THE WEDDERBURN PRINCIPAL THEOREM FOR A GENERALIZATION OF ALTERNATIVE ALGEBRAS

BY

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ABSTRACT. A generalized alternative ring I is a nonassociative ring \( R \) in which the identities
\[
(wx,y,z) + (w,x,[y,z]) - w(x,y,z) - (w,y,z)x; ([w,x],y,z) + (w,x,yz) - y(w,x,z) - (w,x,y)z; \text{ and } (x,x,x) \text{ are identically zero. It is here demonstrated that if } A \text{ is a finite-dimensional algebra of this type over a field } F \text{ of characteristic } \neq 2, 3, \text{ then } A \text{ a nilalgebra implies } A \text{ is nilpotent.}
\]

A generalized alternative ring II is a nonassociative ring \( R \) in which the identities
\[
(wx,y,z) + (w,x,[y,z]) - w(x,y,z) - (w,y,z)x \text{ and } (x,y,x) \text{ are identically zero. Let } A \text{ be a finite-dimensional algebra of this type over a field } F \text{ of characteristic } \neq 2. \text{ Then it is here established that (1) } A \text{ a nilalgebra implies } A \text{ is nilpotent; (2) } A \text{ simple with no nonzero idempotent other than } 1 \text{ and } F \text{ algebraically closed imply } A \text{ itself is a field; and (3) the standard Wedderburn principal theorem is valid for } A.
\]

1. Preliminaries. Let \( R \) be a nonassociative ring. As is customary, for \( x, y, z \in R \) we denote by \((x,y,z)\) the associator \((x,y,z) = (xy)z - x(yz)\) and by \([x,y]\) the commutator \([x,y] = xy - yx\). A straightforward verification shows that the following identity, known as the Teichmüller identity, holds for all \( w, x, y, z \in R \):
\[
(T) \quad (wx,y,z) - (w,xy,z) + (w,x,yz) = w(x,y,z) + (w,x,y)z.
\]

A nonassociative ring \( R \) is called power-associative if for every \( x \in R \) the subring generated by \( x \) is associative.

A. Generalized alternative rings I. In [4] Kleinfeld defines a generalized alternative ring I to be a nonassociative ring \( R \) such that for all \( w, x, y, z \in R \) the following identities are satisfied:
\[
(1.1) \quad (wx,y,z) + (w,x,[y,z]) - w(x,y,z) - (w,y,z)x = 0,
\]
\[
(1.2) \quad ([w,x],y,z) + (w,x,yz) - y(w,x,z) - (w,x,y)z = 0,
\]
\[
(1.3) \quad (x,x,x) = 0.
\]

That such a ring is power-associative can be readily verified as follows:
Theorem 1.1. A generalized alternative ring I is power-associative.

Proof. Define \( x^n = x^{n-1}x \). We need to show \( x^i x^j = x^{i+j} \) for any \( i, j > 0 \). From (1.3) we have \( x^3 = x^2 x = xx^2 \). Also, (1.1) and (1.3) yield \( (x^2, x, x) + (x, x, [x, x]) = x(x, x, x) + (x, x, x)x \) or \( (x^2, x, x) = 0 \), which implies \( x^4 = x^3 x = x^2 x^2 \); while (1.2) and (1.3) yield \( ([x, x], x, x) + (x, x, x^2) = x(x, x, x) + (x, x, x)x \) or \( (x, x, x^2) = 0 \), which implies \( x^2 x^2 = xx^3 \).

The proof is now by induction. We assume \( x^i x^j = x^{i+j} \) for \( i + j < n; i, j > 0 \) and \( n > 4 \). Then (1.1) gives \( (x^2, x^{n-2-i}, x^j) + (x, x, [x^{n-2-i}, x^j]) = x(x, x^{n-2-i}, x^j) + (x, x^{n-2-i}, x^j)x \) or \( (x^2, x^{n-2-i}, x^j) = 0 \), using the induction assumption. Thus, except possibly for \( i = n - 1 \), \( x^{n-i} x^j = x^2 x^{n-2} \). But, again using the induction assumption, (1.1) gives \( (x^2, x^{n-3}, x^i) + (x, x, [x^{n-3}, x]) = x(x, x^{n-3}, x) + (x, x^{n-3}, x)x \) or \( (x^2, x^{n-3}, x^i) = 0 \), that is \( x^n = x^{n-1} x = x^2 x^{n-2} \). Thus \( x^{n-i} x^j = x^n \), except possibly for \( i = n - 1 \). Finally, (1.2) and the induction assumption yield \( ([x, x^{n-3}], x, x) + (x, x^{n-3}, x^2) = x(x, x^{n-3}, x) + (x, x^{n-3}, x)x \) or \( (x, x^{n-3}, x^2) = 0 \), that is \( x x^{n-1} = x^{n-2} x^2 = x^n \). This completes the induction.

Let \( R \) be a generalized alternative ring I. If one defines a new multiplication for \( R \) by \( x * y = yx \), then a straightforward verification shows that under this new multiplication identity (1.1) is converted to identity (1.2) and vice versa. Thus, since identity (1.3) is left unchanged, the resulting ring is itself a generalized alternative ring I. We henceforth refer to this procedure as passing to the anti-isomorphic copy of \( R \).

In this work we consider generalized alternative algebras I over fields of characteristic \( \neq 2, 3 \). In addition to the above defining identities, we also make repeated use of the following:

\begin{align*}
(1.4) \quad (y, x, x) + (x, y, x) + (x, x, y) &= 0, \\
(1.5) \quad (w, xy, z) - (w, x, zy) + (w, x, y)z - (w, y, z)x &= 0, \\
(1.6) \quad (z, yx, w) - (yz, x, w) + z(y, x, w) - x(z, y, w) &= 0, \\
(1.7) \quad (x, x, yx) &= (x, x, y)x, \\
(1.8) \quad (x, xy, x) &= x(x, y, x), \\
(1.9) \quad (x^2, y, x) &= 2x(x, y, x), \\
(1.10) \quad (x^2, x, y) &= (x, x^2, y) = 2(x, x, yx).
\end{align*}

Identity (1.4) is obtained from linearization of (1.3). Identity (1.5) is obtained by subtracting (T) from (1.1), after which (1.6) follows from (1.5) by passing to the anti-isomorphic copy of \( R \). Identities (1.7), (1.8), and (1.9) are established in [4]. To see that \( (x^2, x, w) = (x, x^2, w) \), we let \( z = y = x \) in (1.6); while to see that \( (x, x^2, z) = 2(x, x, x) \), we let \( w = y = x \) in (1.5) and then apply (1.3) and (1.7). Now taking \( w, z \) to be \( y \), these last two equations together give (1.10).

B. Generalized alternative rings II. A generalized alternative ring II is defined
by Kleinfeld in [5] to be a nonassociative ring \( R \) such that for all \( w, x, y, z \in R \) the following identities are satisfied:

\[
\begin{align*}
(2.1) \quad & (wx, y, z) + (w, x, [y, z]) = w(x, y, z) + (w, y, z)x, \\
(2.2) \quad & (x, y, x) = 0.
\end{align*}
\]

From these identities one easily generates:

\[
\begin{align*}
(2.3) \quad & (x, y, z) = -(z, y, x), \\
(2.4) \quad & (w, xy, z) - (w, x, zy) + (w, x, y)z - (w, y, z)x = 0, \\
(2.5) \quad & (wx, z, z) = w(x, z, z) + (w, z, z)x, \\
(2.6) \quad & (z, x, z^2) = 0, \\
(2.7) \quad & z(z, z, x) = (z, z, zx) = (z, z, xz) = (z, z, x)z.
\end{align*}
\]

Identity (2.3) follows from linearization of (2.2). Identity (2.4) is obtained by subtracting (T) from (2.1). If one takes \( y = z \) in (2.1), one obtains (2.5). Letting \( w = y = z \) in (2.4) and applying (2.2), one obtains (2.6).

To see that \( z(z, z, x) = (z, z, zx) \), we let \( w = y = z \) in (2.1) and then apply (2.2) and (2.3). To see that \( (z, z, zx) = (z, z, x)z \), we take \( w, x, z \) to be \( z \) and \( y \) to be \( x \). in (2.4) and then apply (2.2). Finally, to see that \( (z, z, x)z = (z, z, xz) \), we take \( x, y, z \) to be \( z \) and \( w \) to be \( x \) in (2.1) and then apply (2.2) and (2.3). This establishes (2.7).

A nonassociative ring which satisfies identities (2.2) and (2.6) is called noncommutative Jordan. From [10] an algebra of this type over a field of characteristic \( \neq 2 \) is known to be power-associative.

2. Finite-dimensional nilalgebras. Let \( A \) be a power-associative algebra. An element \( x \in A \) is said to be nilpotent if there exists an integer \( k > 0 \) for which \( x^k = 0 \). Should every element of the algebra \( A \) be nilpotent, then \( A \) is called a nilalgebra. For any algebra \( A \) one obtains a derived series of subalgebras \( A^{(0)} \supseteq A^{(1)} \supseteq \cdots \) by defining inductively \( A^{(i)} = A, A^{(i+1)} = (A^{(i)})^2 \). \( A \) is called solvable in case \( A^{(m)} = 0 \) for some integer \( m > 0 \). A nonassociative algebra \( A \) is called nilpotent in case there exists an integer \( n > 0 \) such that any product \( x_1 x_2 \cdots x_n \) of \( n \) elements \( x_i \in A \), no matter how associated, is zero.

For an algebra \( A \) and \( x \in A \), the linear operators on \( A \) of right and left multiplication by \( x \) are denoted by \( R_x \) and \( L_x \), respectively. Let \( M(A) \) denote the subalgebra generated by all right and left multiplications of \( A \) in the associative algebra of all linear operators on \( A \). If \( B \) is any subset of \( A \), we shall write \( B^* \) for the subalgebra of \( M(A) \) generated by all right and left multiplications of \( A \) which correspond to elements of \( B \).
A. Generalized alternative algebras I.

Lemma 1.1. Let $B$ be a generalized alternative algebra I over a field $F$ of characteristic $\neq 2, 3$. Suppose $B = Fb + C$ where $C$ is a subalgebra of $B$ such that $B^2 \subseteq C$. If $H = B^* C^* + C^*$, then $S_x S_y S_z \in H$ for $S = L$ or $R$ and all $x, y, z \in B$.

Proof. We begin by making some reductions. First, since every product of three operators of right or left multiplication corresponding to elements of $B = Fb + C$ may be expressed as a linear combination of products $S_x S_y S_z$ where each of $x, y, z$ is either in $C$ or equal to $b$, it suffices to verify only that products of this latter form belong to $H$. In particular, since $c \in C$ clearly implies $S_x S_y S_z \in H$, we need only consider products of the form $S_x S_y S_b$ where each of $x$ and $y$ is either in $C$ or equal to $b$.

Secondly, should $S_x S_y S_z \in H$, then by passing to the anti-isomorphic copy of $B$ one sees that $S_x' S_y' S_z' \in H$, where $S_x' = R_x(L_x)$ if $S_x = L_x(R_x)$ for $i = 1, 2, 3$.

Henceforth let $c, c' \in C$. From (1.1), $(w_c, y, b) + (w, c, [y, b]) = w(c, y, b) + (w, y, b)c$, we have $R_c R_y R_b - R_c R_{by} + R_{cy} = R_{cy} + R_y R_c - R_{by} R_c$. Since by assumption $B^2 \subseteq C$, this yields $R_c R_y R_b \in H$ or

(1.a) $R_c R_b R_b, R_c R_b R_b \in H$.

Now applying our second reduction, (1.a) in turn yields

(1.b) $L_c L_b L_b, L_c L_b L_b \in H$.

From (1.2), $[b, x]_L, y, c) + (b, x, y)c = y(b, x, c) + (b, x, y)c$, we have $-L_y R_c + R_c L_y L_b = R_{y,c} - L_y L_b R_c$. Again using the assumption $B^2 \subseteq C$, as we will continually do throughout, this yields $R_c L_x L_b \in H$ or

(1.c) $R_c L_b L_b, R_c L_b L_b \in H$.

Again applying our second reduction, as we also will continually do throughout, (1.c) in turn yields

(1.d) $L_c R_b R_b, L_c R_b R_b \in H$.

From (1.1), $(c'b, y, c) + (c'b, [y, c]) = c'(b, y, c) + (c', y, c)b$, using (1.d) we now obtain

(1.e) $R_c L_c R_b, L_c R_c L_b \in H$.

Adding (1.1), $(b, x, c, c') + (b, x, [c, c']) = b(x, c, c') + (b, c, c')x$, to (1.2), $[b, x], c', c') + (b, x, c') = c'(b, x, c) + (b, x, c')c$, we have

(1.f) $R_c L_c R_b, L_c R_c R_b \in H$.

Linearization of (1.10), $(b^2, b, y) = 2(b, b, yb)$, gives

$$2(b, c, y) + (bc, b, y) + (cb, b, y) = 2((b, b, yc) + (b, c, yb) + (c, b, yb)).$$

Using (1.c) this yields $2R_b L_c L_b \in H$ or

(1.g) $R_b L_c L_b, L_b R_c R_b \in H$.

From (1.6) we obtain $(z, bc, b) - (bz, c, b) + z(b, c, b) - c(z, b, b) = 0$. Using (1.g) this implies $R_{bc} R_b \in H$. Then from (1.1), $(wb, b, c) + (w, b, [b, c]) = w(b, b, c) + (w, b, c)b$, we have
(1.h) $R_b R_c R_b$, $L_b L_c L_b \in H$.

Next from (1.5), $(b, c, y, b) - (b, c, hy) + (b, c, y)b - (b, y, b)c = 0$, if we use (1.h) we obtain $L_b R_c R_b - L_c R_b L_b \in H$. Since from (1.6), $(c, bx, b) - (bc, x, b) + c(b, x, b) - xc(b, b) = 0$, one has $L_c R_b R_b - L_b L_c R_b \in H$, adding this to $L_b R_c R_b - L_c R_b L_b$ gives

(1.1) $L_b L_c R_b - L_c R_b L_b \in H$.

From (1.4), $(cx, b, b) + (b, cx, b) + (b, b, cx) = 0$, using (1.b) and (1.d) one also has

(1.i) $L_b L_c R_b R_b L_a R_b L_b \in H$.

If we now linearize (1.8) to obtain $(c, by, b) + (b, cy, b) + (b, by, c) = c(b, y, b) + b(c, y, b) + b(b, y, c)$, then using (1.c), (1.g), (i.1), and (i.2) we have $2L_b R_c L_b \in H$ or

(1.j) $L_b L_c R_b R_b R_c L_b \in H$.

From linearization of (1.4) we obtain $(b, x, b) + (b, c, bx) + (c, bx, b) + (b, b, bx) = 0$. Using (1.g), (1.h), and (i.1) this yields

(1.k) $L_b L_c R_b R_b L_c R_b \in H$.

Next (1.4), $(xc, b, b) + (x, xc, b) + (b, b, xc) = 0$, together with (1.a), (1.c), and (1.k) gives

(1.m) $R_c R_b L_b L_b L_b \in H$.

From (1.10), $(b^2, b, y) = 2(b, b, yb)$, we obtain $2R_b L_b L_b \in H$, that is

(1.n) $R_b L_b L_b, L_b R_b R_b \in H$.

We next add (1.1), $(b^2, x, b) + (b, b, [x, b]) = b(b, x, b) + (b, x, b)b$, to (1.2), $(b, x, b) + (b, x, b^2) = b(b, x, b) + (b, x, b)b$ to derive $(b^2, x, b) + (b, x, b^2) + (b, b, [x, b]) + ([b, x], b, b) = 2b(b, x, b) + 2(b, x, b)b$. Since (1.9) gives $2b(b, x, b) = b^2, x, b$, and since by passing to the anti-isomorphic copy of $B$ this in turn gives $2(b, x, b)b = (b, x, b^2)$, our last equation simplifies to $(b, b, [x, b]) + ([b, x], b, b) = 0$. Then using (1.6), $(xb, b, b) = (x, b, b) + (x, b, b) - (b, b, x)$, we have $(b, b, [x, b]) + (b, b, b) = (b, x, b) + (b, x, b) - (b, b, x)$. Now by passing to the anti-isomorphic copy of $B$, (1.8) and (1.7) become $(b, x, b) = (b, x, b)b$ and $b(b, x, b) = (b, x, b)b$, respectively. Hence our equation again simplifies, this time to $(b, b, [x, b]) = ([b, x], b, b)$. Using (1.n) this gives

(1.p) $L_b L_b L_b + R_b L_b R_b + L_b R_b L_b \in H$.

From (1.4), $(b^2, x, b) + (b, b, bx) + (b, bx, b) = 0$, using (1.8) we obtain $(b, b, b) = -(b, b, bx) - b(b, x, b)$. If we use (1.n), this yields

(1.q) $L_b L_b L_b - L_b R_b R_b \in H$.

Now subtracting (1.q) from (1.p) we have

(2.q) $R_b L_b R_b + 2L_b R_b L_b \in H$.

Next from linearization of (1.4) we obtain $(x, b, b^2) + (b, b^2, x) + (b^2, x, b) + (b, b^2, b) + (b, b, x) + (b, x, b^2) = 0$ or $2(x, b, b^2) + 2(b^2, b, x) + (b^2, x, b) + (b, x, b^2) = 0$, since (1.10) implies $(b, b^2, x) = (b^2, b, x)$, and since by passing to
the anti-isomorphic copy of $B$ this in turn implies $(x, b^2, b) = (x, b, b^2)$. Thus we have

\[(p.4) \ (L_{b^2}b - (R_{b^2})L_b) \in H.\]

Now (1.9) gives $(b^2, x, b) = 2b(b, x, b)$, and by passing to the anti-isomorphic copy of $B$ this in turn gives $(b, x, b^2) = 2(b, x, b)b$. Hence using (1.n) we also have

\[(p.5) \ 2L_bR_bL_b - (L_{b^2})R_b \in H, \quad \text{and} \]
\[(p.6) \ 2R_bL_bR_b - (R_{b^2})L_b \in H. \]

If we now subtract (p.4) and (p.5) from (p.6), we obtain

\[(p.7) \ 2R_bL_bR_b - 2L_bR_bL_b \in H. \]

Lastly, adding (p.7) to (p.3) we have $3R_bL_bR_b \in H$ or

\[(1.p) \ R_bL_bR_b, \ L_bR_bL_b \in H. \]

In addition, (p.2) and (1.p) together also show

\[(1.q) \ L_bL_bL_b, \ R_bR_bR_b \in H. \]

Finally, by passing to the anti-isomorphic copy of $B$, from (1.8) we obtain as before $(b, xb, b) = (b, x, b)b$. Using (1.n) and (1.p) we then have

\[(1.r) \ R_bR_bL_b, \ L_bL_bR_b \in H. \]

This completes the proof of the lemma.

From Schafer's proof of Theorem 3 in [15], which proof in turn is modelled on that of Albert for standard algebras in [2], it follows that Lemma 1.1 is sufficient to obtain the following result.

**Theorem 1.2.** Let $A$ be a finite-dimensional generalized alternative algebra over a field $F$ of characteristic $\neq 2, 3$. If $B$ is a solvable subalgebra of $A$, then $B^*$ is nilpotent.

**Lemma 1.2.** Let $A$ be a generalized alternative algebra $I$ over a field $F$ of characteristic $\neq 2, 3$; and let $B$ be a subalgebra of $A$. If $x \in A$ is such that $xB \subseteq B, Bx \subseteq B$, then $(x^2B)B \subseteq B, x^2B^2 \subseteq B, B(x^2B) \subseteq B, B^2x^2 \subseteq B$, and $(x^2B^2)B \subseteq B$.

**Proof.** We assume throughout that $b_i \in B$ for $i = 1, 2, 3$. From (1.5) we have

\[(x, b_1x, b_2) - (x, b_1, b_2x) + (x, b_1, x)b_2 - (x, x, b_2)b_1 = 0 \quad \text{or} \quad (x^2B)B \subseteq B. \]

Then from (1.5) we also have $(x, xb_1, b_2) - (x, x, b_2b_1) + (x, x, b_1)b_2 - (x, b_1, b_2)x = 0$ or $x^2B^2 \subseteq B$. Next (1.6) gives $(x, xb_2, b_1) - (x^2, b_2, b_1) + x(x, b_2, b_1) - b_2(x, x, b_1) = 0$ or $B(x^2B) \subseteq B$. Now (1.6) also gives $(b_1, xb_1, x) - (xb_1, b_1, x) + b_1(x, b_1, x) - b_2(b_1, x, x) = 0$ or $B(Bx^2) \subseteq B$, whence (1.6) yields $(b_1, b_2x, x) - (b_2b_1, x, x) + b_1(b_2, x, x) - x(b_1, b_2, x) = 0$ or $B^2x^2 \subseteq B$.

There remains only to show $(x^2B^2)B \subseteq B$. We first observe that from (1.5) we have

\[(b_1, b_2x, x) - (b_1, b_2, x^2) + (b_1, b_2, x)x - (b_1, x, x)b_2 = 0 \quad \text{or} \quad (Bx^2)B \subseteq B. \]

Also, (1.6) then implies $(x^2, b_1(b_2x^2), b_3) - (b_1x^2, b_2x^2, b_3) + x^2(b_1, b_2x^2, b_3) - (b_2x^2)(x^2, b_1, b_2) = 0$ or $(Bx^2)^2B \subseteq B$. Now using (1.4), (1.1) gives

\[
[(x^2b_1)b_2]x^2 - (x^2b_1)(x^2b_2) + x^2[b_1(x^2b_2)] - x^2[(b_1b_2)x^2] \\
= (x^2, b_2, x^2)b_1 = -(b_2, x^2, x^2)b_1 - (x^2, x^2, b_2)b_1;
\]
and (1.2) gives
\[
-[(b \cdot x^2)b_2]x^2 + (b \cdot x^2)(b_2 \cdot x^2) - x^2[b_1(b_2 \cdot x^2)] + [x^2(b_1b_2)]x^2
\]
\[
= b_2(x^2, b_1, x^2) = -b_2(x^2, x^2, b_1) - b_2(b_1, x^2, x^2).
\]

Adding these last two equations and using (1.1), (1.2), and (1.4), we have
\[
[(x^2b_1)b_2]x^2 - (x^2b_1)(x^2b_2) + x^2[b_1(x^2b_2)] - x^2[(b_1b_2)x^2]
\]
\[
- [(b_1 \cdot x^2)b_2]x^2 + (b_1 \cdot x^2)(b_2 \cdot x^2) - x^2[b_1(b_2 \cdot x^2)] + [x^2(b_1b_2)]x^2
\]
\[
= -b_2(b_1, x^2, x^2) - (b_2, x^2, x^2)b_1 - b_2(x^2, x^2, b_1) - (x^2, x^2, b_2)b_1
\]
\[
= -(b_2b_1, x^2, x^2) - (x^2, x^2, b_2b_1)
\]
\[
= (x^2, b_2b_1, x^2).
\]

Finally, multiplication of this last equation on the right by \(b_3\) yields \((x^2B)^2B \subseteq B\).

**Lemma 13.** Let \(A\) be a generalized alternative algebra I over a field \(F\) of characteristic \(\neq 2, 3\); and let \(B\) be a subspace of \(A\). If \(x \in A\) is such that \(xB \subseteq B, Bx \subseteq B, x^2B \subseteq B\), then \(xkB \subseteq B, Bxk \subseteq B\) for \(k = 1, 2, 3, \ldots\).

**Proof.** Let \(b \in B\). We note that (1.4), \((b, x, x) + (x, b, x) + (x, x, b) = 0\), implies \(Bx^2 \subseteq B\). Hence we have \(x^kB \subseteq B, Bx^k \subseteq B\) for \(k = 1, 2\). The proof now is by induction. We assume \(x^kB \subseteq B, Bx^k \subseteq B\) for \(k < n, n > 2\). From (1.10) and (1.7) we have \((x, x^2, y) = 2(x, x, y)x\). Linearization of this identity gives
\[
(x^{n-2}, x, b) + (x, x^{n-2}x, b) + (x, xx^{n-2}, b)
\]
\[
= 2[(x^{n-2}, x, b)x + (x, x^{n-2}, b)x + (x, x, b)x^{n-2}].
\]

Applying the induction assumption, we now have \(3x^nB \subseteq B\) or \(x^nB \subseteq B\). Next linearization of (1.4) gives \((b, x, x^{n-1}) + (b, x^{n-1}, x) + (x, b, x^{n-1}) + (x^{n-1}, b, x) + (x, x^{n-1}, b) + (x^{n-1}, x, b) = 0\). Again applying the induction assumption, we have \(2bx^n \subseteq B\) or \(Bx^n \subseteq B\), and our induction is complete.

The proof of the following theorem is now the same as that of Theorem 4 in [15], with the one exception that, since a generalized alternative algebra I is not necessarily noncommutative Jordan, we need to make use of our Lemma 1.3 in addition to Theorem 1.2 and Lemma 1.2 above.

**Theorem 13.** Let \(A\) be a finite-dimensional generalized alternative algebra I over a field \(F\) of characteristic \(\neq 2, 3\). If \(A\) is a nilalgebra, then \(A\) is nilpotent.

**B. Generalized alternative algebras II.**

**Lemma 2.1.** Let \(B\) be a generalized alternative algebra II over a field \(F\) of characteristic \(\neq 2\). Suppose \(B = Fb + C\) where \(C\) is a subalgebra of \(B\) such that
If $H = B^*C^* + C^*$, then $S_x S_y S_z \in H$ for $S = L$ or $R$ and all $x, y, z \in B$.

**Proof.** As in the proof of Lemma 1.1, it suffices to verify only that $H$ contains products of the form $S_x S_y S_b$ where each of $x$ and $y$ is either in $C$ or equal to $b$.

In addition, we note that (2.2) implies $L_b R_b = R_b L_b$.

Throughout we assume $c, c' \in C$. From (2.1), $(w, y, b) + (w, c, [y, b]) = w(c, y, b) + (w, y, b)c$, we have $R_c R_y R_b - R_c R_{by} + R_{cy} b = R_{cy} b + R_y R_b R_c - R_{yb} R_c$.

Since by assumption $B^2 \subseteq C$, this last equation yields $R_c R_y R_b \in H$ or

(2.a) $R_c R_c R_b, R_c R_b R_b \in H$.

From (2.1), $(c, x, y, b) + (c, x, [y, b]) = c(x, y, b) + (c, y, b)x$, we also obtain $L_c R_y R_b - L_c R_{by} + R_{by} L_c = R_y R_b L_c + L(c, y, b)$. Again using the assumption $B^2 \subseteq C$, as we will continually do throughout, this yields $L_c R_y R_b \in H$ or

(2.b) $L_c R_c R_b, L_c R_b R_b \in H$.

Now (2.3), $(x, y, b) = (b, y, x)$, gives $R_c R_y R_b - R_c R_{yb} = -R_c L_{by} + R_c L_y L_b$. Using (2.a) this implies $R_c L_y L_b \in H$ or

(2.c) $R_c L_c L_b, R_c L_b L_b \in H$.

From (2.3), $(c, w, y, b) = (b, y, c, w)$, we also have $L_c R_y R_b - L_c R_{by} = -L_c L_{by} + L_c L_y L_b$. Using (2.b) this gives $L_c L_y L_b \in H$ or

(2.d) $L_c L_c L_b, L_c L_b L_b \in H$.

If we take $w = x = y$ in (2.4) and apply (2.2), we obtain

(2.8) $(y, y^2, z) = (y, y, zy) + (y, y, z) y$.

Taking $x = y = z$ in (2.4) and applying (2.3), we also have

(2.9) $(z^2, z, w) = (z, z^2, w)$.

Now (2.8), (2.9), (2.7), and (2.3) together imply

(2.10) $(b^2, b, x) = 2(b, b, b x) = -2(x b, b, b)$,

whence $2L_b L_b L_b, 2R_b R_b R_b \in H$ or

(2.e) $L_b L_b L_b, R_b R_b R_b \in H$.

From (2.7), $(b, b, b x) = (b, b, x)b$, we have $L_b L_{b2} - L_b L_b L_b = (L_{b2}) R_b - L_b L_b R_b$. Since (2.6) and (2.3) imply $(b^2, x, b) = 0$ or $(L_{b2}) R_b = R_b (L_{b2})$, this last equation is equivalent to $L_b L_{b2} - L_b L_b L_b = R_b L_{b2} - L_b L_b R_b$. Using (2.e) we now have

(2.f) $L_b L_b R_b = L_b R_b L_b = R_b L_b L_b \in H$.

From (2.7) and (2.3) we next obtain $(x b, b, b) = (b x, b, b)$. If we again use (2.e), this gives

(2.g) $L_b R_b R_b = R_b L_b R_b = R_b R_b L_b \in H$.

From (2.1) and (2.3) we have $-(c, x, c') + (c', b, [x, c]) = c'(b, x, c) + (c', x, c) b$.

Using (2.b) this yields

(2.h) $R_c L_c, R_b \in H$.

Then from (2.3), $(b, x c, c') = -(c', x c, b)$, using (2.h) we obtain
(2.i) \( R_c R_c, L_b \in H \).

From (2.1) and (2.3) we also have \(-(c, x, bc') + (b, c', [x, c]) = b(c', x, c) + (b, x, c)c'\). Using (2.d) this yields

(2.j) \( L_c, R_c L_b \in H \).

Then from (2.3), \((b, cx, c') = -(c', cx, b)\), using (2.j) we obtain

(2.k) \( L_c L_c, R_b \in H \).

Linearization of (2.10) gives

\[
(b^2, c, x) + (bc, b, x) + (cb, b, x) = 2[(b, b, cx) + (b, c, bx) + (c, b, bx)]
\]

\[
= -2[(xb, b, c) + (xb, c, b) + (xc, b, b)].
\]

If we now use (2.a) and (2.d), we have \(2L_b L_c L_b, 2R_b R_c R_b \in H\) or

(2.m) \( L_b L_c L_b, R_b R_c R_b \in H \).

Then from (2.3), \((b, c, bx) = -(bx, c, b)\) and \((b, c, xb) = -(xb, c, b)\), using (2.m) we obtain

(2.n) \( L_b R_c R_b, R_b L_c L_b \in H \).

Next (2.4), (2.2), and (2.3) yield \((bc, x, b) = -(b, x, c)b\); while (2.1), (2.2), and (2.3) yield \((bc, x, b) + (b, c, [x, b]) = -b(b, x, c)\). Subtracting the first of these equations from the second, we have \((b, c, [x, b]) = -b(b, x, c) + (b, x, c)b\). Also, from (2.1) one has \((b^2, x, c) + (b, b, [x, c]) = b(b, x, c) + (b, x, c)b\). Adding this equation to the one just prior, we obtain \((b^2, x, c) + (b, b, [x, c]) + (b, c, [x, b]) = 2(b, x, c)b\). Using (2.c), (2.d), (2.m), and (2.n) this now gives \(2R_c L_b R_b \in H\) or

(2.p) \( R_c L_b R_b = R_c R_b L_b \in H \).

From (2.4) and (2.3) we have \((c, xb, b) + (b^2, x, c) - (b, x, c)b + (b, b, c)x = 0\).

If we use (2.n) and (2.p), this yields

(2.q) \( R_b L_c R_b \in H \).

Then (2.3), \((b, xb, c) = -(c, xb, b)\), using (2.q) gives

(2.r) \( R_b R_c L_b \in H \).

From (2.4) and (2.3) we also have \((b, bx, c) + (cx, b, b) - (x, b, b)c + (c, x, b)b = 0\). If we use (2.b) and (2.q), this yields

(2.s) \( L_b R_c L_b \in H \).

Then (2.3), \((b, bx, c) = -(c, bx, b)\), using (2.s) gives

(2.t) \( L_b L_c R_b \in H \).

From (2.4) and (2.3) we next obtain \((x, b^2, c) - (xb, b, c) + (b, b, c)x - (x, b, c)b = 0\). Using (2.m) this gives

(2.u) \( R_b R_b \in H \).

Again using (2.4) and (2.3) we have \((b, xc, b) - (bc, x, b) + (c, x, b)b - (b, c, b)x = 0\), that is \((bc, x, b) = (c, x, b)b\) using (2.2). If we use (2.b) and (2.q), this gives

(2.v) \( L_b L_b R_b \in H \).

Lastly, (2.4) and (2.3) imply \((x, bc, b) + (b, b, xc) - (b, b, c)x + (b, c, x)b = 0\).

Using (2.c), (2.u), and (2.v) we have

(2.w) \( L_c L_b R_b = L_c R_b L_b \in H \).

This completes the proof of the lemma.
As in the case of generalized alternative algebras I, using Lemma 2.1 the following result now follows from the proof of Theorem 3 in [15].

Theorem 2.1. Let $A$ be a finite-dimensional generalized alternative algebra II over a field $F$ of characteristic $\neq 2$. If $B$ is a solvable subalgebra of $A$, then $B^*$ is nilpotent.

Corollary. Let $A$ be a generalized alternative algebra II over a field $F$ of characteristic $\neq 2$. If $x$ is a nilpotent element of $A$, then $R_x$ is nilpotent.

Lemma 2.2. Let $A$ be a generalized alternative algebra II over a field $F$ of characteristic $\neq 2$, and let $B$ be a subalgebra of $A$. If $x \in A$ is such that $xB \subseteq B$, $Bx \subseteq B$, then $(x^2B)B \subseteq B$, $x^2B^2 \subseteq B$, $Bx^2 \subseteq B$, $B(x^2B) \subseteq B$, and $(x^2B)^2B \subseteq B$.

Proof. We assume throughout that $b_i \in B$ for $i = 1, 2, 3$. First using (2.4) and (2.2) we have $(x, b_1x, b_2) - (x, b_1, b_2x) = (x, x, b_2)b_1 = 0$ or $(x^2B)B \subseteq B$. Now from (2.4) we obtain $(x, xb_1, b_2) - (x, x, b_2 b_1) + (x, x, b_1 b_2) - (x, b_1, b_2)x = 0$ or $x^2B^2 \subseteq B$. Then (2.3), $(b_1 b_2, x, x) = -(x, x, b_1 b_2)$, gives $B^2 x^2 \subseteq B$. Next from (2.5) and (2.3) we have $(b_1 b_2, x, x) = -b_1(x, x, b_2) - (x, x, b_1)b_2$ or $B(x^2B) \subseteq B$. Finally, (2.4), (2.2), and (2.3) give $(x, b_1x, b_2) - (x, b_1, b_2x) + (b_2, x, x)b_1 = 0$ or $(Bx^2)B \subseteq B$. Since (2.4) and (2.2) yield $(x^2, b_1, x^2 b_2) = (x^2, b_1, b_2)x^2$, this in turn gives $(x^2, b_1, x^2 b_2)b_3 = [(x^2, b_1, b_2)x^2]b_3$ in $(Bx^2)B \subseteq B$. But then $[x^2(b_1(x^2 b_2))]b_3$ in $(x^2B)B \subseteq B$ implies $[(x^2b_1)(x^2 b_2)]b_3 \in B$, that is $(x^2B)^2B \subseteq B$.

Using Theorem 2.1 and Lemma 2.2, the proof of the following theorem is now the same as the proof of Theorem 4 in [15].

Theorem 2.2. Let $A$ be a finite-dimensional generalized alternative algebra II over a field $F$ of characteristic $\neq 2$. If $A$ is a nilalgebra, then $A$ is nilpotent.

Theorem 2.3. Let $A$ be a simple, finite-dimensional, generalized alternative algebra II over an algebraically closed field $F$ of characteristic $\neq 2$. If $A$ has no nonzero idempotent other than 1, then $A$ is itself a field.

Proof. Since $A$ a simple algebra implies $A^2 = A$, $A$ cannot be nilpotent. Thus the finite-dimensionality of $A$ and Theorem 2.2 imply that $A$ is not a nilalgebra. Proposition 3.3 on p. 32 of [13] then ensures the existence in $A$ of a nonzero idempotent, which by assumption must be 1. Now if characteristic $F = 0$, from [6] it is known that $A$ is itself a field. On the other hand, if characteristic $F \neq 0$ and $A$ is not a field, then $A$ is a nodal algebra, that is $A = F1 + N$ where $N$ consists of nilpotent elements but is not a subalgebra of $A$. Now since from our earlier corollary we know that $x$ nilpotent implies $R_x$ nilpotent, it follows from Lemma 3 of [12] that $A$ cannot be nodal. Hence $A$ must be a field.

3. The Wedderburn principal theorem. Let $A$ be a power-associative algebra over a field $F$ of characteristic $\neq 2$ and define $x \circ y = \frac{1}{2}(xy + yx)$ for $x, y \in A$. If $A$ contains an idempotent $e$, then Albert has shown in [2] that $A = A_1 + A_{1/2}$
$+ A_0$ where $A_i = \{x \in A: x \circ e = ix\}$. In fact, $ex = x = xe$ for $x \in A_1$ and $ex = 0 = xe$ for $x \in A_0$. This decomposition of $A$ is known as the Albert decomposition.

Suppose now one also has $(A,e,e) = (e,A,e) = (e,e,A) = 0$. If, as in the associative case, one takes $x = exe + (ex - exe) + (xe - exe) + (x - ex - xe + exe)$, one sees that $A = A_{11} + A_{10} + A_{01} + A_{00}$ where $A_{ij} = \{x \in A: ex = ix, xe = jx\}$. This further decomposition of $A$ is referred to as the Peirce decomposition.

Let $A$ be a generalized alternative algebra II over a field $F$ of characteristic $\neq 2$. When $A$ contains an idempotent $e$, we will make use of the following results established by Kleinfeld in [5]:

(i) $I = (A,e,e)$ is an ideal of $A$ such that $I^2 = 0$.

(ii) If $A$ permits a Peirce decomposition, then for $i, j, k, t = 0$ or $1$ we have $A_{ij}A_{kt} = 0$, when $j \neq k$, except for $A_{01}A_{01} \subseteq A_{10}$ and $A_{10}A_{10} \subseteq A_{01}$. Also $A_{ij}A_{jk} \subseteq A_{ik}$.

**Lemma 2.3.** Let $A$ be a generalized alternative algebra II over a field $F$. If $B$ is an ideal of $A$, then $AB^2 + B^2 = B^2A + B^2$ and $B^3$ are also ideals of $A$.

**Proof.** Throughout we assume $a, a_i \in A$ and $b_i \in B$ for $i = 1, 2, 3$. Using (2.3) and the fact that $B$ is an ideal of $A$, we first observe that $(b_1b_2)a = (b_1, b_2, a) + b_1(b_2a) = -(a, b_2, b_1) + b_1(b_2a) = -(ab_2)b_1 + a(b_2b_1) + b_1(b_2a)$ implies $B^2A \subseteq A^2B + B^2$. Analogously one has $A^2B \subseteq B^2A + B^2$, and so $AB^2 + B^2 = B^2A + B^2$.

Now from (2.4), $(b_1, b_2a_1, a_2) - (b_1, b_2, a_1a_2) + (b_1, b_2a_1) - (b_1, a_1a_2)b_2 = 0$, we obtain $(B^2A)A \subseteq B^2A + B^2$; whence we have $(B^2A + B^2)A \subseteq B^2A + B^2$. Next (2.4) and (2.3) together give $(a_1, a_2b_1, b_2) - (a_1, a_2, b_2b_1) + (a_1, a_2, b_2) + (b_2, b_1, a_2) = 0$. Since we have just shown $(B^2A)A \subseteq B^2A + B^2 = A^2B + B^2$, we have $A(AB^2) \subseteq A^2B + B^2$; whence $A(A^2B + B^2) \subseteq A^2B + B^2$. Thus $A^2B + B^2 = B^2A + B^2$ is an ideal of $A$.

To show $B^3$ an ideal of $A$, one needs to show $[(b_1b_2)a_3, (b_1b_2)a_3, a_3b_3, a_3b_3] \in B^3$. From (2.4), $(b_1b_2a_3, a_3) - (b_1b_2, a_3a_3) + (b_1b_2, a_3) = 0$, we first obtain $[(b_1b_2)a_3b_3] \in B^3$ or $(B^2A)B \subseteq B^3$. This and (2.3) then give

$$[(b_1b_2)a]b_3 = [(b_1b_2, a) + b_1(b_2a)]b_3 = [-a, b_1, b_2] + b_1(b_2a)b_3$$

or $(A^2B)B \subseteq B^3$. Similarly $[(b_1b_2a)a_3] = (b_1b_2, a, a_3) + (b_1b_2)(a_3a_3) = -(b_3, a, b_1b_2) + (b_1b_2)(a_3)$ implies $B(A^2B) \subseteq B^3$, which with (2.3) in turn gives

$$b_3[a(b_1b_2)] = b_3[-a, b_1, b_2] + (ab_1)b_2 = b_3[(b_2, b_1, a) + (ab_1)b_2]$$

or $B(B^2A) \subseteq B^3$.

We are now ready to show $B^3$ an ideal of $A$. Since we have just verified that $B(B^2A)$ and $(B^2A)B$ are contained in $B^3$, from (2.1), $(b_1b_2, b_3, a) + (b_1, b_2, b_3, a) = b_1(b_2, b_3, a) + (b_1, b_3, a)b_2$, we have $[(b_1b_2)b_3a] \in B^3$. This and (2.3) then give
\[(b_2 b_3)b_1 a = (b_2 b_3, b_1, a) + (b_2 b_3)(b_1 a) = -(a, b_1, b_2 b_3) + (b_2 b_3)(b_1 a)\] or that \[a[b_1(b_2 b_3)] \in B^3.\] Next, since we have now shown \((A B^2) B\) and \(A B B^2\) to be contained in \(B^3\), from (2.4) we obtain \((a, b_1, b_2) - (a, b_1, b_3) + (a, b_1, b_2) b_3 - (a, b_2, b_3)b_1 = 0\) or \(a[(b_2 b_3)b_1] = -(a, b_1, b_2, b_3) + [a(b_2 b_3)]b_1 = (b_1, b_2, b_3, a) + [a(b_2 b_3)]b_1,\) whence \([b_1(b_2 b_3)]a \in B^3.\] This completes the proof of the lemma.

Now, as in the case for standard algebras in [14], let \(B\) be any ideal in \(A\), a generalized alternative algebra II. We define \(B^{(i)}\) inductively by \(B^{(0)} = B, B^{(i+1)} = A(B^{(i)})^2 + (B^{(i)})^2.\) By Lemma 2.3 this gives a descending chain \(B^{(0)} \supseteq B^{(1)} \supseteq \cdots \supseteq B^{(k)} \supseteq \cdots\) of ideals of \(A\) which we call a Penico sequence. We shall call \(B\) Penico solvable in case there is some integer \(k \geq 0\) for which \(B^{(k)} = 0.\)

**Lemma 2.4.** Let \(A\) be a generalized alternative algebra II over a field \(F.\) An ideal \(B\) of \(A\) is Penico solvable if and only if \(B\) is solvable.

**Proof.** If \(B\) is Penico solvable, then \(B\) is clearly solvable since \(B^{(i)} \supseteq B^{(i)}.\) On the other hand, suppose for any ideal \(B\) of \(A\) one has \(B^{(2)} \subseteq B^{(1)}\). Then, as in the proof of Theorem 3 in [14], induction shows \(B^{(2k)} \subseteq B^{(k)},\) since \(B^{(2k+1)} = (B^{(2k)})^2 \subseteq (B^{(2k)})^{(1)} \subseteq (B^{(k)})^{(1)} = B^{(k+1)}.\) Hence, if \(B\) is solvable, then \(B^{(k)} = 0\) for some \(k,\) that is \(B\) is Penico solvable. Now by definition \(B^{(2)} = A(A B^2 + B^2)^2 + (A B^2 + B^2)^2.\) Since \(B^3\) is an ideal of \(A,\) to show \(B^{(2)} \subseteq B^3\) it suffices to verify \((A B^2 + B^2)^2 = (A B^2)(A B^2) + (A B^2)B^2 + B^2(A B^2) + B^2B^2 \subseteq B^3.\) But, since \(B\) an ideal of \(A\) implies that \(A B^2\) and \(B^2\) are contained in \(B,\) one has \((A B^2)B^2, B^2(A B^2),\) and \(B^2B^2\) contained in \(B^3.\) Furthermore, since it has been demonstrated in the proof of Lemma 2.3 above that \((A B^2)B \subseteq B^3,\) one has \((A B^2)(A B^2) \subseteq (A B^2)B \subseteq B^3.\) Thus for any ideal \(B\) of \(A\) we have \(B^{(2)} \subseteq B^3 \subseteq B = B^{(1)},\) and the proof of the lemma is now complete.

**Lemma 2.5.** Let \(A\) be a generalized alternative algebra II over a field \(F\) of characteristic \(\neq 2.\) If \(A\) contains an idempotent \(e,\) then the ideal \(I = (A, e, e)\) satisfies \([A, I] = A_{1/2} I = (A_{1/2})^2 I = 0.\)

**Proof.** As observed in (i), Kleinfeld has shown \(I\) to be an ideal of \(A\) such that \(I^2 = 0.\) We also make use of the following observations. From (2.3) it follows that \((A, e, e) = I = (e, e, A)\) and \((A, e, e) = (e, e, x) + (e, e, x)e = e(e, e, x) + (e, e, x)e \in I \subseteq A_{1/2} .\) In particular, since (2.7) implies \(e(e, e, x) = (e, e, x)e,\) we have

\[(2.v) ek = \frac{1}{k} k = ke \text{ for } k \in I.\]

Next let \((e, y, z) = a_1 + a_{1/2} + a_0 \text{ and } (e, e, [y, z]) = b_{1/2} \text{ where } a_i, b_i \in A_i \text{ for } i = 0, \frac{1}{2}, 1.\] Then (2.1) yields \((e, y, z) + (e, e, [y, z]) = e(e, y, z) + (e, y, z)e \text{ or } a_1 + a_{1/2} + a_0 + b_{1/2} = a_1 + ea_{1/2} + a_1 + a_{1/2} e,\) whence \(a_1 = a_0 = b_{1/2} = 0\) or

\[(2.w) (e, e, [y, z]) = 0,\]

\[(2.x) (e, y, z) \in A_{1/2} \text{ for } y, z \in A.\]

Since from [2] we know \(y, z \in A_{1/2}\) implies \(y \circ z \in A_1 + A_0,\) we also have \(0 = (e, e, yz + zy) + (e, e, yz - zy) = 2(e, e, yz)\) or
(2.7) \( (e,e,yz) = 0 \) for \( y, z \in A_{1/2} \).

Suppose now we are given \( x \in A \). Let \( x = x_1 + x_{1/2} + x_0 \) where \( x_i \in A_i \) for \( i = 0, 1, 2 \). Then using (2.5), (2.3), (2.7), and (2.8) one has for \( i = 0, 1 \) and \( k \in I \) that \( x_i k = x_i (e,e,4k) + (e,e,x_i)(4k) = 4(e,e,x_i k) = 4(e,e,kx_i) = (4k)(e,e,x_i) + (e,e,4k)x_i = kx_i \). Also using (2.7) and the fact that \( I^2 = 0 \), one has in addition that \( 0 = (e,e,x_{1/2})(k = x_{1/2} (e,e,k) + (e,e,x_{1/2})k = x_{1/2} (e,e,k) = \frac{1}{2} k x_{1/2} \). Thus \( [A, I] = 0 \) and, in particular, \( A_{1/2} I = 0 \).

Next let \( x, y \in A_{1/2} \) and \( k \in I \). Then (2.1) gives \( (xy,e,k) + (x,y,[e,k]) = x(y,e,k) + (x,e,k)y \). But \( [A, I] = 0 \) implies \( (x,y,[e,k]) = 0 \), while \( I \) an ideal of \( A \) with \( A_{1/2} I = 0 \) implies \( x(y,e,k) = 0 \). Hence (xy,e,k) = 0. Let \( xy = a_1 + a_{1/2} + a_0 \) where \( a_i \in A_i \) for \( i = 0, 1 \). Then \( 0 = (xy,e,k) = [(xy)e]k - \frac{1}{2} (xy)k = (a_1 + a_{1/2}e)k - \frac{1}{2} a_1 k - \frac{1}{2} a_0 k = a_1 k - \frac{1}{2} a_0 k, \) using the fact from [7] that for noncommutative Jordan algebras \( A_{1/2} A_i, A_i A_{1/2} \subseteq A_{1/2} \) for \( i = 0, 1 \). Thus we have shown

(2.2) \( (xy)x_k = (xy)x_0k \) for \( x, y \in A_{1/2} \) and \( k \in I \).

Continuing as above we have \( (e,x,y) = (e) y - e(xy) = (ex)y - a_i - ex_{1/2} \).

Since, by (2.8), \( (e,x,y) \in A_{1/2} \), this gives \( [(ex)y]_1 = a_1 \) and \( [(ex)y]_0 = 0 \). Then \( a_1 + a_{1/2} + a_0 = xy = (ex)y + (xe)y \) implies \( [(xe)y]_1 = 0 \). Thus \( (ex)y \in A_1 + A_{1/2} \) while \( (xe)y \in A_{1/2} + A_0 \). Now since from [7], as noted above, we know \( xe \in A_{1/2} \), (2.2) gives \( [(xe)y]_1 k = [(xe)y]_0 k \). But \( [(xe)y]_1 = 0 \), so \( [(xe)y]_1 k = 0 \) \( = [(xe)y]_0 k \). Hence \( [(xe)y]_1 k = [(xe)y]_0 k + [(xe)y]_{1/2} k + [(xe)y]_0 k = 0 \), since \( A_{1/2} I = 0 \). In similar fashion we have \( [(ex)y]_1 k = 0 \). But then \( x, y \in A_{1/2} \) gives \( (xy)k = [(x + ex)y]k = [(xe)y]_0 k + [(xe)y]_1 k = 0 \) or \( (A_{1/2})^2 = 0 \). This completes the proof of the lemma.

In proving the next theorem, we will use [8, Lemma 2.1, Theorem 2.1, and Theorem 2.2]. We make note that the exclusion in these results of characteristic 3 is not necessary [16].

**Theorem 2.4 (Wedderburn principal theorem).** Let \( A \) be a finite-dimensional generalized alternative algebra II over a field \( F \) of characteristic \( \neq 2 \), and let \( N \) be the nil radical of \( A \). If \( A/N \) is separable, then \( A = S + N \) (vector space direct sum) where \( S \) is a subalgebra of \( A \) such that \( S \cong A/N \).

**Proof.** As in the proof of Theorem 23 on p. 47 of [1], it suffices to prove that \( A \) contains a subalgebra \( S \cong A/N \). Since our result is true trivially for \( N = 0 \) or \( N = A \), it is certainly true if \( A \) has dimension one. We make an induction on the dimension of \( A \) and assume the result true for algebras of dimension less than that of \( A \).

From the proof of Theorem 23 on page 47 of [1], it now also follows that one may assume \( N \) does not properly contain an ideal of \( A \). Thus we may argue, as in the proof of Theorem 4 of [14], that \( N^2 = 0 \), for suppose \( N^{(k)} = N \). Since \( N \) is solvable by Theorem 2.2, \( N \) is Peano solvable by Lemma 2.4. Hence \( N = N^{(1)} = N^{(2)} = \cdots = N^{(k)} = 0 \) for some \( k \), and our result is immediate.
Since, by Lemma 2.3, \( N^{(1)} \subseteq N \) is an ideal of \( A \), we must then have \( 0 = N^{(1)} = AN^2 + N^2 \), that is \( N^2 = 0 \).

At this point, an argument analogous to that used for Jordan algebras on page 289 of [3] shows one may also assume the field \( F \) to be algebraically closed.

Suppose next that \( A/N \) is not a simple algebra. If \( B \) is a nodal subalgebra of \( A/N \), then from [11] we know that \( B \) has a homomorphic image which is a simple nodal algebra. Since our Theorem 2.3 denies this possibility, we have from Theorem 4 of [11] that \( A/N \) semisimple implies \( A/N = B_1 + \cdots + B_t \) (algebra direct sum) where each \( B_i \) is a simple ideal. Since Theorem 3 of [5] and our Theorem 2.3 imply each \( B_i \) is alternative, each \( B_i \) must have a unity element. Furthermore, from [7] we know \( A \) a noncommutative Jordan algebra implies that \( A_1 \) and \( A_0 \) are subalgebras of \( A \) for any idempotent \( e \in A \). From Theorem 2.1 in [8] it now follows that it will suffice to consider the case \( A/N \) a simple algebra.

As a final reduction we note, as in the proof of Theorem 2.2 of [8], that if there exists a primitive idempotent \( e \) such that our result holds for the ideal \( H \) generated by \( A_{1/2} \), then it holds for \( A \) as well.

Now \( A/N \) not nil implies by Proposition 3.3 on p. 32 of [13] that \( A/N \) contains a nonzero idempotent \( e' \). Should this be the only nonzero idempotent in \( A/N \), then \( e' \) is a unit element for \( A/N \), and Theorem 2.3 implies \( A/N = Fe' \). By Lemma 2.1 in [8], \( e' \) lifts to an idempotent \( e \in A \), and so we have \( Fe \) a subalgebra of \( A \) such that \( Fe \equiv A/N \). Hence we may assume that \( A/N \) contains a nontrivial idempotent \( e' \). Again \( e' \) lifts to an idempotent \( e \in A \). In particular, \( e \) must be nontrivial and, since \( A \) is finite-dimensional, one may assume that \( e \) is primitive.

We now let \( I = (e,e,A) \). By (i) and (2.3), \((e,e,A) = I = (A,e,e) \) is an ideal of \( A \) such that \( I^2 = 0 \). Since, as earlier observed, we may assume \( N \) not to properly contain an ideal of \( A \), we must have either \( I = 0 \) or \( I = N \).

If we suppose first that \( I = 0 \), then \( A \) has a Peirce decomposition relative to \( e \), since, by (2.2), \((e,A,e) = 0 \). Let \( w_{ij}, x_{ij}, y_{ij}, z_{ij} \in A_{ij} \) for \( i,j = 0 \) or 1 and consider \( H = A_{10}A_{01} + A_{10} + A_{01} + A_{01}A_{10} \). Using the multiplication table described by (ii), it follows that to show \( H \) an ideal of \( A \) it suffices to show \( A_{10}A_{01} \) an ideal of \( A_{11} \) and \( A_{01}A_{10} \) an ideal of \( A_{00} \). Using (2.3) and the multiplication table described by (ii), one can compute as follows:

\[
(x_{10}y_{01})z_{11} = (x_{10},y_{01},z_{11}) + x_{10}(y_{01}z_{11})
\]

\[
= -(z_{11},y_{01},x_{10}) + x_{10}(y_{01}z_{11})
\]

\[
= x_{10}(y_{01}z_{11}) \in A_{10}A_{01}
\]

and

\[
z_{11}(x_{10}y_{01}) = -(z_{11},x_{10},y_{01}) + (z_{11}x_{10})y_{01}
\]

\[
= (y_{01},x_{10},z_{11}) + (z_{11}x_{10})y_{01}
\]

\[
= (z_{11}x_{10})y_{01} \in A_{10}A_{01}.
\]
Thus \( A_0 A_0 \) is an ideal of \( A_{11} \). Similarly one may show \( A_0 A_{10} \) to be an ideal of \( A_{00} \), and hence \( H \) is an ideal of \( A \). In particular, \( H \) must be the ideal generated by \( A_{1/2} = A_{10} + A_{01} \).

Now \( H \) a proper ideal implies by the induction hypothesis that our result is valid for \( H \). Thus our final reduction applies, and we may conclude that our result is valid for \( A \) itself. On the other hand, should \( H = A \) then \( A_{11} = A_{10} A_0 \) and \( A_{00} = A_0 A_{10} \). Using (2.3) and the multiplication table described by (ii), we have

\[
[x_{11}(y_{10}z_{01})]_{w11} = -[(x_{11}, y_{10}, z_{01}) + (x_{11}, y_{10})z_{01}]_{w11}
\]

\[
= [(z_{01}, y_{10}, x_{11}) + (x_{11}, y_{10})z_{01}]_{w11} = [(x_{11}, y_{10})z_{01}]_{w11}
\]

\[
= (x_{11}, y_{10}, z_{01}, w_{11}) + (x_{11}, y_{10})(z_{01}, w_{11})
\]

\[
= -(w_{11}, z_{01}, x_{11}, y_{10}) + (x_{11}, y_{10})(z_{01}, w_{11}) = (x_{11}, y_{10})(z_{01}, w_{11})
\]

\[
= (x_{11}, y_{10}, z_{01}, w_{11}) + x_{11}[y_{10}(z_{01}, w_{11})]
\]

\[
= -(z_{01}, w_{11}, y_{10}, x_{11}) + x_{11}[y_{10}(z_{01}, w_{11})] = x_{11}[y_{10}(z_{01}, w_{11})]
\]

\[
= x_{11}[-(y_{10}, z_{01}, w_{11}) + (y_{10}, z_{01}, w_{11})]
\]

\[
= x_{11}[(w_{11}, z_{01}, y_{10}) + (y_{10}, z_{01}, w_{11})]
\]

\[
= x_{11}[(y_{10}z_{01})]_{w11}.
\]

Since \( A_{11} = A_{10} A_0 \), these calculations show \( A_{11} \) to be associative. Similarly one may show \( A_{00} \) to be associative. If one then joins the calculations on p. 337 of [5], one may conclude that \( A \) itself is an alternative algebra. But then from [9] our result is known to be valid for \( A \), and the induction is complete.

Consider now the second alternative, namely \( I = N \), and take \( k = (e, e, e) \neq 0 \). We recall that, since \( A \) is noncommutative Jordan, one has from [7] that \( A_{1/2} A_i, A_i A_{1/2} \subseteq A_{1/2} \) for \( i = 0, 1 \). In particular, this says that \( N = I = (e, e, e, A) \subseteq A_{1/2} \). Let \( H \) be the ideal in \( A \) generated by \( A_{1/2} \), then \( H = A_{1/2} + (A_{1/2})^2 \). To see this, let \( x_i, y_i, z_i \in A_i \) for \( i = 0, 1/2, 1 \). Then for \( i = 0, 1 \) we have \( x_{1/2} y_{1/2} z_i = (x_{1/2}, y_{1/2}, z_i) \) \( + x_{1/2} (y_{1/2} z_i) = (x_{1/2}, y_{1/2} + z_i, y_{1/2} + z_i) - (x_{1/2}, y_{1/2}, y_{1/2}) \)

\[
- (x_{1/2}, z_i, z_i) - (x_{1/2}, z_i, y_{1/2}) + x_{1/2} (y_{1/2} z_i) \text{ is in } N + (A_{1/2})^2 \subseteq A_{1/2} + (A_{1/2})^2,
\]

since \( A/N \) simple implies as before that \( A/N \) is alternative or that \( (a, b, b) \in N \) for all \( a, b \in A \). Similarly one has \( z_i x_{1/2} y_{1/2} z_i \in A_{1/2} + (A_{1/2})^2 \) for \( i = 0, 1 \). Since the cases for \( i = 1/2 \) are immediate if one writes \( x_{1/2} y_{1/2} = a_1 + a_{1/2} + a_0 \) where \( a_i \in A_i \), we have \( H = A_{1/2} + (A_{1/2})^2 \) as claimed. Now, by Lemma 2.5, \( Hk = 0 \), while by (2.v) of Lemma 2.5 \( ek = 1/2 k \neq 0 \). Thus, since \( e \notin H \), \( H \) is a proper ideal of \( A \). Our final reduction now applies to complete the induction and the proof of the theorem.

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