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LINEAR TRANSFORMATIONS ON MATRICES

BY
D. Ž. DJOKOVIĆ(1)

ABSTRACT. The real orthogonal group $O(n)$, the unitary group $U(n)$ and the symplectic group Sp (n) are embedded in a standard way in the real vector space of $n \times n$ real, complex and quaternionic matrices, respectively. Let F be a nonsingular real linear transformation of the ambient space of matrices such that $F(G) \subset G$ where G is one of the groups mentioned above. Then we show that either $F(x) = a\sigma(x)b$ or $F(x) = a\sigma(x^*)b$ where $a, b \in G$ are fixed, x^* is the transpose conjugate of the matrix x and σ is an automorphism of reals, complexes and quaternions, respectively.

1. Introduction. In his survey paper [4], M. Marcus has stated seven conjectures. In this paper we shall study the last two of these conjectures.

Let D be one of the following real division algebras **R** (the reals), **C** (the complex numbers) or **H** (the quaternions). As usual we assume that $\mathbf{R} \subset \mathbf{C} \subset \mathbf{H}$. We denote by $M_n(D)$ the real algebra of all $n \times n$ matrices over D. If $\xi \in D$ we denote by $\bar{\xi}$ its conjugate in D. Let A be the automorphism group of the real algebra D. If $D = \mathbf{R}$ then A is trivial. If $D = \mathbf{C}$ then A is cyclic of order 2 with conjugation as the only nontrivial automorphism. If $D = \mathbf{H}$ then the conjugation is not an isomorphism but an anti-isomorphism of **H**. Every automorphism of **H** is inner (by the Noether-Skolem theorem) and every automorphism of **H** commutes with conjugation.

If $x \in M_n(D)$ then we let $\bar{x}$ be the conjugate matrix, ${}^t x$ the transpose of x and x^* the conjugate transpose of x. The rule $(xy)^* = y^* x^*$ is valid for any $x, y \in M_n(D)$. If $x \in M_n(D)$ and $\sigma \in A$ we let $\sigma(x)$ be the matrix obtained from x by applying σ to each of its entries.

Let $e \in M_n(D)$ be the identity matrix and put $G = \{x \in M_n(D) \mid xx^* = e\}$. We have

$$G = O(n) \text{ the real orthogonal group,} \qquad \text{if } D = \mathbf{R};$$

$$G = U(n) \text{ the unitary group,} \qquad \text{if } D = \mathbf{C};$$

$$G = \mathrm{Sp}\,(n) \text{ the symplectic group,} \qquad \text{if } D = \mathbf{H}.$$

These are classical compact Lie groups. When $D = \mathbf{R}$ then $G = O(n)$ has two components and its identity component is the so called special orthogonal group SO (n), also called the rotation group.

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Now, we can state a slightly modified version of Conjecture 6:

Conjecture 6. *Let F: $M_n(D) \to M_n(D)$ be an* **R**-*linear transformation such that $F(G) \subset G$. Then F must have one of the following two forms: Either* (1) $F(x) = a\sigma(x)b$ *for all* $x \in M_n(D)$, *or* (2) $F(x) = a\sigma(x^*)b$ *for all* $x \in M_n(D)$, *where* $a, b \in G$ *and* $\sigma \in A$. *Also, if* $D = \mathbf{R}$ *and* $F(\mathrm{SO}\,(n)) \subset \mathrm{SO}\,(n)$ *then* (1) *or* (2) *holds.*

In his earlier paper [3] M. Marcus has shown that if $D = \mathbf{C}$ and F is also supposed to be a **C**-linear map then $F(G) \subset G$ implies that either

$$F(x) = axb \qquad \text{for all } x \in M_n(\mathbf{C}), \quad \text{or}$$

$$F(x) = a^t xb \qquad \text{for all } x \in M_n(\mathbf{C}),$$

where $a, b \in U(n)$. These results agree with the conjecture because if we want F to be **C**-linear we must take σ = identity in (1) and σ = conjugation in (2).

If $D = \mathbf{H}$ note that $x \in \mathrm{Sp}\,(n)$ implies that $\sigma(x^*) \in \mathrm{Sp}\,(n)$ because $\sigma \in A$ commutes with the conjugation.

One can equip the real vector space $M_n(D)$ with a positive definite scalar product as follows:

$$\langle x,y\rangle = \mathrm{Re}\,\mathrm{tr}(xy^*). \tag{3}$$

The Conjecture 6, as stated above, has been proved in [2] under additional assumption that F is orthogonal when $M_n(D)$ is equipped with the scalar product (3).

Here we shall prove Conjecture 6 when F is assumed to be nonsingular.

To introduce the next conjecture let V be a finite dimensional complex Hilbert space and let $V \otimes_{\mathbf{C}} V$ be equipped with the scalar product which satisfies: $\langle x \otimes y, a \otimes b\rangle = \langle x,a\rangle\langle y,b\rangle$ for all x, y, a, b in V.

Conjecture 7. *If u: $V \otimes_{\mathbf{C}} V \to V \otimes_{\mathbf{C}} V$ is a complex linear map such that*

$$\|u(a \otimes b)\| = \|a \otimes b\| \quad \textit{for all } a, b \in V$$

then u must be unitary.

We shall show that this conjecture is true. It is worthwhile to remark that the analogous conjecture for real Hilbert spaces is false. I am indebted to M. Marcus for a counterexample to the latter.

2. Matrix equation $ax = xa^*$.

Theorem 1. *Let $a \in O(n)$ have the eigenvalues*

$$\exp(\pm i\theta_r) \textit{ of multiplicity } k_r, 1 \le r \le s;$$

$$+1 \textit{ of multiplicity } p;$$

$$-1 \textit{ of multiplicity } q;$$

where $0 < \theta_r < \pi$ are distinct, p, q, s are nonnegative integers and $2(k_1 + \cdots + k_s) + p + q = n$.

Then the real vector space of skew-symmetric matrices $x \in M_n(\mathbf{R})$ satisfying the equation

$$ax = x^t a \tag{4}$$

has dimension $\sum_{r=1}^{s} k_r(k_r - 1) + \frac{1}{2}p(p-1) + \frac{1}{2}q(q-1)$.

Proof. By performing a similarity transformation by a suitable orthogonal matrix we can assume that a has the canonical quasi-diagonal form. The diagonal blocks of a are the matrices

$$a_r = \begin{pmatrix} \cos\theta_r & -\sin\theta_r \\ \sin\theta_r & \cos\theta_r \end{pmatrix}, \qquad 1 \le r \le s,$$

each a_r being repeated k_r times, followed by p one's and q minus one's.

We partition the unknown skew-symmetric matrix x into blocks of the same size as the corresponding blocks of a.

Now we make the following observations.

(i) If y is a 2×2 diagonal block of x corresponding to an a_r then $y = 0$. Indeed (4) implies that $a_r y = y^t a_r$. Since y is skew-symmetric and $\sin\theta_r \neq 0$ we must have $y = 0$.

(ii) If

$$\begin{pmatrix} 0 & y \\ -^t y & 0 \end{pmatrix}$$

is a principal submatrix of x with y some 2×2 block of x and if the corresponding submatrix of a is

$$\begin{pmatrix} a_r & 0 \\ 0 & a_m \end{pmatrix}, \qquad r \neq m$$

then $y = 0$.

In this case (4) implies that $a_r y = y^t a_m$. Since a_r and a_m are not similar we see that y must be singular. Hence we can write $y = \left(\begin{smallmatrix} \lambda\alpha & \lambda\beta \\ \mu\alpha & \mu\beta \end{smallmatrix}\right)$ where $\lambda, \mu, \alpha, \beta \in \mathbf{R}$. By inspection of the first columns of $a_r y$ and $y^t a_m$ we find that

$$\begin{aligned} \alpha(\lambda\cos\theta - \mu\sin\theta) &= \lambda(\alpha\cos\phi - \beta\sin\phi), \\ \alpha(\lambda\sin\theta + \mu\cos\theta) &= \mu(\alpha\cos\phi - \beta\sin\phi) \end{aligned} \tag{5}$$

where $\theta_r = \theta$, $\theta_m = \phi$.

From (5) we deduce that $\alpha(\lambda\mu\cos\theta - \mu^2\sin\theta) = \alpha(\lambda^2\sin\theta + \lambda\mu\cos\theta)$ and then $\alpha(\lambda^2 + \mu^2)\sin\theta = 0$. But $\sin\theta \neq 0$. If $\lambda^2 + \mu^2 = 0$ then $\lambda = \mu = 0$ and $y = 0$. Otherwise $\alpha = 0$, then from (5) we see that $\beta = 0$ and then $y = 0$ follows.

(iii) If

$$\begin{pmatrix} 0 & y \\ -{}^t y & 0 \end{pmatrix}$$

is a principal submatrix of x where y is a 2×2 block of x and the corresponding submatrix of a is

$$\begin{pmatrix} a_r & 0 \\ 0 & a_r \end{pmatrix}$$

then the equation $a_r y = y\,{}^t a_r$ (which follows from (4)) is satisfied only by the matrices y of the form $\left(\begin{smallmatrix} \alpha & \beta \\ \beta & -\alpha \end{smallmatrix}\right)$. This can be seen by a simple computation.

(iv) If

$$\begin{pmatrix} 0 & y \\ -{}^t y & 0 \end{pmatrix}$$

is a principal submatrix of x where y is a 2×1 block of x and the corresponding submatrix of a is

$$\begin{pmatrix} a_r & 0 \\ 0 & \pm 1 \end{pmatrix}$$

then $y = 0$.

Indeed, (4) implies that $a_r y = \pm y$ but a_r has no real eigenvalues.

(v) If $\left(\begin{smallmatrix} 0 & \alpha \\ -\alpha & 0 \end{smallmatrix}\right)$ is a principal submatrix of x and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is the corresponding submatrix of a then $\alpha = 0$.

This is immediate from (4).

After these observations the formula for the dimension of the space of skew-symmetric solutions of (4) follows by a simple counting.

Theorem 2. *Let $a \in U(n)$ have the eigenvalues*

$$\exp(i\theta_r) \text{ of multiplicity } k'_r;\ 1 \le r \le s,$$

$$\exp(-i\theta_r) \text{ of multiplicity } k''_r;\ 1 \le r \le s,$$

$$+1 \text{ of multiplicity } p;$$

$$-1 \text{ of multiplicity } q;$$

where k'_r, k''_r, p, q, s are nonnegative integers, the θ_r's are distinct and lie in the interval $0 < \theta < \pi$, and $\sum_{r=1}^{s} (k'_r + k''_r) + p + q = n$.

Then the real vector space of skew-hermitian matrices $x \in M_n(\mathbf{C})$ satisfying the equation $ax = xa^*$ has dimension $2\sum_{r=1}^{s} k'_r k''_r + p^2 + q^2$.

Proof. We can assume that a has diagonal form. It is easy to verify that the (r,m)-entry of x must be 0 if the rth and mth diagonal entry of a are not conjugate; otherwise this entry may be arbitrary if $r \neq m$ and arbitrary purely imaginary number if $r = m$.

Then a simple counting will complete the proof.

Before stating the next theorem we need to introduce some more terminology. We say that two quaternions ξ and η are similar if there exists $\sigma \in A$ such that $\sigma(\xi) = \eta$. It follows easily from the Noether-Skolem theorem (see [1]) that ξ and $\bar{\xi}$ are always similar. It is well known that the eigenvalues of $u \in \mathrm{Sp}\,(n)$ are determined uniquely up to similarity.

Theorem 3. *Let* $a \in \mathrm{Sp}\,(n)$ *have the eigenvalues*

$$\lambda_r \text{ with multiplicity } k_r, 1 \leq r \leq s;$$

$$+1 \text{ with multiplicity } p,$$

$$-1 \text{ with multiplicity } q,$$

where $\lambda_r \in \mathbf{H}$ *are nonsimilar and not real;* p, q, s *are nonnegative integers and* $(k_1 + \cdots + k_s) + p + q = n$. *Then the real vector space of skew-hermitian matrices* $x \in M_n(\mathbf{H})$ *satisfying the equation*

$$ax = xa^* \tag{6}$$

has the dimension $\sum_{r=1}^{s} k_r(k_r + 1) + p(2p + 1) + q(2q + 1)$.

Proof. By performing a similarity transformation by a suitable sympletic matrix we can assume that a has diagonal form with the above listed eigenvalues on the diagonal.

Let

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \gamma \end{pmatrix}, \qquad \alpha + \bar{\alpha} = \gamma + \bar{\gamma} = 0,$$

be a principal submatrix of x and

$$\begin{pmatrix} \lambda_r & 0 \\ 0 & \lambda_m \end{pmatrix}$$

the corresponding submatrix of a. If $r \neq m$ we claim that $\beta = 0$. Indeed, (6) gives

$$\lambda_r \beta = \beta \bar{\lambda}_m \,. \tag{7}$$

Since λ_r and $\bar{\lambda}_m$ are not similar we must have $\beta = 0$.

If $r = m$ we claim that the solution space of Equation (7) has dimension 2. Indeed, we can assume that $\lambda_r = i$ is one of the basic units of $\mathbf{H}$ and then our assertion is obvious because the solution space is spanned by the other two basic units j and k, say.

For α we have the equation $\lambda_r \alpha = \alpha \bar{\lambda}_r$ and again the space of solutions has dimension 2.

Similarly, if the corresponding submatrix of a is

$$\begin{pmatrix} \lambda_r & 0 \\ 0 & \pm 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

we get again $\beta = 0$.

Finally, if the corresponding submatrix of a is $\pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ then β may be arbitrary and α and γ are subject only to the condition $\alpha + \bar{\alpha} = \gamma + \bar{\gamma} = 0$.

Thus x must be a quasi-diagonal matrix with diagonal blocks corresponding to the blocks of equal diagonal entries of a.

Simple counting gives us the required formula for the dimension.

3. **Conjecture 6 (General case).** From now on we shall assume that F is nonsingular and satisfies $F(G) \subset G$. Without loss of generality we can assume that $F(e) = e$. Indeed, if $F(e) = a$ then it suffices to replace F by F_1 which is defined by $F_1(x) = a^{-1}F(x)$ since then $F_1(e) = e$.

We consider G as an analytic submanifold of $M_n(D)$. As usual, we shall identify the tangent vectors of G at some point of G with the corresponding matrices in $M_n(D)$. In this sense we can say that the tangent space to G at e is the real vector space of skew-hermitian matrices in $M_n(D)$. Let us denote this vector space by S, i.e., $S = \{x \in M_n(D) \mid x^* = -x\}$. Then the tangent space to G at some point $a \in G$ will be the space $a \cdot S = \{a \cdot x \mid x \in S\}$. Since F is linear it coincides with its own differential. Thus F must map the tangent space of G at $a \in G$ bijectively to the tangent space of G at $F(a)$, i.e., we have

$$F(a \cdot S) = F(a) \cdot S, \quad \text{for all } a \in G. \tag{8}$$

For $a = e$ we get $F(S) = S$ because $F(e) = e$.

We have $F(a \cdot S \cap S) = F(a) \cdot S \cap S$ for $a \in G$ and in particular,

$$\dim (F(a) \cdot S \cap S) = \dim (a \cdot S \cap S), \qquad a \in G. \tag{9}$$

It is easy to see that $aS \cap S$ is precisely the space whose dimension was calculated in §2.

It is easy to verify Conjecture 6 when $n = 1$. Indeed, the case $D = \mathbf{R}$ is trivial. If $D = \mathbf{C}$ then we identify G with the unit circle in $\mathbf{C}$. Since $F(G) = G$ and $F(1) = 1$ we see that F is orthogonal and so either it is identity or conjugation. If $D = \mathbf{H}$ then $G = \mathrm{Sp}\,(1)$ can be identified with the unit 3-sphere of $\mathbf{H}$. Since $F(G) = G$ it follows that F is orthogonal. From $F(1) = 1$ it follows that F preserves the pure quaternions. Since A is transitive on the unit sphere of the space of pure quaternions we can assume further that $F(i) = i$. Since the unit circle group of $\mathbf{C}$ acts transitively by inner automorphisms on the unit circle of the plane $j\mathbf{R} + k\mathbf{R}$ we can further assume that $F(j) = j$. Then we have either

$F(k) = k$ or $F(k) = -k$. In the second case it is easy to see that F is the map $\xi \mapsto k^{-1}\xi k$.

Instead of these arguments we could quote the paper [2] because it is obvious that these F are orthogonal maps.

From now on we shall assume that $n \geq 2$. Let $a \in G$, $a \neq \pm e$. Then if we assume that $n \geq 3$ we deduce easily from Theorems 1–3 that $\dim(aS \cap S)$ is maximal if and only if a or $-a$ is a reflection, i.e., has the following eigenvalues

$$(10) \qquad \begin{aligned} &+1 \text{ with multiplicity } n-1, \\ &-1 \text{ with multiplicity } 1. \end{aligned}$$

This fact and (9) imply that if $a \in G$ is a reflection then either $F(a)$ or $-F(a)$ is again a reflection.

Now, let e_r, $1 \leq r \leq n$ be the diagonal matrix whose rth diagonal entry is -1 and all other diagonal entries are $+1$. Then we have $e_1 + \cdots + e_n = (n-2)e$, and

$$(11) \qquad F(e_1) + \cdots + F(e_n) = (n-2)e.$$

Since (10) are the eigenvalues of $F(e_r)$ or of $-F(e_r)$ we have $\operatorname{Re}\operatorname{tr} F(e_r) = \pm(n-2)$. From (11) we get $\sum_{r=1}^{n} \operatorname{Re}\operatorname{tr} F(e_r) = n(n-2)$. Hence, we must have $\operatorname{Re}\operatorname{tr} F(e_r) = n-2$ for every r. Thus $F(e_r)$ is a reflection for every r. By performing a similarity transformation by some matrix in G we can assume that $F(e_1) = e_1$. The equation (11) now implies that $F(e_2), \ldots, F(e_n)$ each have 1 in the upper left corner. Since $F(e_r) \in G$ this must be the only nonzero entry of $F(e_r)$ lying in the first row or column. By continuing this argument we see that we can assume that $F(e_r) = e_r$ for $1 \leq r \leq n$.

Now, let $n = 2$. If $D = \mathbf{R}$ then $S \cap G$ is invariant under F. Thus if $a = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ then either $F(a) = a$ or $F(a) = -a$. By composing F with transposition we may assume that $F(a) = a$. Thus F fixes every element of SO(2). Hence, $F(e_1)$ must be a reflection and we can assume as above that $F(e_1) = e_1$. Then, of course, $F(e_2) = e_2$ because $e_2 = -e_1$.

If $D = \mathbf{C}$, $a \in G$ and $\dim(a \cdot S \cap S) = 2$ then it follows from Theorem 2 that a is either a reflection or else it is unitarily similar to a matrix $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{smallmatrix}\right)$, where $\lambda \in \mathbf{C}$ is such that $|\lambda| = 1$ and $\lambda \notin \mathbf{R}$. Thus (9) implies that $F(e_1)$ is either a reflection or else we can assume that $F(e_1)$ is the above matrix. But since $F(e_1) \in \mathbf{R}e + S$, $e_1 \notin \mathbf{R}e + S$ and $\mathbf{R}e + S$ is F-invariant subspace it follows that the second case cannot occur. Hence, $F(e_1)$ is a reflection and we can assume that $F(e_1) = e_1$. Since $e_2 = -e_1$ we will also have $F(e_2) = e_2$.

If $D = \mathbf{H}$, $a \in G$ and $\dim(a \cdot S \cap S) = 6$ then it follows from Theorem 3 that a is either a reflection or else it is similar by some matrix in G to a matrix λe where $\lambda \in \mathbf{H}$, $|\lambda| = 1$ but $\lambda \notin \mathbf{R}$. Again the latter case cannot occur because we can in that case assume that $F(e_1) = \lambda e$ but $\lambda e \in \mathbf{R}e + S$, $e_1 \notin \mathbf{R}e + S$ and the subspace $\mathbf{R}e + S$ is F-invariant. It follows that $F(e_1)$ is a reflection and we can then assume that $F(e_1) = e_1$, $F(e_2) = e_2$.

4. Conjecture 6 (Orthogonal group).

Theorem 4. *Conjecture 6 is true when G is the orthogonal group and F is nonsingular.*

Proof. We claim that if $u \in M_n(\mathbf{R})$ has rank 1 then also $F(u)$ has rank 1.

Let m_{pq} be the matrix whose (p,q)-entry is 1 and all other entries are zero. By performing a similarity transformation by an orthogonal matrix a we can assume without loss of generality that $u = \alpha m_{11} + \beta m_{21}$. By performing another similarity transformation by $b \in G$ we can in addition assume that $F(e_r) = e_r$, $1 \leq r \leq n$.

Indeed, this reduction means that we are replacing F, which is supposed to satisfy $F(e) = e$, by the transformation F_1 defined as follows: $F_1(x) = bF(axa^{-1})b^{-1}$.

It is easy to see that the subspace

$$(12) \qquad \left(\bigcap_{r=3}^{n} e_r \cdot S\right) \cap S$$

has dimension 1 and the matrix $m_{12} - m_{21}$ lies in it. It follows from (8) that the subspace (12) is an invariant subspace of the linear transformation F. Thus $F(m_{12} - m_{21}) = \lambda(m_{12} - m_{21})$. But $(m_{12} - m_{21}) + \frac{1}{2}(e_1 + e_2) \in G$ and applying F we get

$$\lambda(m_{12} - m_{21}) + \tfrac{1}{2}(e_1 + e_2) \in G.$$

Therefore, $\lambda = \pm 1$.

Similarly, the subspace $(\bigcap_{r=3}^{n} e_1 e_r S) \cap e_1 S \cap e_2 S$ is F-invariant, has dimension 1 and $m_{12} + m_{21}$ lies in it. Therefore,

$$F(m_{12} + m_{21}) = \mu(m_{12} + m_{21})$$

and we get $\mu = \pm 1$ by a similar argument.

Since $u = \frac{1}{2}\alpha(e - e_1) + \frac{1}{2}\beta[(m_{21} - m_{12}) + (m_{21} + m_{12})]$ we have

$$F(u) = \tfrac{1}{2}\alpha(e - e_1) + \tfrac{1}{2}\beta[\lambda(m_{21} - m_{12}) + \mu(m_{21} + m_{12})].$$

It follows that $F(u)$ has rank 1 because $\lambda = \pm 1$ and $\mu = \pm 1$.

Since F preserves the matrices of rank 1 we must have (see [5]) either

$$F(x) = axb \qquad \text{for } x \in M_n(\mathbf{R}), \quad \text{or}$$
$$F(x) = a\,{}^t x b \qquad \text{for } x \in M_n(\mathbf{R}),$$

where a and b are some nonsingular matrices. From $F(e) = e$ we deduce that $b = a^{-1}$. In both cases we must have $aya^{-1} \in G$ for $y \in G$, i.e.,

$$(aya^{-1}) \cdot {}^t(aya^{-1}) = e, \qquad {}^t aay = y\,{}^t aa.$$

This implies that ${}^t aa = \rho e$ where $\rho > 0$. Thus we have that $a/\sqrt{\rho} \in G$ which completes the proof.

Remark about the rotation group. It is obvious that the Conjecture 6 is also true for the case of the rotation group if n is odd. Indeed, in that case $-e \notin \mathrm{SO}(n)$. If $F(\mathrm{SO}(n)) \subset \mathrm{SO}(n)$ and $x \in O(n)$ but $x \notin \mathrm{SO}(n)$ we have $-x = x \cdot (-e) \in \mathrm{SO}(n)$ and $F(x) = -F(-x) = (-e) \cdot F(-x) \in O(n)$, i.e., $F(O(n)) \subset O(n)$. Thus, we can apply Theorem 4.

The Conjecture 6 is false for the rotation group if $n = 2$. Indeed, in that case we can define F as follows: F fixes the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and it maps

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then F is nonsingular, it preserves SO(2) but not $O(2)$.

5. Conjecture 6 (Unitary group).

Theorem 5. *Conjecture 6 is true when G is the unitary group and F is assumed to be nonsingular.*

Proof. As shown in §3 we can assume that $F(e_r) = e_r$ for $1 \leq r \leq n$. Note that F is assumed only to be $\mathbf{R}$-linear map of $M_n(\mathbf{C})$ into itself. Our next objective is to reduce the problem to the case when F is $\mathbf{C}$-linear.

We define the matrices m_{pq} as in the previous section. The vector space $(\bigcap_{r=1}^{n} e_r S) \cap S$ is F-invariant and consists of diagonal matrices with purely imaginary diagonal entries. Thus, for instance,

$$F(im_{11}) = i \sum_{r=1}^{n} \alpha_r m_{rr}, \qquad \alpha_r \in \mathbf{R}.$$

But $e + (i-1)m_{11} \in G$ implies that

$$e - m_{11} + i \sum_{r=1}^{n} \alpha_r m_{rr} \in G.$$

Therefore, $\alpha_1 = \pm 1$ and $\alpha_r = 0$ for $2 \leq r \leq n$. Hence, in general,

$$F(\xi m_{rr}) = \xi m_{rr} \qquad \text{for } \xi \in \mathbf{C}, \quad \text{or}$$

$$F(\xi m_{rr}) = \bar{\xi} m_{rr} \qquad \text{for } \xi \in \mathbf{C},$$

where, a priori, the alternative may depend on r.

Assume that, say, we have $F(\xi m_{11}) = \xi m_{11}$, $F(\xi m_{22}) = \bar{\xi} m_{22}$. Then by Theorem 2 we have

$$\dim(a \cdot S \cap S) = 2 + (n-2)^2 \quad \text{and} \quad \dim(F(a) \cdot S \cap S) = (n-2)^2$$

where

$$a = e + (i-1)m_{11} - (i+1)m_{22}, \quad F(a) = e + (i-1)m_{11} + (i-1)m_{22}.$$

This contradicts Equation (9). Thus the alternative mentioned above is independent of r. By composing F with conjugation we may assume that $F(\xi m_{rr}) = \xi m_{rr}$, $\xi \in \mathbf{C}$, $1 \leq r \leq n$. Let, for instance, $\omega = \exp(i\pi/4)$ and define $b = \omega e + (i-\omega)m_{11} - (i+\omega)m_{22}$. Then $F(b) = b$ and the subspace $b \cdot S \cap S$ is F-invariant and consists of the matrices $\xi m_{12} - \bar{\xi} m_{21}$, $\xi \in \mathbf{C}$.

Since $e_1 b \in G$ is diagonal we have $F(e_1 b) = e_1 b$. Thus the subspace $e_1(b \cdot S \cap S) = (e_1 b) \cdot S \cap e_1 \cdot S$ is also F-invariant and consists of the matrices $\xi m_{12} + \bar{\xi} m_{21}$, $\xi \in \mathbf{C}$. Thus there exist real linear transformations $u_1, u_2 \colon \mathbf{C} \to \mathbf{C}$ such that

$$F(\xi m_{12} - \bar{\xi} m_{21}) = u_1(\xi) m_{12} - \overline{u_1(\xi)} m_{21},$$
$$F(\xi m_{12} + \bar{\xi} m_{21}) = u_2(\xi) m_{12} + \overline{u_2(\xi)} m_{21}.$$

Hence,

$$F(\xi m_{12}) = v_1(\xi) m_{12} + \overline{v_2(\xi)} m_{21}, \qquad F(\bar{\xi} m_{21}) = v_2(\xi) m_{12} + \overline{v_1(\xi)} m_{21},$$

where $2v_1 = u_2 + u_1$, $2v_2 = u_2 - u_1$.

Since, for $|\lambda| = |\mu| = 1$, $\lambda m_{12} + \bar{\mu} m_{21} + e - m_{11} - m_{22} \in G$, its image under F is in G. This implies that $|v_1(\lambda) + v_2(\mu)| = 1$ whenever $|\lambda| = |\mu| = 1$. It follows that v_1 or v_2 is zero and the other one is an orthogonal transformation of $\mathbf{C}$.

Assume first that $v_2 = 0$. Then either $v_1(\xi) = \alpha\xi$, $|\alpha| = 1$, or $v_1(\xi) = \alpha\bar{\xi}$, $|\alpha| = 1$. In the first case we have $F(\xi m_{12}) = \xi F(m_{12})$. Suppose that the second case is valid. Then we have $F(\xi m_{12}) = \alpha\bar{\xi} m_{12}$, $F(\xi m_{21}) = \bar{\alpha}\bar{\xi} m_{21}$. The quasi-diagonal matrix having

$$\begin{pmatrix} \cos\theta & i\sin\theta \\ \sin\theta & -i\cos\theta \end{pmatrix}$$

in the upper left corner and ones on the remaining diagonal places is unitary. By applying F we get a matrix which is not unitary if $\sin\theta\cos\theta \neq 0$. This is a contradiction.

Now, assume that $v_1 = 0$. Then either

$$v_2(\xi) = \alpha\bar{\xi}, \quad |\alpha| = 1, \quad \text{or} \quad v_2(\xi) = \alpha\xi, \quad |\alpha| = 1.$$

In the first case we have $F(\xi m_{12}) = \xi F(m_{12})$. Suppose that the second case is valid. Then we have $F(\xi m_{12}) = \bar{\alpha}\bar{\xi} m_{21}$, $F(\xi m_{21}) = \alpha\bar{\xi} m_{12}$. By applying F to the quasi-diagonal matrix mentioned above we get again a matrix which is not unitary if $\sin\theta\cos\theta \neq 0$. This is again a contradiction.

It follows that we must have $F(\xi m_{12}) = \xi F(m_{12})$ for $\xi \in \mathbf{C}$.

Similarly $F(\xi m_{pq}) = \xi F(m_{pq})$ for all p, q and $\xi \in \mathbf{C}$, in other words F must be $\mathbf{C}$-linear.

Now, we can complete the proof as in the case of the orthogonal group, or else we can use the result of M. Marcus [3].

6. Conjecture 6 (Symplectic Group).

Theorem 6. *Conjecture* 6 *is true when G is the symplectic group and F is assumed to be nonsingular.*

Proof. From §3 we know that we can assume that $F(e_r) = e_r, 1 \leq r \leq n$. The subspace $(\bigcap_{r=1}^{n} e_r \cdot S) \cap S$ is F-invariant and consists of diagonal matrices with pure quaternionic entries. As in the case of the unitary group we obtain that $F(\xi e_r) = \sigma_r(\xi)e_r$, $1 \leq r \leq n$, $\xi \in \mathbf{H}$, where $\sigma_r \in A$. For every r there exists a quaternion α_r of unit norm such that

$$\sigma_r(\xi) = \alpha_r \xi \alpha_r^{-1} = \alpha_r \xi \bar{\alpha}_r, \qquad \xi \in \mathbf{H}.$$

Let a be the symplectic diagonal matrix with diagonal entries $\alpha_1, \ldots, \alpha_n$. Then replacing F by F_1 which is defined as follows $F_1(x) = F(axa^*)$ we can assume that F, in addition, satisfies $F(\xi e_r) = \xi e_r$, $1 \leq r \leq n$, for all $\xi \in \mathbf{H}$.

Let $S' = S \cap M_n(\mathbf{R})$ be the space of real skew-symmetric matrices and S'' the space of real symmetric matrices. Then S is a direct sum

$$S = S' + iS'' + jS'' + kS''$$

where $1, i, j, k$ are the standard units of $\mathbf{H}$. It is obvious that

$$S'' = iS \cap jS \cap kS, \qquad iS'' = S \cap jS \cap kS,$$

$$jS'' = S \cap iS \cap kS, \qquad kS'' = S \cap iS \cap jS.$$

Since, say, $i \cdot S = (ie) \cdot S$ and $ie \in G$, $F(ie) = ie$ then we can use formula (8) to get

$$F(S'') = S'', \quad F(iS'') = iS'', \quad F(jS'') = jS'', \quad F(kS'') = kS''.$$

The space

$$\left(\bigcap_{r=3}^{n} e_r \cdot S\right) \cap S \tag{13}$$

consists of skew-hermitian matrices whose nonzero entries occur only on the diagonal or at places (1, 2), (2, 1). We shall use the fact that this space is F-invariant.

Let $|\xi| = 1$ and $|\xi_r| = 1$, $3 \leq r \leq n$ where $\xi, \xi_r \in \mathbf{H}$. Then

$$\xi m_{12} - \bar{\xi} m_{21} + \xi_3 m_{33} + \cdots + \xi_n m_{nn} \in G$$

and by applying F we get

$$F(\xi m_{12} - \bar{\xi} m_{21}) + \xi_3 m_{33} + \cdots + \xi_n m_{nn} \in G.$$

Now, the fact that the above space is F-invariant implies that the nonzero entries of $F(\xi m_{12} - \bar{\xi} m_{21})$ are concentrated in the 2×2 block in the upper left corner of that matrix and that this block is a symplectic matrix.

Let

$$b = (i - 1)m_{11} - (i + 1)m_{22} + e,$$
$$c = (j - 1)m_{11} - (j + 1)m_{22} + e,$$
$$d = (k - 1)m_{11} - (k + 1)m_{22} + e.$$

Then $b, c, d \in G$ are diagonal, $F(b) = b$, $F(c) = c$, $F(d) = d$ and the space

$$b \cdot S \cap c \cdot S \cap d \cdot S \tag{14}$$

is F-invariant. The intersection of the spaces (13) and (14) consists of the matrices of the form

$$\begin{pmatrix} 0 & \alpha & & & \\ -\alpha & 0 & & & \\ & & \xi_3 & & \\ & & & \ddots & \\ & & & & \xi_n \end{pmatrix}, \quad \bar{\xi}_r = -\xi_r, \tag{15}$$

where α is real. Therefore, $F(m_{12} - m_{21}) = \pm(m_{12} - m_{21})$.

In general, $F(m_{pq} - m_{qp}) = \pm(m_{pq} - m_{qp})$ which means that $F(S') = S'$. Since also $F(S'') = S''$ we conclude that $M_n(\mathbf{R})$ is F-invariant. Thus F preserves $O(n)$ and by Theorem 4 we may assume, in addition to other properties of F, that F is identity on $M_n(\mathbf{R})$.

Let us multiply on the left by i each of the subspaces $e_r \cdot S$, S, $b \cdot S$, $c \cdot S$, $d \cdot S$ which occur in (13) and (14). Then by taking their intersection we get an F-invariant subspace which consists of the matrices ix where x has form (15). Hence we have

$$F(im_{12} - im_{21}) = \lambda i(m_{12} - m_{21}) + i(\eta_3 m_{33} + \cdots + \eta_n m_{nn})$$

where $\lambda \in \mathbf{R}$ and $\bar{\eta}_r = -\eta_r$.

If $|\xi_r| = 1$ for $3 \leq r \leq n$ then

$$i(m_{12} - m_{21}) + i(\xi_3 m_{33} + \cdots + \xi_n m_{nn}) \in G$$

and also its image by F is in G. This implies that $\lambda = \pm 1$ and $\eta_r = 0$ for $3 \leq r \leq n$. Hence,

$$F(i(m_{12} - m_{21})) = \pm i(m_{12} - m_{21})$$

and similarly,

$$F(i(m_{pq} - m_{qp})) = \pm i(m_{pq} - m_{qp}).$$

Thus $F(iS') = iS'$ and by similar arguments $F(jS') = jS'$, $F(kS') = kS'$. It follows that the subspaces $iM_n(\mathbf{R})$, $jM_n(\mathbf{R})$, $kM_n(\mathbf{R})$ are also F-invariant. Thus F maps $U(n)$ onto itself and now by Theorem 5 we can conclude that the restriction of F to the subspace $M_n(\mathbf{R}) + iM_n(\mathbf{R})$ is orthogonal with respect to the scalar product (3). Using two other embeddings of $\mathbf{C}$ into $\mathbf{H}$ we see that also the restrictions of F to the subspaces $M_n(\mathbf{R}) + jM_n(\mathbf{R})$ and $M_n(\mathbf{R}) + kM_n(\mathbf{R})$ are orthogonal. This implies that the linear transformation F is orthogonal for the same scalar product. Hence we can now use the result [2] to complete the proof.

7. Conjecture 7.

Theorem 7. *Conjecture 7 is true.*

Proof. Let $x, y, z, t \in V$. Then we have

$$\langle u((x+y) \otimes z), u((x+y) \otimes z)\rangle = \langle (x+y) \otimes z, (x+y) \otimes z\rangle,$$

$$\langle u(x \otimes z) + u(y \otimes z), u(x \otimes z) + u(y \otimes z)\rangle = \langle x \otimes z + y \otimes z, x \otimes z + y \otimes z\rangle,$$

$$\langle u(x \otimes z), u(y \otimes z)\rangle + \langle u(y \otimes z), u(x \otimes z)\rangle = \langle x \otimes z, y \otimes z\rangle + \langle y \otimes z, x \otimes z\rangle,$$

$$\operatorname{Re}\langle u(x \otimes z), u(y \otimes z)\rangle = \operatorname{Re}\langle x \otimes z, y \otimes z\rangle.$$

By replacing x with ix we get also

$$\operatorname{Im}\langle u(x \otimes z), u(y \otimes z)\rangle = \operatorname{Im}\langle x \otimes z, y \otimes z\rangle.$$

Thus $\langle u(x \otimes z), u(y \otimes z)\rangle = \langle x \otimes z, y \otimes z\rangle$. Now, replace z by $z + t$. Then we get

$$\langle u(x \otimes z) + u(x \otimes t), u(y \otimes z) + u(y \otimes t)\rangle = \langle x \otimes z + x \otimes t, y \otimes z + y \otimes t\rangle,$$

$$\langle u(x \otimes z), u(y \otimes t)\rangle + \langle u(x \otimes t), u(y \otimes z)\rangle = \langle x \otimes z, y \otimes t\rangle + \langle x \otimes t, y \otimes z\rangle.$$

Replacing z by iz we get

$$\langle u(x \otimes z), u(y \otimes t)\rangle - \langle u(x \otimes t), u(y \otimes z)\rangle = \langle x \otimes z, y \otimes t\rangle - \langle x \otimes t, y \otimes z\rangle.$$

Therefore, $\langle u(x \otimes z), u(y \otimes t)\rangle = \langle x \otimes z, y \otimes t\rangle$ which means that u is unitary.

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DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF WATERLOO, WATERLOO, ONTARIO, CANADA