(x) + \mu h(x) = 0$ for $x < 0$ and $x \in \mathcal{C}$. $\phi(\mu, x)$ is monotone decreasing for $x < 0$.

For $1 < \mu < 6/5$, there will be continuous boundary curves (of types I and II for $A < 0$ and of type II for $A > 0$) emanating from the Cantor set of boundary points for $\mu = 1$ and they must decrease in the case $A > 0$ to the μ-axis between $1 < \mu < 6/5$. For $\mu > 6/5$ there are no periodic solutions.

Note that using a Cantor set of positive linear measure, it is possible to obtain a boundary set of positive 2-dimensional measure.

Example 2. Consider the equation

$$x'' + x + \mu \sum_{i=1}^{n} C_i x^{2i} = 0, \qquad x(0) = A > 0,\ x'(0) = 0. \tag{25}$$

Theorem 7. *Let*

$$\theta(\mu, A) = 2\left[\sum_{i=1}^{n} (\mu C_i)^{2/(2i-1)}\right]\phi(\mu, A). \tag{26}$$

If $C_1 > 0$, $C_i \geq 0$ $(i = 2, \cdots, n-1)$, $C_n > 0$, *then* $\{(\mu, A) \mid \mu > 0, A > 0, \theta(\mu, A) < 1/3\} \subset \mathfrak{A}$. *Further if* $\alpha > 1/3$, *there exists* $(\mu_0, A_0) \in R^2 \backslash \mathfrak{A}$ *such that* $\theta(\mu_0, A_0) = \alpha$.

We first need the following lemma.

Lemma 2. *Let*

$$\tilde{\theta}(w) = \left(\sum_{i=1}^{n} w_i^{2/(2i-1)}\right)\left(1 - \sum_{i=1}^{n} \frac{2}{2i+1} w_i\right),$$

where $w = (w_1, w_2, \cdots, w_n)$. *If* $w_i \geq 0$ *and* $\Sigma_{i=1}^{n} w_i = 1$, *then* $\min \tilde{\theta}(w) = 1/3$.

Proof. If $n = 1$, then $\tilde{\theta}(w) = w_1^2(1 - 2/3)w_1$. $w_1 = 1$ implies that $\tilde{\theta}(w) = 1/3$. If $n = 2$, $w_1 + w_2 = 1$, then

$$\begin{aligned}
\tilde{\theta}(w) - \frac{1}{3} &= (w_1^2 + w_2^{2/3})\left(1 - \frac{2}{3}w_1 - \frac{2}{5}w_2\right) - \frac{1}{3} = [(1 - w_2)^2 + w_2^{2/3}]\left[\frac{1}{3} + \frac{4}{15}w_2\right] - \frac{1}{3} \\
&= [(1 - w_2)^2 + w_2^{2/3}]\frac{4}{15}w_2 - \frac{1}{3}[1 - (1 - w_2)^2 - w_2^{2/3}] \\
&= \frac{1}{15}[5w_2^{2/3} - 6w_2 + 4w_2^{5/3} - 3w_2^2 + 4w_2^3] \geq \frac{1}{15}[5w_2^{2/3} - 6w_2 + 4w_2^{5/3} - 3w_2^2] \\
&= \frac{1}{15}[w_2^{2/3}(1 - w_2^{1/3})(5 - w_2^{1/3} - w_2^{2/3}) + 3w_2^{5/3}(1 - w_2^{1/3})] \geq 0
\end{aligned}$$

since $0 \leq w_2 \leq 1$, and $\tilde{\theta}(w) = 0$ for $w_2 = 0$. Hence the lemma is true for $n = 2$.

Suppose now the lemma is true for $1 \leq n \leq N$. Let $w_i \geq 0$, $i = 1, \cdots, N+1$, $\Sigma_{i=1}^{N+1} w_i = 1$. Let $w_N + w_{N+1} = v_N$. Then

$$\begin{aligned}
&\left(1 - \frac{2}{3}w_1 - \frac{2}{5}w_2 - \cdots - \frac{2}{2N+3}w_{N+1}\right) \\
&\qquad \geq \left(1 - \frac{2}{3}w_1 - \cdots - \frac{2}{2N-1}w_{N-1} - \frac{2}{2N+1}v_N\right)
\end{aligned}$$

and

$$w_1^2 + w_2^{2/3} + \cdots + w_{N+1}^{2/(2N+1)} \geq w_1^2 + \cdots + w_{N-1}^{2N-3} + v_N^{2/(2N-1)}$$

since

$$(w_N + w_{N+1})^{2/(2N-1)} \leq w_N^{2/(2N-1)} + w_{N+1}^{2/(2N-1)} \leq w_N^{2/(2N-1)} + w_{N+1}^{2/(2N+1)}$$

$$\text{for } 0 \leq w_{N+1} \leq 1.$$

This is equivalent to the case $n = N$ and hence the lemma is proved by induction, since in each case $\tilde{\theta}(w) = 1/3$ for $w_1 = 1$ and $w_i = 0$, $i > 1$.

Proof of Theorem 7. By [7] we know that $(\mu, A) \in \mathfrak{A}$ if

$$x\left(x + \mu \sum_{i=1}^{n} C_i x^{2n}\right) > 0, \qquad B \le x \le A, \tag{27}$$

where $B < 0$ is given by $\phi(\mu, B) = \phi(\mu, A)$. From (27) we have that

$$\sum_{i=1}^{n} \mu C_i (-B)^{2i-1} < 1 \tag{28}$$

and also that

$$\theta(\mu, A) = \theta(\mu, B). \tag{29}$$

Let $w_i = \mu C_i (-x)^{2i-1}$, $i = 1, 2, \cdots, n$. Then

$$\theta(\mu, x) = 2\left[\sum_{i=1}^{n} w_i^{2/(2i-1)} \cdot \frac{1}{x^2}\right]\left[\frac{x^2}{2} - \sum_{i=1}^{n} \frac{1}{2i+1} x^2 w_i\right]$$

$$= \left[\sum_{i=1}^{n} w_i^{2/(2i-1)}\right]\left[1 - \sum_{i=1}^{n} \frac{2w_i}{2i+1}\right] = \tilde{\theta}(w).$$

Further, the constraint (28) becomes, for $x = B$,

$$\sum_{i=1}^{n} w_i < 1. \tag{30}$$

Let $0 < \theta_0 < 1/3$ be given. Let $\mu > 0$ and $A > 0$ be such that $\theta(\mu, A) = \theta_0$. Define Γ to be the manifold

$$\frac{w_1}{\mu C_1} = \left(\frac{w_2}{\mu C_2}\right)^{1/3} = \cdots = \left(\frac{w_n}{\mu C_n}\right)^{1/(2n-1)}, \qquad w_i \ge 0 \tag{31}$$

(eliminating those terms for which $C_i = 0$). This is a connected manifold which intersects the origin ($\tilde{\theta}(0) = 0$) and the boundary $\Sigma_{i=1}^{n} w_i = 1$ ($\tilde{\theta} \ge 1/3$). Then by continuity, there exists $w_1^*, \cdots, w_n^*$ on Γ such that $\tilde{\theta}(w^*) = \theta_0$. Hence there exists $B < 0$ such that $\theta(\mu, B) = \theta_0$ with $B = (-w_i^*/\mu C_i)^{1/(2i-1)}$ and B satisfying (28).

Suppose now $\theta_0 > 1/3$. Let $(\mu_0, A_0) \in R_+^2$ with

$$\theta(\mu_0, A_0) = \theta_0 \quad \text{and} \quad \mu_0 > \max_{2 \le i \le n} \left\{1, \frac{n}{3\theta_0 - 1} \frac{C_i^{2/(2i-1)}}{C_i^2}\right\}.$$

By the above, for all w satisfying (30),

$$\frac{w_i^{2/(2i-1)}}{w_1^2} = \frac{(C_i \mu_0)^{2/(2i-1)}}{C_1^2 \mu_0^2} < \frac{C_i^{2/(2i-1)}}{C_1^2} \mu_0^{-1} < \frac{3\theta_0 - 1}{n}, \qquad i = 2, \cdots, n.$$

Hence $\tilde{\theta}(w) < 3\theta_0 w_1^2(1 - 2w_1/3 - \cdots - 2w_n/(2n+1)) \leq 3\theta_0 w_1^2(1 - 2w_1/3) \leq \theta_0$. Hence for such μ_0, $B < 0$ having the required properties for periodic solutions does not exist. This proves the theorem.

Remark 3. A similar result holds for the equation $x'' + x + \mu \sum_{i=1}^{n} C_i x^{2(i+k)} = 0$, $x(0) = A > 0$, $x'(0) = 0$, $k \geq 0$, $C_i \geq 0$, C_1, $C_n > 0$. Let

$$\theta_k(\mu, x) = 2\left[\sum_{i=1}^{n} (\mu C_i)^{2/(2(k+i)-1)}\right]\phi(\mu, x).$$

Then the condition $\phi(\mu, A) < 1/3$ becomes $\theta_k(\mu, A) < 1 - 2/(2k+3)$.

Example 3. This is a special case of Example 2.

$$x'' + x + \mu x^{2n} = 0, \quad x(0) = A > 0, \quad x'(0) = 0. \tag{32}$$

The condition $\theta_k(\mu; A) < 1 - 2/(2k+3)$ becomes

$$2\mu^{2/(2n-1)}(A^2/2 + \mu A^{2n+1}/(2n+1)) < 1 - 2/(2n+1). \tag{33}$$

This is satisfied if

$$\mu A^{2n-1} < \gamma_n, \tag{34}$$

where γ_n is a positive solution of

$$\gamma_n^2(1 + 2\gamma_n/(2n+1))^{2n-1} = ((2n-1)/(2n+1))^{2n-1}. \tag{35}$$

Suppose now that $\mu A^{2n-1} = \gamma_n$. For a periodic solution of equation (32) to exist, there must exist $B < 0$ such that

$$B^2\left(1 + \frac{2}{2n+1}\mu B^{2n-1}\right) = A^2\left(1 + \frac{2\mu A^{2n-1}}{2n+1}\right), \tag{36}$$

or

$$\begin{aligned}(\mu B^{2n-1})^2\left(1 + \frac{2}{2n+1}\mu B^{2n-1}\right)^{2n-1} &= \gamma_n^2\left(1 + \frac{2\gamma_n}{2n+1}\right)^{2n-1} \\ &= \left(\frac{2n-1}{2n+1}\right)^{2n-1} = \left(1 - \frac{2}{2n+1}\right)^{2n-1}.\end{aligned} \tag{37}$$

It is easily shown that the unique negative solution (up to multiplicities) of the equation

$$\gamma^2(1 + 2\gamma/(2n+1))^{2n-1} = ((2n-1)/(2n+1))^{2n-1} \tag{38}$$

is $\gamma = -1$. Hence $\mu B^{2n-1} = -1$, since $B < 0$. Hence $g(B) + \mu h(B) = B - B^{2n}/B^{2n-1} = 0$, and (μ, A) is on $\partial\mathfrak{A}$. This means that the curve

$$\mu A^{2n-1} = \gamma_n \tag{39}$$

is a boundary curve (of type II) for $\mathfrak{A}$.

Remark 4. It can be shown that γ_n is strictly decreasing and that $\lim_{n\to\infty} \gamma_n = \gamma$, where γ is the unique positive root of

$$x^2 e^{2x} = e^{-2}. \tag{40}$$

To prove this, we first consider $(1 - 2/(2n+1))^{2n-1}$. Let $y = (1 - 2/(x+2))^x$. Then

$$y'/y = \log(x/(x+2)) + 2/(x+2) = \log(1-\epsilon) + \epsilon < 0,$$

where $\epsilon = 2/(x+2)$, $x \geq 1$. Hence, since $y > 0$, $y' < 0$ and y is decreasing and so $((2n-1)/(2n+1))^{2n-1}$ is decreasing with n.

Now consider $(1 + 2\gamma_n/(2n+1))^{2n-1}$. Let $y = (1 + \delta/(x+2))^x$, where $\delta > 0$. Here

$$\frac{y'}{y} = \log\left(1 + \frac{\delta}{x+2}\right) - \frac{\delta x}{(x+2)(x+2+\delta)} > \frac{\delta}{x+2} - \frac{1}{2}\frac{\delta^2}{(x+2)^2} - \frac{\delta x}{(x+2)(x+\delta+2)}$$

$$= \frac{\delta}{x+2}\left(1 - \frac{1}{2}\frac{\delta}{x+2} - \frac{x}{x+\delta+2}\right) \geq \frac{\delta}{x+2}\left(1 - \frac{1}{2(x+2)} - \frac{x}{x+2}\right) = \frac{3\delta}{2(x+2)^2} > 0.$$

This means that $(1 + 2x/(2n+1))^{2n-1}$ is increasing with n for each fixed positive x.

Consider now the equation

$$x^2(1 + 2x/(2n+1))^{2n-1} = ((2n-1)/(2n+1))^{2n-1}.$$

If $x = \gamma_n$ is a solution, then as n is replaced by $n+1$, for fixed x, the right side decreases and the left side increases. Clearly x must decrease then to preserve equality and so $\gamma_{n+1} < \gamma_n$.

Remark 5. The special case $n = 1$ in Example 3 gives Maekawa's result [9].

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