LINEAR TRANSFORMATIONS ON MATRICES

BY

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ABSTRACT. The real orthogonal group $O(n)$, the unitary group $U(n)$ and the symplectic group $Sp(n)$ are embedded in a standard way in the real vector space of $n \times n$ real, complex and quaternionic matrices, respectively. Let $F$ be a nonsingular real linear transformation of the ambient space of matrices such that $F(G) \subseteq G$ where $G$ is one of the groups mentioned above. Then we show that either $F(x) = ao(x)b$ or $F(x) = ao(x^*)b$ where $a, b \in G$ are fixed, $x^*$ is the transpose conjugate of the matrix $x$ and $o$ is an automorphism of reals, complexes and quaternions, respectively.

1. Introduction. In his survey paper [4], M. Marcus has stated seven conjectures. In this paper we shall study the last two of these conjectures.

Let $D$ be one of the following real division algebras $R$ (the reals), $C$ (the complex numbers) or $H$ (the quaternions). As usual we assume that $R \subseteq C \subseteq H$. We denote by $M_n(D)$ the real algebra of all $n \times n$ matrices over $D$. If $\xi \in D$ we denote by $\bar{\xi}$ its conjugate in $D$. Let $A$ be the automorphism group of the real algebra $D$. If $D = R$ then $A$ is trivial. If $D = C$ then $A$ is cyclic of order 2 with conjugation as the only nontrivial automorphism. If $D = H$ then the conjugation is not an isomorphism but an anti-isomorphism of $H$. Every automorphism of $H$ is inner (by the Noether-Skolem theorem) and every automorphism of $H$ commutes with conjugation.

If $x \in M_n(D)$ then we let $\bar{x}$ be the conjugate matrix, $x^t$ the transpose of $x$ and $x^*$ the conjugate transpose of $x$. The rule $(xy)^* = y^*x^*$ is valid for any $x, y \in M_n(D)$. If $x \in M_n(D)$ and $o \in A$ we let $o(x)$ be the matrix obtained from $x$ by applying $o$ to each of its entries.

Let $e \in M_n(D)$ be the identity matrix and put $G = \{x \in M_n(D) \mid xx^* = e\}$. We have

$$
G = O(n) \text{ the real orthogonal group,} \quad \text{if } D = R;
$$

$$
G = U(n) \text{ the unitary group,} \quad \text{if } D = C;
$$

$$
G = Sp(n) \text{ the symplectic group,} \quad \text{if } D = H.
$$

These are classical compact Lie groups. When $D = R$ then $G = O(n)$ has two components and its identity component is the so called special orthogonal group $SO(n)$, also called the rotation group.

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Now, we can state a slightly modified version of Conjecture 6:

**Conjecture 6.** Let $F: M_n(D) \to M_n(D)$ be an $\mathbb{R}$-linear transformation such that $F(G) \subset G$. Then $F$ must have one of the following two forms: Either (1) $F(x) = \sigma(x)b$ for all $x \in M_n(D)$, or (2) $F(x) = \sigma(x^*)b$ for all $x \in M_n(D)$, where $a, b \in G$ and $\sigma \in A$. Also, if $D = \mathbb{R}$ and $F(SO(n)) \subset SO(n)$ then (1) or (2) holds.

In his earlier paper [3] M. Marcus has shown that if $D = \mathbb{C}$ and $F$ is also supposed to be a $\mathbb{C}$-linear map then $F(G) \subset G$ implies that either

$$F(x) = ax\sigma$$

for all $x \in M_n(C)$, or

$$F(x) = a'x\sigma$$

for all $x \in M_n(C)$,

where $a, b \in U(n)$. These results agree with the conjecture because if we want $F$ to be $\mathbb{C}$-linear we must take $\sigma = \text{identity}$ in (1) and $\sigma = \text{conjugation}$ in (2).

If $D = \mathbb{H}$ note that $x \in \text{Sp}(n)$ implies that $\sigma(x^*) \in \text{Sp}(n)$ because $\sigma \in A$ commutes with the conjugation.

One can equip the real vector space $M_n(D)$ with a positive definite scalar product as follows:

(3) $\langle x, y \rangle = \Re \text{tr}(xy^*)$.

The Conjecture 6, as stated above, has been proved in [2] under additional assumption that $F$ is orthogonal when $M_n(D)$ is equipped with the scalar product (3).

Here we shall prove Conjecture 6 when $F$ is assumed to be nonsingular.

To introduce the next conjecture let $V$ be a finite dimensional complex Hilbert space and let $V \otimes_C V$ be equipped with the scalar product which satisfies:

$$\langle x \otimes y, a \otimes b \rangle = \langle x, a \rangle \langle y, b \rangle$$

for all $x, y, a, b \in V$.

**Conjecture 7.** If $u: V \otimes_C V \to V \otimes_C V$ is a complex linear map such that

$$\|u(a \otimes b)\| = \|a \otimes b\| \text{ for all } a, b \in V$$

then $u$ must be unitary.

We shall show that this conjecture is true. It is worthwhile to remark that the analogous conjecture for real Hilbert spaces is false. I am indebted to M. Marcus for a counterexample to the latter.

2. Matrix equation $ax = xa^*$.

**Theorem 1.** Let $a \in O(n)$ have the eigenvalues

$$\exp(\pm i\theta_r) \text{ of multiplicity } k_r, 1 \leq r \leq s;$$

$+1 \text{ of multiplicity } p;$

$-1 \text{ of multiplicity } q;$
where $0 < \theta_r < \pi$ are distinct, $p, q, s$ are nonnegative integers and $2(k_1 + \cdots + k_s) + p + q = n$.

Then the real vector space of skew-symmetric matrices $x \in M_n(\mathbb{R})$ satisfying the equation

\begin{equation}
ax = x'a
\end{equation}

has dimension $\sum_{r=1}^{s} k_r(k_r - 1) + \frac{1}{2}p(p - 1) + \frac{1}{2}q(q - 1)$.

**Proof.** By performing a similarity transformation by a suitable orthogonal matrix we can assume that $a$ has the canonical quasi-diagonal form. The diagonal blocks of $a$ are the matrices

\[
\begin{pmatrix}
\cos \theta_r & -\sin \theta_r \\
\sin \theta_r & \cos \theta_r
\end{pmatrix}, \quad 1 \leq r \leq s,
\]

each $a_r$ being repeated $k_r$ times, followed by $p$ one's and $q$ minus one's.

We partition the unknown skew-symmetric matrix $x$ into blocks of the same size as the corresponding blocks of $a$.

Now we make the following observations.

(i) If $y$ is a $2 \times 2$ diagonal block of $x$ corresponding to an $a_r$ then $y = 0$. Indeed (4) implies that $ary = y'a_r$. Since $y$ is skew-symmetric and $\sin \theta_r \neq 0$ we must have $y = 0$.

(ii) If

\[
\begin{pmatrix}
0 & y \\
-\bar{y} & 0
\end{pmatrix}
\]

is a principal submatrix of $x$ with $y$ some $2 \times 2$ block of $x$ and if the corresponding submatrix of $a$ is

\[
\begin{pmatrix}
a_r & 0 \\
0 & a_m
\end{pmatrix}, \quad r \neq m
\]

then $y = 0$.

In this case (4) implies that $a_r y = y'a_m$. Since $a_r$ and $a_m$ are not similar we see that $y$ must be singular. Hence we can write $y = (\lambda \mu, \lambda \beta)$ where $\lambda, \mu, \alpha, \beta \in \mathbb{R}$. By inspection of the first columns of $a_r y$ and $y'a_m$ we find that

\begin{align*}
\alpha(\lambda \cos \theta - \mu \sin \theta) &= \lambda(\alpha \cos \phi - \beta \sin \phi), \\
\alpha(\lambda \sin \theta + \mu \cos \theta) &= \mu(\alpha \cos \phi - \beta \sin \phi)
\end{align*}

where $\theta_r = \theta, \theta_m = \phi$.

From (5) we deduce that $\alpha(\lambda \mu \cos \theta - \mu^2 \sin \theta) = \alpha(\lambda^2 \sin \theta + \lambda \mu \cos \theta)$ and then $\alpha(\lambda^2 + \mu^2) \sin \theta = 0$. But $\sin \theta \neq 0$. If $\lambda^2 + \mu^2 = 0$ then $\lambda = \mu = 0$ and $y = 0$. Otherwise $\alpha = 0$, then from (5) we see that $\beta = 0$ and then $y = 0$ follows.
(iii) If
\[
\begin{pmatrix} 0 & y \\ -y & 0 \end{pmatrix}
\]
is a principal submatrix of \(x\) where \(y\) is a \(2 \times 2\) block of \(x\) and the corresponding submatrix of \(a\) is
\[
\begin{pmatrix} a_r & 0 \\ 0 & a_r \end{pmatrix}
\]
then the equation \(a_r y = y a_r\) (which follows from (4)) is satisfied only by the matrices \(y\) of the form \((\beta \delta \alpha \beta)\). This can be seen by a simple computation.

(iv) If
\[
\begin{pmatrix} 0 & y \\ -y & 0 \end{pmatrix}
\]
is a principal submatrix of \(x\) where \(y\) is a \(2 \times 1\) block of \(x\) and the corresponding submatrix of \(a\) is
\[
\begin{pmatrix} a_r & 0 \\ 0 & \pm 1 \end{pmatrix}
\]
then \(y = 0\).

Indeed, (4) implies that \(a_r y = \pm y\) but \(a_r\) has no real eigenvalues.

(v) If \((0 \delta 0)\) is a principal submatrix of \(x\) and \((0 0 \gamma)\) is the corresponding submatrix of \(a\) then \(a = 0\).

This is immediate from (4).

After these observations the formula for the dimension of the space of skew-symmetric solutions of (4) follows by a simple counting.

**Theorem 2.** Let \(a \in U(n)\) have the eigenvalues
\[
\exp(i \theta_r) \text{ of multiplicity } k'_r; \quad 1 \leq r \leq s,
\]
\[
\exp(-i \theta_r) \text{ of multiplicity } k''_r; \quad 1 \leq r \leq s,
\]
\(+1\) of multiplicity \(p\);
\(-1\) of multiplicity \(q\);

where \(k'_r, k''_r, p, q, s\) are nonnegative integers, the \(\theta_r\)'s are distinct and lie in the interval \(0 < \theta < \pi\), and \(\sum_{r=1}^{s} (k'_r + k''_r) + p + q = n\).

Then the real vector space of skew-hermitian matrices \(x \in M_n(\mathbb{C})\) satisfying the equation \(ax = xa^*\) has dimension \(2 \sum_{r=1}^{s} k'_r k''_r + p^2 + q^2\).

**Proof.** We can assume that \(a\) has diagonal form. It is easy to verify that the \((r, m)\)-entry of \(x\) must be 0 if the \(r\)th and \(m\)th diagonal entry of \(a\) are not
conjugate; otherwise this entry may be arbitrary if \( r \neq m \) and arbitrary purely imaginary number if \( r = m \).

Then a simple counting will complete the proof.

Before stating the next theorem we need to introduce some more terminology. We say that two quaternions \( \xi \) and \( \eta \) are similar if there exists \( \sigma \in A \) such that \( \sigma(\xi) = \eta \). It follows easily from the Noether-Skolem theorem (see [1]) that \( \xi \) and \( \bar{\xi} \) are always similar. It is well known that the eigenvalues of \( u \in \text{Sp}(n) \) are determined uniquely up to similarity.

**Theorem 3.** Let \( a \in \text{Sp}(n) \) have the eigenvalues

\[
\lambda_r \text{ with multiplicity } k_r, 1 \leq r \leq s;
+1 \text{ with multiplicity } p,
-1 \text{ with multiplicity } q,
\]

where \( \lambda_r \in \mathbb{H} \) are nonsimilar and not real; \( p, q, s \) are nonnegative integers and \((k_1 + \cdots + k_s) + p + q = n\). Then the real vector space of skew-hermitian matrices \( x \in M_n(\mathbb{H}) \) satisfying the equation

\[
ax = xa^* \tag{6}
\]

has the dimension \( \sum_{r=1}^{s} k_r(k_r + 1) + p(2p + 1) + q(2q + 1) \).

**Proof.** By performing a similarity transformation by a suitable sympletic matrix we can assume that \( a \) has diagonal form with the above listed eigenvalues on the diagonal.

Let

\[
\left( \begin{array}{cc}
\alpha & \beta \\
-\bar{\beta} & \gamma
\end{array} \right), \quad \alpha + \bar{\alpha} = \gamma + \bar{\gamma} = 0,
\]

be a principal submatrix of \( x \) and

\[
\left( \begin{array}{cc}
\lambda_r & 0 \\
0 & \lambda_m
\end{array} \right)
\]

the corresponding submatrix of \( a \). If \( r \neq m \) we claim that \( \beta = 0 \). Indeed, (6) gives

\[
\lambda_r \beta = \beta \bar{\lambda}_m. \tag{7}
\]

Since \( \lambda_r \) and \( \bar{\lambda}_m \) are not similar we must have \( \beta = 0 \).

If \( r = m \) we claim that the solution space of Equation (7) has dimension 2. Indeed, we can assume that \( \lambda_r = i \) is one of the basic units of \( \mathbb{H} \) and then our assertion is obvious because the solution space is spanned by the other two basic units \( j \) and \( k \), say.
For $\alpha$ we have the equation $\lambda \alpha = a \lambda$, and again the space of solutions has dimension 2.

Similarly, if the corresponding submatrix of $a$ is

$$
\begin{pmatrix}
\lambda & 0 \\
0 & \pm 1
\end{pmatrix}
$$

or

$$
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
$$

we get again $\beta = 0$.

Finally, if the corresponding submatrix of $a$ is $\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then $\beta$ may be arbitrary and $\alpha$ and $\gamma$ are subject only to the condition $\alpha + \alpha = \gamma + \gamma = 0$.

Thus $x$ must be a quasi-diagonal matrix with diagonal blocks corresponding to the blocks of equal diagonal entries of $a$.

Simple counting gives us the required formula for the dimension.

3. Conjecture 6 (General case). From now on we shall assume that $F$ is nonsingular and satisfies $F(G) \subset G$. Without loss of generality we can assume that $F(e) = e$. Indeed, if $F(e) = a$ then it suffices to replace $F$ by $F_l$ which is defined by $F_l(x) = a^{-1} F(x)$ since then $F_l(e) = e$.

We consider $G$ as an analytic submanifold of $M_n(D)$. As usual, we shall identify the tangent vectors of $G$ at some point of $G$ with the corresponding matrices in $M_n(D)$. In this sense we can say that the tangent space to $G$ at $e$ is the real vector space of skew-hermitian matrices in $M_n(D)$. Let us denote this vector space by $S_i$, i.e., $S = \{ x \in M_n(D) \mid x^* = -x \}$. Then the tangent space to $G$ at some point $a \in G$ will be the space $a \cdot S = \{ a \cdot x \mid x \in S \}$. Since $F$ is linear it coincides with its own differential. Thus $F$ must map the tangent space of $G$ at $a \in G$ bijectively to the tangent space of $G$ at $F(a)$, i.e., we have

$$
F(a \cdot S) = F(a) \cdot S, \quad \text{for all } a \in G.
$$

For $a = e$ we get $F(S) = S$ because $F(e) = e$.

We have $F(a \cdot S \cap S) = F(a) \cdot S \cap S$ for $a \in G$ and in particular,

$$
\text{dim } (F(a) \cdot S \cap S) = \text{dim } (a \cdot S \cap S), \quad a \in G.
$$

It is easy to see that $aS \cap S$ is precisely the space whose dimension was calculated in §2.

It is easy to verify Conjecture 6 when $n = 1$. Indeed, the case $D = \mathbb{R}$ is trivial. If $D = \mathbb{C}$ then we identify $G$ with the unit circle in $\mathbb{C}$. Since $F(G) = G$ and $F(1) = 1$ we see that $F$ is orthogonal and so either it is identity or conjugation. If $D = \mathbb{H}$ then $G = \text{Sp}(1)$ can be identified with the unit 3-sphere of $\mathbb{H}$. Since $F(G) = G$ it follows that $F$ is orthogonal. From $F(1) = 1$ it follows that $F$ preserves the pure quaternions. Since $A$ is transitive on the unit sphere of the space of pure quaternions we can assume further that $F(i) = i$. Since the unit circle group of $\mathbb{C}$ acts transitively by inner automorphisms on the unit circle of the plane $j\mathbb{R} + k\mathbb{R}$ we can further assume that $F(j) = j$. Then we have either
$F(k) = k$ or $F(k) = -k$. In the second case it is easy to see that $F$ is the map $\xi \mapsto k^{-1}\xi k$.

Instead of these arguments we could quote the paper [2] because it is obvious that these $F$ are orthogonal maps.

From now on we shall assume that $n \geq 2$. Let $a \in G$, $a \neq \pm e$. Then if we assume that $n \geq 3$ we deduce easily from Theorems 1–3 that $\dim (aS \cap S)$ is maximal if and only if $a$ or $-a$ is a reflection, i.e., has the following eigenvalues

\begin{equation}
+1 \text{ with multiplicity } n - 1,
-1 \text{ with multiplicity } 1.
\end{equation}

This fact and (9) imply that if $a \in G$ is a reflection then either $F(a)$ or $-F(a)$ is again a reflection.

Now, let $e_r$, $1 \leq r \leq n$ be the diagonal matrix whose $r$th diagonal entry is $-1$ and all other diagonal entries are $+1$. Then we have $e_1 + \cdots + e_n = (n - 2)e$, and

\begin{equation}
F(e_1) + \cdots + F(e_n) = (n - 2)e.
\end{equation}

Since (10) are the eigenvalues of $F(e_r)$ or of $-F(e_r)$ we have $\text{Re } \text{tr } F(e_r) = \pm (n - 2)$. From (11) we get $\sum_{r=1}^{n} \text{Re } \text{tr } F(e_r) = n(n - 2)$. Hence, we must have $\text{Re } \text{tr } F(e_r) = n - 2$ for every $r$. Thus $F(e_r)$ is a reflection for every $r$. By performing a similarity transformation by some matrix in $G$ we can assume that $F(e_1) = e_1$. The equation (11) now implies that $F(e_2), \ldots, F(e_n)$ each have $1$ in the upper left corner. Since $F(e_r) \in G$ this must be the only nonzero entry of $F(e_r)$ lying in the first row or column. By continuing this argument we see that we can assume that $F(e_r) = e_r$ for $1 \leq r \leq n$.

Now, let $n = 2$. If $D = \mathbb{R}$ then $S \cap G$ is invariant under $F$. Thus if $a = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ then either $F(a) = a$ or $F(a) = -a$. By composing $F$ with transposition we may assume that $F(a) = a$. Thus $F$ fixes every element of $SO(2)$. Hence, $F(e_1)$ must be a reflection and we can assume as above that $F(e_1) = e_1$. Then, of course, $F(e_2) = e_2$ because $e_2 = -e_1$.

If $D = \mathbb{C}$, $a \in G$ and $\dim (a \cdot S \cap S) = 2$ then it follows from Theorem 2 that $a$ is either a reflection or else it is unitarily similar to a matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, where $\lambda \in \mathbb{C}$ is such that $|\lambda| = 1$ and $\lambda \notin \mathbb{R}$. Thus (9) implies that $F(e_1)$ is either a reflection or else we can assume that $F(e_1)$ is the above matrix. But since $F(e_1) \in \mathbb{R} + S$, $e_1 \notin \mathbb{R} + S$ and $\mathbb{R} + S$ is $F$-invariant subspace it follows that the second case cannot occur. Hence, $F(e_1)$ is a reflection and we can assume that $F(e_1) = e_1$. Since $e_2 = -e_1$ we will also have $F(e_2) = e_2$.

If $D = \mathbb{H}$, $a \in G$ and $\dim (a \cdot S \cap S) = 6$ then it follows from Theorem 3 that $a$ is either a reflection or else it is similar by some matrix in $G$ to a matrix $\lambda e$ where $\lambda \in \mathbb{H}$, $|\lambda| = 1$ but $\lambda \notin \mathbb{R}$. Again the latter case cannot occur because we can in that case assume that $F(e_1) = \lambda e$ but $\lambda e \notin \mathbb{R} + S$, $e_1 \notin \mathbb{R} + S$ and the subspace $\mathbb{R} + S$ is $F$-invariant. It follows that $F(e_1)$ is a reflection and we can then assume that $F(e_1) = e_1$, $F(e_2) = e_2$.  

4. Conjecture 6 (Orthogonal group).

**Theorem 4.** Conjecture 6 is true when \( G \) is the orthogonal group and \( F \) is nonsingular.

**Proof.** We claim that if \( u \in M_n(\mathbb{R}) \) has rank 1 then also \( F(u) \) has rank 1.

Let \( m_{pq} \) be the matrix whose \((p, q)\)-entry is 1 and all other entries are zero. By performing a similarity transformation by an orthogonal matrix \( a \) we can assume without loss of generality that \( u = am_{11} + \beta m_{21} \). By performing another similarity transformation by \( b \in G \) we can in addition assume that \( F(e_r) = e_r \), \( 1 \leq r \leq n \).

Indeed, this reduction means that we are replacing \( F \), which is supposed to satisfy \( F(e) = e \), by the transformation \( F_1 \) defined as follows: \( F_1(x) = bF(axa^{-1})b^{-1} \).

It is easy to see that the subspace

\[
(\bigcap_{r=3}^n e_r \cdot S) \cap S
\]

has dimension 1 and the matrix \( m_{12} - m_{21} \) lies in it. It follows from (8) that the subspace (12) is an invariant subspace of the linear transformation \( F \). Thus \( F(m_{12} - m_{21}) = \lambda(m_{12} - m_{21}) \). But \( (m_{12} - m_{21}) + \frac{1}{2}(e_1 + e_2) \in G \) and applying \( F \) we get

\[
\lambda(m_{12} - m_{21}) + \frac{1}{2}(e_1 + e_2) \in G.
\]

Therefore, \( \lambda = \pm 1 \).

Similarly, the subspace \((\bigcap_{r=3}^n e_r \cdot S) \cap e_1 S \cap e_2 S \) is \( F \)-invariant, has dimension 1 and \( m_{12} + m_{21} \) lies in it. Therefore,

\[
F(m_{12} + m_{21}) = \mu(m_{12} + m_{21})
\]

and we get \( \mu = \pm 1 \) by a similar argument.

Since \( u = \frac{1}{2}a(e - e_1) + \frac{1}{2}\beta[(m_{21} - m_{12}) + (m_{21} + m_{12})] \) we have

\[
F(u) = \frac{1}{2}a(e - e_1) + \frac{1}{2}\beta[\lambda(m_{21} - m_{12}) + \mu(m_{21} + m_{12})].
\]

It follows that \( F(u) \) has rank 1 because \( \lambda = \pm 1 \) and \( \mu = \pm 1 \).

Since \( F \) preserves the matrices of rank 1 we must have (see [5]) either

\[
F(x) = axb \quad \text{for} \quad x \in M_n(\mathbb{R}), \quad \text{or}
\]

\[
F(x) = a'xb \quad \text{for} \quad x \in M_n(\mathbb{R}),
\]

where \( a \) and \( b \) are some nonsingular matrices. From \( F(e) = e \) we deduce that \( b = a^{-1} \). In both cases we must have \( aya^{-1} \in G \) for \( y \in G \), i.e.,

\[
(aya^{-1}) \cdot (aya^{-1}) = e, \quad aya = yaa.
\]
This implies that \( a a = p e \) where \( p > 0 \). Thus we have that \( a' \sqrt{p} \in G \) which completes the proof.

**Remark about the rotation group.** It is obvious that the Conjecture 6 is also true for the case of the rotation group if \( n \) is odd. Indeed, in that case \( -e \notin SO(n) \). If \( F(SO(n)) \subset SO(n) \) and \( x \in O(n) \) but \( x \notin SO(n) \) we have \( -x = x \cdot (-e) \in SO(n) \) and \( F(x) = -F(-x) = (-e) \cdot F(-x) \in O(n) \), i.e., \( F(O(n)) \subset O(n) \). Thus, we can apply Theorem 4.

The Conjecture 6 is false for the rotation group if \( n = 2 \). Indeed, in that case we can define \( F \) as follows: \( F \) fixes the matrices

\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\]

and it maps

\[
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\mapsto
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\mapsto
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}.
\]

Then \( F \) is nonsingular, it preserves \( SO(2) \) but not \( O(2) \).

5. Conjecture 6 (Unitary group).

**Theorem 5.** Conjecture 6 is true when \( G \) is the unitary group and \( F \) is assumed to be nonsingular.

**Proof.** As shown in §3 we can assume that \( F(e_r) = e_r \) for \( 1 \leq r \leq n \). Note that \( F \) is assumed only to be \( R \)-linear map of \( M_n(C) \) into itself. Our next objective is to reduce the problem to the case when \( F \) is \( C \)-linear.

We define the matrices \( m_{pq} \) as in the previous section. The vector space \( (\bigcap_{r=1}^n e_r S) \cap S \) is \( F \)-invariant and consists of diagonal matrices with purely imaginary diagonal entries. Thus, for instance,

\[
F(im_{11}) = i \sum_{r=1}^n \alpha_r m_{rr}, \quad \alpha_r \in R.
\]

But \( e + (i - 1)m_{11} \in G \) implies that

\[
e - m_{11} + i \sum_{r=1}^n \alpha_r m_{rr} \in G.
\]

Therefore, \( \alpha_1 = \pm 1 \) and \( \alpha_r = 0 \) for \( 2 \leq r \leq n \). Hence, in general,

\[
F(\xi m_{rr}) = \xi m_{rr} \quad \text{for} \ \xi \in C, \quad \text{or}
\]

\[
F(\xi m_{rr}) = \xi m_{rr} \quad \text{for} \ \xi \in C,
\]

where, a priori, the alternative may depend on \( r \)
Assume that, say, we have $F(\xi m_{11}) = \xi m_{11}$, $F(\xi m_{22}) = \xi m_{22}$. Then by Theorem 2 we have

$$\dim(a \cdot S \cap S) = 2 + (n - 2)^2 \quad \text{and} \quad \dim(F(a) \cdot S \cap S) = (n - 2)^2$$

where

$$a = e + (i - 1)m_{11} - (i + 1)m_{22}, \quad F(a) = e + (i - 1)m_{11} + (i - 1)m_{22}.$$ 

This contradicts Equation (9). Thus the alternative mentioned above is independent of $r$. By composing $F$ with conjugation we may assume that $F(\xi m_r) = \xi m_r$, $\xi \in \mathbb{C}$, $1 \leq r \leq n$. Let, for instance, $\omega = \exp(i\pi/4)$ and define $b = \omega e + (i - \omega)m_{11} - (i + \omega)m_{22}$. Then $F(b) = b$ and the subspace $b \cdot S \cap S$ is $F$-invariant and consists of the matrices $\xi m_{12} - \xi m_{21}$, $\xi \in \mathbb{C}$. Since $e_1 b \in G$ is diagonal we have $F(e_1 b) = e_1 b$. Thus the subspace $e_1(b \cdot S \cap S) = (e_1 b) \cdot S \cap e_1 \cdot S$ is also $F$-invariant and consists of the matrices $\xi m_{12} + \xi m_{21}$, $\xi \in \mathbb{C}$. Thus there exist real linear transformations $u_1$, $u_2 : \mathbb{C} \to \mathbb{C}$ such that

$$F(\xi m_{12} - \xi m_{21}) = u_1(\xi) m_{12} - \overline{u_1(\xi)} m_{21},$$

$$F(\xi m_{12} + \xi m_{21}) = u_2(\xi) m_{12} + \overline{u_2(\xi)} m_{21}.$$ 

Hence,

$$F(\xi m_{12}) = v_1(\xi) m_{12} + \overline{v_1(\xi)} m_{21}, \quad F(\xi m_{21}) = v_2(\xi) m_{12} + \overline{v_2(\xi)} m_{21},$$

where $2v_1 = u_2 + u_1$, $2v_2 = u_2 - u_1$.

Since, for $|\lambda| = |\mu| = 1$, $\lambda m_{12} + \overline{\mu} m_{21} + e - m_{11} - m_{22} \in G$, its image under $F$ is in $G$. This implies that $|v_1(\lambda) + v_2(\mu)| = 1$ whenever $|\lambda| = |\mu| = 1$. It follows that $v_1$ or $v_2$ is zero and the other one is an orthogonal transformation of $\mathbb{C}$.

Assume first that $v_2 = 0$. Then either $v_1(\xi) = a \xi$, $|a| = 1$, or $v_1(\xi) = a \xi$, $|a| = 1$. In the first case we have $F(\xi m_{12}) = \xi F(m_{12})$. Suppose that the second case is valid. Then we have $F(\xi m_{12}) = a \xi m_{12}$, $F(\xi m_{21}) = a \xi m_{21}$. The quasi-diagonal matrix having

$$\begin{pmatrix}
\cos \theta & i \sin \theta \\
\sin \theta & -i \cos \theta
\end{pmatrix}$$

in the upper left corner and ones on the remaining diagonal places is unitary. By applying $F$ we get a matrix which is not unitary if $\sin \theta \cos \theta \neq 0$. This is a contradiction.

Now, assume that $v_1 = 0$. Then either

$$v_2(\xi) = a \xi, \quad |a| = 1, \quad \text{or} \quad v_2(\xi) = a \xi, \quad |a| = 1.$$ 

In the first case we have $F(\xi m_{12}) = \xi F(m_{12})$. Suppose that the second case is valid. Then we have $F(\xi m_{12}) = a \xi m_{21}$, $F(\xi m_{21}) = a \xi m_{12}$. By applying $F$ to the
quasi-diagonal matrix mentioned above we get again a matrix which is not unitary if \( \sin \theta \cos \theta \neq 0 \). This is again a contradiction.

It follows that we must have \( F(\xi m_{12}) = \xi F(m_{12}) \) for \( \xi \in \mathbb{C} \).

Similarly \( F(\xi m_{pq}) = \xi F(m_{pq}) \) for all \( p, q \) and \( \xi \in \mathbb{C} \), in other words \( F \) must be \( \mathbb{C} \)-linear.

Now, we can complete the proof as in the case of the orthogonal group, or else we can use the result of M. Marcus [3].

6. Conjecture 6 (Symplectic Group).

**Theorem 6.** Conjecture 6 is true when \( G \) is the symplectic group and \( F \) is assumed to be nonsingular.

**Proof.** From §3 we know that we can assume that \( F(e_r) = e_r, 1 \leq r \leq n \). The subspace \( (\cap_{r=1}^n e_r \cdot S) \cap S \) is \( F \)-invariant and consists of diagonal matrices with pure quaternionic entries. As in the case of the unitary group we obtain that \( F(\xi e_r) = \alpha_r(\xi)e_n, 1 \leq r \leq n, \xi \in \mathbb{H} \), where \( \alpha_r \in \mathbb{A} \). For every \( r \) there exists a quaternion \( \alpha_r \) of unit norm such that

\[
\alpha_r(\xi) = \alpha_r\xi\alpha_r^{-1} = \alpha_r\xi\alpha_r, \quad \xi \in \mathbb{H}.
\]

Let \( a \) be the symplectic diagonal matrix with diagonal entries \( \alpha_1, \ldots, \alpha_n \). Then replacing \( F \) by \( F_1 \) which is defined as follows \( F_1(x) = F(axa^*) \) we can assume that \( F \), in addition, satisfies \( F(\xi e_r) = \xi e_n, 1 \leq r \leq n \), for all \( \xi \in \mathbb{H} \).

Let \( S' = S \cap M_n(\mathbb{R}) \) be the space of real skew-symmetric matrices and \( S'' \) the space of real symmetric matrices. Then \( S \) is a direct sum

\[
S = S' + iS'' + jS'' + kS''
\]

where \( 1, i, j, k \) are the standard units of \( \mathbb{H} \). It is obvious that

\[
S'' = iS \cap jS \cap kS, \quad iS'' = S \cap jS \cap kS,
\]

\[
jS'' = S \cap iS \cap kS, \quad kS'' = S \cap iS \cap jS.
\]

Since, say, \( i \cdot S = (ie) \cdot S \) and \( ie \in G, F(ie) = ie \) then we can use formula (8) to get

\[
F(S'') = S'', \quad F(iS'') = iS'', \quad F(jS'') = jS'', \quad F(kS'') = kS''.
\]

The space

\[
(\cap_{r=3}^n e_r \cdot S) \cap S
\]

consists of skew-hermitian matrices whose nonzero entries occur only on the diagonal or at places \((1, 2), (2, 1)\). We shall use the fact that this space is \( F \)-invariant.

Let \( |\xi| = 1 \) and \( |\xi_r| = 1, 3 \leq r \leq n \) where \( \xi, \xi_r \in \mathbb{H} \). Then
\[ \xi m_{12} - \xi m_{21} + \xi_3 m_{33} + \cdots + \xi_n m_{nn} \in G \]

and by applying \( F \) we get

\[ F(\xi m_{12} - \xi m_{21}) + \xi_3 m_{33} + \cdots + \xi_n m_{nn} \in G. \]

Now, the fact that the above space is \( F \)-invariant implies that the nonzero entries of \( F(\xi m_{12} - \xi m_{21}) \) are concentrated in the 2 \( \times \) 2 block in the upper left corner of that matrix and that this block is a symplectic matrix.

Let

\[
\begin{align*}
b &= (i - 1)m_{11} - (i + 1)m_{22} + e, \\
c &= (j - 1)m_{11} - (j + 1)m_{22} + e, \\
d &= (k - 1)m_{11} - (k + 1)m_{22} + e.
\end{align*}
\]

Then \( b, c, d \in G \) are diagonal, \( F(b) = b, F(c) = c, F(d) = d \) and the space

\[ (14) \quad b \cdot S \cap c \cdot S \cap d \cdot S \]

is \( F \)-invariant. The intersection of the spaces (13) and (14) consists of the matrices of the form

\[
\begin{pmatrix}
0 & \alpha \\
-\alpha & 0 \\
\xi_3 & \cdots \\
& \xi_n
\end{pmatrix}, \quad \bar{\xi}_r = -\xi_r,
\]

where \( \alpha \) is real. Therefore, \( F(m_{12} - m_{21}) = \pm(m_{12} - m_{21}) \).

In general, \( F(m_{pq} - m_{qp}) = \pm(m_{pq} - m_{qp}) \) which means that \( F(S') = S' \). Since also \( F(S'') = S'' \) we conclude that \( M_n(R) \) is \( F \)-invariant. Thus \( F \) preserves \( O(n) \) and by Theorem 4 we may assume, in addition to other properties of \( F \), that \( F \) is identity on \( M_n(R) \).

Let us multiply on the left by \( i \) each of the subspaces \( e_r \cdot S, S, b \cdot S, c \cdot S, d \cdot S \) which occur in (13) and (14). Then by taking their intersection we get an \( F \)-invariant subspace which consists of the matrices \( ix \) where \( x \) has form (15). Hence we have

\[
F(im_{12} - im_{21}) = \lambda i(m_{12} - m_{21}) + i(\eta_3 m_{33} + \cdots + \eta_n m_{nn})
\]

where \( \lambda \in R \) and \( \bar{\eta}_r = -\eta_r \).

If \( |\xi_r| = 1 \) for \( 3 \leq r \leq n \) then

\[
i(m_{12} - m_{21}) + i(\xi_3 m_{33} + \cdots + \xi_n m_{nn}) \in G
\]

and also its image by \( F \) is in \( G \). This implies that \( \lambda = \pm 1 \) and \( \eta_r = 0 \) for \( 3 \leq r \leq n \). Hence,
and similarly,

\[ F(i(m_{pq} - m_{pq})) = \pm i(m_{pq} - m_{pq}). \]

Thus \( F(iS') = iS' \) and by similar arguments \( F(jS') = jS' \), \( F(kS') = kS' \). It follows that the subspaces \( iM_n(\mathbb{R}), jM_n(\mathbb{R}), kM_n(\mathbb{R}) \) are also \( F \)-invariant. Thus \( F \) maps \( U(n) \) onto itself and now by Theorem 5 we can conclude that the restriction of \( F \) to the subspace \( M_n(\mathbb{R}) + iM_n(\mathbb{R}) \) is orthogonal with respect to the scalar product (3). Using two other embeddings of \( \mathbb{C} \) into \( \mathbb{H} \) we see that also the restrictions of \( F \) to the subspaces \( M_n(\mathbb{R}) + jM_n(\mathbb{R}) \) and \( M_n(\mathbb{R}) + kM_n(\mathbb{R}) \) are orthogonal. This implies that the linear transformation \( F \) is orthogonal for the same scalar product. Hence we can now use the result [2] to complete the proof.


**Theorem 7.** Conjecture 7 is true.

**Proof.** Let \( x, y, z, t \in V \). Then we have

\[ \langle u((x + y) \otimes z), u((x + y) \otimes z) \rangle = \langle (x + y) \otimes z, (x + y) \otimes z \rangle, \]
\[ \langle u(x \otimes z) + u(y \otimes z), u(x \otimes z) + u(y \otimes z) \rangle = \langle x \otimes z + y \otimes z, x \otimes z + y \otimes z \rangle, \]
\[ \langle u(x \otimes z), u(y \otimes z) \rangle + \langle u(y \otimes z), u(x \otimes z) \rangle = \langle x \otimes z, y \otimes z \rangle + \langle y \otimes z, x \otimes z \rangle, \]
\[ \text{Re} \langle u(x \otimes z), u(y \otimes z) \rangle = \text{Re} \langle x \otimes z, y \otimes z \rangle. \]

By replacing \( x \) with \( ix \) we get also

\[ \text{Im} \langle u(x \otimes z), u(y \otimes z) \rangle = \text{Im} \langle x \otimes z, y \otimes z \rangle. \]

Thus \( \langle u(x \otimes z), u(y \otimes z) \rangle = \langle x \otimes z, y \otimes z \rangle \). Now, replace \( z \) by \( z + t \). Then we get

\[ \langle u(x \otimes z) + u(x \otimes t), u(y \otimes z) + u(y \otimes t) \rangle = \langle x \otimes z + x \otimes t, y \otimes z + y \otimes t \rangle, \]
\[ \langle u(x \otimes z), u(y \otimes t) \rangle + \langle u(x \otimes t), u(y \otimes z) \rangle = \langle x \otimes z, y \otimes t \rangle + \langle x \otimes t, y \otimes z \rangle. \]

Replacing \( z \) by \( iz \) we get

\[ \langle u(x \otimes z), u(y \otimes i) \rangle - \langle u(x \otimes i), u(y \otimes z) \rangle = \langle x \otimes z, y \otimes i - (x \otimes t, y \otimes z). \]

Therefore, \( \langle u(x \otimes z), u(y \otimes i) \rangle = \langle x \otimes z, y \otimes i \rangle \) which means that \( u \) is unitary.

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**References**


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