

## CONSISTENCY THEOREMS FOR ALMOST CONVERGENCE

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**ABSTRACT.** The concept of almost convergence of a sequence of real or complex numbers was introduced by Lorentz, who developed a very elegant theory. The purpose of the present paper is to continue Lorentz's investigations and obtain consistency theorems for almost convergence; this is achieved by studying certain locally convex topological vector spaces.

**1. Introduction** The concept of almost convergence of a sequence of real or complex numbers was introduced, after an idea of Banach, by Lorentz [13] who developed a very elegant theory. Further studies of almost convergence and its relationship with general summability methods have since been carried out in [12], [17] and [19]. The purpose of the present paper is to obtain consistency theorems for almost convergence by studying certain locally convex topological vector spaces.

We adopt the following notation:

$\omega$  denotes the space of all scalar (real or complex) sequences;

$e, e^{(k)} \in \omega$  are given by

$$e = (1, 1, \dots),$$

$$e^{(k)} = (0, \dots, 0, 1, 0, \dots) \text{ with the one in the } k\text{th position;}$$

$\varphi$  is the linear span of  $\{e^{(k)} : k = 1, 2, \dots\}$ ;

$m = \{x \in \omega : \|x\|_\infty = \sup_j |x_j| < \infty\}$ ;

$c = \{x \in \omega : \lim x = \lim_{j \rightarrow \infty} x_j \text{ exists}\}$ ;

$c_0 = \{x \in \omega : \lim x = 0\}$ ;

$l = \{x \in \omega : \|x\|_l = \sum_{j=1}^{\infty} |x_j| < \infty\}$ ;

$bv = \{x \in \omega : \|x\|_{bv} = \sum_{j=1}^{\infty} |x_j - x_{j+1}| + \lim_{j \rightarrow \infty} |x_j| < \infty\}$ ;

$bv_0 = bv \cap c_0$ ;

$bs = \{x \in \omega : \|x\|_{bs} = \sup_n |\sum_{j=1}^n x_j| < \infty\}$ .

A vector subspace of  $\omega$  is called a *sequence space*. If  $E$  is a sequence space with a locally convex topology  $\tau$  then  $(E, \tau)$  is a *K-space* provided that the linear functionals

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$$x \rightarrow x_j \quad (j = 1, 2, \dots)$$

are continuous on  $E$ . If, in addition,  $(E, \tau)$  is complete and metrizable (respectively normable) then  $(E, \tau)$  is called an *FK-space* (respectively *BK-space*). For  $x \in \omega$  we write

$$P_n x = (x_1, x_2, \dots, x_n, 0, \dots).$$

$(E, \tau)$  is an *AK-space* if  $P_n x$  converges to  $x$  for every  $x \in E$ .

If  $E$  and  $F$  are sequence spaces containing  $\varphi$  such that the bilinear form  $\langle x, y \rangle = \sum_{j=1}^{\infty} x_j y_j$  converges whenever  $x \in E$  and  $y \in F$ , then topologies of the dual pairing  $\langle E, F \rangle$  provide examples of  $K$ -space topologies. In particular, we shall be interested in the weak topology  $\sigma(E, F)$ , the Mackey topology  $\tau(E, F)$  and the strong topology  $\beta(E, F)$  (following the notation of Schaefer [18]).

We shall also consider matrix maps and matrix methods of limitation. Let  $A = (a_{ij})_{i,j=1}^{\infty}$  be an infinite matrix with scalar entries; we denote by  $\omega_A$  the set of  $x \in \omega$  such that  $\sum_{j=1}^{\infty} a_{ij} x_j$  converges for each  $i$ . For  $x \in \omega_A$  we write

$$(Ax)_i = \sum_{j=1}^{\infty} a_{ij} x_j$$

so that  $A: \omega_A \rightarrow \omega$  is a linear map. If  $E$  is a sequence space,

$$E_A = \{x \in \omega_A : Ax \in E\}.$$

If  $E$  is an *FK*-space then Zeller [24, Theorem 4.10(a)] has shown that  $E_A$  is also an *FK*-space when topologized by means of the seminorms:

$$x \rightarrow x_j \quad (j = 1, 2, \dots),$$

$$x \rightarrow \sup_n \left| \sum_{j=1}^n a_{ij} x_j \right| \quad (i = 1, 2, \dots),$$

and

$$x \rightarrow q(Ax),$$

where  $q$  runs through the continuous seminorms on  $E$ . A matrix  $A$  defines a method of limitation, viz: if  $x \in c_A$ , we write  $\lim_A x = \lim(Ax)$ .  $A$  is called *conservative* if  $c \subseteq c_A$  or, equivalently (see [26]),

$$(1) \quad \sup_i \sum_{j=1}^{\infty} |a_{ij}| < \infty,$$

$$(2) \quad \lim_{i \rightarrow \infty} a_{ij} = a_j \text{ exists} \quad (j = 1, 2, \dots),$$

and

$$(3) \quad \lim_{i \rightarrow \infty} \sum_{j=1}^{\infty} a_{ij} \text{ exists.}$$

We then write

$$\chi(A) = \lim_{i \rightarrow \infty} \sum_{j=1}^{\infty} a_{ij} - \sum_{j=1}^{\infty} a_j,$$

and say that  $A$  is *conull* when  $\chi(A) = 0$ .  $A$  is called *regular* if  $\lim_A x = \lim x$  whenever  $x \in c$ ; for regularity it is necessary and sufficient (see [26]) to have (1), (2) and (3) with  $a_j = 0$  ( $j = 1, 2, \dots$ ) and  $\chi(A) = 1$ .

**2. Properties of almost convergence.** In this section we develop the theory of almost convergence, deriving the original characterization of almost convergent sequences given by Lorentz [13], as well as several other useful properties of the space  $ac_0$  (to be defined below). Since our approach is from the viewpoint of functional analysis, and therefore differs slightly from Lorentz's, we shall give a complete development of the subject.

The linear functional  $\lim$  on  $c$  has norm one, i.e.

$$|\lim x| \leq \|x\|_{\infty} \quad (x \in c)$$

and so by the Hahn-Banach theorem possesses extensions  $L$ , of norm one, defined on all of  $m$ . We call such a functional  $L$  an *extended limit*. If  $x \in \omega$ , we write

$$Tx = \{x_{n+1}\}_{n=1}^{\infty}$$

and say that an extended limit  $L$  is a *Banach limit* if

$$L(Tx) = L(x) \quad (x \in m).$$

(Some authors insist that a Banach limit should also satisfy  $L(x) \geq 0$  whenever  $x_n \geq 0$  for all  $n$ , or even  $\lim_{n \rightarrow \infty} \sup x_n \geq L(x) \geq \lim_{n \rightarrow \infty} \inf x_n$ . It is clear, however, that any extended limit has these properties.)

The existence of Banach limits was proved by Banach [2]; another proof can be found in Theorem 1 below. If  $x \in m$  is such that for every Banach limit  $L$ ,  $L(x)$  assumes a common value, then we write  $\text{Lim } x$  for this value, and say that  $x$  is *almost convergent to*  $\text{Lim } x$ . The set of almost convergent sequences is denoted by  $ac$ , and the subset  $\{x \in ac : \lim x = 0\}$  is denoted by  $ac_0$ .  $ac_0$  is a hyperplane in  $ac$  and  $ac = ac_0 + \{e\}$ ; it is also easy to show that  $ac$  and  $ac_0$  are closed subspaces of  $m$ . Our first result (Theorem 1) characterizes these spaces.

**Lemma 1.** *If  $L$  is a continuous linear functional on  $m$  with*

- (i)  $\|L\| = 1$ ,
- (ii)  $L(e) = 1$ , and
- (iii)  $L(bs) = 0$ ,

*then  $L$  is a Banach limit.*

**Proof.** Since  $\varphi \subseteq bs$ , it follows from (iii) that  $L(\varphi) = 0$ , and by continuity that

$L(c_0) = 0$ ; therefore  $L$  is an extended limit. Moreover, for  $x \in m$ ,  $x - Tx \in bs$  and so  $L(x) = L(Tx)$ .

**Lemma 2.** *If  $x \in m \setminus c_0$ , then there exists an extended limit  $L$  with  $L(x) \neq 0$ .*

**Proof.** Since  $x \in m \setminus c_0$ , we may choose an increasing sequence  $\{n_k\}_{k=1}^{\infty}$  of positive integers such that

$$\lim_{k \rightarrow \infty} x_{n_k} = \alpha \neq 0.$$

Define  $L$  by

$$Ly = \lim_{k \rightarrow \infty} y_{n_k}$$

where this limit exists, and extend  $L$  to  $m$  by the Hahn-Banach theorem.

**Theorem 1 (Lorentz) [13].**  *$x \in \omega$  is almost convergent (to  $\alpha$ ) if and only if*

$$(4) \quad \lim_{p \rightarrow \infty} \frac{1}{p} (x_n + \cdots + x_{n+p-1}) = \alpha$$

*uniformly in  $n$ .*

**Proof.** Without loss of generality we may assume that  $\alpha = 0$ . Let  $\{n_p\}_{p=1}^{\infty}$  be any increasing sequence of positive integers, and define the matrix map  $A: m \rightarrow m$  by

$$(Ax)_p = \frac{1}{p} (x_{n_p} + \cdots + x_{n_p+p-1}) \quad (x \in m).$$

Then we have  $Ae = e$ ,  $A(bs) \subset c_0$ ,  $\|A\|_{\infty} = 1$ .

If  $L$  is an extended limit, then, by Lemma 1,

$$(5) \quad LA \text{ is a Banach limit}$$

and so, for  $x \in ac_0$ , we have

$$L(Ax) = 0.$$

By Lemma 2 we have  $Ax \in c_0$  so that

$$\lim_{p \rightarrow \infty} \frac{1}{p} (x_{n_p} + \cdots + x_{n_p+p-1}) = 0.$$

Since this is true for any sequence  $\{n_p\}_{p=1}^{\infty}$ , we conclude that

$$\lim_{p \rightarrow \infty} \sup_n \left| \frac{1}{p} (x_n + \cdots + x_{n+p-1}) \right| = 0,$$

which is (4).

Conversely, (4) implies that

$$\lim_{p \rightarrow \infty} \left\| \frac{1}{p} (Tx + \cdots + T^p x) \right\|_{\infty} = 0.$$

Thus, for any Banach limit  $L$ , we have  $L(x) = 0$ , so that  $x \in ac_0$ .

We remark that (5) gives what is perhaps the easiest proof of the existence of Banach limits. Banach's original proof [2] also uses the Hahn-Banach theorem, but involves a rather sophisticated sublinear functional; Day's elegant proof [9, p.83], using fixed point theory, requires considerably more machinery.

Our next result, which follows at once from Theorem 1, shows that  $ac_0$  and  $ac$  are "large" subspaces of  $m$ .

**Corollary.**  $(ac_0, \|\cdot\|_{\infty})$  is a nonseparable BK-space.

We now come to a series of results which relate various properties of  $ac_0$  to those of more familiar sequence spaces.

**Theorem 2.** If  $\{x^{(n)}\}_{n=1}^{\infty}$  is a sequence of points in  $l$ , and  $x \in l$ , then the following conditions are equivalent:

- (i)  $\{x^{(n)}\}_{n=1}^{\infty}$  is  $\sigma(l, ac_0)$ -convergent to  $x$ ;
- (ii)  $\{x^{(n)}\}_{n=1}^{\infty}$  is  $\sigma(l, bs + c_0)$ -convergent to  $x$ ;
- (iii)  $\sup_n \|x^{(n)}\|_1 < \infty$  and  $\lim_{n \rightarrow \infty} \|x^{(n)} - x\|_{bv} = 0$ .

**Proof.** Without loss of generality we may assume that  $x = 0$ .

(i)  $\Rightarrow$  (ii) follows since  $bs + c_0 \subset ac_0$ .

(ii)  $\Rightarrow$  (iii). If  $x^{(n)} \rightarrow 0$   $\sigma(l, bs + c_0)$ , then  $x^{(n)} \rightarrow 0$   $\sigma(l, c_0)$  so that

$$\sup_n \|x^{(n)}\|_1 < \infty.$$

Also,  $x^{(n)} \rightarrow 0$   $\sigma(l, bs)$  so that  $x^{(n)} \rightarrow 0$   $\sigma(bv_0, bs)$ ; this is the weak topology on  $bv_0$ , and, since  $bv_0$  is isomorphic to  $l$ , we may use Schur's theorem [2, p. 137] to deduce that

$$\lim_{n \rightarrow \infty} \|x^{(n)}\|_{bv} = 0.$$

(iii)  $\Rightarrow$  (i). Let  $f \in ac_0$  and  $\epsilon > 0$  be fixed. By Theorem 1 we may choose a positive integer  $p$  so that

$$\left\| \frac{1}{p} (Tf + \cdots + T^p f) \right\|_{\infty} < \epsilon/2 \left( 1 + \sup_n \|x^{(n)}\|_1 \right).$$

We then have, for every  $n$ ,

$$\begin{aligned} & \left| \sum_{k=1}^{\infty} \frac{1}{p} (f_{k+1} + \cdots + f_{k+p}) x_k^{(n)} \right| \\ & \leq \|x^{(n)}\|_1 \left\| \frac{1}{p} (Tf + \cdots + T^p f) \right\|_{\infty} < \frac{\epsilon}{2}. \end{aligned}$$

Furthermore, fixing  $p$ , we may choose a positive integer  $N$  so that

$$\|x^{(n)}\|_{bv} < \frac{\varepsilon}{2(p+1)(1 + \|f\|_\infty)}$$

whenever  $n \geq N$ .

Now

$$\begin{aligned} \left| \sum_{k=1}^{\infty} (f_{k+s} - f_k) x_k^{(n)} \right| &= \left| \sum_{k=1}^{\infty} f_k (x_{k-s}^{(n)} - x_k^{(n)}) \right| \quad (\text{putting } x_m^{(n)} = 0 \text{ if } m \leq 0) \\ &\leq \sum_{k=1}^{\infty} |f_k| \sum_{r=1}^s |x_{k-r+1}^{(n)} - x_{k-r}^{(n)}| \\ &\leq s \|f\|_\infty (\|x^{(n)}\|_{bv} + |x_1^{(n)}|) \\ &\leq 2s \|f\|_\infty \|x^{(n)}\|_{bv}. \end{aligned}$$

Therefore

$$\begin{aligned} \left| \sum_{k=1}^{\infty} \frac{1}{p} (f_{k+1} + \cdots + f_{k+p}) x_k^{(n)} - \sum_{k=1}^{\infty} f_k x_k^{(n)} \right| &\leq \frac{1}{p} \frac{p(p+1)}{2} 2 \|f\|_\infty \|x^{(n)}\|_{bv} \\ &< \frac{\varepsilon}{2} \quad \text{whenever } n \geq N. \end{aligned}$$

Thus, for  $n \geq N$ , we have

$$\left| \sum_{k=1}^{\infty} f_k x_k^{(n)} \right| < \varepsilon,$$

i.e.,  $x^{(n)} \rightarrow 0$   $\sigma(l, ac_0)$ .

We remark that condition (iii) of Theorem 2 identifies sequential convergence in  $\sigma(l, ac_0)$  with a two-norm topology. For details concerning this type of topology we refer the reader to [1], [6], [22] and [23].

**Corollary 1.**  $l$  is sequentially complete under both the topologies  $\sigma(l, ac_0)$  and  $\sigma(l, bs + c_0)$ .

**Proof.** If  $\{x^{(n)}\}_{n=1}^{\infty}$  is a  $\sigma(l, bs + c_0)$ -Cauchy sequence, the proof of Theorem 2 shows that  $\{x^{(n)}\}_{n=1}^{\infty}$  is a Cauchy sequence in  $bv_0$  and bounded in  $l$ . Since  $bv_0$  is complete, there exists  $x \in bv_0$  such that

$$\lim_{n \rightarrow \infty} \|x^{(n)} - x\|_{bv} = 0.$$

But this implies that

$$x_k = \lim_{n \rightarrow \infty} x_k^{(n)} \quad (k = 1, 2, \dots)$$

so that

$$\sum_{k=1}^{\infty} |x_k| \leq \sup_n \sum_{k=1}^{\infty} |x_k^{(n)}| < \infty,$$

and  $x \in l$ . It then follows from Theorem 2, (iii)  $\Rightarrow$  (i), that  $x^{(n)} \rightarrow 0$  in  $\sigma(l, ac_0)$ , giving the desired result.

**Corollary 2.** *For a subset  $C$  of  $l$ , the following conditions are equivalent:*

- (i)  $C$  is  $\sigma(l, ac_0)$ -relatively compact;
- (ii)  $C$  is  $\sigma(l, bs + c_0)$ -relatively compact;
- (iii)  $C$  is  $\|\cdot\|_1$ -bounded and  $\lim_{n \rightarrow \infty} \sup_{x \in C} \|x - P_n x\|_{bv} = 0$ .

**Proof.** A subset of a  $K$ -space is relatively compact if and only if it is relatively sequentially compact (see [10]) and hence Theorem 2 shows that (i) and (ii) are equivalent. Using the sequential completeness of  $l$  in the two-norm convergence defined in (iii) of Theorem 2, it is clear that (i) and (ii) are equivalent to “ $C$  is  $\|\cdot\|_1$ -bounded and  $\|\cdot\|_{bv_0}$ -relatively compact.” However, by a general theorem on bases (see [16]) this is equivalent to (iii).

We note from Corollary 2 that the closed convex hull of a  $\sigma(l, ac_0)$ -compact set is also  $\sigma(l, ac_0)$ -compact (using (iii)); hence the Mackey topology,  $\tau(ac_0, l)$ , is the topology of uniform convergence on  $\sigma(l, ac_0)$ -compact sets.

We now turn to the relationship between  $ac_0$  and  $bs$ .

**Theorem 3.** (i)  $ac_0 = \overline{bs}$ , the closure of  $bs$  in  $m$ .

- (ii) If  $x \in bs + c_0$ , then  $\sup_p \limsup_{n \rightarrow \infty} |x_{n+1} + \dots + x_{n+p}| < \infty$ .
- (iii)  $ac_0 \neq bs + c_0$ .

**Proof.** (i) Clearly  $\overline{bs} \subseteq ac_0$ . Conversely, if  $x \in ac_0$  and  $\epsilon > 0$  are given, we may choose a positive integer  $p$  so that

$$|x_{n+1} + \dots + x_{n+p}| < p\epsilon \quad (n = 1, 2, \dots).$$

In particular,

$$(6) \quad x_{mp+1} + \dots + x_{(m+1)p} = p\delta_m \quad (m = 0, 1, 2, \dots)$$

where  $|\delta_m| \leq \epsilon$ . Letting  $y$  be defined by

$$y_{mp+k} = x_{mp+k} - \delta_m \quad (k = 1, 2, \dots, p; m = 0, 1, 2, \dots),$$

it is clear that  $\|x - y\|_\infty \leq \epsilon$ ; we complete the proof of (i) by showing that  $y \in bs$ .

Now

$$\begin{aligned} \sum_{i=1}^{mp+k} y_i &= \sum_{n=0}^{m-1} \sum_{j=1}^p (x_{np+j} - \delta_n) + \sum_{i=1}^k x_{mp+i} - k\delta_m \\ &= \sum_{i=1}^k x_{mp+i} - k\delta_m \quad \text{by (6).} \end{aligned}$$

Consequently

$$\left| \sum_{i=1}^q y_i \right| \leq p(\|x\|_\infty + \varepsilon)$$

for every  $q$ , and  $y \in bs$ .

(ii) If  $x \in bs + c_0$ , then  $x = y + z$  for some  $y \in bs$  and  $z \in c_0$ . Then

$$|x_{n+1} + \cdots + x_{n+p}| \leq |y_{n+1} + \cdots + y_{n+p}| + |z_{n+1} + \cdots + z_{n+p}|$$

so that

$$\limsup_{n \rightarrow \infty} |x_{n+1} + \cdots + x_{n+p}| = \limsup_{n \rightarrow \infty} |y_{n+1} + \cdots + y_{n+p}| \leq 2\|y\|_{bs},$$

giving the desired result.

(iii) By (ii) we may construct  $x \in ac_0 \setminus (bs + c_0)$  directly; let

$$\begin{aligned} x_k &= 1 && \text{if } k = 2^n + 2^m \text{ for } n \geq m \geq 1, \\ &= 0 && \text{otherwise.} \end{aligned}$$

Then  $x$  does not satisfy (ii), yet it is easy to check that  $x \in ac_0$ .

It is interesting to note that  $bs + c_0$  is a *BK*-space which is *B*-invariant in the sense of Garling [10], and  $c_0 \subset bs + c_0 \subset m$ , yet  $bs + c_0$  is not closed in  $m$ .

**Theorem 4.** (i)  $(ac_0, \tau(ac_0, l))$  is a complete AK-space.

(ii)  $\tau(bs + c_0, l)$  is the restriction of  $\tau(ac_0, l)$  to  $bs + c_0$  [so that  $(ac_0, \tau(ac_0, l))$  is the completion of  $(bs + c_0, \tau(bs + c_0, l))$ ].

**Proof.** (i) If  $C$  is  $\sigma(l, ac_0)$ -relatively compact, then by (iii) of Corollary 2 to Theorem 2, the set  $P(C) = \{P_n f : f \in C\}$  is  $\sigma(l, ac_0)$ -relatively compact. It follows that the operators  $\{P_n : n = 1, 2, \dots\}$  are  $\tau(ac_0, l) \rightarrow \tau(ac_0, l)$ -equicontinuous, so that the set

$$S = \{x \in ac_0 : P_n x \rightarrow x \text{ in } \tau(ac_0, l)\}$$

is  $\tau(ac_0, l)$ -closed. However,  $S \supseteq \varphi$  and  $\varphi$  is  $\tau(ac_0, l)$ -dense in  $ac_0$  (since  $\varphi$  is  $\sigma(ac_0, l)$ -dense); hence  $S = ac_0$ , showing that  $(ac_0, \tau(ac_0, l))$  is an AK-space.

To show that  $(ac_0, \tau(ac_0, l))$  is complete we use Grothendieck's criterion [6, Proposition 1]. Let  $\theta$  be a linear functional on  $l$  which is  $\sigma(l, ac_0)$ -continuous on each  $\sigma(l, ac_0)$ -compact set. Then  $\theta(x^{(n)}) \rightarrow 0$  whenever  $x^{(n)} \rightarrow 0$  in  $\sigma(l, ac_0)$ . Consequently, from Theorem 2,  $\theta$  is continuous in the two-norm topology. Using the standard characterization of the dual of a two-norm space [1, Theorem 4.2], it follows that  $\theta$  lies in the closure of  $bs$  (the dual of  $(l, \|\cdot\|_b)$ ) in  $m$  (the dual of  $(l, \|\cdot\|_\infty)$ ). Hence, by Theorem 3(i),  $\theta$  takes the form

$$\theta(x) = \sum_{k=1}^{\infty} f_k x_k,$$

where  $f$  is a fixed element from  $ac_0$ . It follows from Grothendieck's criterion that  $(ac_0, \tau(ac_0, l))$  is complete.

(ii) This follows from Corollary 2 to Theorem 2.

**Theorem 5.** Let  $E$  be a separable FK-space containing  $c_0$  and  $bs$ . Then

- (i)  $E$  contains  $ac_0$ ;
- (ii)  $x \in ac_0$  implies that  $P_n x \rightarrow x$  in  $E$ ;
- (iii)  $e^{(n)} \rightarrow 0$  in  $E$ .

**Proof.** (i) and (ii). The space  $(bs + c_0, \tau(bs + c_0, l))$  is a Mackey space whose dual,  $l$ , is  $\sigma(l, bs + c_0)$ -sequentially complete by Corollary 1 to Theorem 2. Thus, by the main result of [11] (see also [7, Theorem 5]), the natural inclusion mapping:  $bs + c_0 \rightarrow E$ , which clearly has closed graph, must be continuous. If  $x \in ac_0$ , then by Theorem 4,  $\{P_n x\}_{n=1}^{\infty}$  is Cauchy in  $(bs + c_0, \tau(bs + c_0, l))$  and hence in  $E$ . Since  $E$  is complete,  $\{P_n x\}_{n=1}^{\infty}$  converges in  $E$ , and its limit must be  $x$  since  $E$  is a K-space. This completes the proof of (i) and (ii).

For (iii), we note that if  $C$  is a  $\sigma(l, bs + c_0)$ -compact subset of  $l$ , then

$$\sup_{f \in C} |f_n| \leq \sup_{f \in C} \|f - P_{n-1} f\|_{bv} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

by Corollary 2 to Theorem 2. Consequently  $e^{(n)} \rightarrow 0$  in  $\tau(bs + c_0, l)$  and hence in  $E$ .

We note that (iii) is true if we only assume that  $E$  contains  $bs$  (see [5, Theorem 5]).

Our next result answers a question left open in [6].

**Corollary 1.** There exists a BK-space  $E$  which is not the intersection of the separable FK-spaces containing it.

**Proof.** Using Theorems 3 and 5 we may take  $E$  to be  $bs + c_0$ .

**Corollary 2.** If  $A$  is a conservative matrix such that  $bs \subset c_A$ , then

- (i)  $ac \subset c_A$ ;
- (ii)  $\lim_{n \rightarrow \infty} \sup_m |a_{mn}| = 0$ ;
- (iii)  $\lim_{m \rightarrow \infty} \sup_n |a_{mn}| = 0$ ;
- (iv) for  $x \in ac$ , we have  $\lim_A x = \sum_{j=1}^{\infty} a_j x_j + \chi(A) \text{Lim } x$ .

**Proof.** (i)  $c_A$  is a separable FK-space ([14, 1.4.1]; [4, Corollary 1 to Theorem 4]) and so, by Theorem 5(i),  $ac_0 \subset c_A$ . Since  $e \in c_A$ , it follows that  $ac \subset c_A$ .

(ii)  $e^{(n)} \rightarrow 0$  in  $c_A$  by Theorem 5(iii), so that  $Ae^{(n)} \rightarrow 0$  in  $c$  by Theorem 4.4(c) of [24]. It follows that

$$\lim_{n \rightarrow \infty} \sup_m |a_{mn}| = 0.$$

(iii) follows from (ii) as in the proof of Proposition 8 of [5].

(iv) If  $x \in ac$ , then  $x - (\text{Lim } x)e \in ac_0$  and, by Theorem 5(ii),

$$x - (\text{Lim } x)e = \sum_{k=1}^{\infty} (x_k - \text{Lim } x)e^{(k)} \quad \text{in } c_A.$$

Now  $\lim_A$  is continuous on  $c_A$  so that

$$\lim_A x - (\text{Lim } x) \lim_A e = \sum_{k=1}^{\infty} (x_k - \text{Lim } x) a_k.$$

Since  $x \in m$ ,  $\sum_{k=1}^{\infty} a_k x_k$  converges and so

$$\begin{aligned}\lim_A x &= \sum_{k=1}^{\infty} a_k x_k + \text{Lim } x \left( \lim_A e - \sum_{k=1}^{\infty} a_k \right) \\ &= \sum_{k=1}^{\infty} a_k x_k + \chi(A) \text{Lim } x.\end{aligned}$$

**3. Consistency theorems.** In [6] we used a technique involving the Orlicz-Pettis theorem on unconditional convergence of series to obtain a new proof of the Mazur-Orlicz-Brudno consistency theorem. In this section we apply the same basic technique to derive similar consistency theorems for almost convergence; the details, however, are much more difficult than those in [6] and we shall need considerable preparation before coming to our main results (Theorems 6 and 8).

We begin by introducing an idea which may be of some interest in a more general setting; we say that a sequence  $\{x^{(n)}\}_{n=1}^{\infty}$  is *superconvergent to  $x$*  (in a locally convex space  $E$ ) if  $\{x^{(n)}\}_{n=1}^{\infty}$  converges to  $x$  and

$$\sum_{k=1}^{\infty} (x_{n_k} - x_{n_{k-1}})$$

converges in  $E$  for every increasing sequence  $\{n_k\}_{k=1}^{\infty}$  of positive integers.

Our first result is elementary and its proof is omitted.

**Lemma 3.** *Every subsequence of a superconvergent sequence is superconvergent.*

The validity of the next result is one of the main reasons for studying superconvergence.

**Lemma 4.** *Let  $E$  be a locally convex space with dual  $E'$ . If a sequence  $\{x^{(n)}\}_{n=1}^{\infty}$  superconverges in the weak topology  $\sigma(E, E')$ , then  $\{x^{(n)}\}_{n=1}^{\infty}$  converges in the topology  $\lambda(E, E')$  of uniform convergence on  $\sigma(E', E)$ -compact sets.*

**Proof.** Direct application of the general Orlicz-Pettis theorem (see [6], [15] or [21]).

**Lemma 5.** *Let  $E$  be a Fréchet space and suppose that  $x^{(n)} \rightarrow x$  in  $E$ . Then there is a subsequence  $\{z^{(n)}\}_{n=1}^{\infty}$  of  $\{x^{(n)}\}_{n=1}^{\infty}$  that superconverges to  $x$ .*

**Proof.** Suppose that  $\{p_k\}_{k=1}^{\infty}$  is an increasing sequence of seminorms defining the topology on  $E$ . Choose an increasing sequence  $\{n_k\}_{k=1}^{\infty}$  of positive integers so that

$$p_k(x - x^{(n)}) \leq 1/2^k$$

whenever  $n \geq n_k$ . Putting  $z^{(k)} = x^{(n_k)}$ ,  $k = 1, 2, \dots$ , it is easy to see that

$$\sum_{k=1}^{\infty} (z^{(k)} - z^{(k-1)})$$

converges absolutely, so that  $\{z^{(k)}\}_{k=1}^{\infty}$  superconverges to  $x$  in  $E$ .

**Lemma 6.**  $x^{(n)} \rightarrow x$   $\sigma(m, l)$  if and only if  $x^{(n)} \rightarrow x$   $\sigma(m, \varphi)$  and  $\sup_n \|x^{(n)}\|_{\infty} < \infty$ .

**Proof.** A simple compactness argument (cf. [6, Lemma 3]). Alternatively, a neat proof may be given by using Lebesgue's dominated convergence theorem.

**Lemma 7.** If  $x^{(n)} \rightarrow 0$   $\sigma(c_0, l)$ , then there exists a subsequence  $\{z^{(n)}\}_{n=1}^{\infty}$  of  $\{x^{(n)}\}_{n=1}^{\infty}$  such that

$$\left\| \frac{1}{n} (z^{(1)} + z^{(2)} + \cdots + z^{(n)}) \right\|_{\infty} \rightarrow 0.$$

**Proof.** In view of Lemma 6 the hypotheses are equivalent to

$$(7) \quad \lim_{j \rightarrow \infty} x_j^{(n)} = 0 \quad (n = 1, 2, \dots),$$

$$(8) \quad \lim_{n \rightarrow \infty} x_j^{(n)} = 0 \quad (j = 1, 2, \dots),$$

and

$$(9) \quad \sup_{n,j} |x_j^{(n)}| = M < \infty.$$

We choose increasing sequences  $\{s_m\}_{m=1}^{\infty}$  and  $\{t_m\}_{m=1}^{\infty}$  of positive integers as follows. Let  $s_1 = 1$ ,  $t_0 = 0$ , and suppose that  $s_1, \dots, s_m$  and  $t_1, \dots, t_{m-1}$  have been chosen. Using (7), choose  $t_m > t_{m-1}$  so that

$$(10) \quad \max_{1 \leq n \leq s_m} |x_j^{(n)}| \leq 2^{-m}$$

whenever  $j > t_m$ . Next, using (8), choose  $s_{m+1} > s_m$  so that

$$(11) \quad |x_j^{(s_{m+1})}| \leq 2^{-m}$$

whenever  $1 \leq j \leq t_m$ .

If  $t_m < j \leq t_{m+1}$  and  $m \geq 1$ , then

$$\begin{aligned} |x_j^{(s_1)} + x_j^{(s_2)} + \cdots + x_j^{(s_n)}| &\leq n \cdot 2^{-m} \quad \text{if } n \leq m \text{ by (10),} \\ &\leq m \cdot 2^{-m} + |x_j^{(s_{m+1})}| + \sum_{k=m+1}^{n-1} 2^{-k} \\ &\quad \text{if } n > m \text{ by (10) and (11),} \\ &\leq M + 1 \quad \text{by (9).} \end{aligned}$$

Consequently, putting  $z^{(n)} = x^{(s_n)}$ ,  $n = 1, 2, \dots$ , we have

$$\begin{aligned} \left\| \frac{z^{(1)} + z^{(2)} + \cdots + z^{(n)}}{n} \right\|_{\infty} &= \sup_m \sup_{t_m < j \leq t_{m+1}} \left| \frac{x_j^{(s_1)} + x_j^{(s_2)} + \cdots + x_j^{(s_n)}}{n} \right| \\ &\leq (M + 1)/n \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Lemma 7 says that in the Banach space  $c_0$  every weakly convergent sequence has a subsequence whose arithmetic means converge in norm. This property, the so-called *Banach-Saks property*, is also known to hold for the spaces  $l^p$  and  $L^p(0, 1)$  (see [3]). We remark here that not every Banach space has this property.

**Lemma 8.** *Let  $x^{(n)} \in c_0$ ,  $n = 1, 2, \dots$ , and suppose that  $x^{(n)} \rightarrow x$  in  $\sigma(m, l)$ . Then there exists a subsequence  $\{z^{(n)}\}_{n=1}^\infty$  of  $\{x^{(n)}\}_{n=1}^\infty$  that is superconvergent to  $x$  in  $\sigma(m, l)$ .*

**Proof.** Without loss of generality we may assume that  $x = 0$ . The hypotheses are then the same as in Lemma 7 and we may choose  $\{s_n\}_{n=1}^\infty$  and  $\{t_n\}_{n=1}^\infty$  as before so that (9) and (10) are satisfied. It is easily seen that

$$\sup_j \sum_{n=1}^\infty |x_j^{(s_n)} - x_j^{(s_{n-1})}| < \infty,$$

and so, in view of Lemma 6, we may take  $z^{(n)} = x^{(s_n)}$ ,  $n = 1, 2, \dots$ .

**Lemma 9.** *If  $x \in ac_0$  and  $\{s_n\}_{n=1}^\infty$  is a strictly increasing sequence of positive integers then the sequence*

$$y^{(n)} = \frac{1}{n}(P_{s_1}x + P_{s_2}x + \dots + P_{s_n}x)$$

*is superconvergent to  $x$  in  $\sigma(ac_0, l)$ .*

**Proof.** We define

$$Q_0 = 0; \quad Q_n = \frac{1}{n}(P_{s_1} + P_{s_2} + \dots + P_{s_n}), \quad n = 1, 2, \dots;$$

$$R_n = Q_n - Q_{n-1}.$$

For any finite subset  $M$  of the positive integers  $\mathbf{Z}$  we write

$$S_M = \sum_{k \in M} R_k = \sum_{k=1}^\infty \delta_M(k) R_k$$

where  $\delta_M$  is the characteristic function of  $M$ . We show that the collection  $\{S_M : M \text{ is a finite subset of } \mathbf{Z}\}$  is  $\tau(ac_0, l) \rightarrow \tau(ac_0, l)$ -equicontinuous. Since

$$\sum_{k=1}^\infty (S_M f)_k x_k = \sum_{k=1}^\infty f_k (S_M x)_k,$$

we must show that if  $C$  is  $\sigma(l, ac_0)$ -compact then  $S(C) = \{S_M f : f \in C, M \subset \mathbf{Z}\}$  is  $\sigma(l, ac_0)$ -relatively compact.

For  $x \in l$ , we have

$$(Q_n x)_p = (1 - k/n)x_p \quad \text{if } s_k < p \leq s_{k+1}, \quad 1 \leq k \leq n,$$

$$= 0 \quad \text{if } p > s_n,$$

and

$$(R_n x)_p = \begin{cases} \frac{k}{n(n-1)} x_p & \text{if } s_k < p < s_{k+1}, 1 \leq k \leq n-1, \\ 0 & \text{if } p > s_n, \end{cases}$$

so that

$$(S_M x)_p = \left\{ \sum_{n=k+1}^{\infty} \frac{k}{n(n-1)} \delta_M(n) \right\} x_p \quad \text{if } s_k < p \leq s_{k+1}.$$

Now if  $C$  is  $\sigma(l, ac_0)$ -relatively compact, then by Corollary 2 to Theorem 2, we have

$$\sup_{x \in C} \|x\|_l = K < \infty,$$

and

$$\sup_{x \in C} \sum_{p=n}^{\infty} |x_p - x_{p+1}| = \varepsilon_n \rightarrow 0.$$

Now

$$|(S_M x)_p| \leq |x_p| \sum_{n=k+1}^{\infty} \frac{k}{n(n-1)} \leq |x_p|$$

so that  $\|S_M x\|_l \leq K$ . If  $s_k < p < s_{k+1}$ ,

$$(S_M x)_p - (S_M x)_{p+1} = \left( \sum_{n=k+1}^{\infty} \frac{k}{n(n-1)} \delta_M(n) \right) (x_p - x_{p+1})$$

so that

$$|(S_M x)_p - (S_M x)_{p+1}| \leq |x_p - x_{p+1}|;$$

while, if  $p = s_k$ ,

$$\begin{aligned} (S_M x)_p - (S_M x)_{p+1} &= \left\{ \sum_{n=k}^{\infty} \frac{k-1}{n(n-1)} \delta_M(n) \right\} x_p - \left\{ \sum_{n=k+1}^{\infty} \frac{k}{n(n-1)} \delta_M(n) \right\} x_{p+1} \\ &= \left\{ \sum_{n=k}^{\infty} \frac{k-1}{n(n-1)} \delta_M(n) \right\} (x_p - x_{p+1}) - \left\{ \sum_{n=k+1}^{\infty} \frac{1}{n(n-1)} \delta_M(n) \right\} x_{p+1} \\ &\quad + \frac{1}{k} \delta_M(k) x_{p+1}, \end{aligned}$$

so that

$$|(S_M x)_p - (S_M x)_{p+1}| \leq |x_p - x_{p+1}| + \frac{1}{k} |x_{p+1}|.$$

Consequently, if  $s_k \leq n < s_{k+1}$ ,

$$\begin{aligned} \sum_{p=n+1}^{\infty} |(S_M x)_p - (S_M x)_{p+1}| &\leq \frac{K}{k} + \sum_{p=n+1}^{\infty} |x_p - x_{p+1}| \\ &\leq \frac{K}{k} + \varepsilon_n \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

It follows from Corollary 2 to Theorem 2 that  $S(C)$  is indeed  $\sigma(l, ac_0)$ -compact and so the collection  $\{S_M : M \subseteq \mathbf{Z}\}$  is equicontinuous on  $(ac_0, \tau(ac_0, l))$ . In particular, if  $N$  is an infinite subset of  $\mathbf{Z}$ , the operators

$$\sum_{k=1}^n \delta_N(k) R_k \quad (n = 1, 2, \dots)$$

are equicontinuous, and so, since  $ac_0$  is  $\tau(ac_0, l)$ -complete (Theorem 4(i)) the set of  $x \in ac_0$  for which  $\sum_{k=1}^{\infty} \delta_N(k) R_k x$  converges is closed. However if  $x \in \varphi$  this is clearly so, and so we conclude for all  $x \in ac_0$  and all  $N \subseteq \mathbf{Z}$  that  $\sum_{k=1}^{\infty} \delta_N(k) R_k x$  converges. Hence the sequence  $\{Q_n x\}_{n=1}^{\infty}$  superconverges in  $(ac_0, \tau(ac_0, l))$ .

**Lemma 10.** *Let  $x^{(n)} \in c_0$ ,  $n = 1, 2, \dots$ , and suppose that  $x^{(n)} \rightarrow x$   $\sigma(ac_0, l)$ . Then there exists a subsequence  $\{z^{(n)}\}_{n=1}^{\infty}$  of  $\{x^{(n)}\}_{n=1}^{\infty}$  such that some subsequence  $\{w^{(n)}\}_{n=1}^{\infty}$  of  $\{(z^{(1)} + z^{(2)} + \dots + z^{(n)})/n\}_{n=1}^{\infty}$  superconverges to  $x$  in  $\sigma(ac_0, l)$ .*

**Proof.** Since  $P_n x \rightarrow x$   $\sigma(ac_0, l)$ , we have

$$x^{(n)} - P_n x \rightarrow 0 \quad \sigma(c_0, l).$$

By Lemma 7 we may take a subsequence  $\{z^{(n)}\}_{n=1}^{\infty} = \{x^{(s_n)}\}_{n=1}^{\infty}$  of  $\{x^{(n)}\}_{n=1}^{\infty}$  such that

$$\left\| \frac{1}{n} (z^{(1)} + \dots + z^{(n)}) - \frac{1}{n} (P_{s_1} x + \dots + P_{s_n} x) \right\|_{\infty} \rightarrow 0.$$

Taking a subsequence again, we may suppose that, for each integer  $n$ ,

$$\left\| \frac{1}{m_n} (z^{(1)} + \dots + z^{(m_n)}) - \frac{1}{m_n} (P_{s_1} x + \dots + P_{s_{m_n}} x) \right\|_{\infty} \leq \frac{1}{2^n},$$

so that the sequence

$$\left\{ \frac{1}{m_n} (z^{(1)} + \dots + z^{(m_n)}) - \frac{1}{m_n} (P_{s_1} x + \dots + P_{s_{m_n}} x) \right\}_{n=1}^{\infty}$$

is superconvergent to 0 in  $c_0$ . However, by Lemmas 3 and 9, the sequence

$$\left\{ \frac{1}{m_n} (P_{s_1} x + \dots + P_{s_{m_n}} x) \right\}_{n=1}^{\infty}$$

is superconvergent to  $x$  in  $\sigma(ac_0, l)$ . Hence, with

$$w^{(n)} = \frac{1}{m_n} (z^{(1)} + \cdots + z^{(m_n)}) \quad (n = 1, 2, \dots),$$

$\{w^{(n)}\}_{n=1}^{\infty}$  is superconvergent to  $x$  in  $\sigma(ac_0, l)$ .

Before stating our next result we recall the following notation. For an infinite matrix  $A$ ,  $ac_A$  denotes the set

$$ac_A = \{x \in \omega : Ax \in ac\}.$$

If  $x \in ac_A$ , we write  $\text{Lim}_A x$  in place of  $\text{Lim}(Ax)$ , and denote by  $(ac_0)_A$  the subspace of  $(ac)_A$  on which  $\text{Lim}_A$  vanishes.

**Theorem 6.** *Let  $A$  be a matrix such that*

- (i)  $\sup_i \sum_{j=1}^{\infty} |a_{ij}| < \infty$ , and
- (ii)  $\lim_{i \rightarrow \infty} a_{ij} = 0$  for  $j = 1, 2, \dots$ . Then  $l$  is  $\sigma(l, (ac_0)_A \cap m)$ -sequentially complete.

**Proof.** Let  $x \in (ac_0)_A \cap m$  be fixed. We construct a sequence  $\{z^{(n)}\}_{n=1}^{\infty}$  of elements of  $\varphi$  such that  $\{z^{(n)}\}_{n=1}^{\infty}$  superconverges to  $x$  in  $\sigma((ac_0)_A \cap m, l)$ . To do this, we first observe that

$$AP_n x \rightarrow Ax \quad \sigma(\omega, \varphi).$$

Condition (i) ensures that  $A: m \rightarrow m$  is continuous and hence

$$\|AP_n x\|_{\infty} \leq \|A\| \|x\|_{\infty}.$$

Lemma 6 gives

$$AP_n x \rightarrow Ax \quad \sigma(m, l).$$

Now condition (ii) implies that  $AP_n x \in A(\varphi) \subset c_0$  so we may apply Lemma 10 to deduce the existence of a sequence  $\{v^{(k)}\}_{k=1}^{\infty}$  such that  $\{Av^{(k)}\}_{k=1}^{\infty}$  superconverges to  $Ax$  in  $\sigma(ac_0, l)$  and  $\{v^{(k)}\}_{k=1}^{\infty}$  takes the form

$$v^{(k)} = \frac{1}{m_k} (u^{(1)} + \cdots + u^{(m_k)})$$

where  $\{u^{(k)}\}_{k=1}^{\infty}$  is some subsequence of  $\{P_n x\}_{n=1}^{\infty}$ . Clearly we have

$$\sup_k \|v^{(k)}\|_{\infty} \leq \|x\|_{\infty}$$

and  $v^{(k)} \rightarrow x$  in  $\sigma(\omega, \varphi)$  so that  $v^{(k)} \rightarrow x$  in  $\sigma(m, l)$  by Lemma 6.

Furthermore, since  $Av^{(k)} \rightarrow Ax$  in  $\omega$ , we have

$$v^{(k)} \rightarrow x \quad \text{in } \omega_A.$$

We now apply Lemmas 3, 5 and 8 to obtain a subsequence  $\{z^{(n)}\}_{n=1}^{\infty}$  of  $\{v^{(n)}\}_{n=1}^{\infty}$  such that  $\{z^{(n)}\}_{n=1}^{\infty}$  superconverges to  $x$  in both  $\omega_A$  and  $\sigma(m, l)$ ; it is also clear that

$\{Az^{(n)}\}_{n=1}^{\infty}$  superconverges in  $(ac_0, \sigma(ac_0, l))$ . Now suppose that  $\{\varepsilon_n\}_{n=1}^{\infty}$  is a sequence taking only the values 1 and 0; for each  $k$  let

$$y_k = \sum_{n=1}^{\infty} \varepsilon_n (z_k^{(n)} - z_k^{(n-1)}) \quad (\text{where } z^{(0)} = 0).$$

Since  $\{z^{(n)}\}_{n=1}^{\infty}$  superconverges in both  $\omega_A$  and  $(m, \sigma(m, l))$ , the series

$$\sum_{n=1}^{\infty} \varepsilon_n (z^{(n)} - z^{(n-1)})$$

converges to  $y$  in both  $\omega_A$  and  $(m, \sigma(m, l))$ ; therefore  $y \in m \cap \omega_A$ . Now  $A: \omega_A \rightarrow \omega$  is continuous [24, Theorem 4.4(c)] and so

$$Ay = \sum_{n=1}^{\infty} \varepsilon_n (Az^{(n)} - Az^{(n-1)}) \quad \sigma(\omega, \varphi).$$

However,  $\{Az^{(n)}\}_{n=1}^{\infty}$  superconverges in  $(ac_0, \sigma(ac_0, l))$  so that

$$Ay = \sum_{n=1}^{\infty} \varepsilon_n (Az^{(n)} - Az^{(n-1)}) \quad \sigma(ac_0, l),$$

and  $Ay \in ac_0$ , i.e.,  $y \in (ac_0)_A$ . Thus  $\{z^{(n)}\}_{n=1}^{\infty}$  superconverges to  $x$  in  $((ac_0)_A \cap m, \sigma((ac_0)_A \cap m, l))$ .

We now repeat the argument used in the proof of Theorem 3 of [6]. Consider the topology  $\lambda((ac_0)_A \cap m, l)$  on  $(ac_0)_A \cap m$  of uniform convergence on the  $\sigma(l, (ac_0)_A \cap m)$ -compact subsets of  $l$ ; by Lemma 4 we have

$$z^{(n)} \rightarrow x \quad \lambda((ac_0)_A \cap m, l).$$

Suppose now that  $\psi$  is a linear functional on  $(ac_0)_A \cap m$  whose restrictions to  $\lambda((ac_0)_A \cap m, l)$ -precompact sets are  $\lambda$ -continuous. Then  $\psi$  is  $\lambda$ -sequentially continuous and since  $\lambda \leq \beta((ac_0)_A \cap m, l) \leq \beta(c_0, l)$ ,  $\psi$  is  $\|\cdot\|_{\infty}$ -continuous on  $c_0$  so that

$$\sum_{j=1}^{\infty} |f_j| < \infty$$

where  $\psi(e^{(j)}) = f_j$  ( $j = 1, 2, \dots$ ). Now

$$\psi(z^{(n)}) = \sum_{j=1}^{\infty} z_j^{(n)} f_j$$

since  $z^{(n)} \in \varphi$ , and

$$\lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} z_j^{(n)} f_j = \sum_{j=1}^{\infty} x_j f_j$$

since  $z^{(n)} \rightarrow x$   $\sigma(m, l)$ . Consequently, for each  $x \in (ac_0)_A \cap m$ , we have

$$\psi(x) = \sum_{j=1}^{\infty} x_j f_j.$$

It follows (as in the proof of Theorem 3 of [6]) by Grothendieck's completeness theorem that the topology  $\rho$  on  $l$ , of uniform convergence on  $\lambda$ -precompact subsets of  $(ac_0)_A \cap m$ , must be complete. Furthermore,  $\rho$  defines the same convergent and Cauchy sequences as  $\sigma(l, (ac_0)_A \cap m)$  so that  $l$  is  $\sigma(l, (ac_0)_A \cap m)$ -sequentially complete.

We now come to our first consistency theorem.

**Theorem 7.** *Let  $A$  and  $B$  be regular matrices and suppose that  $ac_A \cap m \subset c_B$ . Then  $\lim_B x = \text{Lim}_A x$  whenever  $x \in ac_A \cap m$ .*

**Proof.** Since  $A$  is regular, the conditions of Theorem 6 are satisfied. Let  $b^{(n)} \in l$ ,  $n = 1, 2, \dots$ , be defined by

$$b_k^{(n)} = b_{nk} \quad (k = 1, 2, \dots),$$

(so that  $b^{(n)}$  is the  $n$ th row of  $B$ ). Since  $(ac_0)_A \cap m \subseteq c_B$ , we have

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} b_k^{(n)} x_k \text{ exists}$$

whenever  $x \in (ac_0)_A \cap m$ . Hence  $\{b^{(n)}\}_{n=1}^{\infty}$  is  $\sigma(l, (ac_0)_A \cap m)$ -Cauchy and so converges, say  $b^{(n)} \rightarrow b$ , by Theorem 6. Clearly

$$b_k = \lim_{n \rightarrow \infty} b_{nk} = 0,$$

so that  $b^{(n)} \rightarrow 0$   $\sigma(l, (ac_0)_A \cap m)$ .

Now, if  $x \in (ac)_A \cap m$ , then  $x - (\text{Lim}_A x)e \in (ac_0)_A \cap m$ , and so

$$\lim_B \left( x - \left( \text{Lim}_A x \right) e \right) = \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} b_k^{(n)} \left( x_k - \text{Lim}_A x \right) = 0,$$

i.e.  $\lim_B x = \text{Lim}_A x$ .

When  $A$  is the identity matrix, Theorem 7 reduces to the following.

**Corollary.** *Let  $B$  be a regular matrix with  $ac \subset c_B$ . Then  $\lim_B x = \text{Lim}_A x$  whenever  $x \in ac$ .*

This special result may also be derived from Corollary 2 to Theorem 5 and was first obtained by Lorentz [13].

Before stating our next result let us recall the following notation. If  $E$  is an FK-space containing  $\varphi$ , then we write

$$W_E = \{x \in E: P_n x \rightarrow x \text{ weakly in } E\}$$

and

$$S_E = \{x \in E: P_n x \rightarrow x \text{ in } E\}.$$

**Theorem 8.** *Let  $E$  be an FK-space containing  $c_0$ . Then  $l$  is sequentially complete*

under both the topologies  $\sigma(l, W_E \cap ac_0)$  and  $\sigma(l, S_E \cap ac_0)$ .

**Proof.** As with Theorem 6, the proof hinges on ideas developed in Theorem 3 of [6].

Let  $x \in W_E \cap ac_0$  be fixed: by Theorem 2 of [6] (see also [20]) there is a sequence  $\{u^{(n)}\}_{n=1}^{\infty}$  of elements of  $\varphi$  with

$$\tau - \lim_{n \rightarrow \infty} u^{(n)} = x$$

and

$$\sup_n \|u^{(n)}\|_{\infty} \leq \|x\|_{\infty},$$

where  $\tau$  denotes the *FK*-topology on  $E$ . By Lemma 6,

$$\lim_{n \rightarrow \infty} u^{(n)} = x \quad \sigma(ac_0, l)$$

and so, by Lemma 10, there exists a sequence  $\{v^{(n)}\}_{n=1}^{\infty}$ , of arithmetic means of a subsequence of  $\{u^{(n)}\}_{n=1}^{\infty}$ , such that  $\{v^{(n)}\}_{n=1}^{\infty}$  superconverges to  $x$  in  $\sigma(ac_0, l)$ ; clearly

$$\tau - \lim_{n \rightarrow \infty} v^{(n)} = x.$$

By using Lemmas 3 and 5 we may select a subsequence  $\{z^{(n)}\}_{n=1}^{\infty}$  which superconverges to  $x$  in both  $\tau$  and  $\sigma(ac_0, l)$ . Thus every subseries of  $\sum_{n=1}^{\infty} (z^{(n)} - z^{(n-1)})$  converges in  $E \cap ac_0$ ; i.e., if  $\epsilon_n = 0$  or 1 for all  $n$  and

$$y = \sum_{n=1}^{\infty} \epsilon_n (z^{(n)} - z^{(n-1)}) \quad (\text{where } z^{(0)} = 0),$$

then  $y \in E \cap ac_0$ . Since this series converges in  $\sigma(m, l)$ , we have

$$\sup_k \left\| \sum_{n=1}^k \epsilon_n (z^{(n)} - z^{(n-1)}) \right\|_{\infty} < \infty,$$

and since the series converges in  $\tau$  we have  $y \in W_E$  by Theorem 2 of [6]. Thus  $\{z^{(n)}\}_{n=1}^{\infty}$  superconverges to  $x$  in  $\sigma(W_E \cap ac_0, l)$ , and the remaining details follow those of Theorem 6 (or Theorem 3 of [6]).

For the second half of the theorem we observe that [10, pp. 1015–1016]  $S_E = W_F$ , where  $F$  is the *FK*-space defined as follows.

$$F = \{x \in E : \{P_n x\}_{n=1}^{\infty} \text{ is } \tau\text{-bounded}\}$$

with the topology given by the seminorms

$$\nu(x) = \sup_n \nu(P_n x) \quad (x \in F)$$

for each  $\tau$ -continuous seminorm  $\nu$ .

Our next result may be thought of as a generalized consistency theorem.

**Theorem 9.** Let  $E$  be an FK-space containing  $c_0$  and let  $F$  be an FK-space containing no (closed) subspace isomorphic to  $m$ . If  $W_E \cap ac_0 \subset F$ , then  $W_E \cap ac_0 \subset W_F$ .

**Proof.** As in the proof of Theorem 8 we can show that each  $x \in W_E \cap ac_0$  can be written in the form  $x = \sum_{n=1}^{\infty} x^{(n)}$  where  $x^{(n)} \in \varphi$ ,  $n = 1, 2, \dots$ , and the convergence is  $\sigma(W_E \cap ac_0, l)$ -subseries. This observation enables us to replace  $W_E \cap m$  in the statement of Proposition 1 of [8] by  $W_E \cap ac_0$ ; but then the present result follows just as in the proof of Theorem 2 of [8].

In particular, it should be noted that Theorem 9 remains valid when  $F$  is a separable FK-space.

**Theorem 10.** Let  $A$  and  $B$  be regular matrices and suppose that  $ac \cap c_A \subseteq c_B$ . Then there exists a constant  $\alpha$  such that

$$\lim_B x = \alpha \lim_A x + (1 - \alpha) \text{Lim } x$$

whenever  $x \in ac \cap c_A$ .

**Proof.** Since  $A$  is regular we have, by Theorem 3.6 of [25],  $(c_0)_A \cap ac_0 = W_{c_A} \cap ac_0$ . Now  $c_B$  is separable ([14], [4]) so that

$$(c_0)_A \cap ac_0 \subset W_{c_B} \cap ac_0 = (c_0)_B \cap ac_0$$

by Theorem 9. Hence  $\lim_A x = \text{Lim } x = 0$  implies that  $\lim_B x = 0$ , and so

$$\lim_B x = \alpha \lim_A x + \beta \text{Lim } x$$

whenever  $x \in ac \cap c_A$ . However  $1 = \lim_B e = \alpha + \beta$  and the desired conclusion follows.

**Theorem 11.** Let  $A$  and  $B$  be conservative matrices and suppose that  $ac \cap c_A \subseteq c_B$ . Then there exist constants  $\alpha, \beta$  such that

(i)  $\lim_B x - \sum_{j=1}^{\infty} b_j x_j = \alpha(\lim_A x - \sum_{j=1}^{\infty} a_j x_j) + \beta \text{Lim } x$  whenever  $x \in ac \cap c_A$ , and

(ii)  $\chi(B) = \alpha\chi(A) + \beta$ .

**Proof.** This is a simple extension of Theorem 10; we observe that

$$W_{c_A} \cap ac_0 = \left\{ x: \lim_A x = \sum_{j=1}^{\infty} a_j x_j \right\} \cap ac_0$$

and apply the same method.

**Corollary 1.** Let  $A$  be conull and  $B$  be regular and suppose that  $ac \cap c_A \subseteq c_B$ . Then there exists a constant  $\alpha$  such that

$$\lim_B x = \text{Lim } x + \alpha \left( \lim_A x - \sum_{j=1}^{\infty} a_j x_j \right)$$

whenever  $x \in ac \cap c_A$ .

**Corollary 2.** *Let  $A$  be regular and  $B$  be conull and suppose that  $ac \cap c_A \subseteq c_B$ . Then there exists a constant  $\alpha$  such that*

$$\lim_B x = \alpha \left( \lim_A x - \text{Lim } x \right) + \sum_{j=1}^{\infty} b_j x_j$$

whenever  $x \in ac \cap c_A$ .

#### REFERENCES

1. A. Alexiewicz and Z. Semadeni, *Linear functionals on two-norm spaces*, Studia Math. **17** (1958), 121–140. MR **20** #6644.
2. S. Banach, *Théorie des opérations linéaires*, Monografie Mat., PWN, Warsaw, 1932.
3. S. Banach and S. Saks, *Sur la convergence forte dans les champs  $L'$* , Studia Math. **2** (1930), 51–57.
4. G. Bennett, *A representation theorem for summability domains*, Proc. Lond. Math. Soc. (3) **24** (1972), 193–203. MR **45** #776.
5. ———, *A new class of sequence spaces with applications in summability theory*, J. Reine Angew. Math. **266** (1974), 49–75.
6. G. Bennett and N.J. Kalton, *FK-spaces containing  $c_0$* , Duke Math. J. **39** (1972), 561–582.
7. ———, *Inclusion theorems for K-spaces*, Canad. J. Math. (to appear).
8. ———, *Addendum to FK-spaces containing  $c_0$* , Duke Math. J. **39** (1972), 819–821.
9. M.M. Day, *Normed linear spaces*, 2nd rev. ed., Academic Press, New York; Springer-Verlag, Berlin, 1962. MR **26** #2847.
10. D.J.H. Garling, *On topological sequence spaces*, Proc. Cambridge Philos. Soc. **63** (1967), 997–1019. MR **36** #1964.
11. N.J. Kalton, *Some forms of the closed graph theorem*, Proc. Cambridge Philos. Soc. **70** (1971), 401–408.
12. J.P. King, *Almost summable sequences*, Proc. Amer. Math. Soc. **17** (1966), 1219–1225. MR **34** #1752.
13. G.G. Lorentz, *A contribution to the theory of divergent sequences*, Acta Math. **80** (1948), 167–190. MR **10**, 367.
14. S. Mazur and W. Orlicz, *On linear methods of summability*, Studia Math. **14** (1954), 129–160. MR **16**, 814.
15. C.W. McArthur, *On a theorem of Orlicz and Pettis*, Pacific J. Math. **22** (1967), 297–302. MR **35** #4702.
16. C.W. McArthur and J.R. Retherford, *Uniform and equicontinuous Schauder bases of subspaces*, Canad. J. Math. **17** (1965), 207–212. MR **30** #4141.
17. G.M. Petersen, *Almost convergence and the Buck-Pollard property*, Proc. Amer. Math. Soc. **11** (1960), 469–477. MR **22** #2819.
18. H.H. Schaefer, *Topological vector spaces*, Macmillan, New York, 1966. MR **33** #1689.
19. P. Schaefer, *Almost convergent and almost summable sequences*, Proc. Amer. Math. Soc. **20** (1969), 51–54. MR **38** #3649.
20. A.K. Snyder, *Conull and coregular FK-spaces*, Math. Z. **90** (1965), 376–381. MR **32** #2783.
21. I. Tweddle, *Vector-valued measures*, Proc. London Math. Soc. (3) **20** (1970), 469–489. MR **41** #3707.
22. A. Wilansky, *Topics in functional analysis*, Lecture Notes in Math., no. 45, Springer-Verlag, Berlin and New York, 1967. MR **36** #6901.
23. A. Więger, *Linear spaces with mixed topology*, Studia Math. **20** (1961), 47–68. MR **24** #A3490.
24. K. Zeller, *Allgemeine Eigenschaften von Limitierungsverfahren*, Math. Z. **53** (1951), 463–487. MR **12**, 604.

- 25.———, *Abschnittskonvergenz in FK-Räumen*, Math. Z. **55** (1951), 55–70. MR 13, 934.
26. K. Zeller and W. Beekman, *Theorie der Limitierungsverfahren. Zweite, erweiterte und verbesserte Auflage*, Ergebnisse der Math. und ihrer Grenzgebiete, Band 15, Springer-Verlag, Berlin and New York, 1970. MR 41 #8863.

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