COHERENT EXTENSIONS AND RELATIONAL ALGEBRAS([1])

BY

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ABSTRACT. The notion of a lax adjoint to a 2-functor is introduced and some aspects of it are investigated, such as an equivalent definition and a corresponding theory of monads. This notion is weaker than the notion of a 2-adjoint (Gray) and may be obtained from the latter by weakening that of 2-functor and replacing the adjointness equations by adding 2-cells satisfying coherence conditions. Lax monads are induced by and resolve into lax adjoint pairs, the latter via 2-categories of lax algebras. Lax algebras generalize the relational algebras of Barr in the sense that a relational algebra for a monad in $\mathcal{Sets}$ is precisely a lax algebra for the lax monad induced in $\mathcal{Rel}$. Similar considerations allow us to recover the T-categories of Burroni as well. These are all examples of lax adjoints of the "normalized" sort and the universal property they satisfy can be expressed by the requirement that certain generalized Kan extensions exist and are coherent. The most important example of relational algebras, i.e., topological spaces, is analysed in this new light also with the purpose of providing a simple illustration of our somewhat involved constructions.

Introduction. Ever since Kan [9] introduced adjoint functors, several variants of this notion have appeared in the literature. One such is the generalization achieved by replacing the category of sets and mappings by any monoidal category (or "multiplicative category", cf. Bénabou [2]) and by relativizing to it all the ingredients entering into the description of an adjoint situation. We have shown in [3] that the theory of monads (Huber [8] and Eilenberg and Moore [6], therein called "triples") carries over to the relative case. In particular, this applies to 2-monads (or "strong" monads) in 2-categories, as these are the notions relative to $\mathcal{Cat}$.

Weaker types of adjointness for 2-functors have also been considered. Thus, Gray [7] defines "2-adjointness" by weakening the notion of a natural transformation and applies it to the fibred category construction.

In this paper we introduce a notion of "lax adjointness" which encompasses those of strong adjointness and of 2-adjointness. From a formal point of view, we obtain it by weakening not just natural transformations but also the functors involved and by replacing the adjointness identities by adding appropriate 2-cells in their place—all of this tempered by the presence of four coherence conditions. From a universal point of view a special instance called "normalized lax adjointness" has a nice interpretation: it is completely determined by giving a family of generalized Kan extensions which behave coherently.

We arrived at the above definitions not out of a mere wish to generalize but rather out of a desire to incorporate into the theory of 2-categories the notion of a relational algebra due to Barr [1]. Motivated by the same example, Burroni [5] introduced the notion of a "T-category", a more general structure than the relational algebras and liable to a variety of interesting applications. We show here that any lax monad resolves into a lax adjoint pair by means of a category of lax algebras. If the lax monad lies in $Span \, \mathfrak{X}$ for some category $\mathfrak{X}$ and is induced by a monad in $\mathfrak{X}$, its lax algebras are none other than the T-categories. This supplies us, in principle, with many more instances of lax adjointness than those originally envisaged. The details of these applications will not, however, be given here.

The contents of the paper are, briefly, as follows. In §1, we define the notion of a family of 1-cells in a 2-category $\mathfrak{A}$ being coherently closed for U-extensions, where U is a given 2-functor $\mathfrak{B} \to \mathfrak{A}$. The motivating example involves topological spaces (the relational algebras over the monad of ultrafilters in $Sets$, as proved in [1]) and is shown in detail to be part of an instance of the universal property. In §2, lax monads and the corresponding 2-category of lax algebras are defined. In §3 formal lax adjoints come in as a way to resolve lax monads; they also induce them. In §4 it is shown that any family of coherent U-extensions, in the sense of §1, determines a lax adjoint to U. The converse holds if the lax adjoint is "normalized". It is then pointed out that such is the case with the available applications.

Lax functors occur in Bénabou [2] with a reversal of 2-cells and under the name "morphisms of bicategories". We assume, however, that the bicategories are 2-categories. (Recall that the pseudo-functors introduced by Grothendieck to correspond to arbitrary fibrations are of this kind.) Lax natural transformations, called "2-natural" in [7] and "quasi-natural" in [4], are responsible for "2-adjointness" and are due to Gray. The "lax" terminology has been borrowed from Street [12]. Our lax functors, however, are dual to those of [12]; our lax transformations are those which there have been labelled "right". Aside from the fact that only one type of transformation occurs throughout the paper, a reason for

avoiding labels is the relationship which these notions bear to extensions and liftings. Thus, for a 2-functor U, a family $\eta_X: X \to UFX$ with the left extension property makes F a lax functor and η a right lax transformation. But also, a family $\epsilon_X: UFX \to X$ with the left lifting property makes F into a lax functor and ϵ into a left transformation.

1. Coherent U-extensions. We start by giving a definition that generalizes the notion of a left Kan extension, as in Mac Lane [10]. The generalization is two-fold: first, extensions take place in an arbitrary 2-category rather than in $\mathcal{C}\mathit{at}$; secondly, extensions are required to be relative to U in some sense.

(1.1) **Definition.** Let $\mathcal{C}$, $\mathcal{B}$ be 2-categories and $U: \mathcal{B} \to \mathcal{C}$ a 2-functor. Let $\kappa_X: X \to U\overline{X}$ and $f: X \to UY$ be 1-cells of $\mathcal{C}$.

The (left) U-extension of f along κ_X is given by a pair $(\overline{f}; \psi_f)$ consisting of a 1-cell $\overline{f}: \overline{X} \to Y$ and a 2-cell $\psi_f: f \to U\overline{f} \cdot \kappa_X$, i.e., as in the diagram
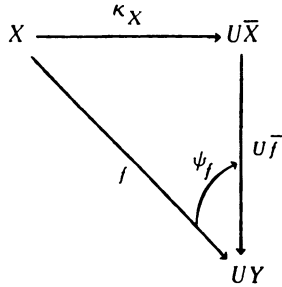
satisfying the following universal property: for any other pair $(g; \phi)$ with $g: \overline{X} \to Y$ and $\phi: f \to Ug \cdot \kappa_X$, there exists a *unique* $\overline{\phi}: \overline{f} \to g$ such that the diagram of the 2-cells in
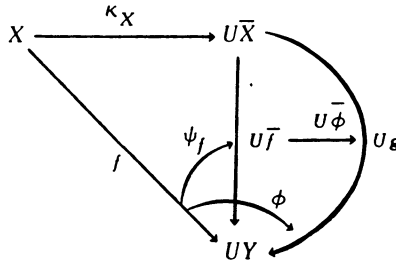
commutes. This says, exactly, that $\phi = [(U\overline{\phi})\kappa_X] \cdot \psi_f$.

Note that the usual notion is recovered with $\mathcal{C} = \mathcal{B} = \mathcal{C}\mathit{at}$ and U the identity 2-functor.

Assume now that for each $X \in |\mathcal{C}|$ we are given $\kappa_X: X \to U\overline{X}$ for some $\overline{X} \in |\mathcal{B}|$. Note then that there is a diagram for any $f: X \to UY$, obtained by composing the U-extension of $(\kappa_{UY} \cdot f)$ with the U-extension of 1_{UY}, assuming these exist.

$$X \xrightarrow{\kappa_X} U\overline{X}$$

$$\psi_{(\kappa_{UY} \cdot f)}$$

$$U\overline{Y} \xrightarrow{\kappa_{UY}} U(\overline{UY})$$
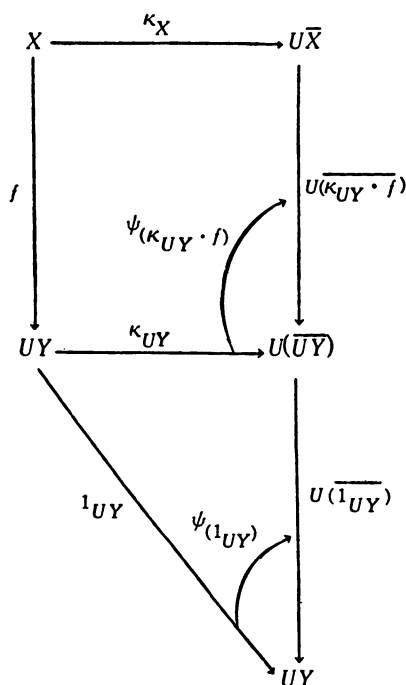
Also note that the U-extension of κ_X along itself, if it exists, is a pair $(\overline{\kappa_X}; \psi_{(\kappa_X)})$, as in

$$X \xrightarrow{\kappa_X} U\overline{X}$$

$$\psi_{(\kappa_X)}$$

$$U(\overline{\kappa_X})$$

$$U\overline{X}$$
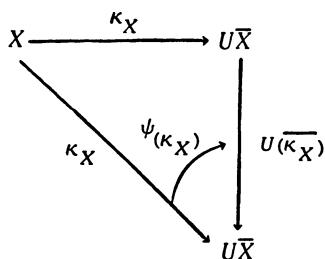
(1.2) Definition. A family of 1-cells $\{\kappa_X : X \to U\overline{X}\}$, indexed by the objects of $\mathfrak{A}$, is said to be *coherently closed for U-extensions* iff the following hold:

(i) for every $f: X \to UY$, the pair $(\overline{f}, \psi_f)$ exists;

(ii) $\overline{1_{UY}} \cdot (\overline{\kappa_{UY} \cdot f}) = \overline{f}$ and $[U(\overline{1_{UY}})]\psi_{(\kappa_{UY} \cdot f)} \cdot [\psi_{(1_{UY})}]f = \psi_f$;

(iii) $\overline{\kappa_X} = 1_{\overline{X}}$ and $\psi_{(\kappa_X)} = 1_{(\kappa_X)}$.

(1.3) Example. Recall the description of the monad β in $\mathcal{S}ets$, whose algebras are the compact T_2-spaces (Manes [11]). For a set X, βX is the set of all ultrafilters on X and a basis for the topology on βX (making it into a compact T_2-space) is given by all sets of the form $\dot{A} = \{\mathfrak{U} \in \beta X : A \in \mathfrak{U}\}$ for subsets A

of X. The unit of the monad, $\eta: 1_{\mathcal{S}ets} \to \beta$ assigns, to a point $x \in X$, the principal ultrafilter on x, i.e., $\dot{x} = \{A \subset X : x \in A\}$. Finally, if $\mathfrak{H} \in \beta\beta X$, the monad multiplication $\mu: \beta\beta \to \beta$ has the effect that $\mu_X(\mathfrak{H}) = \{A \subset X : \dot{A} \in \mathfrak{H}\} \in \beta X$.

In [1], Barr showed that a topological space is a relational algebra for the monad β. If Y is a topological space, let $\theta: \beta Y \to Y$ denote the relation on $\beta Y \times Y$ which is determined by the condition: $(\mathfrak{B}, y) \in \theta$ iff $\mathfrak{B} \to y$ (i.e., $\mathfrak{B}$ converges to y).

Let us extend the functor $\beta: \mathcal{S}ets \to \mathcal{S}ets$ over to a $\beta: \mathcal{R}el \to \mathcal{R}el$, where $\mathcal{R}el$ is the 2-category of sets, relations and inclusions of their graphs as 2-cells. This is done in [1] as follows: given a relation $r : X \to Y$, decompose it as

$$X \xrightarrow{d_r^{-1}} \Gamma_r \xrightarrow{c_r} Y,$$

where $\Gamma_r \subset X \times Y$ is the graph of the relation and where d_r, c_r are the domain and codomain functions. Define $\beta(r): \beta X \to \beta Y$ as the composite

$$\beta X \xrightarrow{(\beta d_r)^{-1}} \beta(\Gamma_r) \xrightarrow{\beta c_r} \beta Y.$$

If we recall that, for a function $f: X \to Y$, $\beta f: \beta X \to \beta Y$ assigns to an ultrafilter $\mathfrak{U} \in \beta X$ the filter generated by sets of the form fA for $A \in \mathfrak{U}$, the latter denoted $f[\mathfrak{U}]$ and automatically an ultrafilter, we have now the following description of βr: let $(\mathfrak{U}, \mathfrak{B}) \in \beta X \times \beta Y$. Then, $(\mathfrak{U}, \mathfrak{B}) \in \beta r$ iff there exists $\mathfrak{B} \in \beta(\Gamma_r)$ such that $d_r[\mathfrak{B}] = \mathfrak{U}$ and $c_r[\mathfrak{B}] = \mathfrak{B}$.

Recall also Barr's observation that, in general, for composable relations r and s one only has $\beta(r \circ s) \leq \beta(r) \cdot \beta(s)$. This will later on be called a "lax functor".

(1.3.1) **Definition.** A relation $r: X \to Y$, where (X, ξ) and (Y, θ) are topological spaces, is called a *lax morphism of topological spaces* iff the following holds:

$$
\begin{array}{ccc}
\beta X & \xrightarrow{\beta r} & \beta Y \\
\xi \downarrow & \geq & \downarrow \theta \\
X & \xrightarrow{r} & Y
\end{array}
$$

i.e., $\theta \cdot \beta_r \leq r \cdot \xi$. By the above, this means that, given $\mathfrak{U} \in \beta X$ and $y \in Y$, if there exists $\mathfrak{B} \in \beta(\Gamma_r)$ such that $d_r[\mathfrak{B}] = \mathfrak{U}$ and $c_r[\mathfrak{B}] \to y$, then there exists $x \in X$ such that $(x, y) \in r$ and $\mathfrak{U} \to x$.

We make some remarks on this notion. First, it follows from the characteri-

zation of continuous functions given in [1] that an inverse of a function, i.e., $r = g^{-1}$ with $g: Y \to X$, is a lax morphism of topological spaces iff g is a continuous function. This suggests that we call a relation $r: X \to Y$ a *continuous relation* whenever the reverse inequality holds, namely, $r \cdot \xi \leq \theta \cdot \beta r$.

Basil Rattray pointed out to us that any lax morphism $f: X \to Y$ with f a function is always a *closed* mapping, as it is easy to prove. He also called our attention to the following observations. There exist closed mappings which do not satisfy the condition of (1.3.1). For example, a constant mapping $X \to \{y\}$ satisfies the condition iff in X every ultrafilter converges. (Needless to say, the condition always holds for continuous functions between compact spaces.) The above example shows that continuous or open mappings are in the same predicament with respect to the condition. But also, the condition does not imply continuity. An example is the following: let $f: X \to Y$ and $g: Y \to X$ be inverse functions with f continuous and g closed but not conversely. Then g satisfies the condition and is not continuous.

Denote by $\mathcal{R}el\mathcal{T}op$ the 2-category of topological spaces, lax morphisms and usual ordering between relations. Let $U: \mathcal{R}el\mathcal{T}op \to \mathcal{R}el$ be the forgetful 2-functor.

(1.3.2) **Proposition.** *The family $\{X \xrightarrow{\eta_X} \beta X\}$, indexed by all sets, is coherently closed for U-extensions.*

Proof. Given $r: X \to Y$ with X a set and Y a topological space, define $\bar{r}: \beta X \to Y$ as follows: $(\mathfrak{U}, y) \in \bar{r}$ iff there exists $\mathfrak{B} \in \beta Y$ such that $(\mathfrak{U}, \mathfrak{B}) \in \beta r$ and such that $\mathfrak{B} \to y$. We show now that $\bar{r}$ is a U-extension of r along η_X (note that 2-cells need not be specified in this example). First, we show that $r \leq \bar{r} \cdot \eta_X$. This statement says: given $(x, y) \in r$ it follows that $(\dot{x}, y) \in \bar{r}$. In order to see that this is so, we only need to observe that, since $\dot{y} \to y$ in any topology, $(\dot{x}, \dot{y}) \in \beta r$. Now, let $\mathfrak{B}$ be the principal ultrafilter on (x, y) in Γ_r. Clearly $d_r[\mathfrak{B}] = \dot{x}$ while $c_r[\mathfrak{B}] = \dot{y}$.

Next, we wish to show that $\bar{r}: \beta X \to Y$ satisfies the condition (1.3.1), i.e., that $\theta \cdot \beta \bar{r} \leq \bar{r} \cdot \mu_X$ holds. To do so, endow $\Gamma_{\bar{r}}$ with a topology ζ in the canonical way so as to have both diagrams below commutative:

Once this is done, the result follows from the way the left-hand side commutative square is affected if one inverts $d_{\overline{r}}$ and $\beta(d_{\overline{r}})$. In general, for functions a, b, c, d, it follows from $a \cdot d = c \cdot b$ that $d \cdot b^{-1} \leq a^{-1} \cdot c$, as the reader may easily verify.

Let us verify the universality of $\overline{r} : \beta X \to Y$ among all relations $s : \beta X \to Y$ for which $(x, y) \in r$ implies $(\dot{x}, y) \in s$, i.e., show that in that case $\overline{r} \leq s$, which means that for all $\mathfrak{U} \in \beta X$ and $y \in Y$, $(\mathfrak{U}, y) \in \overline{r}$ implies $(\mathfrak{U}, y) \in s$.

Given $(\mathfrak{U}, y) \in \overline{r}$, let $\mathfrak{V} \in \beta Y$ be such that $\mathfrak{V} \to y$ and $(\mathfrak{U}, \mathfrak{V}) \in \beta r$. Such a $\mathfrak{V}$ exists by the definition of $\overline{r}$.

Since s satisfies (1.3.1) one knows that $(\mathfrak{U}, y) \in s$ provided one can find some $\mathfrak{H} \in \beta\beta X$ with $\mathfrak{H} \to \mathfrak{U}$ and $(\mathfrak{H}, \mathfrak{V}) \in \beta s$. We claim that $\mathfrak{H} = \eta_X(\mathfrak{U})$ has these properties.

(1)
$$\eta_X(\mathfrak{U}) \to \mathfrak{U}.$$

Let $\mathfrak{U} \in \dot{A}$, for some $A \subset X$. This means simply that $A \in \mathfrak{U}$. We want to show that for some $B \in \mathfrak{U}$, $\dot{A}$ contains $\eta_X(B)$ so that $\mathfrak{N}_\mathfrak{U} \subset \eta_X(\mathfrak{U})$ as required for convergence in the topology of βX. But this is immediate as $\dot{A} \supset \eta_X(A)$. Indeed, $\eta_X(A) = \{\dot{x} : x \in A\}$, and any such $\dot{x}$ is an ultrafilter on X containing A (since $x \in A$); thus, $\dot{x} \in \dot{A}$.

(2)
$$(\eta_X(\mathfrak{U}), \mathfrak{V}) \in \beta s.$$

Since $(\mathfrak{U}, \mathfrak{V}) \in \beta r$, there is $\mathfrak{W} \in \beta(\Gamma_r)$ such that $d_r[\mathfrak{W}] = \mathfrak{U}$ and $c_r[\mathfrak{W}] = \mathfrak{V}$. We want to define $\mathfrak{R} \in \beta(\Gamma_s)$ such that $d_s[\mathfrak{R}] = \eta_X(\mathfrak{U})$ whereas $c_s[\mathfrak{W}] = \mathfrak{V}$. A filter basis for $\mathfrak{R}$ may be given by all sets of the form $\eta_X(A) \times B$ for all pairs (A, B) such that $A \in \mathfrak{U}$, $B \in \mathfrak{V}$ and $A \times B \in \mathfrak{W}$. Then, we are done since $(x, y) \in r$ for $x \in A$ and $y \in B$ implies, by assumption on s, that $(\dot{x}, y) \in s$. The rest is clear.

It remains to check on the coherence of these extensions. For this, observe that the diagrams below are all U-extension diagrams:

for X a set and (Y, θ) a topological space.

Let us show (i): by definition of $\overline{\eta_Y \cdot r} : \beta X \to \beta Y$, $(\mathfrak{U}, \mathfrak{V}) \in \overline{\eta_Y \cdot r}$ iff there exists some $\mathfrak{W} \in \beta\beta Y$ such that $(\mathfrak{U}, \mathfrak{W}) \in \beta(\eta_Y \cdot r)$ and $\mathfrak{W} \to \mathfrak{V}$. Equivalently, there exists some $\mathfrak{V}' \in \beta Y$ with $(\mathfrak{U}, \mathfrak{V}') \in \beta r$ and $\eta_Y(\mathfrak{V}') \to \mathfrak{V}$ (or $\eta_Y(\mathfrak{V}') = \mathfrak{V}$).

The last condition implies that $\mathfrak{B} = \mathfrak{B}'$ and therefore, $(\mathfrak{U}, \mathfrak{B}) \in \overline{\eta_Y \cdot r}$ iff $(\mathfrak{U}, \mathfrak{B}) \in \beta r$.

Conditions (ii) and (iii) are obvious: $(\mathfrak{B}, y) \in \overline{1_Y}$ iff $\mathfrak{B} \to y$, i.e., iff $(\mathfrak{B}, y) \in \theta$, and $(\mathfrak{U}, \mathfrak{U}') \in \overline{\eta_X}$ iff $\eta_X(\mathfrak{U}) \to \mathfrak{U}'$, i.e., iff $\mathfrak{U} = \mathfrak{U}'$.

The second coherence condition in (1.2) is precisely the statement that (iii) above is an extension diagram.

The first coherence condition follows immediately from the observation that an equivalent description of $\bar{r} : \beta X \to Y$, for any given $r : X \to Y$, is the following: $\bar{r} = \beta X \xrightarrow{\beta r} \beta Y \xrightarrow{\theta} Y$. In fact, we could have stated the definition in precisely this way. This completes the proof. $\square$

Let us close the section with some remarks. First, note that the usual forgetful functor $\mathfrak{Top} \to \mathfrak{Sets}$ can be obtained by pulling back the *opposite* of $U : \mathfrak{Rel Top} \to \mathfrak{Rel}$ along the functor $\mathfrak{Sets} \to \mathfrak{Rel}^{op}$ which is the identity on objects and takes a function $f : X \to Y$ into the relation $f^{-1} : Y \to X$. Secondly, note that even if we only tested the universal property with functions $f : X \to Y$ rather than arbitrary relations $r : X \to Y$, relations come out anyway since $\bar{f} : \beta X \to Y$ is a relation not a function, unless Y is compact. Of course, when restricted to compact spaces as well as functions, what we obtain is the universal property of the Stone-Čech compactification functor, i.e., ordinary adjointness (this is easily seen as, for functions f and g, $f \le g$ simply means $f = g$).

2. Lax monads and their algebras.

(2.1) **Definition.** Let $\mathfrak{A}$ and $\mathfrak{B}$ be 2-categories. By a *lax functor* $F : \mathfrak{A} \to \mathfrak{B}$ we mean the following:

to each $X \in |\mathfrak{A}|$, an $FX \in |\mathfrak{B}|$;

to each 1-cell $f : X \to Y$ of $\mathfrak{A}$, a 1-cell $Ff : FX \to FY$ of $\mathfrak{B}$;

to each 2-cell $a : f \to f'$ of $\mathfrak{A}$, a 2-cell $Fa : Ff \to Ff'$ of $\mathfrak{B}$;

to each $X \in |\mathfrak{A}|$ a 2-cell $e_X^F : F(1_X) \to 1_{FX}$;

to each composable pair of 1-cells g, f of $\mathfrak{A}$ a 2-cell $c_{g,f}^F : F(gf) \to Fg \cdot Ff$ of $\mathfrak{B}$; satisfying the following conditions

$$(2.1.1) \qquad\qquad (Ff \cdot e_X^F) c_{f,1_X}^F = 1_{Ff}.$$

$$(2.1.2) \qquad\qquad (e_Y^F \cdot Ff) c_{1_Y,f}^F = 1_{Ff}.$$

$$(2.1.3) \qquad (Fb \cdot c_{g,f}^F) c_{b,gf}^F = (c_{b,g}^F \cdot Ff) c_{bg,f}^F,$$

for any pair of composable g, f.

(2.1.4) For any $f: X \rightarrow Y$, $F(1_f) = 1_{Ff}$.

(2.1.5) For any $a: f \rightarrow f'$, $a': f' \rightarrow f''$, $F(a' \cdot a) = Fa' \cdot Fa$.

(2.1.6) For any composable g, f and g', f' and 2-cells $a: f \rightarrow f'$, $b: g \rightarrow g'$ one has $(Fb \cdot Fa)c_{g,f}^{F} = c_{g',f'}^{F} \cdot F(ba)$.

Remarks. A 2-*functor* is a lax functor with e_X and $c_{g,f}$ all identity 2-cells. If $\mathfrak{A}$ and $\mathfrak{B}$ are *bicategories* and if we replace in the above diagrams, identities such as $f = f \cdot 1_X$ or $h(gf) = (hg)f$ by the corresponding 2-cell isomorphisms, a lax functor becomes a dual morphism of bicategories (cf. Bénabou [2]).

(2.2) **Definition.** Let $F, G: \mathfrak{A} \rightarrow \mathfrak{B}$ be lax functors between 2-categories. A *lax natural transformation* $a: F \rightarrow G$ is given by any family $\{a_X: FX \rightarrow GX\}$ of 1-cells of $\mathfrak{B}$ indexed by $|\mathfrak{A}|$, and a family $\{a_f: a_Y \cdot Ff \rightarrow Gf \cdot a_X\}$ of 2-cells of $\mathfrak{B}$ indexed by the 1-cells of $\mathfrak{A}$, satisfying:

(2.2.1) $(e_X^G \cdot a_X)a_{1_X} = a_X \cdot e_X^F: a_X \cdot F(1_X) \rightarrow a_X.$

(2.2.2) $(Gg \cdot a_F)(c_{g,f}^G \cdot a_X)a_{gf} = (a_g \cdot Ff)(a_Z \cdot c_{g,f}^F): a_Z \cdot F(gf) \rightarrow Gg \cdot a_Y \cdot Ff.$

(2.2.3)   For each $a: f \rightarrow f'$, $(Ga \cdot a_X)a_f = a_{f'}(a_Y \cdot Fa).$

Remarks. If $F, G: \mathfrak{A} \rightarrow \mathfrak{B}$ are 2-functors, and if one requires that each 2-cell a_f be the identity, the definition reduces to the commutativity of the diagrams of 1-cells and 2-cells of the form:

for any f, f' and $a: f \rightarrow f'$. This says exactly that $a: F \rightarrow G$ is a *strongly natural transformation* or a $\mathcal{C}at$-*natural transformation* (cf. [3]).

(2.3) **Definition.** Let $\mathfrak{A}$ be a 2-category. By a *lax monad* in $\mathfrak{A}$ we mean

a lax functor $T: \mathfrak{A} \rightarrow \mathfrak{A}$;

a lax natural transformation $\eta: 1_{\mathfrak{A}} \rightarrow T$;

a lax natural transformation $\mu: T \cdot T \rightarrow T$;

(note that there is a canonical way to make the composite $T \cdot T$ into a lax functor)

families $\{\lambda_X\}, \{\rho_X\}, \{a_X\}$ of 2-cells of $\mathcal{C}$ where $\lambda_X: \mu_X \cdot T\eta_X \to 1_{TX}; \rho_X: 1_{TX} \to \mu_X \cdot \eta_{TX}; a_X: \mu_X \cdot T\mu_X \to \mu_X \cdot \mu_{TX}$; indexed by the objects of $\mathcal{C}$, satisfying:

(2.3(1.1)) $$(a_X \cdot \eta_{TTX})(\mu_X \cdot \eta_{\mu_X})(\rho_X \cdot \mu_X)$$

$$= \mu_X \cdot \rho_{TX}: \mu_X \to \mu_X \cdot \mu_{TX} \cdot \eta_{TTX}.$$

(2.3(1.2)) $$(\mu_f \cdot \eta_{TX})(\mu_{X'} \cdot \eta_{Tf})(\rho_{X'} \cdot Tf)$$

$$= Tf \cdot \rho_X: Tf \to Tf \cdot \mu_X \cdot \eta_{TX}, \quad \text{for any } f: X \to X'.$$

(2.3(1*)) $$(Tf \cdot \lambda_X)(\mu_f \cdot T\eta_X)(\mu_{X'} \cdot c^T)(\mu_{X'} \cdot T(\eta_f))$$

$$= (\lambda_{X'} \cdot Tf)(\mu_{X'} \cdot c^T): \mu_{X'} \cdot T(\eta_{X'} \cdot f) \to Tf, \quad \text{for any } f: X \to X'.$$

(2.3(a)) $$(\mu_X \cdot \lambda_{TX})(a_X \cdot T\eta_{TX})(\mu_X \cdot c^T)(\mu_X \cdot T(\rho_X))$$

$$= \mu_X \cdot e^T: \mu_X \cdot T(1_{TX}) \to \mu_X \cdot 1_{TTX}.$$

(2.3(2*)) $$(\lambda_X \cdot \eta_X)(\mu_X \cdot \eta_{\eta_X})(\rho_X \cdot \eta_X) = 1_{\eta_X}.$$

(2.3(3.1)) $$(\lambda_X \cdot \mu_X)(\mu_X \cdot \mu_{\eta_X})(a_X \cdot TT\eta_X)(\mu_X \cdot c^T)$$

$$= (\mu_X \cdot e^T)(\mu_X \cdot T(\lambda_X)): \mu_X \cdot T(\mu_X \cdot T\eta_X) \to \mu_X \cdot T(1_{TX}).$$

(2.3(3.2)) $$(a_X \cdot \mu_{TTX})(\mu_X \cdot \mu_{\mu_X})(a_X \cdot TT\mu_X)(\mu_X \cdot c^T)$$

$$= (\mu_X \cdot a_{TX})(a_X \cdot T\mu_{TX})(\mu_X \cdot c^T)(\mu_X \cdot Ta_X):$$

$$\mu_X \cdot T(\mu_X \cdot T\mu_X) \to \mu_X \cdot \mu_{TX} \cdot \mu_{TTX}.$$

(2.3(3.3)) $$(\mu_f \cdot \mu_{TX})(\mu_{X'} \cdot \mu_{Tf})(a_X \cdot TTTf)(\mu_X \cdot c^T)$$

$$= (Tf \cdot a_X)(\mu_f \cdot T\mu_X)(\mu_{X'} \cdot c^T)(\mu_{X'} \cdot T\mu_f):$$

$$\mu_{X'} \cdot T(\mu_{X'} \cdot TTf) \to Tf \cdot \mu_X \cdot \mu_{TX} \text{ for any } f: X \to X'.$$

Remarks. If T is a 2-functor, η and μ strongly natural and if all the 2-cells λ_X, ρ_X and α_X are identities we obtain precisely a strong monad. One could also assume that T is a 2-functor and that the 2-cells λ_X, ρ_X and α_X are identities, leaving the transformations to be lax. The resulting notion of monad corresponds to the 2-adjointness notion given by Gray.

(2.4) **Definition.** Let $T = \langle T, \eta, \mu, (\lambda_X), (\rho_X), (\alpha_X) \rangle$ be a lax monad in the 2-category $\mathfrak{A}$. A T-*lax algebra* X_ξ consists of

an object X of $\mathfrak{A}$;

a 1-cell $\xi \colon TX \to X$ of $\mathfrak{A}$;

a 2-cell $\iota_\xi \colon 1_X \to \xi \cdot \eta_X$ of $\mathfrak{A}$;

a 2-cell $\kappa_\xi \colon \xi \cdot T\xi \to \xi \cdot \mu_X$ of $\mathfrak{A}$;

satisfying:

(2.4.1) $(\kappa_\xi \cdot \eta_{TX})(\xi \cdot \eta_\xi)(\iota_\xi \cdot \xi) = \xi \cdot \rho_X \colon \xi \to \xi \cdot \mu_X \cdot \eta_{TX}.$

(2.4.2) $(\xi \cdot \lambda_X)(\kappa_\xi \cdot T\eta_X)(\xi \cdot c^T)(\xi \cdot T\iota_\xi) = \xi \cdot e^T \colon \xi \cdot T(1_X) \to \xi \cdot 1_{TX}.$

(2.4.3)
$$(\xi \cdot \alpha_X)(\kappa_\xi \cdot T\mu_X)(\xi \cdot c^T)(\xi \cdot T\kappa_\xi)$$
$$= (\kappa_\xi \cdot \mu_{TX})(\xi \cdot \mu_\xi)(\kappa_\xi \cdot TT\xi)(\xi \cdot c^T) :$$
$$\xi \cdot T(\xi \cdot T\xi) \to \xi \cdot \mu_X \cdot \mu_{TX}.$$

Remarks. Let $\mathfrak{X}$ be any category, T a monad on $\mathfrak{X}$. Let $\mathfrak{A} = \mathfrak{Span}\, \mathfrak{X}$, the bicategory of spans (cf. Burroni [5]). In the "same" way that a monad on $\mathfrak{Set}$ induces a lax monad on $\mathfrak{Rel}$ (cf. Barr [1]) one can show that T induces a lax monad, also called T, on $\mathfrak{Span}\, \mathfrak{X}$. The lax T-algebras in this case are precisely the T-*categories* of Burroni. We leave the details to the reader. Note that this application requires the slight generalization of lax functors, etc., suggested in the remarks after (2.1), unless a choice of pullbacks is made in $\mathfrak{X}$ so as to have $\mathfrak{Span}\, \mathfrak{X}$ a 2-category.

(2.5) **Definition.** Let X_ξ and Y_θ be lax T-algebras for some lax monad T on a 2-category $\mathfrak{A}$. A *lax T-homomorphism* $X_\xi \to Y_\theta$ is given by any pair $(f; \phi)$ where $f \colon X \to Y$ is a 1-cell of $\mathfrak{A}$ and $\phi \colon \theta \cdot Tf \to f \cdot \xi$ is a 2-cell of $\mathfrak{A}$, i.e., one has

$$
\begin{array}{ccc}
TX & \xrightarrow{\ Tf\ } & TY \\
\xi \downarrow & \overset{\phi}{} & \downarrow \theta \\
X & \xrightarrow{\ f\ } & Y
\end{array}
$$

satisfying:

(2.5.1) $(\phi \cdot \eta_X)(\theta \cdot \eta_f)(\iota_\theta \cdot f) = f \cdot \iota_\xi : f \to f \cdot \xi \cdot \eta_X.$

(2.5.2) $(f \cdot \kappa_\xi)(\phi \cdot T\xi)(\theta \cdot c^T)(\theta \cdot T\phi)$

$$= (\phi \cdot \mu_X)(\theta \cdot \mu_f)(\kappa_\theta \cdot TTf)(\theta \cdot c^T): \theta \cdot T(\theta \cdot Tf) \to f \cdot \xi \cdot \mu_X.$$

Remarks. For X_ξ and Y_θ topological spaces, i.e., lax β-algebras for the induced monad β in $\mathcal{S}eth$, a lax β-morphism was analysed in §1. Note that in this example the coherence conditions are automatic as, in $\mathcal{R}el$, each category $\mathcal{R}el(X, Y)$ is a preorder. One may, in general, i.e., for a lax monad T, consider T-algebras and T-homomorphisms, the former by requiring the 2-cells ι_ξ and κ_ξ to be identities; the latter by assuming that ϕ is the identity.

(2.6) Proposition. *There exists, for any lax monad* T *on a 2-category* $\mathcal{A}$, *a 2-category* $\mathcal{A}^T$ *whose objects are the lax* T*-algebras, whose 1-cells are the lax* T*-homomorphisms, and a 2-functor* $U^T: \mathcal{A}^T \to \mathcal{A}$ *which is faithful on 2-cells.*

Proof. Define *composition of 1-cells* as follows: Let $(f; \phi): X_\xi \to Y_\theta$ and $(g; \gamma): Y_\theta \to Z_\zeta$ be given. Define

$$(g; \gamma) \cdot (f; \phi) =_{df} (gf; [g\phi \cdot \gamma(Tf) \cdot \zeta c^T_{g,f}]).$$

Note that

$$\gamma * \phi = \zeta \cdot T(gf) \xrightarrow{\zeta \cdot c^T_{g,f}} \zeta \cdot Tg \cdot Tf \xrightarrow{\gamma \cdot Tf} g \cdot \zeta \cdot Tf \xrightarrow{g \cdot \phi} gf\xi$$

has the correct domain and codomain for $(gf; \gamma * \phi)$ to be a lax T-homomorphism $X_\xi \to Z_\zeta$. It remains to verify the two conditions on a lax T-homomorphism.

(2.6.1) *The pair* $(gf; \gamma * \phi)$ *satisfies* (2.5.1). This follows from the following commutative diagram:

The reasons for the commutativity of each of the subdiagrams is indicated as follows: a number such as (2.5.1) indicates that this condition is used essentially but it leaves unspecified which is the lax T-morphism in question, the reader can, however, identify it easily. Above, e.g., (2.2.2) refers clearly to η. The notation $\smile$ will be used when a diagram commutes by no special reason, e.g., above it does because both composites are clearly equal to $\gamma \cdot \eta_f$: $\zeta \cdot Tg \cdot \eta_Y \cdot f \to g \cdot \theta \cdot Tf \cdot \eta_X$. The word "def" means "by definition". Similar conventions will be used in the sequel.

(2.6.2) *The pair* $(gf; \gamma * \phi)$ *satisfies* (2.5.2). This follows from the following commutative diagram:

$$[\zeta \cdot T(\zeta \cdot T(g \cdot f))] \xrightarrow{c^T} [\zeta \cdot T\zeta \cdot TT(g \cdot f)] \xrightarrow{\kappa_\zeta} [\zeta \cdot \mu_Z \cdot TT(g \cdot f)]$$

with intermediate terms and arrows labelled by:

c^T (2.1.6), c^T, def, c^{TT}, $\smile$, c^{TT}, μ_{gf}

$$[\zeta \cdot T(\zeta \cdot Tg \cdot Tf)] \xrightarrow{c^T} [\zeta \cdot T\zeta \cdot T(Tg \cdot Tf)] \xrightarrow{c^T} [\zeta \cdot T\zeta \cdot TTg \cdot TTf] \xrightarrow{\kappa_\zeta} [\zeta \cdot \mu_Z \cdot TTg \cdot TTf] \quad [\zeta \cdot T(g \cdot f) \cdot \mu_X]$$

c^T, (2.1.3), c^T, $[\zeta \cdot T(\zeta \cdot Tg) \cdot TTf]$, μ_g, (2.2.2), c^T

γ, (2.1.6), γ, (2.5.2)

$$[\zeta \cdot T(g \cdot \theta \cdot Tf)] \xrightarrow{c^T} [\zeta \cdot T(g \cdot \theta) \cdot TTf] \xrightarrow{c^T} [\zeta \cdot Tg \cdot T\theta \cdot TTf] \qquad [\zeta \cdot Tg \cdot \mu_Y \cdot TTf] \xrightarrow{\mu_f} [\zeta \cdot Tg \cdot Tf \cdot \mu_X]$$

c^T, (2.1.3), c^T, $[\zeta \cdot Tg \cdot T(\theta \cdot Tf)]$, (2.1.6)

γ, $[g \cdot \theta \cdot T(\theta \cdot Tf)]$

ϕ, ϕ, c^T, γ, γ

$[g \cdot \theta \cdot T\theta \cdot TTf]$

(2.1.6), $\smile$, ϕ, κ_θ

$$[\zeta \cdot T(g \cdot \gamma \cdot \xi)] \xrightarrow{c^T} [\zeta \cdot Tg \cdot T(f \cdot \xi)] \xrightarrow{\gamma} [g \cdot \theta \cdot T(f \cdot \xi)] \qquad [g \cdot \theta \cdot \mu_Y \cdot TTf] \xrightarrow{\mu_f} [g \cdot \theta \cdot Tf \cdot \mu_X]$$

c^T, (2.1.3), c^T, $\smile$, c^T, (2.5.2), ϕ

$$[\zeta \cdot T(g \cdot f) \cdot T\xi] \xrightarrow{c^T} [\zeta \cdot Tg \cdot Tf \cdot T\xi] \xrightarrow{\gamma} [g \cdot \theta \cdot Tf \cdot T\xi] \xrightarrow{\phi} [g \cdot f \cdot \xi \cdot T\xi] \xrightarrow{\kappa_\xi} [g \cdot f \cdot \xi \cdot \mu_X]$$

Define the pair $(1_X; \xi \cdot e_X^T): X_\xi \to X_\xi$ to be the *unit*, a fact which we proceed to verify. First, it is a morphism.

(2.6.3) *The pair* $(1_X; \xi \cdot e_X^T)$ *satisfies* (2.5.1). Look at the corresponding diagram, which is:

$$
\begin{array}{c}
\xi \cdot \eta_X \\
\end{array}
$$
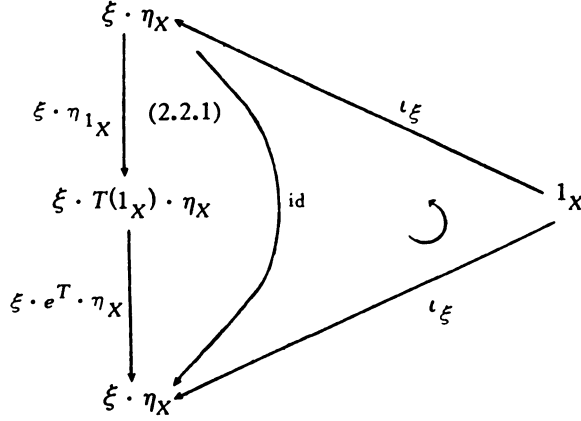
(2.6.4) *The pair* $(1_X; \xi \cdot e_X^T)$ *satisfies* (2.5.2). In this case, the appropriate diagram is:
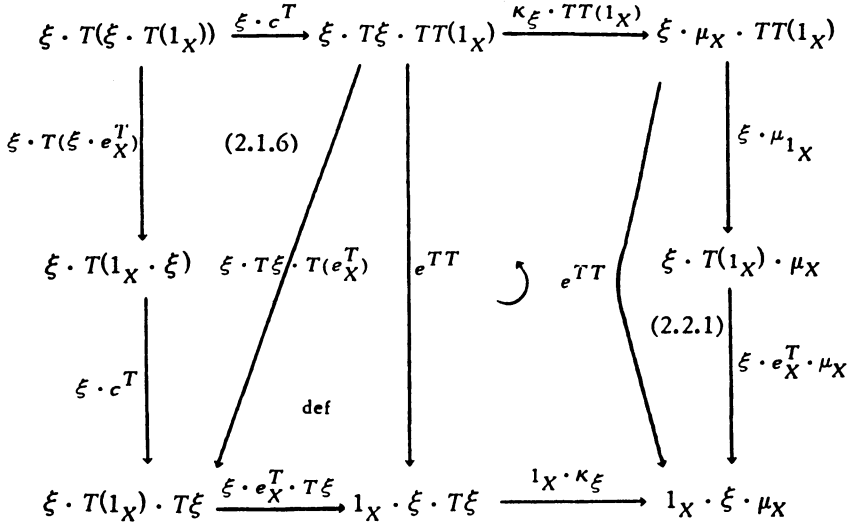
(2.6.5) 1_{X_ξ} *is a left and right unit for composition.*

By definition, $(f; \phi) \cdot (1_X; \xi \cdot e_X^T) = (f \cdot 1_X; (f \cdot \xi \cdot e_X^T) \cdot (\phi \cdot T(1_X)) \cdot (\theta \cdot c_{f,1_X}^T))$. Clearly, the result follows from (2.1.1) since one has the diagram

$$\theta \cdot T(f \cdot 1_X)$$

$$\theta \cdot c_{f,1_X}^T \Big\downarrow \qquad (2.2.1)$$

$$\theta \cdot Tf \cdot T(1_X) \xrightarrow{\ \theta \cdot Tf \cdot e_X^T\ } \theta \cdot Tf \cdot 1_{TX}$$

$$\phi \cdot T(1_X) \Big\downarrow \qquad\qquad\qquad \Big\downarrow \phi \cdot 1_{TX}$$

$$f \cdot \xi \cdot T(1_X) \xrightarrow{\ f \cdot \xi \cdot e_X^T\ } f \cdot \xi \cdot 1_{TX}$$

Similarly, using (2.2.2) one can check that $(1_Y; \theta \cdot e_Y^T) \cdot (f; \phi) = (f; \phi)$. We leave this to the reader.

(2.6.6) *Composition of 1-cells is associative.* Let $(f; \phi): X_\xi \to Y_\theta$; $(g; \gamma): Y_\theta \to Z_\zeta$ and $(h; \chi): Z_\zeta \to U_\nu$. We compute:

$$(h; \chi) \cdot [(g; \gamma) \cdot (f; \phi)]$$

$$= (h(gf); (hg\phi) \cdot (h\gamma(Tf))(h \cdot \zeta \cdot c_{g,f}^T)(\chi \cdot T(gf)) \cdot (\nu c_{h,gf}^T))$$

and

$$[(h; \chi) \cdot (g; \gamma)] \cdot (f; \phi)$$

$$= ((hg)f; (hg\phi)(h\gamma(Tf))(\chi \cdot Tg \cdot Tf)(\nu[(c_{h,g}^T \cdot Tf)(c_{hg,f}^T)]))$$

The result now follows from (2.1.3) for T.

It remains to specify the 2-cells of $\mathfrak{A}^T$ and to show that it is a 2-category.

(2.6.7) Given 1-cells $(f; \phi)$ and $(f'; \phi')$ from X_ξ to Y_θ, a 2-cell $a: f \to f'$ of $\mathfrak{A}$ is a 2-cell of $\mathfrak{A}^T$ provided it satisfies the following coherence condition: the diagram

$$\theta \cdot Tf \xrightarrow{\ \phi\ } f \cdot \xi$$

$$\theta \cdot Ta \Big\downarrow \qquad\qquad \Big\downarrow a \cdot \xi$$

$$\theta \cdot Tf' \xrightarrow{\ \phi'\ } f' \cdot \xi$$

should be commutative.

Let them compose in the same way as in $\mathfrak{A}$, the unit in $\mathfrak{A}^T$ is then also the unit in $\mathfrak{A}$. It is, of course clear that composites of 2-cells satisfying (2.6.7) again satisfy this condition and that the identity 2-cell always does.

Define now $U^T: \mathfrak{A}^T \to \mathfrak{A}$ to be forgetful, i.e., it sends X_ξ into X; $(f; \phi)$ into f and a into a. From the definitions of compositions and unit U^T is a 2-functor (even though we started with a lax functor T). It is clearly faithful on 2-cells as $U^T(a) = a$. This completes the proof. $\square$

3. Lax monads generation and resolution.

(3.1) Definition. Let $U: \mathcal{B} \rightarrow \mathcal{C}$ be a 2-functor. A *formal lax adjoint* to U is given by

a lax functor $F: \mathcal{C} \rightarrow \mathcal{B}$;

a lax natural transformation $\eta: 1_{\mathcal{C}} \rightarrow UF$;

a lax natural transformation $\epsilon: FU \rightarrow 1_{\mathcal{B}}$;

for each $X \in |\mathcal{C}|$ a 2-cell $L_X: \epsilon_{FX} \cdot F\eta_X \rightarrow 1_{FX}$;

for each $Y \in |\mathcal{B}|$ a 2-cell $R_Y: 1_{UY} \rightarrow U\epsilon_Y \cdot \eta_{UY}$,

satisfying

(3.1.1) For each $g: Y \rightarrow Y'$ in $\mathcal{B}$,

$$(U\epsilon_g \cdot \eta_{UY})(U\epsilon_{Y'} \cdot \eta_{Ug})(R_{Y'} \cdot Ug)$$

$$= Ug \cdot R_Y: Ug \rightarrow Ug \cdot U\epsilon_Y \cdot \eta_{UY}.$$

(3.1.1*) For each $f: X \rightarrow X'$ in $\mathcal{C}$,

$$(Ff \cdot L_X)(\epsilon_{Ff} \cdot F\eta_X)(\epsilon_{FX'} \cdot c^F)(\epsilon_{FX'} \cdot F\eta_f)$$

$$= (L_{X'} \cdot Ff)(\epsilon_{FX'} \cdot c^F): \epsilon_{FX'} \cdot F(\eta_{X'} \cdot f) \rightarrow Ff.$$

(3.1.2) $$(\epsilon_Y \cdot L_{UY})(\epsilon_{\epsilon_Y} \cdot F\eta_{UY})(\epsilon_Y \cdot C^F)(\epsilon_Y \cdot F(R_Y))$$

$$= \epsilon_Y \cdot e^F: \epsilon_Y \cdot F(1_{UY}) \rightarrow \epsilon_Y \cdot 1_{FUY}.$$

(3.1.2*) $$(U(L_X) \cdot \eta_X)(U\epsilon_{FX} \cdot \eta_{\eta_X})(R_{FX} \cdot \eta_X) = 1_{\eta_X}.$$

Remark. This definition could have been stated more generally with U a dual (or right) lax functor. In that case the symmetry of the conditions would be more apparent. However, as we shall see, any lax monad resolves into a pair F, U with U a 2-functor and F a lax adjoint to U in the above sense.

The above definition yields immediately:

(3.2) Proposition. *Let $\langle F, \eta, \epsilon, (L_X), (R_Y) \rangle$ be the data for a formal lax adjoint to a 2-functor $U: \mathcal{B} \rightarrow \mathcal{C}$. Then, the data $\langle UF, \eta, U\epsilon F, (U(L_X)), (R_{(FX)}),$ $(U\epsilon_{(\epsilon_{FX})}) \rangle$ is that of a lax monad in $\mathcal{C}$ said to be generated by U and the given lax adjoint to U.*

Proof. We split up the proof into the verification of the required conditions.

(3.2.1) $T = UF: \mathfrak{A} \to \mathfrak{A}$ *is a lax functor.* Define $e_X^T = U(e_X^F)$ and $c_{g,f}^T = U(c_{g,f}^F)$. Verification of the axioms is immediate and follows from F lax. It is essential here, as in what follows, that U is a 2-functor and not just lax in either direction.

(3.2.2) $\mu = U\epsilon F$ *is a lax natural transformation.* In this case there is a little more work to be done but it follows essentially from ϵ being a lax natural transformation. It is clear that it will be enough to show that ϵF is a lax natural transformation, since U is a 2-functor.

Condition (2.2.1) for ϵF holds by virtue of the following commutative diagram:

$$
\begin{array}{ccc}
\epsilon_{FX} \cdot FUF(1_X) & \xrightarrow{\epsilon_{F(1_X)}} & F(1_X) \cdot \epsilon_{FX} \\
{\scriptstyle \epsilon_{FX} \cdot FUe_X^F} \downarrow & (2.2.3) & \downarrow {\scriptstyle e_X^F \cdot \epsilon_{FX}} \\
\epsilon_{FX} \cdot FU(1_{FX}) & \xrightarrow{\epsilon_{1_{FX}}} & 1_{FX} \cdot \epsilon_{FX} \\
\end{array}
$$

$$\epsilon_{FX} \cdot e_{UFX}^F \searrow \qquad (2.2.1) \qquad \parallel$$

$$\epsilon_{FX} \cdot 1_{FUFX}$$

To see the above note that the lax structure on FUF is given by

$$FUF(1_X) \xrightarrow{FUe_X^F} FU(1_{FX}^\bullet) = F(1_{UFX}) \xrightarrow{e_{UFX}^F} 1_{FUFX}$$

and by

$$FUF(gf) \xrightarrow{FUc_{g,f}^F} FU(Fg, Ff) = F(UFg, UFf) \xrightarrow{c_{UFg,UFf}^F} FUFg \cdot FUFf.$$

Similarly, the reader can check that (2.2.2) for ϵF follows from an application of both (2.2.2) and (2.2.3) for ϵ, plus U a 2-functor. Finally, all that is needed to obtain (2.2.3) for ϵF is the corresponding one for ϵ, this since for any $a: f \to f'$, $Fa: Ff \to Ff'$ implies that

$$
\begin{array}{ccc}
U(\epsilon_{FY} \cdot FUFf) & \xrightarrow{U\epsilon_{Ff}} & U(Ff \cdot \epsilon_{FX}) \\
{\scriptstyle U(\epsilon_{FY} \cdot FUFa)} \downarrow & (2.2.3) & \downarrow {\scriptstyle U(Fa \cdot \epsilon_{FX})} \\
U(\epsilon_{FY} \cdot FUFf') & \xrightarrow{U\epsilon_{Ff'}} & U(Ff' \cdot \epsilon_{FX}) \\
\end{array}
$$

commutes.

(3.2.3) *It remains to verify the coherence conditions on a lax monad.*

Condition (2.3(1.1)) is precisely

$$
\begin{array}{ccc}
U\epsilon_{FX} \cdot \eta_{UFX} \cdot U\epsilon_{FX} & & \\
\Big\downarrow \scriptstyle U\epsilon_{FX} \cdot \eta_{U\epsilon_{FX}} & & \searrow \scriptstyle R_{FX} \cdot U\epsilon_{FX} \\
U\epsilon_{FX} \cdot UFU\epsilon_{FX} \cdot \eta_{UFUFX} & & U\epsilon_{FX} \\
\Big\downarrow \scriptstyle U\epsilon_{FX} \cdot \eta_{UFUFX} & & \nearrow \scriptstyle U\epsilon_{FX} \cdot R_{FUFX} \\
U\epsilon_{FX} \cdot U\epsilon_{FUFX} \cdot \eta_{UFUFX} & &
\end{array}
$$

and commutes by an application of (3.1.1) with $g = \epsilon_{FX}: FUFX \rightarrow FX$.

Let us indicate how to obtain the remaining conditions on a lax monad:

(2.3(1.2)) follows also from application of (3.1.1), this time with $g = Ff: FX \rightarrow FX'$;

(2.3(1*)) follows from (3.1.1*) simply by applying U to the diagram;

(2.3(2)) is a consequence of (3.1.2) and it is obtained by applying U again;

(2.3(2*)) follows from (3.1.2*), in fact: it is the very same condition.

The remaining conditions, i.e., (3.1), (3.2) and (3.3) do not depend on the coherence axioms for a lax adjoint but only on the lax naturality of ϵ. We shall be more explicit here since the diagrams may be not easy to find on a first try.

(2.3(3.1)) follows from

$$
\begin{array}{ccccc}
\epsilon_{FX} \cdot F(U\epsilon_{FX} \cdot UF\eta_X) & \xrightarrow{\epsilon_{FX} \cdot c^F} & \epsilon_{FX} \cdot FU\epsilon_{FX} \cdot FUF\eta_X & \xrightarrow{\epsilon_{FX} \cdot FUF\eta_X} & \epsilon_{FX} \cdot \epsilon_{FUFX} \cdot FUF\eta_X \\
\Big\downarrow \scriptstyle \epsilon_{FX} \cdot F(UL_X) & & \searrow \scriptstyle \epsilon_{(\epsilon_{FX} \cdot F\eta_X)} & (2.2.2) & \Big\downarrow \scriptstyle \epsilon_{FX} \cdot \epsilon_{F\eta_X} \\
& & (2.2.3) & & \epsilon_{FX} \cdot F\eta_X \cdot \epsilon_{FX} \\
\Big\downarrow \scriptstyle \epsilon_{FX} \cdot F(U1_{FX}) & & \searrow \scriptstyle \epsilon_{(1_{FX})} & & \Big\downarrow \scriptstyle L_X \cdot \epsilon_{FX} \\
\| & & (2.2.1) & & \\
\epsilon_{FX} \cdot F(1_{UFX}) & \xrightarrow[\epsilon_{FX} \cdot e^F_{UFX}]{} & & & 1_{FX} \cdot \epsilon_{FX}
\end{array}
$$

Note that (2.2.3) is applied with $a = L_X: \epsilon_{FX} \cdot F\eta_X \rightarrow 1_{FX}$. Similarly, the reader can show that (2.3(3.2)) is a consequence of twice an application of (2.2.2), namely for the composites $\epsilon_{FX} \cdot FU\epsilon_{FX}$ and $\epsilon_{FX} \cdot \epsilon_{FUFUFX}$, and then applying (2.2.3) with $a = \epsilon_{\epsilon_{FX}}: \epsilon_{FX} \cdot FU\epsilon_{FX} \rightarrow \epsilon_{FX} \cdot \epsilon_{FUFX}$.

Finally, (2.3(3.3)) is an application of (2.2.3) for ϵ with $a = \epsilon_{Ff}: \epsilon_{FX'} \cdot FUFf \rightarrow Ff \cdot \epsilon_{FX}$ plus a double application of (2.2.2) one for each of the composites which constitute the domain and codomain of the 2-cell a. The proof is now finished. □

(3.3) Proposition. *For any lax monad* T *on a 2-category* $\mathfrak{A}$, *the 2-functor* $U^T : \mathfrak{A}^T \to \mathfrak{A}$ *has a formal lax adjoint together with which it generates* T.

Proof. The data for a lax adjoint to U^T is given as follows.

(3.3.1) *A lax functor* $F^T : \mathfrak{A} \to \mathfrak{A}^T$ given in this way. For an object X of $\mathfrak{A}$, let $F^T(X) = (TX)_{(\mu_X)}$ with $\iota_{(\mu_X)} = \rho_X$ and $\kappa_{(\mu_X)} = \alpha_X$. That this is the data for a lax T-algebra follows directly from (2.3(1.1)), (2.3(2)) and (2.3(3.2)).

If $f : X \to X'$ is any 1-cell in $\mathfrak{A}$, define $F^T(f) = (Tf; \mu_f)$. That this is a lax T-homomorphism $F^T(X) \to F^T(X')$ is a direct consequence of (2.3(1.2)) and (2.3(3.3)).

Let $F^T(a) = Ta$ for any 2-cell $a : f \to f'$. That Ta is also a 2-cell in $\mathfrak{A}^T$, i.e., that (2.6.7) is satisfied for any a, follows from (2.2.3) for lax natural μ.

The rest of the data is given by $e_X^{F^T} = e_X^T$ and $c_{g,f}^{F^T} = c_{g,f}^T$. That these are well defined can be shown by using (2.2.1) and (2.2.2) respectively. E.g., since $e_X^T : T(1_X) \to 1_{TX}$ and since $F^T(1_X) = (T(1_X); \mu_{(1_X)})$ and $1_{F^T(X)} = (1_{TX}; \mu_X \cdot e_{TX}^T)$, in order for e_X^T to be a 2-cell $F^T(1_X) \to 1_{F^T(X)}$ in $\mathfrak{A}^T$, the following, i.e., (2.6.7) must be satisfied.

$$
\begin{array}{ccc}
\mu_X \cdot TT(1_X) & \xrightarrow{\mu_{(1_X)}} & T(1_X) \cdot \mu_X \\
\Big\downarrow{\mu_X \cdot T(e_X^T)} & \searrow^{\mu_X \cdot e_X^{TT}} \quad (2.2.1) & \Big\downarrow{e_X^T \cdot \mu_X} \\
& & 1_{TX} \cdot \mu_X \\
\mu_X \cdot T(1_{TX}) & \xrightarrow[\mu_X \cdot e_{TX}^T]{} & \Vert \\
& & \mu_X \cdot 1_{TTX}
\end{array}
$$

The reasons for the commutativity are indicated inside the diagram. We leave to the reader a similar verification with regard to $c_{g,f}^{F^T}$.

That F^T is a lax functor now follows directly from F a lax functor and U a 2-functor. Also, their composite is T.

(3.3.2) Since there is available *a lax natural transformation* $\eta : 1_{\mathfrak{A}} \to T = U^T F^T$, we must now produce *a lax natural transformation* $\epsilon : F^T U^T \to 1_{(\mathfrak{A}^T)}$.

For any object X_ξ of $\mathfrak{A}^T$, let $\epsilon_{(X_\xi)} : F^T U^T(X_\xi) \to X_\xi$ be given by $\epsilon_{(X_\xi)} = (\xi; \kappa_\xi) : TX_{(\mu_X)} \to X_\xi$. Note that conditions (2.5.1) and (2.5.2) for $(\xi; \kappa_\xi)$ translate into conditions (2.4.1) and (2.4.3) of the lax T-algebra X_ξ and say that $(\xi; \kappa_\xi)$ is a lax T-homomorphism. So, $\epsilon_{(X_\xi)}$ is well defined.

For a 1-cell $(f; \phi) : X_\xi \to Y_\theta$ of $\mathfrak{A}^T$, let

$$\epsilon_{(f;\phi)} \colon \epsilon_{(Y_\theta)} \cdot [F^T U^T(f;\phi)] \to [1_{\mathbb{Q}^T}(f;\phi)] \cdot \epsilon_{(X_\xi)}$$

be given by $\epsilon_{(f;\phi)} = \phi \colon \theta \cdot Tf \to f \cdot \xi$. Since we should have

$$\epsilon_{(f;\phi)} \colon (\theta \cdot Tf; (\theta\mu_f) \cdot (\kappa_\theta(TTf)) \cdot (\theta \cdot c_{\theta,Tf}^T)) \to (f\xi; (f\kappa_\xi)(\phi \cdot T\xi)(\theta \cdot c_{f,\xi}^T))$$

after computing the domain and codomain of $\epsilon_{(f;\phi)}$, we must have (2.6.7) satisfied for ϕ. This says precisely that $(f;\phi)$ satisfies (2.5.2), which is one of the two conditions stating that it is a lax T-homomorphism. Thus, also $\epsilon_{(f;\phi)}$ is well defined.

It remains to check that the definitions of $\epsilon_{(X_\xi)}$ and $\epsilon_{(f;\phi)}$ given above make ϵ a lax natural transformation. This is immediately observed by simply writing down what this means. We let the reader carry out these computations.

(3.3.3) Finally, we want *families* (L_X) *and* $(R_{(X_\xi)})$ *of 2-cells*, as in (3.1).

Define $L_X \colon \epsilon_{F^T(X)} \cdot F_{(\eta_X)}^T \to 1_{F^T(X)}$ by letting $L_X = \lambda_X \colon \mu_X \cdot T\eta_X \to 1_{TX}$. In order to see that this is well defined, we observe that the domain of L_X should be the composite

$$(\mu_X; \alpha_X) \cdot (T\eta_X; \mu_{(\eta_X)})$$

$$= (\mu_X \cdot T\eta_X; (\mu_X \cdot \mu_{(\eta_X)}) \cdot (\alpha_X(TT\eta_X)) \cdot (\mu_X \cdot c_{\mu_X, T\eta_X})),$$

and that its codomain should be, simply, $(1_{TX}; \mu_X \cdot e_{TX}^T)$. That λ_X verifies (2.6.7) and is thus a 2-cell in $\mathbb{Q}^T$, is spelled out in the condition (2.3(3.1)) on the lax monad T.

Define $R_{(X_\xi)} \colon 1_{U^T(X_\xi)} \to U^T \epsilon_{(X_\xi)} \cdot \eta_{U^T(X_\xi)}$ by letting $R_{(X_\xi)} = \iota_\xi \colon 1_X \to \xi \cdot \eta_X$. This is clearly well defined.

We verify, lastly, the axioms for the formal lax adjointness of $\langle F^T, \eta, \epsilon, L_X, R_{(X_\xi)} \rangle$ to $U^T \colon \mathbb{Q}^T \to \mathbb{Q}$, as in (3.1).

Let us mention briefly how this is done, letting the reader convince himself by writing down the appropriate diagrams:

(3.1.1) with $g = (f;\phi) \colon X_\xi \to Y_\theta$ is verified by condition (2.5.1) on a lax T-homomorphism;

(3.1.1*) for $f \colon X \to X'$ in $\mathbb{Q}$, results in (2.3(1*)) on the lax monad T;

(3.1.2) with a lax T-algebra Y_θ is precisely condition (2.4.2) on the latter;

(3.1.2*) is just (2.3(2*)) on the lax monad T.

This completes the proof, once we make sure that the lax monad induced in the sense of (3.2) is indeed T. We have seen already that the lax functor $U^T F^T = T$ and η is the same in both. But also, $U^T(\epsilon_{(F^T(X))}) = U^T(\mu_X; \alpha_X) = \mu_X$, whereas $U^T(\epsilon_{(F^T f)}) = U^T(\mu_f) = \mu_f$. Also, $U^T(L_X) = U^T(\lambda_X) = \lambda_X$ and $R_{F^T(X)} =$

$R_{((TX)_{\mu_X})} = {}^{\iota}(\mu_X) = \rho_X.$ We are done. $\square$

Further investigations into other aspects of a theory of lax monads, such as an analogue of Beck's theorem (cf. [10]), are out of the scope of this paper.

4. The universal property of a lax adjoint.

(4.1) Theorem. *Let $U: \mathcal{B} \to \mathcal{Q}$ be a 2-functor. Let us be given, for each $X \in |\mathcal{Q}|$, an object FX of $\mathcal{B}$ and a 1-cell $\eta_X: X \to UFX$ of $\mathcal{Q}$.*

Then, if the family $\{\eta_X\}$ is coherently closed for U-extensions, it follows that there is determined a structure of a lax functor on F, as well as a lax natural transformation $\epsilon: FU \to 1_{\mathcal{B}}$ and families of 2-cells (L_X) and (R_Y) and the data $\langle F, \eta, \epsilon, (L_X), (R_Y) \rangle$ is that of a formal lax adjoint to U.

Proof. (4.1.1) *F can be extended to a lax functor and η becomes lax natural.*

(1) *Define*, for $f: X \to X'$ in $\mathcal{Q}$, $Ff: FX \to FX' \in \mathcal{B}$ as the U-extension of $(\eta_X; f): X \to UFX'$ along $\eta_X: X \to UFX$. Let η_f be the 2-cell in

i.e., let $Ff = \overline{\eta_{X'} \cdot f}$ and $\eta_f = \psi_{(\eta_{X'} \cdot f)}$ in the terminology of (1.1) with $\kappa_X = \eta_X$.

(2) *Given* $a: f \to f'$ in $\mathcal{Q}$, *define* $Fa: Ff \to Ff'$ as follows: $Fa = \overline{(\eta_{X'} \cdot a) \cdot \eta_{f'}}$, also following the terminology of (1.1) with $\phi = (\eta_{X'} \cdot a)\eta_{f'}$. Indeed, one has, from

that

By the universal property, Fa is *unique* so that the diagram

$$
\begin{array}{ccc}
\eta_{X'} \cdot f & \xrightarrow{\;\;\eta_f\;\;} & UFf \cdot \eta_X \\
\Big\downarrow{\scriptstyle \eta_{X'} \cdot a} & & \Big\downarrow{\scriptstyle UFa \cdot \eta_X} \\
\eta_{X'} \cdot f' & \xrightarrow[\;\;\eta_{f'}\;\;]{} & UFf' \cdot \eta_X
\end{array}
$$

commutes.

Note that this is already the condition (2.2.3) for (η_X, η_f) to become a *lax natural transformation* $\eta: 1_{\mathcal{C}} \longrightarrow UF$.

(3) The diagram below commutes (we write the identity 2-cell to indicate this)

$$
\begin{array}{ccc}
X & \xrightarrow{\;\;\eta_X\;\;} & UFX \\
\Big\downarrow{\scriptstyle 1_X} & {\scriptstyle 1_{\eta_X}} \nearrow & \Big\downarrow{\scriptstyle U(1_{FX})} \\
X & \xrightarrow[\;\;\eta_X\;\;]{} & UFX
\end{array}
$$

By the universal property there is a *unique* 2-cell $e_X^F =_{df} \overline{1_{(\eta_X)}}: F(1_X) \longrightarrow 1_{FX}$ satisfying

$$
\begin{array}{ccc}
\eta_X & \xrightarrow{\;\;\eta_{(1_X)}\;\;} & UF(1_X) \cdot \eta_X \\
\| {\scriptstyle 1_{(\eta_X)}} & \curvearrowright & \Big\uparrow{\scriptstyle U e_X^F \cdot \eta_X} \\
& & 1_{UFX} \cdot \eta_X
\end{array}
$$

The commutativity of this diagram is condition (2.2.1) on lax naturality for η. This is indeed so, letting $e_X^{UF} = U(e_X^F)$ and $e_X^{1_{\mathcal{C}}} = 1_X$.

Next we look at the diagram:

$$
\begin{array}{ccc}
X & \xrightarrow{\;\;\eta_X\;\;} & UFX \\
\Big\downarrow{\scriptstyle f} & {\scriptstyle \eta_f}\nearrow & \Big\downarrow{\scriptstyle UFf} \\
Y & \xrightarrow{\;\;\eta_Y\;\;} & UFY \\
\Big\downarrow{\scriptstyle g} & {\scriptstyle \eta_g}\nearrow & \Big\downarrow{\scriptstyle UFg} \\
Z & \xrightarrow[\;\;\eta_Z\;\;]{} & UFZ
\end{array}
$$

It follows that there exists a *unique* 2-cell $c_{g,f}^{F} =_{df} [(\eta_g \cdot f) \cdot (UFg)\eta_f]: F(gf) \to$ $Fg \cdot Ff$, so that

$$(UFg \cdot \eta_f)(\eta_g \cdot f) = (Uc_{g,f}^{F} \cdot \eta_X)\eta_{gf}: \eta_Z \cdot gf \to UFg \cdot UFf \cdot \eta_X)$$

a condition which is none other than (2.2.2) for a lax natural transformation η. Thus, as all three conditions for η lax natural have been established and e^F, c^F defined, all we must now do is check F is a lax functor with these.

(4) We verify the conditions of (2.1) of F. For these, the *uniqueness* part in the definition of the extensions will prove the essential tool.

Proof of (2.1.1). Note that the diagram

is commutative (setting $c_{f,g}^{UF} = U(c_{f,g}^{F})$). But also $(U(1_{Ff}) \cdot \eta_X)\eta_f = \eta_f$. By uniqueness and universality of the pair (Ff, η_f), it follows that $(Ff \cdot e_X^F) \cdot c_{f,1_X}^{F} = 1_{Ff} = \overline{\eta_f}$.

Proof of (2.1.2). Similar to the previous one and left to the reader.

Proof of (2.1.3). Begin by observing the commutativity of the diagrams below for any 1-cells $X \xrightarrow{f} Y$, $Y \xrightarrow{g} Z$, $Z \xrightarrow{h} W$:

and

$$\eta_W \cdot hgf \xrightarrow{\;\;\eta_{hgf}\;\;} UF(hgf) \cdot \eta_X$$

(diagram (2.2.2))

$$\eta_W \cdot hgf \xrightarrow{\;\;\eta_{hgf}\;\;} UF(hgf) \cdot \eta_X$$

Left edge top: $\eta_{hg} \cdot f$, right edge top: $Uc^F_{hg,f} \cdot \eta_X$

$$UF(hg) \cdot \eta_Y \cdot f \xrightarrow{\;UF(hg) \cdot \eta_{gf}\;} UF(hg) \cdot UFf \cdot \eta_X$$

Left edge: $Uc^F_{h,g} \cdot \eta_Y \cdot f$, right edge: $Uc^F_{h,g} \cdot UFf \cdot \eta_X$

$$UFh \cdot UFg \cdot \eta_Y \cdot f \xrightarrow{\;UFh \cdot UFg \cdot \eta_f\;} UFh \cdot UFg \cdot UFf \cdot \eta_X$$

Next, observe that another application of (2.2.2) for η delivers:

$$(UFh \cdot UFg \cdot \eta_f) \cdot (UFh \cdot \eta_g \cdot f) \cdot (\eta_h \cdot gf)$$

$$= (UFh \cdot UFg \cdot \eta_f) \cdot (Uc^F_{h,g} \cdot \eta_Y \cdot f)(\eta_{hg} \cdot f).$$

Calling this 2-cell β, uniqueness yields

$$(Fh \cdot c^F_{g,f})(c^F_{h,gf}) = \overline{\beta} = (c^F_{h,g} \cdot Ff) \cdot c^F_{hg,f}.$$

Proof of (2.1.4). Observe that $\eta_f \cdot (\eta_{X'} \cdot 1_f) = (U(1_{Ff}) \cdot \eta_X)\eta_f$ for any $f: X \to X'$, trivially. Hence, $F(1_f) =_{df} \overline{(\eta_{X'} \cdot 1_f) \cdot \eta_f} = 1_{Ff}$, by uniqueness.

Proof of (2.1.5). Let $a: f \to f'$, $a': f' \to f''$. By definition, $F(a' \cdot a)$ is unique so that $\eta_{f''}(\eta_{X'}(a' \cdot a)) = (UF(a'a) \cdot \eta_X)\eta_f$. Since also

$$\eta_{f''} \cdot (\eta_{X'}(a'a)) = \eta_{f''}(\eta_{X'} \cdot a')(\eta_{X'} \cdot a) = (UFa' \cdot \eta_X)(\eta_{f'})(\eta_{X'} \cdot a)$$

$$= (UFa' \cdot \eta_X)(UFa \cdot \eta_X) \cdot \eta_f = (U(Fa' \cdot Fa) \cdot \eta_X) \cdot \eta_f,$$

one has that $F(a' \cdot a) = Fa' \cdot Fa$.

Proof of (2.1.6). Let $a: f \to f'$, $b: g \to g'$ and g, f composable, g', f' composable.

Consider the commutative diagrams

$$\begin{array}{ccc}
\eta_Z \cdot gf & \xrightarrow{\;\;\eta_{gf}\;\;} & UF(gf) \cdot \eta_X \\
{\scriptstyle \eta_Z \cdot ba}\Big\downarrow & (2.2.3) & \Big\downarrow{\scriptstyle UF(ba) \cdot \eta_X} \\
\eta_Z \cdot g'f' & \xrightarrow{\;\;\eta_{g'f'}\;\;} & UF(g'f') \cdot \eta_X \\
{\scriptstyle \eta_{g'} \cdot f'}\Big\downarrow & (2.2.2) & \Big\downarrow{\scriptstyle Uc^F_{g',f'} \cdot \eta_X} \\
UFg' \cdot \eta_Y \cdot f' & \xrightarrow{\;UFg' \cdot \eta_{f'}\;} & UFg' \cdot UFf' \cdot \eta_X
\end{array}$$

and

$$\begin{array}{ccc}
\eta_Z \cdot gf & \xrightarrow{\qquad\eta_{g,f}\qquad} & UF(gf) \cdot \eta_X \\
{\scriptstyle \eta_g \cdot f}\Big\downarrow & (2.2.2) & \Big\downarrow{\scriptstyle Uc^F_{g,f} \cdot \eta_X} \\
UFg \cdot \eta_Y \cdot f & \xrightarrow{\;\;UFg \cdot \eta_f\;\;} & UFg \cdot UFf \cdot \eta_X \\
{\scriptstyle UFg \cdot \eta_Y \cdot a}\Big\downarrow & (2.2.3) & \Big\downarrow{\scriptstyle UFg \cdot UFa \cdot \eta_X} \\
UFg \cdot \eta_Y \cdot f' & \xrightarrow{\;\;UFg \cdot \eta_{f'}\;\;} & UFg \cdot UFf' \cdot \eta_X \\
{\scriptstyle UFb \cdot \eta_Y \cdot f'}\Big\downarrow & \circlearrowleft & \Big\downarrow{\scriptstyle UFb \cdot UFf' \cdot \eta_X} \\
UFg' \cdot \eta_Y \cdot f' & \xrightarrow{\;\;UFg' \cdot \eta_{f'}\;\;} & UFg' \cdot UFf' \cdot \eta_X.
\end{array}$$

Next, we observe that

$$(UFg' \cdot \eta_{f'}) \cdot (\eta_{g'} \cdot f') \cdot (\eta_Z \cdot ba)$$

$$= (UFg' \cdot \eta_{f'})(UFb \cdot \eta_Y \cdot f')(UFg \cdot \eta_Y \cdot a)(\eta_g \cdot f)$$

since

$$\begin{array}{ccccc}
\eta_Z \cdot gf & \xrightarrow{\;\eta_Z \cdot g \cdot a\;} & \eta_Z \cdot g \cdot f' & \xrightarrow{\;\eta_Z \cdot b \cdot f'\;} & \eta_Z \cdot g' \cdot f' \\
{\scriptstyle \eta_g \cdot f}\Big\downarrow & \circlearrowleft & {\scriptstyle \eta_g \cdot f'}\Big\downarrow \;\; (2.2.3) & & \Big\downarrow{\scriptstyle \eta_{g'} \cdot f'} \\
UFg \cdot \eta_Y \cdot f & \xrightarrow{\;UFg \cdot \eta_Y \cdot a\;} & UFg \cdot \eta_Y \cdot f' & \xrightarrow{\;UFb \cdot \eta_Y \cdot f'\;} & UFg' \cdot \eta_Y \cdot f'
\end{array}$$

The uniqueness now gives: $c^F_{g',f'} \cdot F(ba) = (Fb \cdot Fa) \cdot c^F_{g,f}$.

(4.1.2). *The rest of the data, i.e.,* ϵ, (L_X), (R_Y). So far we have only used the fact that there are U-extensions along the η_X. This gave us F a lax functor, $\eta: 1_{\mathbb{A}} \to UF$ a lax natural transformation. The coherence conditions (ii) and (iii) of (1.2), the definition of a coherently closed family for U-extensions, are indispensable for establishing lax adjointness.

Let $\epsilon_Y = \overline{1_{UY}}$ and $R_Y = \psi_{(1_{UY})}$. Note that both are well defined, as we have

$$UY \xrightarrow{\eta_{UY}} UFUY$$

$$1_{UY} \qquad \psi_{(1_{UY})} \qquad U(\overline{1_{UY}})$$

$$UY$$
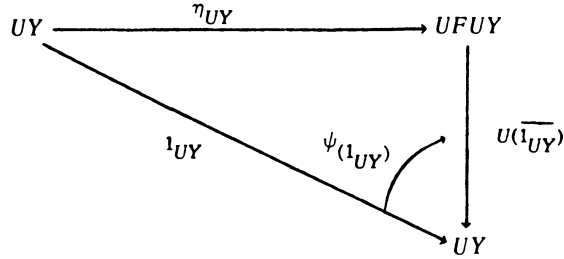
By coherence (ii) and the above definitions, $\overline{f} = \epsilon_Y \cdot Ff$ for any $f: X \to UY$ and $\psi_f = (U\epsilon_Y \cdot \eta_f) \cdot (R_Y \cdot f)$. In particular, $\overline{\eta_X} = \epsilon_{FX} \cdot F\eta_X$ and $\psi_{(\eta_X)} = (U\epsilon_{FX} \cdot \eta_{\eta_X}) \cdot (R_{FX} \cdot \eta_X)$. One then deduces the existence of a unique $L_X = \overline{1_{(\eta_X)}}: \epsilon_{FX} \cdot F\eta_X \to 1_{FX}$, satisfying $U(L_X) \cdot \psi_{(\eta_X)} = 1_{(\eta_X)}$. Before reducing, let us translate: the above is exactly condition $(3.1.2^*)$ of a lax adjoint. It says precisely $(U(L_X) \cdot \eta_X)(U\epsilon_{FX} \cdot \eta_{(\eta_X)}) \cdot R_{FX} \cdot \eta_X = 1_{(\eta_X)}$.

Let us now bring in the coherence (i) into this picture. It says that $\overline{\eta_X} = 1_{FX}$ and that $\psi_{(\eta_X)} = 1_{(\eta_X)}$. But then, $L_X = identity$. This will be one of the conditions on an arbitrary formal lax adjoint when we attempt to recapture the universal property.

Note something else. The diagram

$$X \xrightarrow{\eta_X} UFX$$

$$1_X \qquad \psi_{\eta_X} \qquad U(\overline{\eta_X}) \cdot$$

$$X \xrightarrow{\eta_X} UFX$$
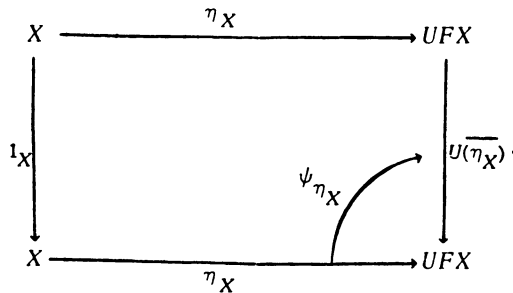
says also that $\overline{\eta_X} = F(1_X)$, by definition, and that $\psi_{(\eta_X)} = \eta_{(1_X)}$. Since $\overline{\eta_X} = 1_{FX}$ one has $F(1_X) = 1_{FX}$, and, since $e^F_X = \overline{1_{(\eta_X)}}$ is unique so that $1_{(\eta_X)} =$

$(Ue_X^F \cdot \eta_X) \cdot \eta_{(1_X)}$ and since $\eta_{(1_X)} =_{df} \psi_{(\eta_X)} = 1_{(\eta_X)}$, one must have $e_X^F =$ *identity as well.* This will be another condition which will allow us to recapture the universal property.

We have not finished defining the data. We need to choose, for each $g\colon Y \to Z \in \mathfrak{B}$, some $\epsilon_g\colon \epsilon_Z \cdot FUg \to g \cdot \epsilon_Y$.

By coherence, $\epsilon_Z \cdot FUg = \overline{Ug}$. Thus, we only need some appropriate diagram of the form

$$
\begin{array}{ccc}
UY & \xrightarrow{\ \eta_{UY}\ } & UFUY \\
& \searrow^{Ug} \quad {}^{\beta} & \big\downarrow^{U(g \cdot \epsilon_Y)} \\
& & UZ
\end{array}
$$

By taking $\beta = Ug \cdot R_Y\colon Ug \to Ug \cdot U\epsilon_Y \cdot \eta_{UY}$, and letting $\epsilon_g = \overline{\beta}$, we obtain as a result a characterization of ϵ_g as the only 2-cell which makes

$$
\begin{array}{ccc}
Ug & \xrightarrow{\ \psi_{Ug}\ } & U(\epsilon_Z \cdot FUg) \cdot \eta_{UY} \\
& \searrow^{Ug \cdot R_Y} & \big\downarrow^{U(\epsilon_g) \cdot \eta_{UY}} \\
& & U(g \cdot \epsilon_Y) \cdot \eta_{UY}
\end{array}
$$

commutative which, upon translation with $\psi_{Ug} = (U\epsilon_Z \cdot \eta_{Ug}) \cdot (R_Z \cdot Ug)$ by coherence (ii), gives precisely the condition (3.1.1) on lax adjoints, with $g\colon Y \to Z$.

(4.1.3) $\epsilon\colon FU \to 1_{\mathfrak{B}}$ *is a lax natural transformation.*

Proof of (2.2.1). By definition, $\epsilon_{(1_Y)} = \overline{U(1_Y) \cdot R_Y} = \overline{R_Y} = \overline{\psi_{(1_{UY})}} = 1_{(\epsilon_Y)}$. This, of course, is true of any $\overline{\psi_f}$. On the other hand we have shown that coherence implies that $e_X^F = 1_{FX}$. Thus the result.

Proof of (2.2.2). By definition, $\epsilon_{hg} = \overline{U(hg) \cdot R_Y}$ for any pair $g\colon Y \to Z$, $h\colon Z \to W$ of 1-cells of $\mathfrak{B}$. This says that ϵ_{hg} is the only 2-cell for which

$$(U(\epsilon_{hg}) \cdot \eta_{UY})(U\epsilon_W \cdot \eta_{U(hg)})(R_W \cdot U(hg)) = U(hg) \cdot R_Y.$$

Thus, in order to establish (2.2.2) all we have to do is show that the composite

$$\epsilon_W \cdot FU(hg) \xrightarrow{\ \epsilon_W \cdot c^F\ } \epsilon_W \cdot FUh \cdot FUg \xrightarrow{\ \epsilon_h \cdot FUg\ } h \cdot \epsilon_Z \cdot FUg \xrightarrow{\ h \cdot \epsilon_g\ } h \cdot g \cdot \epsilon_Y$$

satisfies the same condition as ϵ_{bg} above. This is done below:

$$U(bg) \xrightarrow{R_W \cdot U(bg)} U\epsilon_W \cdot \eta_{UW} \cdot U(bg) \xrightarrow{U\epsilon_W \cdot \eta_{U(bg)}} U\epsilon_W \cdot UFU(bg) \cdot \eta_{UY}$$

(2.2.2)

with vertical map $U\epsilon_W \cdot \eta_{Ub} \cdot Ug$ and $U\epsilon_W \cdot Uc^F \cdot \eta_{UY}$

(3.1.1) $U\epsilon_W \cdot UFUb \cdot \eta_{UZ} \cdot Ug \xrightarrow{U\epsilon_W \cdot UFUb \cdot \eta_{Ug}} U\epsilon_W \cdot UFUb \cdot UFUg \cdot \eta_{UY}$

$U\epsilon_b \cdot \eta_{UZ} \cdot Ug$ $U\epsilon_b \cdot UFUg \cdot \eta_{UY}$

$$Ub \cdot Ug \xrightarrow{Ub \cdot R_Z \cdot Ug} Ub \cdot U\epsilon_Z \cdot \eta_{UZ} \cdot Ug \xrightarrow{Ub \cdot U\epsilon_Z \cdot \eta_{Ug}} Ub \cdot U\epsilon_Z \cdot UFUg \cdot \eta_{UY}$$

(3.1.1) $Ub \cdot U\epsilon_g \cdot \eta_{UY}$

$Ub \cdot Ug \cdot R_Y$

$$Ub \cdot Ug \cdot U\epsilon_Y \cdot \eta_{UY}$$

Note that condition (3.1.1) was established in (4.1.2).

   Proof of (2.2.3). Let $b: g \to g'$ in $\mathcal{B}$, g, $g': Y \to Z$. We want to show that the 2-cells

$$\gamma_1 = \epsilon_Z \cdot FUg \xrightarrow{\epsilon_g} g \cdot \epsilon_Y \xrightarrow{b\epsilon_Y} g' \cdot \epsilon_Y, \quad \text{and}$$

$$\gamma_2 = \epsilon_Z \cdot FUg \xrightarrow{\epsilon_Z \cdot FUb} \epsilon_Z \cdot FUg' \xrightarrow{\epsilon_{g'}} g' \cdot \epsilon_Y$$

are equal. To do so we must find some $\beta: Ug \to U(g' \cdot \epsilon_Y) \cdot \eta_{UY}$ for which it is the case that

$$(U(\gamma_i) \cdot \eta_{UY})(U\epsilon_Z \cdot \eta_{Ug})(R_Z \cdot Ug) = \beta, \quad \text{for } i = 1, 2.$$

We claim that this is so with $\beta = Ug \xrightarrow{Ub} Ug' \xrightarrow{Ug' \cdot R_Y} Ug' \cdot U\epsilon_Y \cdot \eta_{UY}$. The verification is given in the diagrams below:

$$Ug \xrightarrow{R_Z \cdot Ug} U\epsilon_Z \cdot \eta_{UZ} \cdot Ug \xrightarrow{U\epsilon_Z \cdot \eta_{Ug}} U\epsilon_Z \cdot UFUg \cdot \eta_{UY}$$

Ub (3.1.1) $U\epsilon_g \cdot \eta_{UY}$

$Ug \cdot R_Y$

$$Ug' \qquad\qquad Ug \cdot U\epsilon_Y \cdot \eta_{UY}$$

$Ub \cdot U\epsilon_Y \cdot \eta_{UY}$

$Ug' \cdot R_Y$

$$Ug' \cdot U\epsilon_Y \cdot \eta_{UY}$$

and

$$
\begin{array}{ccccc}
Ug & \xrightarrow{R_Z \cdot Ug} & U\epsilon_Z \cdot \eta_{UZ} \cdot Ug' & \xrightarrow{U\epsilon_Z \cdot \eta_{Ug}} & U\epsilon_Z \cdot UFUg \cdot \eta_{UY} \\
\Big\downarrow{\scriptstyle Ub} & \circlearrowleft & \Big\downarrow{\scriptstyle U\epsilon_Z \cdot \eta_{UZ} \cdot Ub} & (2.2.3) & \Big\downarrow{\scriptstyle U\epsilon_Z \cdot UFUb \cdot \eta_{UY}} \\
Ug' & \xrightarrow{R_Z \cdot Ug'} & U\epsilon_Z \cdot \eta_{UZ} \cdot Ug' & \xrightarrow{U\epsilon_Z \cdot \eta_{Ug'}} & U\epsilon_Z \cdot UFUg' \cdot \eta_{UY} \\
& & & (3.1.1) & \Big\downarrow{\scriptstyle U\epsilon_{g'} \cdot \eta_{UY}} \\
& & Ug' \cdot R_Y & & Ug' \cdot U\epsilon_Y \cdot \eta_{UY}
\end{array}
$$

(4.1.4). *The remaining conditions on lax adjointness hold.* We have already established (3.1.2*) and (3.1.1), both in (4.1.2). We need to prove (3.1.1*) and (3.1.2).

Proof of (3.1.2). We want to show that the composite

$$
\gamma = \epsilon_Y \cdot F(1_{UY}) \xrightarrow{\epsilon_Y \cdot F(R_Y)} \epsilon_Y \cdot F(U\epsilon_Y \cdot \eta_{UY}) \xrightarrow{\epsilon_Y \cdot c^F} \epsilon_Y \cdot FU\epsilon_Y \cdot F\eta_{UY}
$$

$$
\xrightarrow{\epsilon_{\epsilon_Y} \cdot F\eta_{UY}} \epsilon_Y \cdot \epsilon_{FUY} \cdot F\eta_{UY}
$$

is the identity. Indeed, this is all that remains of (3.1.2) after the identifications of e^F_{UY} and L_{UY} with identity 2-cells have been made.

The following observations will guide us to find the correct diagram. First, note that $\gamma: \epsilon_Y \to \epsilon_Y$ since by the coherence conditions imposed, it followed that $F(1_{UY}) = 1_{FUY}$ and that $\epsilon_{FUY} \cdot F\eta_{UY} = 1_{FUY}$. Secondly, note that, since $R_Y = \psi_{(1_{UY})}$, $\overline{R_Y} = 1_{\epsilon_Y}$ and therefore that $\gamma = 1_{\epsilon_Y}$ will immediately follow if we could establish the equation

(*) $\qquad\qquad (U\gamma \cdot \eta_{UY}) \cdot R_Y = R_Y.$

Note also that $\eta_{(1_X)} = 1_{(\eta_X)}$ has been established in (4.1.2) using coherence (i), and finally, note that (3.1.2*) reduces, after all the identifications with identity 2-cells, to the equation

(3.1.2*) $\qquad\qquad (U\epsilon_{FX} \cdot \eta_{\eta_X})(R_{FX} \cdot \eta_X) = 1_{\eta_X}.$

We use the above remarks in the proof of (*) below:

Proof of (3.1.1*). This amounts, after reducing the 2-cells which are identities, to showing the equation:

$$(\epsilon_{Ff} \cdot F\eta_X)(\epsilon_{FX'} \cdot c^F)(\epsilon_{FX'} \cdot F\eta_f) = \epsilon_{FX'} \cdot c^F: \eta_{FX'} \cdot F(\eta_{X'} \cdot f) \to Ff.$$

Let $\gamma_1 = \epsilon_{FX'} \cdot c^F$ and γ_2 be the other side of the equation given above. The proof will be achieved by showing that $(U(\gamma_i) \cdot \eta_X)(\psi_{(\eta_{X'} \cdot f)}) = \eta_f$, for $i = 1, 2$. Recall that, since $\eta_{X'} \cdot f: X \to UFX'$,

$$\psi_{(\eta_{X'} \cdot f)} = (U\epsilon_{FX'} \cdot \eta_{(\eta_{X'} \cdot f)}) \cdot (R_{FX'} \cdot \eta_{X'} \cdot f).$$

Let $\gamma = \gamma_1 = \epsilon_{FX'} \cdot c^F_{\eta_{X'}, f}$. The diagram is given below and commutes:

Let $\gamma = \gamma_2 = (\epsilon_{Ff} \cdot F\eta_X) \cdot (\epsilon_{FX'} \cdot c^F) \cdot (\epsilon_{FX'} \cdot F\eta_f)$. The relevant diagram commutes and is the following:

$$
\begin{array}{ccccc}
\eta_{X'} \cdot f & \xrightarrow{R_{FX'} \cdot \eta_{X' \cdot f}} & U\epsilon_{FX'} \cdot \eta_{UFX'} \cdot \eta_{X'} \cdot f & \xrightarrow{U\epsilon_{FX'} \cdot \eta_{(\eta_{X'} \cdot f)}} & U\epsilon_{FX'} \cdot UF(\eta_{X'} \cdot f) \cdot \eta_X
\end{array}
$$

(diagram)

η_f ; η_f ; $U\epsilon_{FX'} \cdot \eta_{UFX'} \cdot \eta_f$; (2.2.3) ; $U\epsilon_{FX'} \cdot UF\eta_f \cdot \eta_X$

$$
UFf \cdot \eta_X \xrightarrow{R_{FX'} \cdot UFf \cdot \eta_X} U\epsilon_{FX'} \cdot \eta_{UFX'} \cdot UFf \cdot \eta_X \xrightarrow{U\epsilon_{FX'} \cdot \eta_{(Ff \cdot \eta_X)}} U\epsilon_{FX'} \cdot UF(UFf \cdot \eta_X) \cdot \eta_X
$$

(3.1.1)

$U\epsilon_{FX'} \cdot \eta_{UFf} \cdot \eta_X$; (2.2.2) ; $U\epsilon_{FX'} \cdot U\epsilon^F \cdot \eta_X$

$UFf \cdot R_{FX} \cdot \eta_X$

$$
U\epsilon_{FX'} \cdot UFUFf \cdot \eta_{UFX} \cdot \eta_X \xrightarrow{U\epsilon_{FX'} \cdot UFUFf \cdot \eta_{\eta_X}} U\epsilon_{FX'} \cdot UFUFf \cdot UF\eta_X \cdot \eta_X
$$

$U\epsilon_{Ff} \cdot \eta_{UFX} \cdot \eta_X$; $U\epsilon_{Ff} \cdot UF\eta_X \cdot \eta_X$

$$
UFf \cdot \eta_X \xrightarrow{UFf \cdot R_{FX} \cdot \eta_X} UFf \cdot U\epsilon_{FX} \cdot \eta_{UFX} \cdot \eta_X \xrightarrow{UFf \cdot U\epsilon_{FX} \cdot \eta_{\eta_X}} UFf \cdot U\epsilon_{FX} \cdot UF\eta_X \cdot \eta_X
$$

(3.1.2*)

$\| \qquad UFf \cdot \eta_X$

This completes the proof of the theorem. □

(4.2) **Definition.** Let $U: \mathcal{B} \to \mathcal{A}$ be a 2-functor with $\langle F, \eta, \epsilon, (L_X), (R_Y) \rangle$ a formal lax adjoint to U. Say that it is a *normalized lax adjoint* provided the following 2-cells are all identities:

(i) $e_X^F: F(1_X) \to 1_{FX}$, for all X;

(ii) $c_{\eta_X, f}^F: F(\eta_X \cdot f) \to F\eta_X \cdot Ff$, for all $f: X' \to X$;

(iii) $\eta_{(\eta_X)}: \eta_{UFX} \cdot \eta_X \to UF\eta_X \cdot \eta_X$ for all X;

(iv) $L_X: \epsilon_{FX} \cdot F\eta_X \to 1_{FX}$, for all X.

We point out that this list can be expanded since (i) implies that also

(v) $\eta_{(1_X)}$; $\epsilon_{(1_Y)}$; $c_{f, 1_X}$ and $c_{1_Y, f}$ are all identities, and (iii) and (iv) yield

(vi) $R_{FX}: 1_{UFX} \to U\epsilon_{FX} \cdot \eta_{UFX}$ is the identity, for all X. Note however that nothing in the world gives arbitrary R_Y to be the identity and that, in general, the η_f and ϵ_g need not be so either. Same for the arbitrary $c_{g, f}^F$. In other words, the matter does not trivialize.

(4.3) **Theorem.** *Let* $U: \mathcal{B} \to \mathcal{A}$ *be a 2-functor, and* $\langle F, \eta, \epsilon, (L_X), (R_Y) \rangle$ *a normalized lax adjoint. Then, the family* $\{\eta_X: X \to UFX\}$ *is coherently closed for U-extensions.*

Proof. Given any $f: X \to UY$, define $\bar{f}: FX \to Y$ to be $\bar{f} = \epsilon_Y \cdot Ff$, and $\psi_f: f \to U\bar{f} \cdot \eta_X$ to be $\psi_f = [(U\epsilon_Y)\eta_f] \cdot [R_Y \cdot f]$.

Assuming we have shown the extension property, let us establish the coherence, independently. Here, the normalization is the key.

(4.3.1) *Coherence of the extensions.* The conditions (ii) say that one should have

$$\bar{f} = \overline{1_{UY}} \cdot \overline{\eta_{UY} \cdot f} \quad \text{and} \quad \psi_f = [U\overline{(1_{UY})} \cdot \psi_{(\eta_{UY} \cdot f)}] \cdot [\psi_{(1_{UY})} \cdot f].$$

By definition, $\overline{1_{UY}} = \epsilon_Y \cdot F(1_{UY}) = \epsilon_Y$ and $\overline{\eta_{UY} \cdot f} = \epsilon_{FUY} \cdot F(\eta_{UY} \cdot f) =$
$\epsilon_{FUY} \cdot F\eta_{UY} \cdot Ff = Ff$. Note that the above uses that: e^F_{UY}, $c^F_{\eta_{UY},f}$ and L_X are
identities. Therefore, it is true that $\overline{1_{UY}} \cdot \overline{\eta_{UY} \cdot f} = \bar{f}$.

Also by definition, $\psi_{(\eta_{UY} \cdot f)} = (U\epsilon_{FUY} \cdot \eta_{(\eta_{UY} \cdot f)}) \cdot (R_{FUY} \cdot \eta_{UY} \cdot f)$. But
$\eta_{(\eta_{UY} \cdot f)} = (UF\eta_{UY})\eta_f$, since $c^F_{\eta_{UY},f}$ and $\eta_{\eta_{UY}}$ are identities; also, $R_{FUY} =$
identity. Thus, $\psi_{(\eta_{UY} \cdot f)} = \eta_f$. As for $\psi_{(1_{UY})}$, by definition $\psi_{(1_{UY})} =$
$[(U\epsilon_Y)\eta_{(1_{UY})}][R_Y \cdot 1_{UY}] = R_Y$, since $\eta_{(1_{UY})}$ is the identity. Therefore, their
composite yields $[(U\epsilon_Y)\eta_f][R_Y \cdot f] =_{df} \psi_f$.

We now verify the coherence condition (i), which says that $\bar{\eta}_X = .1_{FX}$ and
$\psi_{(\eta_X)} = 1_{(\eta_X)}$.

By definition, $\bar{\eta}_X = \epsilon_{FX} \cdot F\eta_X$. On the other hand, since L_X is the identity,
the latter is 1_{FX}. Also, by definition, $\psi_{(\eta_X)} = [(U\epsilon_{FX})\eta_{\eta_X}] \cdot [R_{FX} \cdot \eta_X] =$
identity by conditions (iii) and (vi) of a normalized lax adjoint.

(4.3.2) *Extension property of the pairs* $(\bar{f} ; \psi_f)$. Let $g: FX \to Y$ and $\beta: f \to$
$Ug \cdot \eta_X$ be given. Define then $\bar{\beta}: \epsilon_Y \cdot Ff \to g$ as follows:

$$\bar{\beta} = \epsilon_Y \cdot Ff \xrightarrow{\epsilon_Y \cdot F\beta} \epsilon_Y \cdot F(Ug \cdot \eta_X) \xrightarrow{\epsilon_Y \cdot c^F} \epsilon_Y \cdot FUg \cdot F\eta_X$$

$$\xrightarrow{\epsilon_g \cdot F\eta_X} g \cdot \epsilon_{FX} \cdot F\eta_X.$$

The diagram on the next page establishes that $(U\bar{\beta})\eta_X \cdot \psi_f = \beta$.

Note that, when verifying the commutativity of this diagram, we did not take
advantage of the fact that some of the 2-cells are identities. However, this
made it easier to identify the coherence conditions involved.

The only thing that remains is to establish the uniqueness of $\bar{\beta}$. Thus,
assume $\gamma: \epsilon_Y \cdot Ff \to g$ satisfies the equation $(U\gamma)\eta_X \cdot \psi_f = \beta$. Claim: $\bar{\beta} = \gamma$. In
order to prove it let us write down the definition of $\bar{\beta}$ replacing β by $(U\gamma)\eta_X \cdot \psi_f$
in it. What results, prior to the usual reductions, is the composite:

$$\bar{\beta} = \epsilon_Y \cdot Ff \xrightarrow{\epsilon_Y \cdot F(R_Y \cdot f)} \epsilon_Y \cdot F(U\epsilon_Y \cdot \eta_{UY} \cdot f) \xrightarrow{\epsilon_Y \cdot F(U\epsilon_Y \cdot \eta_f)} \epsilon_Y \cdot F(U\epsilon_Y \cdot UFf \cdot \eta_X)$$

$$\xrightarrow{\epsilon_Y \cdot F(U\gamma \cdot \eta_X)} \epsilon_Y \cdot F(Ug \cdot \eta_X) \xrightarrow{\epsilon_Y \cdot c^F} \epsilon_Y \cdot FUg \cdot F\eta_X \xrightarrow{\epsilon_g \cdot F\eta_X} g \cdot \epsilon_{FX} \cdot F\eta_X.$$

$$f \xrightarrow{\psi_f} U\epsilon_Y \cdot UFf \cdot \eta_X$$

$$f \xrightarrow{R_Y \cdot f} U\epsilon_Y \cdot \eta_{UY} \cdot f \xrightarrow{U\epsilon_Y \cdot \eta_f} U\epsilon_Y \cdot UFf \cdot \eta_X$$

$$\beta \qquad U\epsilon_Y \cdot \eta_{UY} \cdot \beta \qquad (2.2.3) \qquad U\epsilon_Y \cdot UF\beta \cdot \eta_X$$

$$Ug \cdot \eta_X \xrightarrow{R_Y \cdot (Ug \cdot \eta_X)} U\epsilon_Y \cdot \eta_{UY} \cdot Ug \cdot \eta_X \xrightarrow{U\epsilon_Y \cdot \eta_{(Ug \cdot \eta_X)}} U\epsilon_Y \cdot UF(Ug \cdot \eta_X) \cdot \eta_X$$

$$(3.1.1) \qquad U\epsilon_Y \cdot \eta_{Ug} \cdot \eta_X \qquad (2.2.2) \qquad c^F$$

$$U\epsilon_Y \cdot UFUg \cdot \eta_{UFX} \cdot \eta_X \xrightarrow{U\epsilon_Y \cdot UFUg \cdot \eta_{\eta_X}} U\epsilon_Y \cdot UFUg \cdot UF\eta_X \cdot \eta_X$$

$$U\epsilon_g \cdot \eta_{UFX} \cdot \eta_X \qquad U\epsilon_g \cdot UF\eta_X \cdot \eta_X$$

$$Ug \cdot U\epsilon_{FX} \cdot \eta_{UFX} \cdot \eta_X \xrightarrow{Ug \cdot U\epsilon_{FX} \cdot \eta_{\eta_X}} Ug \cdot U\epsilon_{FX} \cdot UF\eta_X \cdot \eta_X$$

$$Ug \cdot R_{FX} \cdot \eta_X \qquad (3.1.2^*) \qquad Ug \cdot UL_X \cdot \eta_X$$

$$Ug \cdot \eta_X =\!=\!=\!=\!=\!=\!= Ug \cdot \eta_X$$

First reduction. The diagram below is commutative:

$$\epsilon_Y \cdot Ff \xrightarrow{\epsilon_Y \cdot F(R_Y \cdot f)} \epsilon_Y \cdot F(U\epsilon_Y \cdot \eta_{UY} \cdot f)$$

$$\epsilon_Y \cdot F(U\epsilon_Y \cdot \eta_f)$$

$$\epsilon_Y \cdot F(U\epsilon_Y \cdot UFf \cdot \eta_X) \xrightarrow{\epsilon_Y \cdot F(U\gamma \cdot \eta_X)} \epsilon_Y \cdot F(Ug \cdot \eta_X)$$

$$\|$$

$$\epsilon_Y \cdot FU\epsilon_Y \cdot FUFf \cdot F\eta_X \qquad \qquad \epsilon_Y \cdot c^F$$

$$\epsilon_Y \cdot c^F \qquad (2.1.6)$$

$$\epsilon_Y \cdot FU(\epsilon_Y \cdot Ff) \cdot F\eta_X \xrightarrow{\epsilon_Y \cdot FU\gamma \cdot F\eta_X} \epsilon_Y \cdot FUg \cdot F\eta_X$$

$$\epsilon_{(\epsilon_Y \cdot Ff)} \cdot F\eta_X \qquad (2.2.3) \qquad \epsilon_g \cdot F\eta_X$$

$$\epsilon_Y \cdot Ff \cdot \epsilon_{FX} \cdot F\eta_X \xrightarrow{\gamma \cdot \epsilon_{FX} \cdot F\eta_X} g \cdot \epsilon_{FX} \cdot F\eta_X$$

$$\| \qquad \qquad \|$$

$$\epsilon_Y \cdot Ff \xrightarrow{\gamma} g$$

Combine this with the following in order to obtain the desired result:

Second reduction. The following diagram commutes:

$$[\epsilon_Y \cdot F(1_{UY} \cdot f)] \xrightarrow{R_Y} [\epsilon_Y \cdot F(U\epsilon_Y \cdot \eta_{UY} \cdot f)]$$

(diagram, labels include: c^F, $(2.1.6)$, $(2.1.3)$, $(2.2.2)$, $(3.1.1^*)$, $(3.1.2)$, η_f, R_Y, e^F, L_X, L_{UY}, $\epsilon_{(\epsilon_Y)}$, $\epsilon_{(\epsilon_Y \cdot Ff)}$, ϵ_{Ff})

Vertices:
$[\epsilon_Y \cdot F(1_{UY} \cdot f)]$
$[\epsilon_Y \cdot F(U\epsilon_Y \cdot \eta_{UY} \cdot f)]$
$[\epsilon_Y \cdot F(U\epsilon_Y \cdot UFf \cdot \eta_X)]$
$[\epsilon_Y \cdot FU(\epsilon_Y \cdot Ff) \cdot F\eta_X]$
$[\epsilon_Y \cdot F U\epsilon_Y \cdot FUFf \cdot F\eta_X]$
$[\epsilon_Y \cdot Ff \cdot \epsilon_{FX} F\eta_X]$
$[\epsilon_Y \cdot Ff]$
$[\epsilon_Y \cdot F(U\epsilon_Y) \cdot F(UFf \cdot \eta_X)]$
$[\epsilon_Y \cdot \epsilon_{FUY} \cdot FUFf \cdot F\eta_X]$
$[\epsilon_Y \cdot \epsilon_{FUY} \cdot F(UFf \cdot \eta_X)]$
$[\epsilon_Y \cdot F(1_{UY}) \cdot Ff]$
$[\epsilon_Y \cdot F(U\epsilon_Y \cdot \eta_{UY}) \cdot Ff]$
$[\epsilon_Y \cdot F(U\epsilon_Y) \cdot F(\eta_{UY} \cdot f)]$
$[\epsilon_Y \cdot FU\epsilon_Y \cdot F\eta_{UY} \cdot Ff]$
$[\epsilon_Y \cdot \epsilon_{FUY} \cdot F(\eta_{UY} \cdot f)]$
$[\epsilon_Y \cdot \epsilon_{FUY} \cdot F\eta_{UY} \cdot Ff]$
$[\epsilon_Y \cdot Ff]$
$[\epsilon_Y \cdot Ff]$

This completes the proof of the theorem. □

(4.4) **How to recover the original example.** Note that any monad in $\mathcal{S}\!ets$ induces a lax monad in $\mathcal{R}\!el$—just follow the indications given by Barr [1].

That the example (1.3), shown directly in (1.3.2) to have the universal property of (1.2), is a consequence of (4.3) follows from the observation that any such induced lax monad **T** on $\mathcal{R}\!el$ resolves (as in (3.3)) into a 2-functor $U^\mathbf{T}$ and a *normalized* lax adjoint $F^\mathbf{T}$ and therefore, by (4.3), it satisfies the universal property of (1.2).

This observation consists of the following remark, which the reader can find in [1]. First, that if $T: \mathcal{R}\!el \to \mathcal{R}\!el$ is induced from $T: \mathcal{S}\!ets \to \mathcal{S}\!ets$ in the manner therein indicated, one always has $T(1_X) = 1_{TX}$, trivially. This gives (i) of (4.2). Next, if r is a function (or if s is an inverse function) then equality holds in $T(r \cdot s) \leq T(r) \cdot T(s)$. Apply this to the pair η_X, f in order to obtain (ii) of (4.2), i.e., use that η_X is a function. Thirdly, observe that lax natural $\alpha: T \to T_1$, i.e., having the property that if $r: X \to Y$ is a relation then $\alpha_Y \cdot T(r) \leq T_1(r) \cdot \alpha_X$, becomes natural relative to functions—clearly, if $r: X \to Y$ is a function, the above inequality becomes an equality. This gives (iii) for η since it is there required that the 2-cell $\eta_{(\eta_X)}$ should be the identity. As for (iv) it is immediate: it says that $\epsilon_{FX} \cdot F\eta_X \leq 1_{FX}$ should be an equality where here $F = F^\mathbf{T}$. Note for this that $\epsilon_{FX} = \mu_X$, $F\eta_X = T\eta_X$ and that equality $\mu_X \cdot T\eta_X = 1_{TX}$ holds for the original monad. Or, should we obtain a 2-cell $\mu_X \cdot T\eta_X \leq 1_{TX}$ from formal considerations, note that an inequality between relations which are functions *must* be an equality. This completes the proof that all conditions of (4.2) hold.

(4.5) **Remark about continuous relations.** We may ask now whether it is also possible to recover the continuous relations in a similar way. The recipe is this. Let the functor $\mathcal{S}\!ets^{op} \to \mathcal{R}\!el$ ($f \mapsto f^{-1}$) act on the monad β in $\mathcal{S}\!ets$. There results a comonad $(\beta, \eta^{-1}, \mu^{-1})$ with β lax but η^{-1}, μ^{-1} dual (or "left") lax. The lax coalgebras are again the topological spaces (view X_ξ as a coalgebra via $\xi^{-1}: X \to \beta X$) but the morphisms are continuous relations, as desired.

The universal property changes; explicitly it is the following. Given a relation $r: X \to Y$ where X_ξ is a topological space and Y is a set, there exists a continuous relation $\bar{r}: X \to \beta Y$ (in fact, $\bar{r} = \beta r \cdot \xi^{-1}$) such that $r \leq \eta_Y^{-1} \cdot \bar{r}$ and such that if s is any other continuous relation satisfying $r \leq \eta_Y^{-1} \cdot s$ it follows that $\bar{r} \leq s$. I.e., in the language of [12], $\bar{r}$ is the *left U-lifting* of r along $\eta_Y^{-1}: Y \to \beta Y$ where $U: \mathcal{T}\!op\,\mathcal{R}\!el \to \mathcal{R}\!el$ is the forgetful and $\mathcal{T}\!op\,\mathcal{R}\!el$ is the 2-category of topological spaces, continuous relations and natural inclusions. Coherence holds and one has that β is a *lax coadjoint* to U.

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