TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 198, 1974

INITIAL-BOUNDARY VALUE PROBLEMS FOR HYPERBOLIC SYSTEMS IN REGIONS WITH CORNERS. II(1)

BY
STANLEY OSHER(2)

ABSTRACT. In the previous paper in this series we obtained conditions equivalent to the validity of certain energy estimates for a general class of hyperbolic systems in regions with corners. In this paper we examine closely the phenomena which occur near the corners if these conditions are violated. These phenomena include: the development of strong singularities (lack of existence), travelling waves which pass unnoticed through the corner (lack of uniqueness), existence and uniqueness if and only if additional conditions are imposed at the corner, and weak solutions which are not strong solutions. We also systematically analyze the conditions for certain important problems. We discuss the physical and computational significance of these results.

I. Introduction. We shall continue our study of initial-boundary value problems for certain hyperbolic partial differential equations in regions with corners. The study began in [5]. We obtained a simple counterexample to well-posedness in [7], and we constructed exact solutions to certain physically oriented problems in [8], and with Kupka in [4].

In this paper we are primarily concerned with the conditions equivalent to a priori estimates, derived in [5]. These conditions are rather complicated; hence we shall analyze them in some detail for certain specific problems. Moreover, when they are violated, certain new phenomena occur. These phenomena include the development of singularities, travelling wave solutions to the homogenous problem with zero initial data, weak solutions which are not strong, and existence and uniqueness iff additional conditions are imposed at the corner.

Our examples in §II are physically oriented. In particular the wave equation and the linearized shallow water equations are discussed. The equations of meteorology are usually hyperbolic, with artificial boundaries put in for computational purposes. These boundaries and corners lead to numerical instabilities, e.g [1], hence the necessity of this theory.

The author would like to thank Professors Björn Engquist and Heinz-Otto Kreiss for many helpful discussions of these problems.

II. Simple examples.

Example 1. Consider the system of equations

Received by the editors April 17, 1973.

AMS (MOS) subject classifications (1970). Primary 35L50, 35L30; Secondary 78A45.

Key words and phrases. Hyperbolic equations, initial boundary conditions, energy estimates, existence and uniqueness.

(1) Research partly supported by NSF grant GP-29.273.

(2) Fellow of the Alfred P. Sloan Foundation.

$$u_t = -c_1 u_x + d_1 u_y, \qquad v_t = c_2 v_x - d_2 v_y, \tag{2.1}$$

with c_1, c_2, d_1, d_2, all positive, to be solved in the region $x, y, t > 0$ for the complex valued functions u, v.

We have initial conditions

$$u(x,y,0) = \Phi(x,y), \qquad v(x,y,0) = \psi(x,y), \tag{2.2}$$

and boundary conditions

$$\begin{aligned} u(0,y,t) &= av(0,y,t) + f(y,t), \\ v(x,0,t) &= bu(x,0,t) + g(x,t), \end{aligned} \tag{2.3}$$

where a and b are arbitrary complex numbers. Denote by $\|u(\ ,t)\|$ the usual L_2 norm over $x, y > 0$, and $\|f(\ ,t)\|_{B_1}$, $\|g(\ ,t)\|_{B_2}$, those taken over $x = 0$ and $y = 0$ respectively with inner products defined analogously. We seek an estimate for the solution at time $t = T > 0$:

$$\begin{aligned} &\|u(\ ,T)\|^2 + \|v(\ ,T)\|^2 \\ &\qquad + C_1 \int_0^T (\|u(\ ,t)\|_{B_1}^2 + \|v(\ ,t)\|_{B_1}^2)\,dt \\ &\qquad + C_2 \int_0^T (\|u(\ ,t)\|_{B_2}^2 + \|v(\ ,t)\|_{B_2}^2)\,dt \\ &\le K(T)(\|\Phi\|^2 + \|\psi\|^2) \\ &\qquad + C_3 \int_0^T \|f(\ ,t)\|^2 dt + C_4 \int_0^T \|g(\ ,t)\|^2 dt. \end{aligned} \tag{2.4}$$

Here, and in what follows, C_j will denote a universal positive constant, $K(T)$ a positive constant depending only on T.

Theorem 2.1. *Estimate* (2.4) *is valid iff*

$$|ab| < \sqrt{(c_2/d_2)(d_1/c_1)} = \zeta^{1/2}.$$

If we weaken our estimate by requiring only that $C_1 = C_2 = 0$ *and considering only* $f \equiv 0 \equiv g$, *then the estimate is valid iff* $|ab| \le \zeta^{1/2}$.

Proof. The "if" part of the proof is straightforward and deferred to the end. We are primarily interested in the counterexamples we obtain when the hypothesis is violated.

Case A. $\zeta > 1$. We consider the pair of functions

$$\begin{aligned} u(x,y,t) &= (d_1 x + c_1 y)^\beta p(t + \gamma_1 x + \gamma_2 y), \\ v(x,y,t) &= b(d_1 x + d_1 c_1 y/d_2)^\beta p(t + \gamma_1 x + \gamma_2 y), \end{aligned} \tag{2.5}$$

with $\beta = -\ln ab/\ln \zeta$; hence $\operatorname{Re} \beta < -\frac{1}{2}$ if $|ab| > \zeta^{1/2}$ and

$$0 < \gamma_1 = \frac{1/c_1 + d_1/c_1 d_2}{\zeta - 1}, \qquad 0 < \gamma_2 = \frac{1/d_2 + c_2/c_1 d_2}{\zeta - 1}.$$

We choose for $p(x)$ any bounded, smooth function with support in the interval $0 < \varepsilon_1 \leq x \leq \varepsilon_2 < \infty$, positive for $\varepsilon_1 < x < \varepsilon_2$. Thus $p(t + \gamma_1 x + \gamma_2 y)$ is a wave which travels into the corner from the interior of the region as t increases. The pair of functions u and v obey (2.1), (2.2), (2.3) with Φ and ψ smooth, bounded, with compact support, and f and g both zero. As $t \to \varepsilon_1$, the estimate (2.4) fails because a singularity develops at the origin. Even if we weaken our estimate by permitting $C_1 = C_2 = 0$ for homogenous boundary conditions, the estimate still fails. If $|ab| = \zeta^{1/2} > 1$, then the weaker estimate is valid but the pair of functions fail to be square integrable on the boundaries as $t \to \varepsilon_1$. Hence, this is not a strong solution to the boundary value problem in the sense of [11], but it is an M-weak solution. This means that if we integrate by parts in the usual fashion against a pair of smooth, bounded functions satisfying the adjoint homogenous boundary conditions, we obtain the desired result. It is also an M-strong solution. This means first that $\exists$ a sequence of smooth bounded pairs of functions whose interior L_2 norms converge to the solution. Also the interior L_2 norm of the differential operator and the boundary L_2 norm of the boundary operator applied to these functions both converge to the appropriate values. This situation occurs, in fact, as long as $-1 < \operatorname{Re} \beta$, or $|\zeta| > |ab| \geq |\zeta|^{1/2}$. These are the values of β for which u and v remain square integrable for all $t > 0$ in the interior.

Case B. If $\zeta < 1$, we consider the same class of functions with the same β, γ_1, γ_2, and $p(x)$. This time, however, if $|ab| > \zeta^{1/2}$, then $\operatorname{Re} \beta > -\frac{1}{2}$, and γ_1 and γ_2 are both negative. Hence the function $p(t + \gamma_1 x + \gamma_2 y)$ represents a wave travelling into the region from the exterior passing through the corner. Thus the solution to the homogenous problem need not only be the zero vector. If $-1 > \operatorname{Re} \beta \geq -\frac{1}{2}$, i.e., $|\zeta| < |ab| \leq \zeta^{1/2}$, the resulting solution is again M-weak, M-strong, but not strong. The estimate (2.4) fails for C_1, $C_2 \neq 0$ with $|ab| = \sqrt{\zeta}$. We can see that by taking as initial data $u(x, y, \varepsilon_2 + \delta)$, $v(x, y, \varepsilon_2 + \delta)$ and letting $\delta \searrow 0$. If $\zeta < |ab| < \zeta^{1/2}$, we have the interesting phenomena of existence and uniqueness of weak and strong solutions, but the nonuniqueness of M-weak and M-strong solutions.

We notice that Cases A and B can be viewed as adjoints of each other with $\zeta \to 1/\zeta$, if we merely integrate by parts over the space region.

Case C. $\zeta = 1$. This case is invariant under the taking of adjoints. We consider the pair of functions:

$$\begin{aligned} u &= (ab)^{((t - x/c_1)/(d_1 x + c_1 y))c_1 d_1 d_2/(d_2 + d_1)} p(d_1 x + c_1 y), \\ v &= b(ab)^{-d_2/(d_1 + d_2)} \cdot (ab)^{((t - y/d_2)/(d_1 x + c_1 y))c_1 d_1 d_2/(d_2 + d_1)} p(d_1 x + c_1 y), \end{aligned} \tag{2.6}$$

where p is as in Case A.

If $|ab| > 1$, we see that as $\varepsilon_1 \downarrow 0$, we have a sequence of smooth solutions with bounded initial data whose norms explode exponentially for any positive time. Thus none of the estimates (2.4) are possible. Because of this exponential blow up, this problem is ill-posed in any reasonable Sobolev space with weight at the corner. If $|ab| = 1$, then we consider the pair of functions

$$\tilde{u} = u \cdot \left[\frac{t - x/c_1}{d_1 x + c_1 y}\right], \qquad \tilde{v} = v \cdot \left[\frac{t - y/d_2}{d_1 x + c_1 y}\right]. \tag{2.7}$$

Then $\tilde{u}$ and $\tilde{v}$ obey (2.1)–(2.3) with ψ, Φ, f and g bounded and smooth, but any estimate (2.4), even with $C_1 = C_2 = 0$, is impossible. Notice that if $f = g = 0$ then the estimate is true with $C_1 = C_2 = 0$, as we shall prove below.

In equation (2.1), we multiply the first equation by $k_1 \bar{u}$, the second by $k_2 \bar{v}$, where k_1 and k_2 are positive numbers to be determined below, then integrate by parts over all space, using the boundary conditions (2.2). We have

$$\begin{aligned} &\frac{d}{dt}\int_0^\infty \int_0^\infty (k_1 |u|^2 + k_2 |v|^2)\, dx\, dy \\ &\qquad = \int_0^\infty k_1 c_1 |u(0,y,t)|^2 dy - \int_0^\infty k_2 c_2 |v(0,y,t)|^2 dy \\ &\qquad\quad - \int_0^\infty k_1 d_1 |u(x,0,t)|^2 dx + \int_0^\infty k_2 d_2 |v(x,0,t)|^2 dx. \end{aligned} \tag{2.8}$$

We then use the boundary conditions. The "if" part of the main theorem follows simply from Schwarz' inequality. We merely choose k_1, k_2 so that

$$k_1 c_1 |a|^2 - k_2 c_2 \le 0, \qquad k_2 d_2 |b|^2 - k_1 d_1 \le 0, \tag{2.9}$$

with $<$ holding if $|ab| < \zeta^{1/2}$, $=$ holding if $|ab| = \zeta^{1/2}$.

The physical interpretation of Case C is rather simple. As t increases we have a wave propagating back and forth from one wall to the other along the lines $d_1 x + c_1 y =$ constant. Each time it does so, it increases in energy by a factor $|ab|$. Thus, as the time of increase diminishes (as the constant $\searrow 0$), energy blows up with arbitrarily high speed.

In Case A, again a wave propagates from one wall to the next, first along $d_2 x + c_2 y =$ constant, then along $d_1 x + c_1 y = \text{constant}/c_2$, then $d_2 x + c_2 y = \text{constant}/c_2 d_1$ and so on. After each bounce, the function value along the characteristic increases in absolute value by a factor $|ab|$. We must then see if the L_2 energy increases after one complete reflection. Of course this is a matter of comparing $|ab|$ to the ratio of the slope of the characteristics $(c_2/d_2)(d_1/c_1)$. We wish the resulting explosion to be sufficiently mild.

In Case B, if we follow the same procedure as in A, we go away from the origin. The lack of uniqueness occurs because energy may slip through the corner undetected, not being killed off unless $|ab|/\zeta^{1/2} < 1$.

We notice that in Cases A and C the problem would be ill-posed if $|ab| > \zeta^{1/2}$

even if we specified the values of u and v at the corner. In Case B, the solution is unique with this additional condition. We prove this uniqueness in Theorem 3.1.

Example 2. Consider the single equation

$$u_t = c(x,y)u_x + d(x,y)u_y \tag{2.10}$$

to be solved in all space, $-\infty < x, y < \infty$, and $t > 0$, except at the axes $x = 0$, $y = 0$.

We have initial conditions

$$u(x,y,0) = \Phi(x,y). \tag{2.11}$$

In each of the four quadrants, c and d are different pairs of nonzero constants.

$$\begin{array}{llll} \text{If} & x, y > 0, & c(x,y) \equiv -c_1, & d(x,y) \equiv d_1; \\ \text{if} & x < 0, y > 0, & c(x,y) \equiv -c_2, & d(x,y) \equiv -d_2; \\ \text{if} & x < 0, y < 0, & c(x,y) \equiv c_3, & d(x,y) \equiv -d_3; \\ \text{if} & x > 0, y < 0, & c(x,y) \equiv c_4, & d(x,y) \equiv d_4. \end{array} \tag{2.12}$$

All the c_i, d_i are positive.

We impose the condition that u be continuous across the axes.

We seek an estimate for the solution at time $t = T > 0$.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u(x,y,T)|^2 dx\, dy \leq K(T) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\Phi(x,y)|^2 dx\, dy. \tag{2.13}$$

Theorem 2.2. *Estimate* (2.13) *is valid iff* $1 \leq (d_1/c_1)(c_2/d_2)(d_3/c_3)(c_4/d_4)$.

Proof. If the condition is violated, then the function

$$\begin{array}{ll} u = p(t - a_1 x - b_1 y), & x, y > 0, \\ u = p(t + a_2 x - b_1 y), & x < 0, y > 0, \\ u = p(t + a_2 x + b_2 y), & x < 0, y < 0, \\ u = p(t - a_1 x + b_2 y), & x > 0, y > 0, \end{array} \tag{2.14}$$

where

$$\begin{array}{ll} a_1 c_1 - b_1 d_1 = 1, & a_2 c_3 - b_2 d_3 = 1, \\ -a_2 c_2 + b_1 d_2 = 1, & -a_1 c_4 + b_2 d_4 = 1 \end{array}$$

satisfy (2.10), (2.11) and the jump condition. Here $\Phi \equiv 0$. It can be easily shown that a_1, a_2, b_1, b_2 are all positive iff $1 > (d_1/c_1)(c_2/d_2)(d_3/c_3)(c_4/d_4)$. We take $p(x)$ to be the same as in (2.5).

Thus the solution is identically zero for $t \leq \varepsilon_1$. As t increases past ε_1, we have a diamond shaped wave in the x, y plane which emerges from the origin. If

$1 \le (d_1/c_1)(c_2/d_2)(d_3/c_3)(c_4/d_4)$ the proof of well-posedness is essentially the same as in the previous example.

We again have a simple interpretation. We follow a signal emanating from the positive x axis along characteristics around the origin. If it lands back on the x axis to the right of its point of incidence we have nonuniqueness of solutions; otherwise the solution is unique. Again we see that energy can slip through the origin iff the slope condition is violated.

Example 3. Consider the wave equation in two dimensions

$$u_{xx} + u_{yy} - u_{tt} = F(x,y,t) \tag{2.15}$$

to be solved in the region $x, y, t > 0$, with initial conditions

$$u(x,y,0) = \Phi(x,y), \qquad u_t(x,y,0) = \psi(x,y), \tag{2.16}$$

and boundary conditions

$$\begin{aligned} u_x &= \alpha u_y + f(y,t), \qquad x = 0, \\ u_y &= \beta u_x + g(x,t), \qquad y = 0, \end{aligned} \tag{2.17}$$

where α and β are real numbers. We begin by analyzing the following special case.

Case A. Let $\beta = 1/\alpha$.

Theorem 2.3. *A unique classical solution exists for C^∞ data with bounded support if $\alpha < 0$. If $\alpha > 0$, then a unique classical solution exists iff we impose the additional condition that $u(0,0,t)$ be given. If $\alpha < 0$, then this additional condition will overdetermine the problem and a solution will not in general exist.*

Proof. We use the classical trick of considering the function

$$v(x,y,t) = u_x(x,y,t) - \alpha u_y(x,y,t). \tag{2.18}$$

Then v satisfies

$$\begin{gathered} v_{xx} + v_{yy} - v_{tt} = F_x - \alpha F_y, \\ v(x,y,0) = \Phi_x - \alpha\Phi_y, \qquad v(0,y,t) = f(y,t), \\ v_t(x,y,0) = \psi_x - \alpha\psi_y, \qquad v(x,0,t) = -\alpha g(x,t). \end{gathered} \tag{2.19}$$

We next apply a Laplace transform in time, then solve the Dirichlet problem for the inhomogenous reduced wave equation in a quadrant. The Green's function for this problem can be obtained by reflecting across the axes. Thus we can construct the unique solution. The inverse transform is easily shown to exist and to be smooth.

To obtain u we merely solve equation (2.18) under the assumption that $u \to 0$ as $x^2 + y^2 \to \infty$. The solution is unique modulo a function of the form $M(\alpha x + y, t)$.

M must satisfy

$$(2.20) \qquad \begin{aligned} &(\alpha^2 + 1)M_{11} - M_{22} = 0, \\ &M(z,t) = q(\sqrt{\alpha^2 + 1}\,t + z) + r(\sqrt{\alpha^2 + 1}\,t - z), \end{aligned}$$

with $M(\alpha x + y, 0) \equiv M_t(\alpha x + y, 0) \equiv 0$ for $t = 0$. Hence

$$(2.21) \qquad \begin{aligned} 0 &\equiv q(\alpha x + y) + r(\alpha x + y), \\ 0 &\equiv q'(\alpha x + y) - r'(\alpha x + y). \end{aligned}$$

If $\alpha > 0$, we merely let $q(x) \equiv 0$ for $x > 0$, and let $r(x) = p(x)$ for $p(x)$ defined in (2.5). Thus, again uniqueness fails because of this travelling wave. If we prescribed $u(0,0,t)$ we would have uniquely determined the function $r(z,t)$, and hence uniquely determined the solution $u(x,y,t)$. If $\alpha < 0$, then in the first quadrant the quantity $ax + y$ takes all real values. Thus (2.21) is possible iff $q \equiv r \equiv 0$. If in this case we were to prescribe $u(0,0,t)$ to be zero, then there would exist no solution to (2.15), (2.16), (2.17) with this additional condition, for $F = 0 = f = g$, $\Phi = p(\alpha x + y)$, $\psi = \sqrt{\alpha^2 + 1}\,p'(\alpha x + y)$. This is true because the unique solution without the additional condition is $p(\alpha x + y + \sqrt{\alpha^2 + 1}\,t)$. Hence for $t > \varepsilon_1/\sqrt{\alpha^2 + 1}$, the function is not zero at the corner.

Physically we can think of u as the amplitude of a stretched membrane constrained to vibrate over the first quadrant with the derivative in the direction $[1, -\alpha]$ given at the boundary. If we rounded the edge of the membrane, the directional derivative would become tangential at some point iff $\alpha > 0$. Thus the additional condition is needed in this case to make the "rounded" problem physically and mathematically well-posed.

Case B. α and β arbitrary real numbers.

Theorem 2.4. *Consider the reduced wave equation*

$$(2.15') \qquad u_{xx} + u_{yy} - s^2 u = F$$

to be solved for $x, y > 0$.

We take $s = \eta + i\xi$ *with* $\eta^2 > \xi^2$. *The boundary conditions are of the form* (2.17). *Then we have the a priori estimate*

$$(2.22) \qquad \begin{aligned} &\|u_x(\cdot,0)\|^2 + \|u_y(\cdot,0)\|^2 + (\eta^2 - \xi^2)\|u(\cdot,\cdot)\|^2 \\ &\qquad + \|u(0,\cdot)\|_{B_1}^2 + \|u(\cdot,0)\|_{B_2}^2 - (\alpha + \beta)|u(0,0)|^2 \\ &\le [\|F(\cdot,\cdot)\|^2 + \|f\|_{B_1}^2 + \|g\|_{B_2}^2]C, \end{aligned}$$

C depends only on $\eta^2 - \xi^2$, α *and* β.

Proof. Multiply (2.15′) by $\bar{u}$, integrate by parts over all space, then use the boundary conditions. We have at first

$$(u_x, u_x) + (u_y, u_y) + s^2(u,u) + (u,u_x)_{B_1} + (u,u_y)_{B_2} = -(u,F) \tag{2.23}$$

or, taking real parts and using the boundary conditions,

$$\begin{aligned}(u_x, u_x) + (u_y, u_y) + (\eta^2 - \xi^2)(u,u) - (\alpha+\beta)(u(0,0))^2 \\ = -(u,f)_{B_1} - (u,g)_{B_2} - (u,F).\end{aligned} \tag{2.24}$$

We use Schwarz' inequality.

$$\begin{aligned}\text{The left side above} \leq \varepsilon_4 \|u\|^2 + \varepsilon_4^{-1}\|F\|^2 \\ + \varepsilon_5(\|u\|_{B_1}^2 + \|u\|_{B_2}^2) \\ + \varepsilon_5^{-1}(\|f\|_{B_1}^1 + \|f\|_{B_2}^2)\end{aligned} \tag{2.25}$$

$\forall \varepsilon_4, \varepsilon_5 > 0.$

Finally, we estimate

$$\begin{aligned}(u,u)_{B_1} + (u,u)_{B_2} &= -2\operatorname{Re}(u,u_x) - 2\operatorname{Re}(u,u_y) \\ &\leq 4(u,u) + 2(u_x,u_x) + 2(u_y,u_y).\end{aligned} \tag{2.26}$$

The result follows directly.

Example 4. We consider the nonstrongly hyperbolic system,

$$\begin{aligned}u_t &= -u_x + u_y + v_x, \\ v_t &= -v_x + v_y, \\ w_t &= w_x - w_y,\end{aligned} \tag{2.27}$$

to be solved for $x, y, t > 0$ with boundary conditions

$$\begin{aligned}u(0,y,t) &= aw(0,y,t), \\ v(0,y,t) &= bw(0,y,t), \\ w(x,0,t) &= cu(x,0,t) + dv(x,0,t),\end{aligned} \tag{2.28}$$

for a, b, c, d real numbers, with initial conditions

$$\begin{aligned}u(x,y,0) &= \Phi(x,y), \\ v(x,y,0) &= \psi(x,y), \\ w(x,y,0) &= \chi(x,y).\end{aligned} \tag{2.29}$$

Theorem 2.5. *Let*

$$bc < 0, \tag{2.30}$$

$$ac + db - 1 > bc. \tag{2.31}$$

Then $\exists$ *a sequence of initial data*

$$\{\Phi_n\}_1^\infty, \quad \{\psi_n\}_1^\infty, \quad \{\chi_n\}_1^\infty, \tag{2.32}$$

with L_p *norm one for any* p, $1 \le p \le \infty$, *for which the resulting* u_n, v_n, w_n *have the property*

$$u_n = e^{nt}\Phi_n, \quad v_n = e^{nt}\psi_n, \quad w_n = e^{nt}\chi_n, \tag{2.33}$$

for any $t > 0$.

Thus we have an exponential explosion of the type associated with either the Cauchy problem for elliptic equations, or half-space hyperbolic problems which violate the weak Kreiss condition [3].

This situation arises here because energy reflects back and forth from the two walls along $x + y = \text{const}$ in arbitrary small time, close to the corner. Each complete reflection results in the loss of one derivative.

Proof. Define the functions

$$\begin{aligned} &K_n(t - x, x + y) \\ &= C_n e^{n(t-x)} e^{2n(x+y)} \exp\left(\frac{1}{bc}\int_0^{x+y}\left(\frac{e^{2rn} - 1}{r}\right) dr\right)(x + y)^{[(1-ac-db)/bc]}, \end{aligned} \tag{2.34}$$

where the C_n act as normalizing constants at $t = 0$.

We let

$$\begin{aligned} w_n(x,y,t) &= K_n(t - y, x + y), \\ v_n(x,y,t) &= bK_n(t - 2x - y, x + y), \\ u_n(x,y,t) &= aK_n(t - 2x - y, x + y) + xb(\partial/\partial x)K_n(t - 2x - y, x + y). \end{aligned} \tag{2.35}$$

The result now follows by inspection.

Example 5. We consider the equations governing linearized shallow water flow:

$$\begin{aligned} u_t + u_0 u_x + v_0 u_y + \eta_x &= 0, \\ v_t + u_0 v_x + v_0 v_y + \eta_y &= 0, \\ \eta_t + u_0 \eta_x + \eta_0 u_x + v_0 \eta_y + \eta_0 v_y &= 0, \end{aligned} \tag{2.36}$$

to be solved in the region $x, y, t > 0$, with initial conditions

$$u(x,y,0) = \Phi(x,y),$$
$$v(x,y,0) = \psi(x,y), \tag{2.37}$$
$$\eta(x,y,0) = \chi(x,y),$$

and boundary conditions specified below.

Here u and v represent the components of velocity in the x and y directions, and η represents the magnitude of the height of the wave.

Numerical calculations recently performed in Sweden by Elvius and Sundström [1] show that boundary conditions must be selected carefully. We present here examples of seemingly reasonable boundary conditions which lead to either nonuniqueness or nonexistence of solutions. Of course, in all our examples both here and above, the corresponding half-plane problems obey the weak Kreiss conditions [3], and are hence at least weakly well-posed on L_2. This means that their solutions may lose a finite number of derivatives for all time.

Thus we impose the boundary conditions

$$B_1 \begin{pmatrix} u \\ v \\ \eta \end{pmatrix}_{x=0} = 0, \quad B_2 \begin{pmatrix} u \\ v \\ \eta \end{pmatrix}_{y=0} = 0, \tag{2.38}$$

with the following specifications:

(2.39) (a) $\eta > \max(u_0^2, v_0^2)$.

(b) B_1 has two linearly independent rows if $u_0 > 0$, one if $u_0 < 0$.

(c) B_2 has two linearly independent rows if $v_0 > 0$, one if $v_0 < 0$.

(d) (i) If $u_0 > 0$, then, for all s, γ with $|s|^2 + |\gamma|^2 = 1$,

$$\left| \det B_1 \begin{bmatrix} u_0 i\gamma & -\xi_+ \\ s + v_0 i\gamma & i\gamma \\ 0 & u_0\xi_+ - s - v_0 i\gamma \end{bmatrix} \right| \geq \delta > 0,$$

where

$$\xi_+ = \frac{-u_0(s + v_0 i\gamma) + \sqrt{u_0^2(s + v_0 i\gamma)^2 + (\eta_0 - u_0^2)[(s + v_0 i\gamma)^2 + \eta_0 \gamma^2]}}{(\eta_0 - u_0^2)}$$

with $s = K + i\omega$, ω real, $K > 0$, γ real and $\operatorname{Re} \xi_+ > 0$.

(ii) If $u_0 < 0$, then, for all s, γ as above,

$$\left| \det B_1 \begin{bmatrix} -\xi_+ \\ i\gamma \\ u_0\xi_+ - s - v_0 i\gamma \end{bmatrix} \right| \geq \delta.$$

(e) (i) If $v_0 > 0$, then, for all s, γ as above,

$$\left|\det B_2\begin{bmatrix} u_0 i\gamma + s & i\gamma \\ v_0 i\gamma & -\tilde{\xi}_+ \\ 0 & -u_0 i\gamma - s + v_0\tilde{\xi}_+ \end{bmatrix}\right| \geq \delta$$

where $\tilde{\xi}_+(u_0, v_0, \eta_0) = \xi_+(v_0, u_0, \eta_0)$.

(ii) If $v_0 < 0$, then, for all s, γ as above,

$$\left|\det B_2\begin{bmatrix} i\gamma \\ -\tilde{\xi}_+ \\ -u_0 i\gamma - s + v_0\tilde{\xi}_+ \end{bmatrix}\right| \geq \delta.$$

Theorem 2.6. *Given the two Kreiss half-space conditions* (2.39) *for this problem* (2.36), (2.37), (2.38), *then the following corner problems are ill-posed*:

($\tilde{a}$) *because of a lack of uniqueness of solutions,*

($\tilde{b}$) *because of a lack of existence of solutions.*

We shall state and prove part ($\tilde{a}$) first.

($\tilde{a}$) (i) B_1 and B_2 both annihilate some fixed vector of the form

$$\begin{pmatrix} a \\ -b \\ 0 \end{pmatrix} \text{with } a, b > 0 \text{ and } \quad 1 = bu_0 + av_0.$$

For example

$$u_0, v_0 > 0 \quad \text{and} \quad B_1 = B_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(ii) B_1 and B_2 both annihilate some fixed vector of the form

$$\begin{pmatrix} K_1 \\ K_2 \\ \sqrt{\eta_0}\sqrt{K_1^2 + K_2^2} \end{pmatrix} \text{ or } \begin{pmatrix} K_1 \\ K_2 \\ -\sqrt{\eta_0}\sqrt{K_1^2 + K_2^2} \end{pmatrix},$$

where K_1 and K_2 are positive with $1 - K_1 u_0 - K_2 u_0 = \pm\sqrt{\eta_0(K_1^2 + K_2^2)}$, e.g., $u_0, v_0 < 0$, $B_1 = B_2 = [v_0, -u_0, 0]$, $\eta_0 > u_0^2 + v_0^2$.

Proof of part (a). We seek a solution to (2.36) with zero initial data of the form

$$\begin{pmatrix} u \\ v \\ \eta \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} p(t - K_1 x - K_2 y), \tag{2.40}$$

where $p(x)$ is the same as in (2.5), and $K_1, K_2 > 0$.

This function must obey the homogenous boundary conditions (2.38). We have a linear system of equations

$$(2.41)\qquad \begin{bmatrix} 1-K_1u_0-K_2v_0 & 0 & -K_1 \\ 0 & 1-K_1u_0-K_2v_0 & -K_2 \\ -K_1\eta_0 & -K_2\eta_0 & 1-K_1u_0-K_2v_0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The determinant condition yields either $1 = K_1u_0 + K_2v_0$, in which case

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} K_2 \\ -K_1 \\ 0 \end{bmatrix}$$

or $1 - K_1u_0 - K_2v_0 = \pm\sqrt{\eta_0}\sqrt{K_1^2+K_2^2}$ in which case

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} K_1 \\ K_2 \\ \pm\sqrt{\eta_0}\sqrt{K_1^2+K_2^2} \end{bmatrix}.$$

The proof is immediate.

We may rewrite (2.36) in an obvious fashion:

$$(2.42)\qquad U_t = AU_x + BU_y.$$

We Laplace transform in time, letting s be the dual variable

$$(2.43)\qquad s\hat{U} = A\hat{U}_x + B\hat{U}_y$$

with boundary conditions (2.38). Under the previous hypothesis, the function

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \exp(-s(K_1x + K_2y))$$

is an eigenfunction of this problem. If we integrate by parts over the x,y-space, it follows that the problem

$$(2.44)\qquad sU = -A^TU_x - B^TU_y + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \exp(-s(K_1x + K_2y))$$

with boundary conditions

$$(2.45)\qquad U(0,y,s) \perp N_{AB_1}, \qquad U(x,0,s) \perp N_{BB_2}$$

has no solution on L_2 of the quarter-plane. (N denotes the null space of the matrices in question.) Make the change of variables

$$(2.46)\qquad U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/\eta_0 \end{bmatrix} V = DV.$$

Thus the problem

(2.47) $$sV = -D^{-1}A^TDV_x - D^{-1}B^TDV_y + \begin{pmatrix} c_1 \\ c_2 \\ -\eta_0 c_3 \end{pmatrix} \exp(-s(K_1 x + K_2 y))$$

has no solution with boundary conditions

(2.48) $$V(0,y,s) \perp N_{DAB_1}, \qquad V(x,0,s) \perp N_{DBB_2}.$$

We merely relabel u_0, v_0 by $-u_0$, $-v_0$, and we have an equation of the type (2.42). Then we inverse Laplace transform, and use Duhamel's principle, obtaining the result.

($\tilde{\text{b}}$) There exists in general no solution to (2.36) (2.37), (2.38) if we impose boundary conditions of the type (2.48), with u_0, v_0 replaced by $-u_0$, $-v_0$,in part ($\tilde{\text{a}}$). For example, let $u_0, v_0 < 0$ and take boundary conditions

$$u_0 u - u_0 v - \eta = 0 \qquad \text{at } x = 0,$$

$$v_0 v - v_0 u + \eta = 0 \qquad \text{at } y = 0,$$

or let $u_0, v_0 > 0$ with boundary conditions

$$\left.\begin{aligned} \eta_0 u - u_0 \eta &= 0 \\ -u_0 u - v_0 v + \eta &= 0 \end{aligned}\right\} \quad \text{at } x = 0,$$

and

$$\left.\begin{aligned} \eta_0 v - v_0 \eta &= 0 \\ -u_0 v - v_0 v + \eta &= 0 \end{aligned}\right\} \quad \text{at } y = 0.$$

This last set of boundary conditions is extremely reasonable if we are linearizing around a set of values for which $\eta_0 = u_0^2 + v_0^2 + \varepsilon_1$. In spite of this the linearized problem will have a solution which explodes at the origin.

III. Analysis of the corner condition, certain simple necessary and sufficient conditions for well-posedness. We briefly review here the new condition for well-posedness of the corner problem which we derived in [5].

Consider the system with constant coefficients

(3.1) $$AU_x + BU_y + \sum_{j=3}^{n} C_j U_{z_j} + DU - U_t = F(x,y,z,t),$$

to be solved in the region $x, y, t > 0$, $-\infty < z_j < \infty$, $j = 3, 4, \ldots, n$, $z = (z_3, z_4, \ldots, z_n)$, for the complex-valued m-vector U. We take initial conditions $U \equiv 0$ for $t = 0$. We assume the system is either strictly hyperbolic or obeys a somewhat weaker condition, i.e., Assumption (2.3) of [5]. Moreover $\det A \neq 0 \neq \det B$. The boundary conditions are of the form

(3.2) $$\begin{aligned} U^{\mathrm{I}} &= SU^{\mathrm{II}} + f \qquad \text{at } x = 0, \\ U^{\mathrm{III}} &= RU^{\mathrm{IV}} + g \qquad \text{at } y = 0. \end{aligned}$$

U^{I} and U^{II} are made up of those components of U which lie in the respectively negative and positive eigenspaces of A, U^{III} and U^{IV} are defined the same way for B. Thus R and S are rectangular constant matrices. We assume that each of the half-space problems are strongly Kreiss well-posed [3].

Finally, we define a new condition as follows. Let $D = 0$, $F = 0$, $g = 0$. Then Laplace transform the equation in time and Fourier transform it in each z_j. Let the dual variable of time be $s = \eta + i\xi$, $\eta > 0$, ξ real, the dual variable of each z_j be ω_j and $Ci\omega = \sum_{j=3}^{n} C_j i\omega_j$. The resulting problem that we must solve is

$$AU_x + BU_y + [Ci\omega - s]U = 0, \qquad x, y > 0, \tag{3.3}$$

$$U^{\mathrm{I}}(0,y,\omega,s) = SU^{\mathrm{II}}(0,y,\omega,s) + f(y,\omega,s), \tag{3.4}$$

$$U^{\mathrm{III}}(x,0,\omega,s) = RU^{\mathrm{IV}}(x,0,\omega,s). \tag{3.5}$$

We extend the function U to the whole upper half-plane

$$V \equiv U \quad \text{for} \quad x \geq 0, \qquad V \equiv 0 \quad \text{for} \quad x < 0. \tag{3.6}$$

Then V satisfies

$$AV_x + BV_y + [Ci\omega - s]V = A\delta(x-0)U(0,y,\omega,s) \tag{3.7}$$

in the upper half-plane, with $U(0,y,\omega,s)$ having the form (3.4), and V obeying (3.5) at $y = 0$ for all x. We can solve this equation uniquely for V because we have assumed that the upper half-space problem is Kreiss well-posed. The solution can be constructed by first applying a Fourier transform in x, then solving the resulting ordinary differential equation in y. The result will depend on the unknown vector $U^{\mathrm{II}}(0,y,\omega,s)$. We next define a function $W(x, y, \omega, s)$ in the right half-plane by

$$W = V \quad \text{for} \quad y \geq 0, \qquad W = 0 \quad \text{for} \quad y < 0. \tag{3.8}$$

Then W satisfies

$$AW_x + BW_y + (Ci\omega - s)W = B\delta(y-0)V(x,0,\omega,s) \tag{3.9}$$

in the right half-plane, with $V(x, 0, \omega, s)$ having the form (3.5) and W obeying (3.4) at $x = 0$, for all y. We then solve this right half-plane problem in an analogous manner as the upper half-plane problem. We thus obtain $W(x, y, \omega, s)$. If a solution to (3.3), (3.4), (3.5) exists, it must be such that

$$W^{\mathrm{II}}(0,y,\omega,s) = U^{\mathrm{II}}(0,y,\omega,s). \tag{3.10}$$

From our construction of W, it follows that $\exists$ linear operators depending on ω, s for which

$$W^{\mathrm{II}}(0, \cdot, \omega, s) = T_{\omega,s} U^{\mathrm{II}}(0, \cdot, \omega, s) + \tilde{T}_{\omega,s} f(\cdot, \omega, s). \tag{3.11}$$

We thus have the corner condition:

(3.12) (*Corner condition.*) $(I - T_{\omega,s})$ is uniformly invertible for all s with $\operatorname{Re} s > 0$, all real ω.

We have deliberately not defined the precise Hilbert spaces into which the inverse of this operator maps $L_2[0, \infty]$. We are merely interested in constructing and analyzing the operator here.

Example 6.

$$AU_x + BU_y - U_t = F(x,y,t) \tag{3.13}$$

with $U(x,y,0) = 0$,

$$A = \begin{pmatrix} -c_1 & 0 & & & & & 0 \\ 0 & -c_2 & & & & & 0 \\ & & \ddots & & & & \\ & & & -c_l & & & \\ & & & & c_{l+1} & & \\ & & & & & \ddots & \\ 0 & & & \cdots & & & c_m \end{pmatrix},$$

(3.14)

$$B = \begin{pmatrix} d_1 & 0 & & & & & 0 \\ 0 & d_2 & & & & & 0 \\ & & \ddots & & & & \\ & & & d_l & & & \\ & & & & -d_{l+1} & & \\ & & & & & \ddots & \\ 0 & & & \cdots & & & -d_m \end{pmatrix}.$$

Each, c_i, d_1 is positive. Thus

$$\begin{aligned} U^{\mathrm{I}} &= U^{\mathrm{IV}} = (u_1, u_2, \ldots, u_l), \\ U^{\mathrm{II}} &= U^{\mathrm{III}} = (u_{l+1}, \ldots, u_m), \end{aligned} \tag{3.15}$$

$S = [S_{ij}]$ is an $l \times (m-l)$ complex matrix, $R = [R_{ij}]$ is an $(m-l) \times l$ complex matrix.

We take the convention that $l \leq m - l$. If this were false, we would merely switch the roles of x and y.

We may follow the procedure used in the case $m = 2$, $l = 1$, in the proof of the if part of Theorem 2.1. Thus we show that the problem is well-posed in the sense of Theorem 2.1 and in the sense of the main theorem of [5], if $\exists$ a sequence of positive numbers $(K_1, K_2, \cdots, K_l)$ such that, if we define

$$K^{\mathrm{I}} = \begin{pmatrix} K_1 & 0 & \cdots & 0 \\ 0 & K_2 & & \\ \vdots & & \ddots & \\ 0 & & & K_l \end{pmatrix}, \qquad C^{\mathrm{I}} = \begin{pmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & & \\ \vdots & & \ddots & \\ 0 & & & c_l \end{pmatrix},$$

$$\tag{3.16} C^{\mathrm{II}} = \begin{pmatrix} c_{l+1} & 0 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & & c_m \end{pmatrix}, \qquad D^{\mathrm{I}} = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & & d_l \end{pmatrix},$$

$$D^{\mathrm{II}} = \begin{pmatrix} d_{l+1} & 0 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & & d_m \end{pmatrix},$$

then

$$(D^{\mathrm{I}})^{-1/2} R^t (D^{\mathrm{II}})^{1/2} S^t (C^{\mathrm{I}})^{1/2} K^{\mathrm{I}} (C^{\mathrm{I}})^{1/2} S (C^{\mathrm{II}})^{1/2} R (D^{\mathrm{I}})^{-1/2} \le K^{\mathrm{I}}.$$

In Theorem 2.1, we have shown that this condition is necessary and sufficient for $m = 2$, $l = 1$. This is false in this general case.

We now improve this condition by examining $I - T_s$.

The general solution to the Laplace transform of (3.13) with $F \equiv 0$ is

$$\tag{3.17} \begin{aligned} u_j &= \exp(-sx/c_j) g_j(d_j x + c_j y), \qquad j = 1, 2, \ldots, l, \\ u_j &= \exp(-sy/d_j) g_j(d_j x + c_j y), \qquad j = l+1, \ldots, m. \end{aligned}$$

The conditions (3.4) and (3.5) become

$$\tag{3.18} \begin{aligned} g_j(c_j y) &= \sum_{K=l+1}^{m} S_{jK} \exp(-sy/d_K) g_K(c_K y) + f_j(y), \qquad j = 1, 2, \ldots, l, \\ g_K(d_K x) &= \sum_{r=1}^{l} R_{Kr} \exp(-sx/c_r) g_r(d_r x), \qquad K = l+1, \ldots, m. \end{aligned}$$

Thus

$$\tag{3.19} \begin{aligned} g_j(y) = \sum_{r=1}^{l} \sum_{K=l+1}^{m} S_{jK} R_{Kr} \exp\left(-sy\left[\frac{1}{c_j d_K} + \frac{c_K}{c_r c_j d_K}\right]\right) g_r\left(\frac{c_K d_r}{d_K c_j} y\right) + f_j\left(\frac{y}{c_j}\right), \\ j = 1, 2, \ldots, l. \end{aligned}$$

Hence we have T_s.

In Example 1, we had $l = n - l = 1$, $S_{11} = a$, $R_{11} = b$, we were concerned with the equation

$$g_1(y) = ab \exp\left(-sy\left[\frac{1}{c_1 d_2} + \frac{c_2}{c_1^2 d_2}\right]\right) g_1(\zeta y) + f_1\left(\frac{y}{c_1}\right). \tag{3.20}$$

However,

$$\int_0^\infty |g_1(\zeta y)|^2 dy = \frac{1}{\zeta}\int_0^\infty |g_1(y)|^2 dy. \tag{3.21}$$

Thus, if $|ab| < \sqrt{\zeta}$, the operator $I - T_s$ is uniformly invertible on $L_2[0, \infty]$. If on the other hand, $|ab| > \sqrt{\zeta}$, then:

(B) If $\zeta < 1$, the function

$$g_1(y) = \exp\left(-\frac{sy}{1-\zeta}\left[\frac{1}{c_1 d_2} + \frac{c_2}{c_1^2 d_2}\right]\right) y^{-(\ln ab)/(\ln \zeta)}$$

is a square integrable solution to the equation (3.20) with $f_1 \equiv 0$.

Thus we have our counterexample to part B in Example 1, if we inverse Laplace transform this.

(A) If $\zeta > 1$, the adjoint of $(I - T_s)$ is of the type in part B.

Thus $(I - T_s)$ is not onto, and the original problem has in general no solution. Moreover $(I - T_s)$ has a non-square-integrable eigenfunction of the type in part B above. If we inverse Laplace transform this we are led to the counterexample to existence of part A, Example 1.

(C) If $\zeta = 1$, then we can solve the equation

$$ab \exp\left(-sy\left[\frac{1}{c_1 d_2} + \frac{c_2}{c_1^2 d_2}\right]\right) = 1$$

for values of $sy = K_0$ with real part positive. We merely consider $g_1(y)$ to be the approximate eigenfunction $\delta(s - K_0/y)$. We are then led to the counterexample in part C of Example 1, by again applying the Laplace transform.

If $|ab| = \zeta$, we merely examine our previous counterexample in some detail in order to finish off the problem.

The situation is much more complicated when $l > 1$. However we have some partial results.

Theorem 3.1. *If all the numbers $c_K d_r/d_K c_j$, for $r = 1, 2, \cdots m, j = 1, 2, \cdots, m$, $K = l+1, \ldots, m$, for which $S_{jk} R_{Kr} \neq 0$ are less than one, then the operator $(I - T_s)$ is not uniformly invertible if the equation*

$$\det\left(\sum_{K=l+1}^{n} S_{jK} R_{Kr}\left(\frac{c_K d_r}{c_j d_K}\right)^B - \delta(j-r)\right) = 0 \tag{3.22}$$

has roots B, with real $B \geq -\frac{1}{2}$.

If such roots exist for Re $B > -\frac{1}{2}$, *then the original initial boundary value problem has nonzero solutions for zero data.*

Suppose there are such roots with Re $B > -\frac{1}{2}$. Let B_0 be one with the largest real part, and let

$$M^0 = (M_1^0, M_2^0, \ldots, M_l^0) \tag{3.23}$$

be an associated eigenfunction of the above matrix. Then we construct an eigenfunction satisfying the equation (3.19) of the form

$$g(y) = y^{B_0} \sum_{j=0}^{\infty} M^j (sy)^j \tag{3.24}$$

in a neighbourhood $0 \leq |sy| < \delta$ of the origin. Plug this into (3.19) with each $f_j \equiv 0$. We have the recursion relationship

$$\begin{aligned} &\left(M_j^p - \sum_{r=1}^{l} \sum_{k=l+1}^{m} S_{jK} R_{Kr} \left(\frac{c_K d_r}{d_K c_j}\right)^{p+B_0} M_r^p\right) \\ &= \sum_{r=1}^{l} \sum_{k=l+1}^{m} S_{jK} R_{Kr} \sum_{r=1}^{p} (-1)^r \frac{[1/c_j d_K + c_K/c_r d_j a_K]^r}{r!} (c_K d_r/d_K c_j)^{p-r+B_0} M_r^{p-r}, \\ &\qquad j = 1, 2, \ldots, l. \end{aligned} \tag{3.25}$$

We can thus solve for each M^p in terms of $M^0, M^1, \ldots, M^{p-1}$. All that remains is to prove the convergence. First we note that by a simple transformation which replaces y by yb for $b > 0$ in (3.19), we can replace the exponentials which appear there by any constant multiple of them. Then it is easy to prove inductively

$$|M_j^p| \leq CK^p p^p/p!, \quad \text{for } C > 0, 0 < K < 1. \tag{3.26}$$

Convergence is thus proven. We then have $g(y)$ in this neighbourhood. We use the equation (3.19) to define it on the whole line. The result follows.

Suppose next there is a root B_0 with real part equal to $-\frac{1}{2}$, and no other root with real part $> -\frac{1}{2}$. Consider next the functions

$$g_\varepsilon(y) = y^{B_0+\varepsilon} \sum_{j=0}^{\infty} M_\varepsilon^j (sy)^j, \quad \text{for } \varepsilon > 0, \tag{3.27}$$

with $M_\varepsilon^0 = M^0$, and the other coefficients defined recursively as above. Then in some neighbourhood $0 \leq |sy| < \delta$, for δ independent of ε, we have

$$[I - T_s]g(y) = C(\varepsilon) y^{B_0+\varepsilon} M^0 \tag{3.28}$$

with $C(\epsilon) \longrightarrow 0$ as $\epsilon \longrightarrow 0$. We then define $g_\epsilon(y)$ for all $y > 0$ using equation (3.19) for $|sy| > \delta/2$.

Thus the functions $g_\varepsilon(y)$ satisfy

$$\|(I - T_s)g_\varepsilon(y)\|/\|g_\varepsilon(y)\| \to 0 \quad \text{as } \varepsilon \to 0. \tag{3.29}$$

Theorem 3.2. *If all the numbers $c_K d_r/d_K c_j$, for $r = 1, 2, \ldots, l, j = 1, 2, \ldots, l$, $K = l + 1, \ldots, m$, for which $S_{jK} R_{Kr} \neq 0$, are greater than one, then the operator $(I - T_s)$ is not uniformly invertible if the equation*

$$\det\left(\sum_{k=l+1}^{n} S_{jK} R_{Kr}\left(\frac{c_K d_r}{c_j d_K}\right)^B - \delta(j - r)\right) = 0$$

has roots B, with $\operatorname{Re} B \leq -\frac{1}{2}$.

If such roots exist for $\operatorname{Re} B < -\frac{1}{2}$, then the original initial-boundary value problem has no solution for certain initial data.

Proof. Merely take the adjoint of $(I - T_s)$. We then have an operator obeying the hypotheses of Theorem 3.1. The result follows.

We conjecture that the above two hypotheses are sufficient as well as necessary for invertibility of $I - T_s$. However, at this time, we can only prove a somewhat weaker result which includes more general cases.

Theorem 3.3. *Let $N(\omega)$ be the matrix*

$$\left[\delta(j - r) - \sum_{k=l+1}^{m} S_{jK} R_{Kr}\left(\frac{c_K d_r}{d_K c_j}\right)^{-i\omega - 1/2}\right] = N_{jr}(\omega) \tag{3.30}$$

for ω real, $-\infty < \omega < \infty$.

Then if $N^{-1}(\omega)$ exists with

$$\sup_\omega |||N^{-1}(\omega)||| < \inf \|Tg\|/\|g\| \quad \forall g \neq 0,$$

$(I - T_s)$ is uniformly invertible. Here $|||A|||$ is just the usual finite dimension Hilbert norm of a matrix

$$\|g\|^2 = \sum_{j=1}^{l} \int_0^\infty |g_j(y)|^2 dy$$

and

$$(Tg)_j(y) = \sum_{r=1}^{l} \sum_{k=l+1}^{m} S_{jK} R_{Kr}\left[\exp\left(-sy\left[\frac{1}{c_j d_K} + \frac{c_r}{c_K c_j d_K}\right]\right) - 1\right] g_r\left(\frac{c_K d_r}{d_K c_j} y\right). \tag{3.31}$$

Proof. Rewrite $(I - T_s)g = f$ as

$$\begin{aligned} g_j(y) &- \sum_{r=1}^{l} \sum_{k=l+1}^{m} S_{jK} R_{Kr} g_r\left(\frac{c_K d_r}{c_j d_K} y\right) \\ &- \sum_{r=1}^{l} \sum_{k=l+1}^{m} S_{jK} R_{Kr}\left[\exp\left(-sy\left[\frac{1}{c_j d_K} + \frac{c_r}{c_K c_j d_K}\right]\right) - 1\right] g_r\left(\frac{c_K d_r}{c_j d_K} y\right) \\ &= f_j(y). \end{aligned} \tag{3.32}$$

Then make the unitary transformation from $L_2(0, \infty)$ to $L_2(-\infty, \infty)$:

$$e^{z/2} g_j(e^z) = \tilde{g}_j(z). \tag{3.33}$$

Thus

$$g_j(y) = \tilde{g}_j(\ln y)/\sqrt{y},$$

$$g_j\left(\frac{c_K d_r}{c_j d_K} y\right) = \frac{\tilde{g}_j(\ln(c_K d_r/c_j d_K) + \ln y)}{\sqrt{(c_K d_r/c_j d_K) y}}.$$

Inner product (3.32) with $\bar{g}_j(y)$ and sum. Apply this unitary transformation on the first part of the left side, then apply a Fourier transform. The result follows immediately.

Theorem 3.4. *If all the numbers $c_K d_r/c_j d_K$, for $r = 1, 2, \ldots, l$, $j = 1, 2, \ldots, l$, for which $S_{jK} R_{Kr} \neq 0$ are equal to one, then the operator $(I - T_s)$ is uniformly invertible iff the matrix*

$$\left[\delta(j - r) - \sum_{r=1}^{l} \sum_{k=l+1}^{n} S_{jK} R_{Kr} \exp\left(-sy\left[\frac{1}{c_j d_K} + \frac{c_K}{c_r c_j d_K}\right]\right)\right] \tag{3.34}$$

is uniformly invertible for Re $sy > 0$. *If this condition fails for $sy = K_0$,* Re $K_0 > 0$, *then there exists a sequence of initial data for which the initial value problem blows up exponentially near the corner.*

Proof. We merely generalize the argument which followed equation (3.21) in the scalar case. The generalization is straightforward.

IV. Conclusions. If we wish to approximate any of these corner problems numerically, we must decide whether or not to impose some extra condition at the corner, or to extrapolate. We should first test for incoming waves as we did in the shallow water equation. If they exist, then the solution can be made unique by imposing extra conditions on the problem. If they exist for the adjoint problem, then some explosion is to be expected near the corner for the original problem, and perhaps we should approximate $r^p U$ for some $p > 0$, if no better boundary conditions are available.

We also mention here that recent work done jointly with Björn Engquist indicates a strong connection between boundary value problems whose boundary is characteristic at isolated points and corner problems.

This will be discussed by us in succeeding papers.

Bibliography

1. T. Elvius and A. Sundström, *Computationally efficient schemes and boundary conditions for a fine mesh barotropic model based on shallow water equations*, Tellus, **25** (1973), 132–156.

2. V. A. Kondrat'ev, *Boundary value problems for elliptic equations in conical regions*, Dokl. Akad. Nauk SSSR **153** (1964), 27–29 = Soviet Math. Dokl. **4** (1964), 1600–1602. MR **28** #1383.

3. H. -O. Kreiss, *Initial boundary value problems for hyperbolic systems*, Comm. Pure. Appl. Math. **23** (1970), 277–298.

4. I. A. K. Kupka and S. Osher, *On the wave equation in a multidimensional corner*, Comm. Pure. Appl. Math. **24** (1971), 381–393.

5. S. Osher, *Initial-boundary value problems for hyperbolic systems in regions with corners*. I, Trans. Amer. Math. Soc. 176 (1973), 141–165.

6.———, *A symmetrizer for certain hyperbolic mixed problems with singular coefficients*, Indiana J. Math. 22 (1973), 667–671.

7.———, *An ill posed problem for a hyperbolic equation near a corner*, Bull. Amer. Math. Soc. (to appear).

8.———, *On a generalized reflection principle and a transmission problem for a hyperbolic equation*, Indiana J. Math. 79 (1973), 1043–1044.

9. J. Ralston, *Note on a paper of Kreiss*, Comm. Pure. Appl. Math. 24 (1971), 759–762.

10. R. Sakomoto, *Mixed problem for hyperbolic equations*. I, II, J. Math. Kyoto Univ. **10** (1970), 349–373, 403–417. MR **44** #632a,b.

11. L. Sarason, *On weak and strong solutions of boundary value problems*, Comm. Pure Appl. Math. **15** (1962), 237–288. MR **27** #460.

DEPARTMENT OF COMPUTER SCIENCES, UPPSALA UNIVERSITY, UPPSALA, SWEDEN

Current address: Department of Mathematics, State University of New York at Stony Brook, Stony Brook, New York 11790