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FORMAL GROUPS AND HOPF ALGEBRAS OVER DISCRETE RINGS

BY

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ABSTRACT. A theory of formal schemes and groups over abitrary rings is presented. The flat formal schemes in this theory have coalgebras of distributions which behave in the usual way. Frobenius and Verschiebung maps are studied.

Introduction. In this paper we provide the details of the results announced in [8].

Throughout k is a commutative ring, all algebras are commutative unless otherwise stated, k-Alg denotes the category of commutative k-algebras, $\otimes$ means $\otimes_k$, etc. Func = Func(k-Alg, Sets) is the category of set valued functors on k-Alg.

The theory of formal groups over fields was classically described for group laws arising from power series algebras [6]. Later, complete local noetherian algebras over fields were treated (cf. [7], which has a comprehensive historical survey). By assuming strong topological properties—namely pseudocompactness—of the base ring, substantial generalization of the field based theory was obtained [4], but complete generality on the base ring still required only power series algebras to be treated [2]. None of these approaches permits consideration, for example, of the functor G on commutative rings given by $G(R) = \{(a, b) \text{ in } R \times R \mid a$ is nilpotent, $b^2 = 0\}$. This functor is formally represented by the truncated power series ring $\mathbf{Z}[[x, y]]/(y^2)$. Even though $\mathbf{Z}$ has no pseudocompact topology, this ring has an extremely pleasant topology, namely the inverse limit topology arising from the system $\{\mathbf{Z}[[x, y]]/(x^n, y^2)\}$.

We assume always our base ring is discrete and treat such functors with two principal tools: a topology on the affine algebra $\mathcal{O}(G)$ of a functor G allows us to form its continuous linear dual, the coalgebra $B(G)$ of distributions on G; a topology on C^*, the full linear dual of an arbitrary coalgebra C, allows recovery of the functor when given the distributions.

Our theory coincides with [4] if k is a field and so in particular coincides with the theory of cocommutative coalgebras and Hopf algebras in case k is a field.

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Our categories of formal schemes (Func_{gf}) and flat formal schemes (Func_{pf}) and various related categories of algebras and coalgebras fit the following:

$$
\begin{tikzcd}
\mathrm{Func} \arrow[r, shift left, "\mathcal{O}"] & {[k\text{-}\mathrm{Alg}_c]^{\mathrm{op}}} \arrow[l, shift left, "(\ )^{\bullet}"] & & k\text{-}\mathrm{Coalg} \\
\mathrm{Func}_{gf} \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, phantom, "\cong"] & {[k\text{-}\mathrm{Alg}_{gf}]^{\mathrm{op}}} \arrow[u, phantom, sloped, "\subseteq"] & & \\
\mathrm{Func}_{pf} \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, phantom, "\cong"] \arrow[rrr, bend right, "B(\ )"'] & {[k\text{-}\mathrm{Alg}_{pf}]^{\mathrm{op}}} \arrow[u, phantom, sloped, "\subseteq"] \arrow[rr, shift left, "(\ )^{\circ}\ \cong"] & & k\text{-}\mathrm{Coalg}_{pf} \arrow[uu, phantom, sloped, "\subseteq"] \arrow[ll, shift left, "(\ )^{*}"]
\end{tikzcd}
$$

A similar diagram holds for the groups in these categories.

1. The affine algebra of a functor. Let G be a set valued functor on k-Alg. Let $\mathcal{O}$: k-Alg $\to$ Sets denote the underlying set functor and define $\mathcal{O}(G) = \mathrm{Func}(G, \mathcal{O})$ the set of natural transformations from G to $\mathcal{O}$.(2) $\mathcal{O}(G)$ is an algebra via pointwise operations, functorially in G. For each pair (x, K) with K a k-algebra and x in $G(K)$, denote by $\chi_{x,K}$ the algebra homomorphism $\mathcal{O}(G) \to K$ given by $\chi_{x,K}(f) = f(K)(x)$. Topologize $\mathcal{O}(G)$ with the coarsest topology making each of these maps continuous (each K having the discrete topology). This makes $\mathcal{O}(G)$ a topological algebra. An ideal I in $\mathcal{O}(G)$ is open if and only if I contains the intersection of a finite collection of kernels of $\chi_{x,K}$'s. If k-Alg_c denotes the category of topological k-algebras with continuous algebra maps, then $\mathcal{O}$: Func $\to$ k-Alg_c is a contravariant functor.

By the Yoneda lemma, it follows that if G is representable by an algebra A then $\mathcal{O}(G) \cong A$, as k-algebras. The situation is similar for topological algebras in a sense we make precise next.

Let $\mathfrak{A}$ be a topological algebra. We write $\mathfrak{A}^{\bullet}$: k-Alg $\to$ Sets for the functor $\mathfrak{A}^{\bullet}(K) = k\text{-}\mathrm{Alg}_c(\mathfrak{A}, K)$ with K regarded as discrete. Thus if k-Alg is considered as a full subcategory of k-Alg_c then these $\mathfrak{A}^{\bullet}$ are just the restrictions of representable functors on k-Alg_c. An example of such a functor which is not representable on k-Alg is the functor which assigns to each algebra K the set of nilpotent elements of K. This is $k[[t]]^{\bullet}$, "topologically represented" by the formal power series ring.

1.1 Proposition [8, §1]. $\mathrm{Func}(G, \mathfrak{A}^{\bullet}) \cong k\text{-}\mathrm{Alg}_c(\mathfrak{A}, \mathcal{O}(G))$, *that is* $(\ )^{\bullet}$ *and* $\mathcal{O}$ *are adjoint on the right.*

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(2) To avoid set theoretic difficulties one should cast this definition in the language of "universes" as in [3, 'Conventions générales']. At the expense of conventional terminology, this language can be employed throughout our work withour significant change in the statements or proofs of the results.

Proof. Let ϕ be in $\mathrm{Func}(G, \mathfrak{A}^{\bullet})$. For each a in $\mathfrak{A}$ define $\Delta\phi(a)$ in $\mathcal{O}(G)$ by

$$\Delta\phi(a)(K)(x) = \phi(K)(x)(a)$$

for any algebra K and any x in $G(K)$. One easily verifies that $\Delta\phi(a)$ is natural. To show that $\Delta\phi\colon \mathfrak{A} \to \mathcal{O}(G)$ is continuous it suffices to show $(\Delta\phi)^{-1}\mathrm{Ker}(\chi_{x,K})$ is open for each pair (x, K). But $(\Delta\phi)^{-1}\mathrm{Ker}(\chi_{x,K}) = \{a \text{ in } \mathfrak{A} | (\Delta\phi)(a)(K)(x) = 0\} = \{a \text{ in } \mathfrak{A} | \phi(K)(x)(a) = 0\} = \mathrm{Ker}(\phi(K)(x))$. Since $\phi(K)(x)$ is a continuous algebra map from $\mathfrak{A}$ to the discrete algebra K, its kernel is open.

Now suppose $f\colon \mathfrak{A} \to \mathcal{O}(G)$ is a continuous algebra map. Define $\Gamma f\colon G \to \mathfrak{A}^{\bullet}$ by

$$\Gamma f(K)(x)(a) = f(a)(K)(x).$$

Again it is straightforward to show Γf is natural. To show $\Gamma f(K)(x)\colon \mathfrak{A} \to K$ is continuous we need only show its kernel is open (K is discrete). This kernel is $\{a$ in $\mathfrak{A} | f(a)(K)(x) = 0\} = \{a \text{ in } \mathfrak{A} | \chi_{x,K}(f(a)) = 0\} = f^{-1}(\mathrm{Ker}\ \chi_{x,K})$ which is open since f is continuous. That Δ and Γ are inverse to one another is immediate from their definitions. The reader may easily verify that all maps which want to be algebra maps are indeed so, and that the arguments above are natural in $\mathfrak{A}$ and G, completing the proof.

The proposition gives rise to two natural adjunction maps: $G \to \mathcal{O}(G)^{\bullet}$ and $\mathfrak{A} \to \mathcal{O}(\mathfrak{A}^{\bullet})$. We set forth next some conditions motivated by the theory of algebraic groups and formal groups [1, p. 101], under which these adjunction maps are isomorphisms.

We consider functors G in Func equipped with a set of pairs $\{(D, p_D) | D \in k\text{-Alg};\ p_D \in G(D)\}$ satisfying the axioms below. Associated with this collection will be a diagram $\mathfrak{D}$ consisting of the D's and $\mathfrak{D}(D, D') = \{\alpha \in k\text{-Alg}(D, D') | G(\alpha)(p_D) = p_{D'}\}$.

(1.2a) Each $\chi_D = \chi_{p_D, D}\colon \mathcal{O}(G) \to D$ is surjective.

(1.2b) Given any A in k-Alg and x in $G(A)$, there is a D and an algebra map $y\colon D \to A$ such that $G(y)(p_D) = x$ [1, p. 101].

(1.3) $\mathfrak{D}$ is filtered.

If (1.2) holds call the p_D *generic points* of G. If $\mathfrak{A}$ is a topological k-algebra, when we refer to one of these axioms as holding for $\mathfrak{A}$ we mean that it should hold for $\mathfrak{A}^{\bullet}$. Thus for any continuous map $x\colon \mathfrak{A} \to A$ there is a D and a unique algebra map $y\colon D \to A$ making

$$\begin{tikzcd} \mathfrak{A} \arrow[rr, "p_D"] \arrow[dr, "x"'] & & D \arrow[dl, "y"] \\ & A & \end{tikzcd}$$

commute (cf. [1, p. 101]).

1.4 Remark. If G has generic points then there is at most one α in $\mathfrak{D}(D, D')$ and it is a surjection. Note that a representable functor (or a discrete algebra) satisfies axioms (1.2)–(1.3) for the trivial diagram containing the representing algebra and the identity. The power series ring $k[[t]]$ satisfies these axioms with respect to the diagram arising naturally from the projections $k[[t]] \to k[[t]]/(t^n)$.

1.5 Lemma. *Let* $\mathfrak{A} = \text{proj lim } \mathfrak{D}$, $\mathfrak{D}$ *a diagram in* k-Alg, *with* $\{p_D : \mathfrak{A} \to D \mid D \in \mathfrak{D}\}$ *the natural projections. If* $\mathfrak{D}$ *satisfies* (1.2) *then the adjunction map* $\mathfrak{A} \to \mathcal{O}(\mathfrak{A}^{\cdot})$ *is an isomorphism in* k-Alg_c.

Proof. By definition of the projective limit topology, the maps p_D lie in $\mathfrak{A}^{\cdot}(D) \equiv k\text{-}\mathrm{Alg}_c(\mathfrak{A}, D)$. Hence by definition of the topology on $\mathcal{O}(\mathfrak{A}^{\cdot})$, the maps $\chi_D = \chi_{p_D, D} : \mathcal{O}(\mathfrak{A}^{\cdot}) \to D$ are continuous. It follows directly from the definition of $\chi_{p_D, D}$ that the diagrams

$$\begin{tikzcd} \mathcal{O}(\mathfrak{A}^{\cdot}) \arrow[rr, "\chi_D"] \arrow[dr, "\chi_{D'}"'] & & D \arrow[dl] \\ & D' & \end{tikzcd}$$

commute, hence there is a continuous algebra map $\psi : \mathcal{O}(\mathfrak{A}^{\cdot}) \to \mathfrak{A}$ making each

$$\begin{tikzcd} \mathcal{O}(\mathfrak{A}^{\cdot}) \arrow[rr, "\chi_D"] \arrow[dr, "\psi"'] & & D \\ & \mathfrak{A} \arrow[ur, "p_D"'] & \end{tikzcd}$$

commute.

Now the adjunction map $\mathfrak{A} \to \mathcal{O}(\mathfrak{A}^{\cdot})$ is the map $\phi : a \mapsto \phi_a$ where ϕ_a is evaluation at a, i.e., $\phi_a(K)(x) = x(a)$ for x in $\mathfrak{A}^{\cdot}(K)$. ϕ is plainly an algebra map, and by standard appeals to the universal mapping property of proj lim one shows that $\psi\phi$ is the identity on $\mathfrak{A}$.

We next show that ψ is injective. Suppose f and g are distinct in $\mathcal{O}(\mathfrak{A}^{\cdot})$. Choose a pair (x, K) which distinguishes them, i.e. $f(K)(x) \neq g(K)(x)$, x in $\mathfrak{A}^{\cdot}(K) = k\text{-}\mathrm{Alg}_c(\mathfrak{A}, k)$. Now let (p_D, D) be a generic point for (x, K). Thus there is an algebra map $h : D \to K$ with $x = hp_D$. Then $f(K)(hp_D) \neq g(K)(hp_D)$, i.e., $f(K)\mathfrak{A}^{\cdot}(h)(p_D) \neq g(K)\mathfrak{A}^{\cdot}(h)(p_D)$. By the naturality of f and g this is the same as saying $\mathcal{O}(h)f(D)(p_D) \neq \mathcal{O}(h)g(D)(p_D)$ hence $f(D)(p_D) \neq g(D)(p_D)$. By the definitions of $\chi_{p_D, D}$ and ψ, this says

$$p_D\psi(f) = \chi_{p_D, D}(f) = f(D)(p_D) \neq g(D)(p_D) = p_D\psi(g).$$

Thus $\psi(f) \neq \psi(g)$.

Finally, we show ψ is an open map. Since $\mathfrak{A}$ has generic points by hypothesis, for any K and x in $\mathfrak{A}^{\cdot}(K)$ there is a diagram

$$\begin{tikzcd} \mathcal{O}(\mathfrak{A}^{\cdot}) \arrow[r, "x_D"] \arrow[d, "\psi"'] & D \arrow[d, "y"] \\ \mathfrak{A} \arrow[ur, "p_D"] \arrow[r, "x"'] & K \end{tikzcd}$$

in which the lower triangle also commutes.

This shows that for any f in $\mathcal{O}(\mathfrak{A}^{\cdot})$, $\chi_{x,K}(f) \equiv f(K)(x) = f(K)(yp_D) = f(K)(\mathfrak{A}^{\cdot}(y)(p_D))$. By the naturality of f and the definition of $\chi_{p_D,D}$, this is in turn equal to $yf(D)(p_D) = y\chi_{p_D,D}(f)$. The commutativity of the diagram thus implies $x\psi = \chi_{x,K}$. Hence $\psi(\text{Ker } \chi_{x,K}) = \text{Ker } x$ is open in $\mathfrak{A}$.

Thus ψ is an open continuous bijective map, hence a homeomorphism with inverse ϕ. This completes the proof.

1.6 Definition. A functor G for which there is a diagram $\mathfrak{D}$ satisfying (1.2)–(1.3) is a *formal scheme* over k. A topological algebra $\mathfrak{A} = \text{proj lim } \mathfrak{D}$ satisfying (1.2)–(1.3) is a *gf-algebra* (denoting generic and filtered). Here the D's are discrete and $\mathfrak{A}$ has the limit topology. Observe that the full subcategory k-Alg of k-Alg$_c$ consists only of gf-algebras and that each representable functor is a formal scheme.

An ideal I in a gf-algebra is open if and only if I contains the kernel of some p_D, since by (1.3) these form a base for the limit topology.

The full subcategory of Func consisting of formal schemes over k will be denoted k-Func$_{gf}$ or simply Func$_{gf}$. The full subcategory of k-Alg$_c$ consisting of gf-algebras will be denoted k-Alg$_{gf}$ (or Alg$_{gf}$) (cf. [4, 0.4.2]). These categories are in fact anti-equivalent under the adjointness of 1.1:

1.7 Theorem [8, §1]. *Let G be a formal scheme with diagram $\mathfrak{D}$. Then the adjunction map $G \to \mathcal{O}(G)^{\cdot}$ is an isomorphism and $\mathcal{O}(G) \cong$ proj lim $\mathfrak{D}$. This provides an anti-equivalence of k-Func$_{gf}$ with k-Alg$_{gf}$.*

Proof. Let x be in $G(K)$, K a k-algebra. The adjunction map $\chi: G \to \mathcal{O}(G)^{\cdot}$ is defined by $\chi(K)(x) = \chi_{x,K}$. Define a map $\psi: \mathcal{O}(G)^{\cdot} \to G$ as follows: for z in $\mathcal{O}(G)^{\cdot}(K)$, (i.e., $z: \mathcal{O}(G) \to K$ a continuous map), Ker z is an open ideal since K is discrete. Hence there are algebras A and a in $G(A)$ with Ker z containing a finite intersection $\bigcap \text{Ker } \chi_{a,A}$. For each (a, A) there is a (p_D, D) and $y: D \to A$ with $G(y)(p_D) = a$. Hence

$$y\chi_{p_D,D}(f) = yf(D)(p_D) = \mathcal{O}(y)f(D)(p_D) = f(A)G(y)(p_D) = f(A)(a) = \chi_{a,A}(f),$$

so that $y\chi_{p_D,D} = \chi_{a,A}$. Since Ker $\chi_{p_D,D} \subseteq$ Ker $\chi_{a,A}$ we have Ker $z \supseteq \bigcap$ Ker $\chi_{p_D,D}$, the intersection taken over the finitely many (p_D, D) corresponding to the (a, A). By (1.3) there is a single $(p_{D'}, D')$ with maps α: $D' \to D$ for each of the D's in question. Then Ker $\chi_{p_{D'},D'} \subseteq$ Ker $\chi_{p_D,D}$ for each D so that Ker $z \supseteq$ Ker$_{p_{D'},D'}$. By (1.2a) there is a unique algebra map w making

$$\begin{tikzcd}
\mathcal{O}(G) \arrow[rr, "z"] \arrow[dr, "\chi_{p_{D'},D'}"'] & & K \\
& D' \arrow[ur, "w"'] &
\end{tikzcd}$$

commute. Set $\psi(K)(z) = G(w)(p_{D'})$ in $G(K)$. The fact that $\mathfrak{D}$ is filtered implies that $\psi(K)(z)$ is independent of the choice of $(p_{D'}, D')$. ψ is natural and is inverse to χ. Thus $G \cong \mathcal{O}(G)^{\bullet}$.

We must now show $\mathcal{O}(G)$ is a gf-algebra, more precisely that $\mathcal{O}(G) \cong$ proj lim $\mathfrak{D}$. Suppose a topological algebra $\mathfrak{B}$ and maps q_D: $\mathfrak{B} \to D$ coherent with the transition maps, are given. For any (x, K) choose (p_D, D) and y: $D \to K$ with $G(y)(p_D) = x$. Define $\phi(b)(K)(x) = yq_D(b)$. Axioms (1.2) and (1.3) imply that this definition is independent of the choice of D and one easily checks that $\phi(b)$ is natural in K. Thus

$$\begin{tikzcd}
\mathfrak{B} \arrow[rr, "\phi"] \arrow[dr, "q_D"'] & & \mathcal{O}(G) \arrow[dl, "\chi_D"] \\
& D &
\end{tikzcd}$$

is a commutative diagram. To show that $\mathcal{O}(G)$ has the universal mapping property of proj lim $\mathfrak{D}$, it remains only to show that ϕ is continuous and is the unique such map rendering this diagram commutative. Continuity is immediate from the fact that $\chi_{x,K}(f) = f(K)G(y)(p_D) = yf(d)(p_D) = y\chi_D(f)$, hence $\chi_{x,K} = y\chi_D$. This shows that $\phi^{-1}(\text{Ker } \chi_{x,K}) = \phi^{-1}(\text{Ker } y\chi_D) \supseteq \phi^{-1}(\text{Ker } \chi_D) = \text{Ker } q_D$ which is an open ideal. Hence $\phi^{-1}(\text{Ker } \chi_{x,K})$ is an open ideal. If ϕ is a second such map then

$$\tilde{\phi}(b)(K)(x) = \tilde{\phi}(b)(K)(G(y)(p_D)) = \mathcal{O}(y)\tilde{\phi}(b)(D)(p_D) = yq_D(b) = \phi(b)(K)(x),$$

hence $\phi = \tilde{\phi}$.

If $\mathfrak{A}$ is a gf-algebra then $\mathfrak{A}^{\bullet}$ is by definition in Func$_{gf}$. Thus Lemma 1.5 and the above isomorphism show that $(\ )^{\bullet}$ and $\mathcal{O}$ are inverse category anti-equivalences to one another, completing the proof.

The identification of G with $\mathcal{O}(G)$ via χ allows us to view p_D in $G(D)$ as a map from $\mathcal{O}(G)$ to D, namely $\chi(D)(p_D) = \chi_D$ by definition of χ. So we may say p_D is a surjection.

That χ_D must be assumed surjective may be seen from the following example pointed out to us by P. Gabriel:

For each cofinite set S of prime integers, let Q_S denote the ring of rational numbers with denominators divisible only by primes in S, and G_S the functor on commutative rings represented by Q_S. The set of S's is directed by inclusion. Let $G = \text{inj lim}\ G_S = \bigcup G_S$. Note that $G_S(A)$ is empty if any prime in S is a non-unit in A, otherwise $G_S(A)$ has one element. Thus, for example, $G(\mathbf{Z})$ is empty. On the other hand $\mathcal{O}(G) = \text{Func}\ (\text{inj lim}\ G_S, \mathcal{O}) = \text{proj lim Func}\,(G_S, \mathcal{O}) = \text{proj lim}\ Q_S = \mathbf{Z}$. Thus G and the functor represented by $\mathbf{Z}$ have the same affine algebra, although they clearly are not isomorphic functors. G does however satisfy axioms (1.2b)–(1.3). Indeed, the y's arising in (1.3) are even unique in this case.

1.8. **Proposition.** *If* $\mathfrak{A} = \text{proj lim}\ \mathfrak{D}$ *is a gf-algebra then* $\mathfrak{A}^\bullet \cong \text{inj lim}\ \mathfrak{D}^\bullet$ *in* Func.

Proof. The proof is of a highly technical nature and we omit many of the details, most of which are applications of the Yoneda lemma.

Suppose G is any set valued functor and $\{\phi_D : D^\bullet \to G \mid D \in \mathfrak{D}\}$ a compatible family of natural transformations. Define $\Phi: \mathfrak{A}^\bullet \to G$ as follows: for x in $\mathfrak{A}^\bullet(K)$ choose D in $\mathfrak{D}$ any $y: D \to K$ with $\mathfrak{A}^\bullet(y)(p_D) = x$ i.e., $yp_D = x$. Define $\Phi(K)(x) = G(y)(\phi_D)$. The description is independent of the choice of D because $\mathfrak{D}$ is filtered. It is straightforward to verify Φ is natural and makes the following diagram commute for each D:

$$\begin{tikzcd} D^\bullet \arrow[rr, "p_D^\bullet"] \arrow[dr, "\phi_D"'] & & \mathfrak{A}^\bullet \arrow[dl, "\Phi"] \\ & G & \end{tikzcd}$$

Now Φ is the unique map rendering these commutative because any such map must have (for x, K and y as in the definition of ϕ)

$$\Phi(K)(x) = \Phi(K)(yp_D) = \Phi(K)(p_D^\bullet(K)(y)) = \phi_D(K)(y) = G(y)(\phi_D)$$

the last equality arising from the Yoneda identification $\text{Func}\,(D^\bullet, G) = G(D)$ (D being discrete $D^\bullet$ is just $k\text{-Alg}\ (D,\ )$), which identifies $G(y)(\phi_D)$ with $G(y)(\phi_D(D)(\mathrm{id}_D))$.

1.9 Definition and proposition. Alg_{gf} has coproducts given by (proj lim $\mathcal{C}$)$\hat{\otimes}$ (proj lim $\mathcal{D}$) = proj lim ($\mathcal{C} \otimes \mathcal{D}$).

By $\mathcal{C} \otimes \mathcal{D}$ we understand the system $\{C \otimes D \mid C \in \mathcal{C}$ and $D \in \mathcal{D}\}$ directed by $C \otimes D \leq C' \otimes D'$ in case $C \leq C'$ and $D \leq D'$. We omit the straightforward proof which depends only on careful application of the universal properties of the tensor product and inverse limit.

Since $\mathfrak{A} \hat{\otimes} \mathfrak{B}$ is the coproduct in Alg_{gf} we get $\mathrm{Alg}_{gf}(\mathfrak{A} \hat{\otimes} \mathfrak{B}, \mathcal{C}) \cong \mathrm{Alg}_{gf}(\mathfrak{A}, \mathcal{C}) \times \mathrm{Alg}_{gf}(\mathfrak{B}, \mathcal{C})$ for all $\mathcal{C}$ in Alg_{gf}. Now the full subcategory Alg in Alg_c sits inside Alg_{gf}, hence

$$(\mathfrak{A} \hat{\otimes} \mathfrak{B})^{\bullet}(K) \cong \mathfrak{A}^{\bullet}(K) \times \mathfrak{B}^{\bullet}(K) = (\mathfrak{A}^{\bullet} \times \mathfrak{B}^{\bullet})(K),$$

or

$$(\mathfrak{A} \hat{\otimes} \mathfrak{B})^{\bullet} \cong \mathfrak{A}^{\bullet} \times \mathfrak{B}^{\bullet}.$$

This shows that the products in Func_{gf} are just the products in Func. Note also that the multiplication $\mathfrak{A} \otimes \mathfrak{A} \to \mathfrak{A}$ on $\mathfrak{A}$ = proj lim $\mathcal{D}$ factors through the unique μ in k-$\mathrm{Alg}_c(\mathfrak{A} \hat{\otimes} \mathfrak{A}, \mathfrak{A})$ induced by the multiplications $\{D \otimes D \to D \mid D$ in $\mathcal{D}\}$. In general there is a natural map $\mathfrak{A} \otimes \mathfrak{B} \to \mathfrak{A} \hat{\otimes} \mathfrak{B}$. The image under this map of $a \otimes b$ will be denoted $a \hat{\otimes} b$.

2. Flat schemes and dualization.

2.1 Definition. A gf-algebra $\mathfrak{A}$ = proj lim $\mathcal{D}$ is *flat* if each D in $\mathcal{D}$ is a finitely generated projective k-module [8, 1.4]. A formal scheme is called *flat* if $\mathcal{O}(G)$ is flat (cf. [4, 0.4.2]).

Note that if $\mathfrak{A}$ and $\mathfrak{B}$ are flat gf-algebras then so is $\mathfrak{A} \hat{\otimes} \mathfrak{B}$. In particular $\mathfrak{A} \hat{\otimes} \mathfrak{B}$ is the coproduct in the full subcategory of Alg_{gf} consisting of flat gf-algebras.

Suppose $\mathfrak{A}$ = proj lim $\mathcal{D}$ is flat. If $a^*: \mathfrak{A} \to k$ is a continuous linear map (k here is discrete), there is a D in $\mathcal{D}$ and a unique linear map $x: D \to k$ making

$$\begin{tikzcd} \mathfrak{A} \arrow[rr, "p_D"] \arrow[dr, "a^*"'] & & D \arrow[dl, "x"] \\ & k & \end{tikzcd}$$

commute; such x exists because Ker a^* is an open neighborhood of 0 so contains some Ker p_D. The uniqueness follows from the surjectivity of p_D.

For any topological algebra $\mathfrak{A}$, define $\mathfrak{A}^\circ = \mathrm{Hom}_c(\mathfrak{A}, k)$ the continuous linear dual.

If $\mathfrak{A}$ = proj lim $\mathcal{C}$, $\mathfrak{B}$ = proj lim $\mathcal{D}$ are gf-algebras then we define a map $\Phi: \mathfrak{A}^\circ \otimes \mathfrak{B}^\circ \to (\mathfrak{A} \hat{\otimes} \mathfrak{B})^\circ$ as follows: for f in $\mathfrak{A}^\circ$, g in $\mathfrak{B}^\circ$, choose generic points $p_C: \mathfrak{A} \to C$ and $p_D: \mathfrak{B} \to D$ with f factoring through p_C, say $f = xp_C$ for some

$x\colon C \to k$ and with $g = yp_D$, say. Define

$$\Phi(f \otimes g) = (x \otimes y)(p_{C\otimes D}) : \mathfrak{A} \otimes \mathfrak{B} \to k \otimes k \simeq k$$

(here $p_{C\otimes D}$ is the canonical map $\mathfrak{A} \hat{\otimes} \mathfrak{B} \to C \otimes D$). It follows from (1.3) that this definition is independent of the choice of C and D and, bearing in mind that $(C \otimes D)^* \cong C^* \otimes D^*$, one can prove the following

2.2 **Theorem.** *If* $\mathfrak{A} = \text{proj lim } \mathfrak{C}$ *and* $\mathfrak{B} = \text{proj lim } \mathfrak{D}$ *are flat gf-algebras then* $\Phi\colon \mathfrak{A}^\circ \otimes \mathfrak{B}^\circ \to (\mathfrak{A} \hat{\otimes} \mathfrak{B})^\circ$ *is bijective.*

If $\mathfrak{A}$ is a flat gf-algebra the map

$$\mathfrak{A}^\circ \xrightarrow{\mu^\circ} (\mathfrak{A} \hat{\otimes} \mathfrak{A})^\circ \simeq \mathfrak{A}^\circ \otimes \mathfrak{A}^\circ$$

makes $\mathfrak{A}^\circ$ a cocommutative coalgebra over k.

2.3 **Definition.** $k\text{-Alg}_{pf}$ denotes the full subcategory of $k\text{-Alg}_c$ consisting of flat gf-algebras. Its objects will be called *pf-algebras.* Func_{pf} is the full subcategory of Func consisting of flat formal schemes, i.e. those formal schemes G with $\mathcal{O}(G)$ a pf-algebra. A *pf-coalgebra* is a cocommutative coalgebra $C = \text{inj lim } C_i$ which is the filtered inductive limit of subcoalgebras C_i which are finitely generated projective and such that the natural maps $C_i \to C$ have k-linear retractions. The full subcategory of k-Coalg consisting of pf-coalgebras is denoted $k\text{-Coalg}_{pf}$.

2.4 **Remark and definition.** $C^* = \text{Hom}(C, k)$ is an algebra with at least two interesting topologies: that with base of neighborhoods of zero consisting of annihilators of finite subsets of C and that given by annihilators of subcoalgebras which are k-module finite. Both make C^* a topological algebra, but in the latter case zero has a neighborhood base consisting of ideals.

If C^* is in Coalg_{pf} these coincide and we call this the *finite annihilator topology.*

Note that if $C = \text{inj lim } D$ is in Coalg_{pf} then the annihilators of the D's in fact form a base of neighborhoods of zero. Note also that there is no ambiguity about the coalgebra structure on D. Under the pf hypothesis $D \otimes D \to C \otimes C$ is a monomorphism so the restriction to D of the comultiplication on C must factor uniquely, if at all, through $D \otimes D$.

2.5 **Lemma.** *Let* $\mathfrak{A} = \text{proj lim } \mathfrak{D}$ *be a pf-algebra. Then* $\mathfrak{A}^\circ \cong \text{inj lim } D^*$ *as coalgebras, the inductive limit taken over* D *in* $\mathfrak{D}$*, and* $\mathfrak{A}^\circ$ *is in* $k\text{-Coalg}_{pf}$. (*Each* D^* *is naturally a coalgebra because* D *is finitely generated projective.*)

Proof. It is straightforward to check that $\mathfrak{A}^\circ$ satisfies the universal property of inj lim.

2.6 **Lemma.** *Let* $C = \text{inj lim } C_i$ *be in* $k\text{-Coalg}_{pf}$*. Then* $C^* \cong \text{proj lim } C_i^*$ *as topological algebras* (*with* C_i^* *discrete*).

Proof. Note that colimits in k-mod and k-Coalg are the same hence there is a natural isomorphism $\mathrm{Hom}_k(\text{inj lim } C_i, k) \cong \text{proj lim } \mathrm{Hom}_k(C_i, k)$. By naturality it is easy to see that this is an algebra isomorphism. So the only nontrivial part is to show that the finite annihilator topology on C^* coincides with the inverse limit topology on proj lim C_i^*. Now the map $C^* \to \text{proj lim } C_i^*$ is the unique map making

$$\begin{tikzcd} C^* \arrow[r, "\alpha"] \arrow[dr, "\alpha_i^*"'] & \text{proj lim } C_i^* \arrow[d] \\ & C_i^* \end{tikzcd}$$

commute, where $\alpha_i\colon C_i \to C$ are the split inclusions of C_i in C. To show α is continuous it thus suffices to show α_i^* is continuous. But Ker α_i^* is the annihilator of C_i so is open in C^*. Since C_i^* is discrete, this shows α is continuous. On the other hand, if E is a finitely generated subcoalgebra, then $E \subseteq C_i$ for some C_i, because $\{C_i\}$ is filtered. Hence if $(\ )^\perp$ denotes annihilators, we have $C_i^\perp \subseteq E^\perp$ whence $\alpha(E^\perp) \supseteq \alpha(C_i^\perp) = \alpha(\mathrm{Ker}\ \alpha_i^*) = \mathrm{Ker}(\text{proj lim } C_i^* \to C_i^*)$, so that $\alpha(E^\perp)$ is open. Thus α is an open map, so a homeomorphism.

2.7 **Remark.** 2.5 and 2.6 say $\mathfrak{A}^{\circ *} \cong \mathfrak{A}$ and $C^{*\circ} \cong C$.

Note that if $C = \text{inj lim } C_i$ is in $k\text{-Coalg}_{pf}$ the proof of 2.6 shows that C^* has generic points, for if $x\colon C^* \to A$ is continuous (A discrete) then Ker $x \supseteq$ Ker$(C^* \to C_i^*)$ for some C_i, since these kernels are basic neighborhoods of 0 in the limit topology. Hence x factors through this C_i^*.

2.8 **Definition.** For a flat formal scheme G, the *coalgebra of distributions* of G is $B(G) = \mathcal{O}(G)^\circ$.

From 1.7, 2.5 and 2.6 it follows that $B = B(\ )$ provides an equivalence between Func_{pf} and $k\text{-Coalg}_{pf}$.

2.9 **Definition.** A *formal group* G is a group object in the category Func_{gf} of formal schemes.

We say a formal group G is flat if it is flat as a formal scheme, i.e., if $\mathcal{O}(G)$ is a flat gf-algebra. A formal group is a group in Func, i.e., a group valued functor. A flat formal scheme is a (flat) formal group if and only if it is a group in Func_{pf}, the category of flat formal schemes. This again is equivalent to being a group valued functor.

If G is a flat formal group, the group law induces a natural topological coalgebra structure Δ in $k\text{-Alg}_c(\mathcal{O}(G), \mathcal{O}(G) \hat{\otimes} \mathcal{O}(G))$ and ϵ in $k\text{-Alg}_c(\mathcal{O}(G), k)$ and in turn a (not necessarily commutative) algebra structure, and hence a Hopf algebra structure, on $B(G)$. $B(G)$ is a discrete set carrying no particular topology and is a Hopf algebra in the usual sense. We say $B(G)$ is the *bialgebra* of G and $\mathcal{O}(G)$ the *affine algebra of* G.

We may summarize the results of this section as:

2.10 **Theorem.** *B defines an equivalence of the category of flat formal groups over k with the category of cocommutative k-Hopf algebras whose coalgebra lies in k-*Coalg_{pf}.

Note that over a field every coalgebra is in k-Coalg_{pf}, so in this case the theory of cocommutative Hopf algebras is equivalent to our theory of formal groups (every formal group is flat) which is in turn equivalent to that of [4].

3. **Base extension.** In this section we sometimes denote Hom_k by k-Hom for notational convenience. The proofs here often depend on the interchange of various limits and Homs. Where this does not result from categorical considerations, it is essentially a consequence of generic points and filtration.

3.1 **Definition.** Let K be a k-algebra. The restriction to K-algebras defines a functor: $(\ )_K$

$$\mathrm{Func}\,(k\text{-Alg, Sets}) \ni G \longmapsto G_K \in \mathrm{Func}\,(K\text{-Alg, Sets}).$$

One checks easily, using Lemma 1.5, that if G is a formal scheme over k then G_K is a formal scheme over K with $\mathcal{O}(G_K) = \mathcal{O}(G) \mathbin{\hat{\otimes}} K$.

Further we have

3.2 **Proposition.** *If G is a flat formal scheme over K then G_K is flat over K and $B(G_K) \cong B(G) \otimes K$.*

Proof. $\mathcal{O}(G) \mathbin{\hat{\otimes}} K$ is clearly a pf-algebra over K. By Lemma 2.5

$$K\text{-}\mathrm{Hom}_c\,(\mathrm{proj\ lim}\,(\mathfrak{D} \otimes K),\ K) = \mathrm{inj\ lim\ Hom}_K\,(\mathfrak{D} \otimes K,\ K)$$

and

$$\mathrm{inj\ lim\ Hom}_k\,(\mathfrak{D},\ k) = k\text{-}\mathrm{Hom}_c\,(\mathrm{proj\ lim}\ \mathfrak{D},\ k)$$

so the assertion about bialgebras follows from the following lemma:

3.3 **Lemma.** *If D is a k-algebra, finitely generated and projective as a k-module, then there is an isormophism ϕ: $\mathrm{Hom}_k(D, k) \otimes K \to \mathrm{Hom}_K(D \otimes K, K)$ of K-coalgebras given by $\phi(f \otimes \lambda)(d \otimes \mu) = f(d) \cdot (\lambda\mu)$.*

Proof. Straightforward.

Note that restriction from K to k of scalars need not preserve flatness, so is not an adjoint to $(\ )_K$. Nevertheless, we have

3.4 Remark. $(\ )_K$ *preserves products.*

Proof. $\mathcal{O}((G\times H)_K) \cong \mathcal{O}(G\times H)\,\hat{\otimes}\,K \cong \mathcal{O}(G)\,\hat{\otimes}\,\mathcal{O}(H)\,\hat{\otimes}\,K \cong (\mathcal{O}(G)\,\hat{\otimes}\,K)\,\hat{\otimes}_K\,(\mathcal{O}(H)\,\hat{\otimes}\,K) \cong \mathcal{O}(G_K\times H_K)$. Hence $(G\times H)_K \cong G_K\times H_K$.

Here we have made use of a number of previously unstated but trivial properties of $\hat{\otimes}$, e.g., $\mathfrak{A}\,\hat{\otimes}_K\,K \cong \mathfrak{A}$.

3.5 Corollary. $(\ )_K$ *preserves formal* (*resp. flat formal*) *groups.*

3.6 Definition. Let G be a commutative flat formal group. The *dual group* G^D of G is the affine group scheme represented by $B(G)$. Let Γ be an affine commutative group scheme whose affine algebra H is in k-Coalg_{pf}. Thus $\mathfrak{A} = H^*$ is a flat gf-algebra and we define $\Gamma^D = \mathfrak{A}^{\bullet}$.

Proposition 3.2 and the remarks preceding it imply, in the above situation, that $(G_K)^D \cong (G^D)_K$ and $(\Gamma_K)^D \cong (\Gamma^D)_K$, so we will write merely G^D_K and Γ^D_K.

3.7 Theorem [8, Theorem 3.1] (Cartier duality). *Let G be a commutative flat formal group over k, K any commutative k-algebra. Then* $\mathrm{Hom}_{K\text{-}Gr}(G_K, G_{mK}) \cong G^D(K)$, *naturally in G and K. Further,* $\mathrm{Hom}_{K\text{-}Gr}(\mathrm{Hom}_{\_\text{-}Gr}(G_{\_}, G_{m\_})_K, G_{mK}) \cong G(K)$.

Here $\mathrm{Hom}_{K\text{-}Gr}$ denotes the homomorphisms in the category of group valued functors on K-Alg. Loosely stated, the second assertion says double dualization is the identity.

Proof. The proof is analogous to the usual one, e.g., for finite group schemes ([9, p. 132], [3, II. 1.2.10]). One shows that

$$\mathrm{Hom}_{K\text{-}Gr}(G_K, G_{mK}) \cong \{a \text{ in } \mathcal{O}(G_K) \mid \Delta(a) = a\,\hat{\otimes}_K\,a,\ \epsilon(a) = 1\}$$

and this in turn is $K\text{-Alg}(B(G_K), K)$, because the algebra structure on $B(G_K) = \mathcal{O}(G_K)^{\circ}$ is induced by the topological comultiplication on $\mathcal{O}(G_K)$. By Proposition 3.2 this group is

$$K\text{-Alg}(B(G)\otimes K, K) \cong k\text{-Alg}(B(G), K) \equiv G^D(K).$$

Similarly, for $\Gamma = G^D$ we get

$$\begin{aligned}
\mathrm{Hom}_{K-Gr}(\Gamma_K, G_{mK}) &\cong \{a \in \mathcal{O}(\Gamma_K) \mid \epsilon(a) = 1 \text{ and } \Delta a = a \otimes_K a\} \\
&\equiv \{a \text{ in } B(\Gamma^D_K) \mid \epsilon(a) = 1 \text{ and } \Delta a = a \otimes_K a\} \\
&= \{a \text{ in } \mathrm{Hom}_c(\mathcal{O}(\Gamma^D_K), K) \mid \epsilon(a) = 1 \text{ and } \Delta(a) = a \otimes_K a)\} \\
&\cong K\text{-Alg}_c(\mathcal{O}(\Gamma^D_K), K) \cong K\text{-Alg}_c(\mathcal{O}(\Gamma^D)\otimes K, K) \\
&\cong k\text{-Alg}_c(\mathcal{O}(\Gamma^D), K) = \Gamma^D(K).
\end{aligned}$$

The second assertion of the theorem now follows from the first by taking $\Gamma = G^D$ and noting that $\Gamma^D \cong G$ in view of Remark 2.7.

4. **Frobenius and Verschiebung.** Let k be a commutative ring and assume $\mathrm{char}(k) = p \neq 0$ is prime. Let A be a k-algebra and denote by $A^{1/p}$ the k-algebra whose ring is A with k-algebra structure $k \xrightarrow{p} k \to A$ where $p(\lambda) = \lambda^p$. $(\ )^{1/p}$ is a functor from k-Alg to k-Alg, and there is a natural transformation $F\colon \mathrm{Id} \to (\ )^{1/p}$ of functors on k-Alg called the *Frobenius map* and given by $F(A)(a) = a^p$ [3, II, 7.1.1].

Now $p\colon k \to k^{1/p}$ is a ring map making $k^{1/p}$ into a k-algebra. Since $k^{1/p} = k$ as rings there are category isomorphisms $(\ )'$ from $k^{1/p}$-Alg (resp. $k^{1/p}$-Coalg) to k-Alg (resp. k-Coalg).

From any gf-algebra $\mathfrak{A}$ over k we write $\mathfrak{A}^{(p)} = (\mathfrak{A} \hat{\otimes} k^{1/p})'$. $\mathfrak{A}^{(p)}$ is clearly a gf-algebra over k.

4.1 **Proposition.** $(\mathfrak{A}^{(p)})^{\cdot}(B) \cong \mathfrak{A}^{\cdot}(B^{1/p})$ *naturally in the k-algebra B.*

Proof. Let $'(\ )\colon k\text{-Alg} \to k^{1/p}\text{-Alg}$ denote the isomorphism inverse to $(\ )'$. Now a map f is in $k^{1/p}\text{-Alg}_c(\mathfrak{A} \hat{\otimes} k^{1/p},\ 'B)$ if and only if

$$\begin{tikzcd}
& k \arrow[dl, "p"'] \arrow[d] \\
{'k = k^{1/p}} \arrow[r] \arrow[dr] & \mathfrak{A} \hat{\otimes} k^{1/p} \arrow[d, "f"] \\
& {'B}
\end{tikzcd}$$

is a commutative diagram of rings.

Such maps are clearly in one to one correspondence with the continuous ring maps $\phi\colon \mathfrak{A} \to {'B}$ making

$$\begin{tikzcd}
k \arrow[d] \arrow[dr] & \\
\mathfrak{A} \hat{\otimes} k^{1/p} \arrow[d] & \mathfrak{A} \arrow[l] \arrow[dl] \\
{'B} &
\end{tikzcd}$$

commute. Such maps are precisely the k-algebra maps. But ${}'B = B^{1/p} = B$ as rings, so $f \leftrightarrow \phi$ establishes an isomorphism $k^{1/p}\text{-Alg}_c(\mathfrak{A} \hat{\otimes} k^{1/p}, {}'B)$ with $k\text{-Alg}_c(\mathfrak{A}, B^{1/p}) \equiv \mathfrak{A}'(B^{1/p})$. Since $(\ )'$ and $'(\ )$ are inverse category isomorphisms we thus have

$$(\mathfrak{A}^{(p)})^{\cdot}(B) \equiv k\text{-Alg}_c((\mathfrak{A} \hat{\otimes} k^{1/p})', B) = k^{1/p}\text{-Alg}_c(\mathfrak{A} \hat{\otimes} k^{1/p}, {}'B) \cong \mathfrak{A}^{\cdot}(B^{1/p}),$$

completing the proof.

There is a natural transformation, which we also call the Frobenius map, $\mathfrak{F}: (\ )^{(p)} \to \text{Id}$ of functors from $k\text{-Alg}_{gf}$ to $k\text{-Alg}_{gf}$ given by composing F with the isomorphism of the proposition; more precisely $\mathfrak{F}$ is defined via the anti-isomorphism $(\ )^{\cdot}$ to be the unique map such that $\mathfrak{F}^{\cdot}(B): \mathfrak{A}^{\cdot}(B) \to \mathfrak{A}^{(p)\cdot}(B)$ is the composition

$$\mathfrak{A}^{\cdot}(B) \xrightarrow{\mathfrak{A}^{\cdot}(F)} \mathfrak{A}^{\cdot}(B^{1/p}) \cong (\mathfrak{A}^{(p)})^{\cdot}(B).$$

If $\mathfrak{A}$ is discrete, so that $\mathfrak{A}^{\cdot}$ is representable (by $\mathfrak{A}$) then $\mathfrak{F}$ has the usual description viz., $\mathfrak{F}(\mathfrak{A})(a \otimes \lambda) = a^p\lambda$. This follows easily from the definitions involved, keeping in mind that $\mathfrak{A}^{(p)} = \mathfrak{A} \otimes k^{1/p}$ as rings. Indeed, if $\mathfrak{A}$ and $\mathfrak{B}$ are both discrete, $\mathfrak{A} \hat{\otimes} \mathfrak{B} = \mathfrak{A} \otimes \mathfrak{B}$.

4.2 **Lemma.** *Let A be a k-algebra, finitely generated and projective as a k-module. Then the map*

$$\phi : A^* \otimes k^{1/p} \longrightarrow \text{Hom}_{k^{1/p}}(A \otimes k^{1/p}, k^{1/p})$$

given by $(\phi(a^* \otimes \mu), a \otimes \lambda) = (a^*, a)^p\mu\lambda$ is an isomorphism of $k^{1/p}$-coalgebras.

Proof. This is a special case of Lemma 3.3.

We may now define a functor $(\ )^{(p)}$ on k-Coalg by $C^{(p)} = (C \otimes k^{1/p})'$.

4.3 **Proposition.** *If $\mathfrak{A}$ is a pf-algebra then $(\mathfrak{A}^{(p)})^{\circ} \cong (\mathfrak{A}^{\circ})^{(p)}$ in k-*Coalg_{pf}.

Proof. $(\mathfrak{A}^{(p)})^{\circ} \cong k\text{-Hom}_c(\mathfrak{A}^{(p)}, k) \cong (k^{1/p}\text{-Hom}_c(\mathfrak{A} \hat{\otimes} k^{1/p}, k^{1/p}))' \cong (k\text{-Hom}_c(\mathfrak{A}, k) \otimes k^{1/p})'$ (by Proposition 3.2). This last is by definition $(\mathfrak{A}^{\circ} \otimes k^{1/p})' = (\mathfrak{A}^{\circ})^{(p)}$ ($\mathfrak{A}^{\circ}$ is discrete).

4.4 **Corollary.** *Let C be in k-*Coalg_{pf}. *Then* $C^{(p)} \cong ((C^*)^{(p)})^{\circ}$.

The corollary allows us to define the *Verschiebung* $\mathfrak{V}$, a natural transformation of functors on $k\text{-Coalg}_{pf}$ from the identity functor to $(\ )^{(p)}$ by dualizing $\mathfrak{F}$.

4.5 **Proposition.** *For any coalgebra C in k-*Coalg_{pf}, *$V = \mathfrak{V}(C)$ is expressed by $V(c) = \Sigma\, c_i \otimes \lambda_i$ where $\Sigma\, c_i \otimes \lambda_i$ is uniquely determined by the property that $(a^p, c) = \Sigma\, (a, c_i)^p\lambda_i$ for any a in C^*.*

Proof. If suffices to assume C is finitely generated projective. Then C^* is discrete and so $F\colon (C^*)^{(p)} \to C^*$ is given by $F(a \otimes \lambda) = a^p\lambda$ for a in C^*, λ in $k^{1/p} = k$ (as rings!). As in Corollary 4.4, we need not distinguish between $((C^*)^{(p)})^*$ and $C^{(p)}$ so that $V = F^*\colon C \cong C^{**} \to ((C^*)^{(p)})^* \cong C^{(p)}$ is given by requiring $(V(c), a \otimes \lambda) = (F(a \otimes \lambda), c) = (a^p\lambda, c) = (a^p, c)\lambda$. This exhibits $V(c)$ as an element of $((C^*)^{(p)})^*$. Now say $V(c) = \Sigma\, c_i \otimes \lambda_i$ in $C^{(p)}$. Applying $(\ )'$ to Lemma 4.2, such $\Sigma\, c_i \otimes \lambda_i$ must act as

$$\left(\sum c_i \otimes \lambda_i, a \otimes \lambda\right) = \sum (c_i, a)^p \lambda \lambda_i$$

proving the result.

Note that our description of $\mathcal{V}$ is independent of any discussion of symmetric tensors (cf. [5] and [3, IV. 3.4.3]). It follows from Proposition 4.5 that our $\mathcal{V}$ coincides with the map $\mathcal{V}$ in the Hopf algebra literature when k is a field [5].

4.6 **Theorem** [8, §4]. *$\mathcal{F}$ and $\mathcal{V}$ are natural transformations of cogroups and groups resp. when restricted to the cogroups and groups resp. in k-Alg_{pf} and k-Coalg_{pf} resp.*

Proof. It follows from Proposition 4.1 and the fact that $(\ )^{\cdot}$ is an anti-equivalence that $(\ )^{(p)}$ preserves coproducts, for

$$\begin{aligned}(\mathfrak{A}^{(p)} \hat{\otimes} \mathfrak{B}^{(p)})^{\cdot}(B) &\cong (\mathfrak{A}^{(p)})^{\cdot}(B) \times (\mathfrak{B}^{(p)})^{\cdot}(B) \cong \mathfrak{A}^{\cdot}(B^{1/p}) \times \mathfrak{B}^{\cdot}(B^{1/p}) \\ &\cong (\mathfrak{A} \hat{\otimes} \mathfrak{B})^{\cdot}(B^{1/p}) \cong ((\mathfrak{A} \hat{\otimes} \mathfrak{B})^{(p)})^{\cdot}(B).\end{aligned}$$

Hence the following diagram commutes

$$\begin{tikzcd}
\mathfrak{A} \arrow[r, "\mathcal{F}(\mathfrak{A})"] \arrow[d, "\delta"'] & \mathfrak{A}^{(p)} \arrow[d, "\delta^{(p)}"] \arrow[dr, "\Delta"] & \\
\mathfrak{A} \hat{\otimes} \mathfrak{A} \arrow[r, "\mathcal{F}(\mathfrak{A}\hat{\otimes}\mathfrak{A})"] & (\mathfrak{A} \hat{\otimes} \mathfrak{A})^{(p)} \arrow[r, "\sim"] & \mathfrak{A}^{(p)} \hat{\otimes} \mathfrak{A}^{(p)}
\end{tikzcd}$$

where δ and Δ are the comultiplication maps of $\mathfrak{A}$ and $\mathfrak{A}^{(p)}$. (The commutativity of the triangle follows from the fact that the comultiplication on $\mathfrak{A}^{(p)} \equiv (\mathfrak{A} \hat{\otimes} k^{1/p})'$ comes entirely from that on $\mathfrak{A}$.) Thus $\mathcal{F}(\mathfrak{A})$ is a cogroup map (a similar argument is needed to show that $\mathcal{F}(\mathfrak{A})$ preserves counits). The result for $\mathcal{V}$ is immediate by categorical duality.

The result says $\mathcal{V}(C)$ is a Hopf algebra map. This recovers [5, 4.1.6].

4.7 Remark. Standard formalisms relating the Verschiebung to sequences of divided powers hold in our generality. Let C^n be the module free on $x_0, \cdots, x_{n-1}$ with $\Delta(x_i) = \Sigma_{i+j=k}\, x_i \otimes x_j$. One easily finds that

$$(*) \qquad \mathcal{V}(C^n)(x_i) = \begin{cases} 0 & \text{if } p \nmid i, \\ x_{i/p} \otimes 1 & \text{if } p \mid i. \end{cases}$$

To see this note that $(C^n)^* \cong k[y]/(y^n)$. (With y^i the dual basis we get easily $y^i y^j = y^{i+j}$.) According to Proposition 4.5, we need to show

$$(y^{jp}, x_{ip+s}) = \begin{cases} (y^j, x_i)^p & \text{if } s = 0, \\ 0 & \text{if } 1 \leq s < p. \end{cases}$$

This is the case since $(y^{jp}, x_{ip+s}) = \delta_{jp,ip+s}$ and since $\delta_{jp,ip} = \delta_{ij} = (\delta_{ij})^p = (y_j, x_i)^p$.

Note that if C is any coalgebra, the divided powers of length n are naturally in one-one correspondence with k-Coalg(C^n, C), so that the above formula $(*)$ holds for any sequence of divided powers in C (provided C is in k-Coalg$_{pf}$, since only then is $\mathcal{V}(C)$ defined. This is of course true for any coalgebra over a field).

REFERENCES

1. P. Cartier, *Groupes algébriques et groupes formels*, Colloq. Théorie des Groupes Algébriques (Bruxelles, 1962), Librairie Universitaire, Louvain; Gauthier-Villars, Paris, 1962, pp. 87–111. MR 26 #6172.

2. ———, *Groupes formels associés aux anneaux de Witt généralisés*, C. R. Acad. Sci. Paris Sér. A–B 265 (1967), A49–A52. MR 36 #1448.

3. M. Demazure and P. Gabriel, *Groupes algébriques*, Tom 1, North-Holland, Amsterdam, 1970.

4. P. Gabriel, *Étude infinitésimale des schémas en groupes: Groupes formels*, Exposé VIIB, Schémas en Groupes 1 (SGA 3) 1962/64, Lecture Notes in Math., vol. 151, Springer-Verlag, Berlin, 1970.

5. R. Heyneman and M. E. Sweedler, *Affine Hopf algebras*. II, J. Algebra 16 (1970), 271–297. MR 42 #342.

6. M. Lazard, *Sur les groupes de Lie formels à un paramètre*, Bull. Soc. Mat. France 83 (1955), 251–274. MR 17, 508.

7. Ju. I. Manin, *Theory of commutative formal groups over fields of finite characteristics*, Uspehi Mat. Nauk 18 (1963), no. 6 (114), 3–90 = Russian Math. Surveys 18 (1963), no. 6, 1–83. MR 28 #1200.

8. R. Morris and B. Pareigis, *Formal groups over discrete rings*, Bull. Amer. Math. Soc. 79 (1973), 449–453.

9. D. Mumford, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Math., no. 5, Oxford Univ. Press, London, 1970. MR 44 #219.

10. M. E. Sweedler, *Hopf algebras*, Math. Lecture Note Series, Benjamin, New York, 1969. MR 40 #5705.

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