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A CHARACTERIZATION OF THE INVARIANT MEASURES FOR AN INFINITE PARTICLE SYSTEM WITH INTERACTIONS. II (1)

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ABSTRACT. Let $p(x,y)$ be the transition function for a symmetric, irreducible Markov chain on the countable set S. Let $\eta(t)$ be the infinite particle system on S with the simple exclusion interaction and one-particle motion determined by p. The present author and Spitzer have determined all of the invariant measures of $\eta(t)$, and have obtained ergodic theorems for $\eta(t)$, under two different sets of assumptions. In this paper, these problems are solved in the remaining case.

1. **Introduction.** Let $p(x,y)$ be the transition probabilities for an irreducible Markov chain on the countable set S. Assume that $p(x,y) = p(y,x)$ for $x, y \in S$. Consider the infinite particle system $\eta(t)$ with the simple exclusion interaction introduced by Spitzer in [5]. This is a strong Markov process with state space $X = \{0,1\}^S$ which describes the random motion of infinitely many particles on S. Each point $\eta \in X$ determines a possible configuration of particles on S: for $x \in S$, $\eta(x) = 1$ means that x is occupied, and $\eta(x) = 0$ that x is vacant. The basic motion of each particle is that of a continuous time Markov chain on S with transition probabilities $p(x,y)$ and holding times which are exponentially distributed with parameter one. Superimposed upon this motion is the interaction of simple exclusion, which prevents a particle from making a transition to an occupied site. With this interaction, the motion of each individual particle ceases to be Markovian.

The existence of $\eta(t)$ as a strong Markov process on X which behaves according to this intuitive prescription was proved in [4]. In [3] and [6], some basic ergodic properties of $\eta(t)$ were studied under two different sets of assumptions on $p(\cdot, \cdot)$. This paper is devoted to the investigation of the same ergodic problems in the case not covered by these two papers.

Using the notation of [3], let $\mathcal{I}$ be the set of all probability measures on X which are invariant for the process $\eta(t)$, and let $\mathcal{I}_e$ be the set of extreme points of $\mathcal{I}$. Put

$$\mathcal{H} = \left\{\alpha(\cdot) \text{ on } S \mid 0 \leq \alpha(x) \leq 1, \text{ and } \sum_y p(x,y)\alpha(y) = \alpha(x) \quad \text{for} \quad x \in S\right\}.$$

The following result characterizes $\mathcal{I}_e$ in terms of $\mathcal{H}$.

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Theorem 1.1. *Assume that $p(x,y)$ is the transition function for a symmetric, irreducible Markov chain on S. Then for each $\alpha \in \mathcal{H}$, there exists a unique $\mu_\alpha \in \mathcal{I}_e$ such that*

$$\mu_\alpha\{\eta \mid \eta(x) = 1\} = \alpha(x)$$

for $x \in S$. Furthermore, $\mathcal{I}_e = \{\mu_\alpha \mid \alpha \in \mathcal{H}\}$. If α is constant, then μ_α is a product measure on X.

In particular, if $\mathcal{H}$ consists only of constants, then $\mathcal{I}$ is exactly the class of symmetric probability measures on X, whose extreme points are the product measures μ_ρ, $0 \leq \rho \leq 1$. This occurs, for example, if p is recurrent, or if S is an Abelian group and p is translation invariant.

Three separate cases arise in the proof of Theorem 1.1. In order to describe this, it is convenient to introduce, for $n \geq 2$, the discrete time Markov chain $X(k)$ on S^n which has transition operator U_n given by

$$U_n h(\vec{x}) = E^{\vec{x}} h(X(1))$$
$$= \frac{1}{n} \sum_{i=1}^{n} \sum_{u} p(x_i, u) h(x_1, \ldots, x_{i-1}, u, x_{i+1}, \ldots, x_n).$$

According to this process, n particles move on S in such a way that at each time, one of the particles is chosen, with probability $1/n$ each, and then it moves to a new site with probabilities determined by $p(\cdot, \cdot)$. Let

$$T_n = \{\vec{x} \in S^n \mid x_i \neq x_j \quad \text{for } i \neq j\},$$

$$f_n(\vec{x}) = P^{\vec{x}}(X(k) \in T_n^c \quad \text{for some } k \geq 1)$$

for $\vec{x} \in S^n$, and define N=number of $k \geq 1$ for which $X(k) \in T_n^c$. Then we have

Case I. $E^{\vec{x}}(N) < \infty$ for all $\vec{x} \in S^n$ and $n \geqslant 2$.

Case II. $E^{\vec{x}}(N) = \infty$, but $P^{\vec{x}}(N < \infty) = 1$ for all $\vec{x} \in S^n$ and $n \geqslant 2$.

Case III. $P^{\vec{x}}(N = \infty) = 1$ for all $\vec{x} \in S^n$ and $n \geqslant 2$.

As will be seen in §2, these three cases exhaust all possibilities. It follows from Lemma 3.1 of [3] that Case I occurs if and only if $p(\cdot, \cdot)$ is transient, and therefore $\mathcal{H}$ consists only of constants in Cases II and III. Furthermore, as will be proved in §2, Case III occurs if and only if $f_n(\vec{x}) = 1$ for all $\vec{x} \in S^n$ and all $n \geq 2$. It was noted in [6] that if S is an Abelian group and $p(x,y) = p(0,y-x)$, then Case II does not occur, and therefore Cases I and III correspond exactly to the transience and recurrence of $p(\cdot, \cdot)$ respectively.

The proof of Theorem 1.1 in Case III is due to Spitzer [6], and in Case I to the present author [3]. The proof in Case II will be presented in §3 of this paper. It will be noted that in all three cases, the proof is obtained by reducing it to a problem concerning the behavior of a finite number of interacting particles. Cases I and II are similar in that a comparison is then made between the interacting and noninteracting finite particle systems. In Case III, on the other hand, the interacting finite particle system is treated directly.

Once all of the invariant measures of $\eta(t)$ have been found, a problem which arises naturally is to determine conditions under which convergence to equilibrium occurs. Let $S(t)$ be the semigroup corresponding to the process $\eta(t)$. The problem is then to find

$$\{\mu \text{ on } X \mid S(t)\mu \to \mu_\alpha \text{ as } t \to \infty\}$$

for each $\alpha \in \mathcal{H}$. In order to give its solution, put

$$p_t(x,y) = e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!} p^{(k)}(x,y),$$

where $p^{(k)}(x,y)$ are the k-step transition probabilities for the Markov chain on S which has transition function $p(x,y)$.

Theorem 1.2. *In addition to the assumptions of Theorem* 1.1, *suppose that Case* I *or Case* II *holds, that μ is a probability measure on X, and that $\alpha \in \mathcal{H}$. Then $S(t)\mu \to \mu_\alpha$ weakly as $t \to \infty$ if and only if*

$$\sum_y p_t(x,y)[\eta(y) - \alpha(y)] \to 0 \tag{1.3}$$

in probability with respect to μ as $t \to \infty$ for each $x \in S$.

This theorem was proved in Case I in [3], and will be proved in Case II in §3 of this paper. A somewhat different necessary and sufficient condition for the convergence of $S(t)\mu$ to μ_α in Case III is given by Spitzer in [6]. However, it is shown there that condition (1.3) is sufficient in this case.

§2 of this paper is devoted to proving preliminary results about finite particle systems, which will be needed in §3 for the proofs of Theorems 1.1 and 1.2 in Case II. In §4, a class of examples is constructed to show that Case II in fact does occur.

2. The finite particle system. We begin by verifying some of the observations made in §1 concerning the three cases.

Lemma 2.1. *Fix $n \geq 2$.*

(a) *If $f_n(\vec{x}) < 1$ for some $\vec{x} \in T_n$, then $f_n(\vec{x}) < 1$ for all $\vec{x} \in T_n$.*

(b) *If $P^{\vec{x}}(N < \infty) = 1$ for some $\vec{x} \in S^n$, then $P^{\vec{x}}(N < \infty) = 1$ for all $\vec{x} \in S^n$.*

(c) *If $E^{\vec{x}}(N) < \infty$ for some $\vec{x} \in S^n$, then $E^{\vec{x}}(N) < \infty$ for all $\vec{x} \in S^n$.*

(d) *For $\vec{x} \in T_n$, $f_n(\vec{x}) < 1$ if and only if $P^{\vec{x}}(N < \infty) = 1$. In this case,* $\lim_{k\to\infty} U_n^k f_n = 0$.

Proof. In order to prove (a), it is necessary to show that for $\vec{x}, \vec{y} \in T_n$,

$$P^{\vec{x}}(\exists\, k \text{ for which } X(k) \in \rho(\vec{y}) \text{ and } X(i) \in T_n \forall i \leq k) > 0$$

where $\rho(\vec{y})$ is the set of all points in T_n obtained by permuting the coordinates of $\vec{y}$. It suffices to show this in case $\vec{x}$ and $\vec{y}$ differ in only one coordinate, say $x_i = y_i$ for $2 \le i \le n$. Since $p(\cdot, \cdot)$ is irreducible, there is a sequence of distinct points $z_1, \ldots, z_m \in S$ different from x_1 and y_1 so that

$$p(x_1, z_1)p(z_1, z_2)\cdots p(z_{m-1}, z_m)p(z_m, y_1) > 0.$$

By reordering $x_2, \ldots, x_n$, we may assume that $z_{k_2} = x_2, \ldots, z_{k_l} = x_l$ where $k_2 < \cdots < k_l$, and that $x_{l+1}, \ldots, x_n$ do not appear in the set $\{z_1, \ldots, z_m\}$. Then beginning at $\vec{x}$, the process $X(k)$ can reach $(x_2, \ldots, x_l, y_1, x_{l+1}, \ldots, x_n)$ with positive probability without passing through T_n^c. The proofs of (b) and (c) are similar, although simpler. To prove (d), note that from

$$P^{\vec{x}}(N = \infty) = \lim_{n\to\infty} U_n^k f_n(\vec{x}), \tag{2.2}$$

it follows that if $f_n \equiv 1$, then $P^{\vec{x}}(N < \infty) = 0$. On the other hand, if $f_n(\vec{x}) < 1$, then $P^{\vec{x}}(N < \infty) > 0$. To show that $P^{\vec{x}}(N < \infty) = 1$, consider two cases. If $p(\cdot, \cdot)$ is transient, then $E^{\vec{x}}(N) < \infty$, so the result follows. If $p(\cdot, \cdot)$ is recurrent then $\mathscr{H}$ consists only of constants, and therefore the only bounded solutions of $U_n h = h$ are constants by Lemma 3.14 of [3]. Therefore $P^{\vec{x}}(N < \infty) = 1$ by Proposition 5.19 of [2]. The final statement now follows from (2.2).

As in [3], it is convenient to define a Markov chain $Y(k)$ on S^n for $n \ge 2$ which describes n particles moving with the simple exclusion interaction on S. It has transition operator V_n given by

$$V_n h(\vec{x}) = E^{\vec{x}} h(Y(1)) = \frac{1}{n}\left[\sum_{\substack{i=1\\ i\neq j}}^{n} \sum_{j=1}^{n} p(x_i, x_j)\right] h(\vec{x}) + \frac{1}{n}\sum_{i=1}^{n} \sum_{u \neq x_j; j\neq i} p(x_i, u) h(x_1, x_2, \ldots, x_{i-1}, u, x_{i+1}, \ldots, x_n)$$

if $\vec{x} \in T_n$, and $V_n h(\vec{x}) = h(\vec{x})$ otherwise.

The remainder of this section has as its aim the proof that in Case II,

$$\lim_{k\to\infty} V_n^k f_n(\vec{x}) = 0 \tag{2.3}$$

for $\vec{x} \in T_n$. We will give two different proofs of this result. The first one is based exclusively on the process $Y(k)$, while the second one depends on a comparison of the processes $Y(k)$ and $X(k)$. This latter proof also applies to Case I, although in that case, (2.3) follows from Theorem 3.6 of [3]. Of course, (2.3) does not hold in Case III, since $f_n \equiv 1$. It is essentially this fact that distinguishes Cases I and II from Case III.

Lemma 2.4. *Assume that* $f_2(\vec{x}) < 1$ *for* $\vec{x} \in S^2$. *Then for each* $x \in S$, $\lim_{y\to\infty} f_2(x,y) = 0$.

Proof. Since $P^{\vec{x}}(N < \infty) = 1$ for $\vec{x} \in S^2$ by Lemma 2.1,

$$
\begin{aligned}
f_2(\vec{x}) &= \sum_{k=1}^{\infty} E^{\vec{x}}[(1 - f_2(X(k))), X(k) \in T_2^c] \\
&= \sum_{k=1}^{\infty} \sum_u [1 - f_2(u,u)] P^{\vec{x}}(X(k) = (u,u)) \\
&= \sum_u [1 - f_2(u,u)] \sum_{i,j=0}^{\infty} \frac{1}{2^{i+j}} \binom{i+j}{i} p^{(i)}(x_1,u) p^{(j)}(x_2,u) \\
&\quad - [1 - f_2(x_1,x_2)] p^{(0)}(x_1,x_2).
\end{aligned} \tag{2.5}
$$

Let J be the set of all functions $\beta(i,u)$ on $Z^+ \times S$ (Z^+ is the set of nonnegative integers) which have finite support. For β, $\gamma \in J$, define

$$(\beta,\gamma) = \sum_{u \in S} [1 - f_2(u,u)] \sum_{i,j=0}^{\infty} \frac{1}{2^{i+j}} \binom{i+j}{i} \beta(i,u)\gamma(j,u).$$

That this is an inner product follows from the fact that the matrix with entries $\binom{i+j}{i}$ is positive definite. This, in turn, follows from

$$\sum_{i,j=0}^{\infty} \binom{i+j}{i} a_i a_j = \sum_{k=0}^{\infty} \left[\sum_{m=k}^{\infty} \binom{m}{k} a_m \right]^2.$$

Let H be the Hilbert space obtained by completing J with respect to the above defined inner product. For $x \in S$, define β_x by $\beta_x(i,u) = p^{(i)}(x,u)$. Then by (2.5), $\beta_x \in H$, $\|\beta_x\| = 1$ for $x \in S$, and $(\beta_x, \beta_y) = f_2(x,y)$ for $x \neq y$. But J is dense in H, and $\lim_{y\to\infty}(\gamma, \beta_y) = 0$ for each $\gamma \in J$. So, since $\{\beta_x\}$ are uniformly bounded in norm, $\beta_y \to 0$ weakly as $y \to \infty$. Therefore $\lim_{y\to\infty}(\beta_x, \beta_y) = 0$ for each x, and the proof is complete.

The following result was suggested by C. Stone.

Lemma 2.6. *Assume that $p(\cdot,\cdot)$ is recurrent. Fix $z \in S$ and define $C = \{\vec{x} \in T_2 \mid x_1 = z$ or $x_2 = z\}$. Then $P^{\vec{x}}(Y(k) \in C$ for infinitely many $k) = 1$ for $\vec{x} \in T_2$.*

Proof. Consider the Markov chain $Z(k)$ on T_2 which has transition operator W given by

$$
\begin{aligned}
Wh(\vec{x}) = E^{\vec{x}}(h(Z(1))) &= \frac{1}{2} \sum_{u \neq x_2} p(x_1,u)h(u,x_2) \\
&\quad + \frac{1}{2} \sum_{u \neq x_1} p(x_2,u)h(x_1,u) + \frac{1}{2} p(x_1,x_2)[h(x_2,x_1) + h(x_1,x_2)].
\end{aligned}
$$

Then each coordinate of $Z(k)$ is Markovian and has transition probabilities $\frac{1}{2}[p^{(0)}(x,y) + p(x,y)]$, although of course the two coordinates are not independent. Therefore, since $p(\cdot,\cdot)$ is recurrent,

$$P^{\vec{x}}(Z_1(k) = z \text{ for infinitely many } k) = 1$$

for $\vec{x} \in T_2$. However, for symmetric functions h, $Wh = V_2 h$, so since C is a symmetric set,

$$P^{\vec{x}}(Y(k) \in C \text{ for infinitely many } k) = 1$$

for $\vec{x} \in T_2$.

The next result is the key to one of our proofs of (2.3). For another application, see [6]. Define the continuous time analogues U_n^t and V_n^t of U_n and V_n respectively by

$$U_n^t h(\vec{x}) = e^{-nt} \sum_{k=0}^{\infty} \frac{(nt)^k}{k!} U_n^k h(\vec{x})$$
$$= \sum_{\vec{y} \in S^n} \prod_{j=1}^{n} p_t(x_j, y_j) h(\vec{y}),$$

and

$$V_n^t h(\vec{x}) = e^{-nt} \sum_{k=0}^{\infty} \frac{(nt)^k}{k!} V_n^k h(\vec{x}).$$

Lemma 2.7. *Let $h(x,y)$ be a bounded, symmetric function on S^2 which is positive definite in the sense that $\sum_{x,y} \gamma(x)\gamma(y)h(x,y) \geq 0$ whenever $\sum_x |\gamma(x)| < \infty$. Then $V_2^t h \leq U_2^t h$ on T_2.*

Proof. For $t \geq 0$, $U_2^t h$ is positive definite since

$$\sum_{x,y} \gamma(x)\gamma(y) U_2^t h(x,y) = \sum_{x,y} \beta(x)\beta(y) h(x,y)$$

where $\beta(x) = \sum_u \gamma(u) p_t(u,x)$. Therefore,

$$2U_2^t h(x,y) \leq U_2^t h(x,x) + U_2^t h(y,y)$$

for all $x, y \in S$, and hence $V_2 U_2^t h \leq U_2 U_2^t h$ on T_2. The desired conclusion then follows from the relation

$$U_2^t - V_2^t = \int_0^t V_2^{t-s}(U_2 - V_2) U_2^s \, ds$$

and the fact that V_2 maps functions which are nonnegative on T_2 into functions which are nonnegative on T_2.

Lemma 2.8. *In Case* II, $\lim_{k\to\infty} V_2^k f_2(\vec{x}) = 0$ *for* $\vec{x} \in T_2$.

First proof. By conditioning on the value of $X(1)$, we obtain the relation

$$f_2(\vec{x}) = p(x_1, x_2) + \frac{1}{2} \sum_{u \neq x_2} p(x_1, u) f_2(u, x_2) + \frac{1}{2} \sum_{u \neq x_1} p(x_2, u) f_2(x_1, u).$$

Therefore

(2.9) $$f_2(\vec{x}) - V_2 f_2(\vec{x}) = p(x_1, x_2)[1 - f_2(\vec{x})] \geq 0$$

for $\vec{x} \in T_2$, and hence $f_2(Y(k))$ is a supermartingale. Since f_2 is bounded, $\lim_{k\to\infty} f_2(Y(k))$ exists with probability one. Now $Y(k)$ is transient, since otherwise f_2 would be constant on T_2, and this is impossible by Lemma 2.1. Therefore, by Lemmas 2.4 and 2.6,

$$\underline{\lim}_{k\to\infty} f_2(Y(k)) = 0$$

with probability one. Since the limit exists, it follows that $\lim_{k\to\infty} f_2(Y(k)) = 0$ with probability one. The required conclusion then follows from the bounded convergence theorem.

Second proof. Using the notation in the proof of Lemma 2.4, define $h(x,y) = (\beta_x, \beta_y)$ for $x, y \in S$. Then h is positive definite, so $V_2^t h \leq U_2^t h$ on T_2 by Lemma 2.7. But $U_2^t h \to 0$ as $t \to \infty$ by part (d) of Lemma 2.1 and the fact that $\lim_{t\to\infty} p_t(x,y) = 0$ for $x, y \in S$, since $f_2 = h$ on T_2. Therefore $V_2^t f_2 \to 0$ on T_2. The required conclusion then follows from the fact that $V_2^k f_2$ is monotonic in k (see (2.9)).

The next result is useful in reducing problems concerning the interacting n-particle system to ones concerning the interacting 2-particle system. Define a transformation A_n from symmetric functions h on T_2 to symmetric functions on T_n by

$$A_n h(x_1, \ldots, x_n) = \sum_{1 \leq i \neq j \leq n} h(x_i, x_j).$$

Lemma 2.10. *For $k \geq 1$,*

(2.11) $$V_n^k A_n = \sum_{i=0}^{k} \binom{k}{i} \left(\frac{2}{n}\right)^i \left(1 - \frac{2}{n}\right)^{k-i} A_n V_2^i.$$

Proof. Fix $i \neq j$, and put $g(x_1, \ldots, x_n) = h(x_i, x_j)$, where h is a symmetric function on T_2. Then

$$\begin{aligned} V_n g(\vec{x}) = {} & (1 - 2/n) h(x_i, x_j) + (2/n) V_2 h(x_i, x_j) \\ & - \frac{1}{n} \sum_{k \neq i,j} [p(x_i, x_k)[h(x_k, x_j) - h(x_i, x_j)] \\ & \qquad + p(x_j, x_k)[h(x_k, x_i) - h(x_i, x_j)]]. \end{aligned}$$

Therefore

$$V_n A_n h(\vec{x}) = (1 - 2/n) A_n h + (2/n) A_n V_2 h,$$

since $\sum p(x_i, x_k)[h(x_k, x_j) - h(x_i, x_j)] = 0$, where the sum is taken over all distinct i, j, k. This gives (2.11) for $k = 1$. The general result now follows easily by induction on k.

Lemma 2.12. *For $n \geq 2$ and $\vec{x} \in T_n$,*

$$(1/n(n-1))A_n f_2(\vec{x}) \leq f_n(\vec{x}) \leq \tfrac{1}{2}A_n f_2(\vec{x}). \tag{2.13}$$

Proof. For $1 \leq i \neq j \leq n$,

$$f_2(x_i, x_j) = P^{\vec{x}}(X_i(k) = X_j(k) \text{ for some } k \geq 1) \leq f_n(\vec{x}),$$

and the first inequality in (2.13) follows by summing on i and j. On the other hand,

$$\begin{aligned} f_n(\vec{x}) &\leq \sum_{1 \leq i < j \leq n} P^{\vec{x}}(X_i(k) = X_j(k) \text{ for some } k \geq 1) \\ &= \sum_{1 \leq i < j \leq n} f_2(x_i, x_j) = \tfrac{1}{2}A_n f_2(\vec{x}). \end{aligned}$$

Remark. It follows from this lemma that the condition $f_n(\vec{x}) < 1$ on T_n is independent of n for $n \geq 2$. To see this, if $f_2 \equiv 1$, then $f_n \equiv 1$ by the first inequality in (2.13). On the other hand, if $f_2 < 1$, then $\lim_{y\to\infty} f_2(x,y) = 0$ for each $x \in S$ by Lemma 2.4, and therefore $f_n < 1$ by the second inequality in (2.13).

Lemma 2.14. *In Case* II, $\lim_{k\to\infty} V_n^k f_n(\vec{x}) = 0$ *for $n \geq 2$ and $\vec{x} \in T_n$.*

Proof. By Lemma 2.12, it suffices to prove that

$$\lim_{k\to\infty} V_n^k A_n f_2(\vec{x}) = 0 \tag{2.15}$$

for $\vec{x} \in T_n$. But by Lemma 2.10,

$$V_n^k A_n f_2(\vec{x}) = \sum_{i=0}^{k} \binom{k}{i} \left(\frac{2}{n}\right)^i \left(1 - \frac{2}{n}\right)^{k-i} A_n V_2^i f_2(\vec{x}).$$

So (2.15) follows from Lemma 2.8.

Finally, we collect the results which will be needed in the next section.

Theorem 2.16. *In Case* II,

(a) $|V_n^k g(\vec{x}) - U_n^k g(\vec{x})| \leq f_n(\vec{x})$ *for* $\vec{x} \in T_n$, $n \geq 2$, $k \geq 1$, *and* $0 \leq g \leq 1$.
(b) $\lim_{k\to\infty} V_n^k f_n(\vec{x}) = 0$ *for* $\vec{x} \in T_n$ *and* $n \geq 2$.
(c) $\lim_{k\to\infty} U_n^k f_n(\vec{x}) = 0$ *for* $\vec{x} \in S^n$ *and* $n \geq 2$.

Proof. Parts (b) and (c) come from Lemmas 2.14 and 2.1 respectively. Part (a) is a consequence of the fact that $X(k)$ and $Y(k)$ behave identically if $X(k)$ never hits T_n^c.

3. The infinite particle system. This section is devoted to the proofs of Theorems 1.1 and 1.2 in Case II, and we assume throughout the section that this case holds. The basic tool which connects the results in §2 concerning finite particle systems with the desired properties of the corresponding infinite particle system $\eta(t)$ is the following lemma due to Spitzer [5]. In order to state it, define operators B_n from the space of probability measures on X to the space of functions on S^n by

$$B_n\mu(x_1, \ldots, x_n) = \mu\{\eta \in X \mid \eta(x_i) = 1 \text{ for } 1 \leq i \leq n\}.$$

Lemma 3.1 (Spitzer). *$B_n S(t) = V_n^t B_n$ on T_n.*

Proof of Theorem 1.1. Since $p(\cdot, \cdot)$ is recurrent in Case II, $\mathscr{H}$ consists only of constants. Therefore, by de Finetti's theorem, it suffices to prove that for a probability measure μ on X, $S(t)\mu = \mu$ for all $t > 0$ if and only if μ is a symmetric measure; in other words, if and only if $B_n\mu(\vec{x})$ is independent of $\vec{x} \in T_n$ for each $n \geq 1$. By Lemma 3.1, $S(t)\mu = \mu$ for all $t > 0$ if and only if $V_n^t B_n\mu = B_n\mu$ for all $t > 0$ and each $n \geq 1$. However, $V_n^t h = h$ for all $t > 0$ if and only if $V_n h = h$, since $V_n^t = \exp[nt(V_n - I)]$. So it is enough to show that for $0 \leq h \leq 1$, $V_n h = h$ implies that h is constant on T_n. For $n = 1$, this is true because $V_1 = U_1$ and $p(\cdot, \cdot)$ is recurrent. For $n \geq 2$, it follows from Theorem 2.16, since if $V_n h = h$ and $0 \leq h \leq 1$,

$$|h - U_n^k h| \leq f_n, \tag{3.2}$$

and therefore $|U_n^i h - U_n^{k+i} h| \leq U_n^i f_n$, so $\lim_{k\to\infty} U_n^k h = g$ exists. Since $U_n g = g$ and $0 \leq g \leq 1$, g is constant in $\vec{x}$ by Lemma 3.14 of [3] and the fact that $\mathscr{H}$ consists only of constants. So $|h - g| \leq f_n$ by (3.2), and since $V_n g = g$, it follows that $|h - g| \leq V_n^k f_n$ for all k. A final application of Theorem 2.16 gives $h = g$, and hence that h is constant in $\vec{x}$.

Proof of Theorem 1.2. Since $\mathscr{H}$ consists only of constants in Case II, it suffices to prove that if $\rho \in [0, 1]$ is constant, $S(t)\mu \to \mu_\rho$ weakly as $t \to \infty$ if and only if $U_n^t B_n\mu \to \rho^n$ as $t \to \infty$ for $n \geq 1$. By Lemma 3.1, $S(t)\mu \to \mu_\rho$ if and only if $V_n^t B_n\mu \to \rho^n$ on T_n as $t \to \infty$ for $n \geq 1$. So, it remains to be shown that if $0 \leq h(\vec{x}) \leq 1$ for $\vec{x} \in S^n$ and γ is a constant, then $U_n^t h \to \gamma$ on S^n if and only if $V_n^t h \to \gamma$ on T_n. But this follows easily from Theorem 2.16 as in the proof of Lemma 5.1 of [3].

4. Examples. This section is devoted to showing that Case II does in fact occur—that is, that there exists a symmetric, irreducible, recurrent Markov chain for which $f_2 < 1$. It will be shown, in fact, that any symmetric, irreducible recurrent Markov chain which has the property that the corresponding two particle motion is transient can be modified in a natural way in order to obtain an example of Case II. In order to do this, we need the following result, whose proof was suggested by C. Stone.

Lemma 4.1. *If $\beta_n > 0$ and $\beta_n \to 0$, then there exists a symmetric irreducible random walk $W(n)$ on Z^1 so that $W(0) = 0$,*

$$\sum_{n=1}^{\infty} P(W(n) = 0) = \infty \quad \textit{and} \quad \sum_{n=1}^{\infty} \beta_n P(W(n) = 0) < \infty.$$

Proof. Define inductively a sequence of symmetric, irreducible, transient random walks $W^k(n)$ on Z^1 (i.e., sums of independent, identically distributed random variables), and an increasing sequence of integers N_k in the following way. Let $W^0(n)$ be any symmetric, irreducible, transient random walk on Z^1, and $N_0 = 0$. Put

$$A = \sum_{n=1}^{\infty} P(W^0(n) = 0).$$

Now choose N_1 so large that

(i) $\beta_n \leq \frac{1}{2}$ for all $n \geq N_1$, and

(ii) $\sum_{n=1}^{N_1} P(W^0(n) = 0) > A/2$.

Next modify the random walk $W^0(n)$ to get $W^1(n)$ in such a way that

(iii) $A/2 < \sum_{n=1}^{N_1} P(W^1(n) = 0) < A$, and

(iv) $\sum_{n=N_1+1}^{\infty} P(W^1(n) = 0) = A$.

The distribution of $W^1(1)$ can be obtained from that of $W^0(1)$ by symmetrically moving some of the mass in towards the origin in the following way. Let $p(x) = P(W^0(1) = x)$ for $x \in Z^1$, and choose x_0 so that if $q(x)$ is any probability distribution on Z^1 so that $p(x) = q(x)$ for $|x| < x_0$, then $A/2 < \sum_{n=1}^{N_1} q^{(n)}(0) < A$. Define $p_\lambda(x)$ for $0 \leq \lambda \leq 1$ by

$$\begin{aligned} p_\lambda(x) &= p(x) && \text{if} \quad |x| < x_0, \\ &= p(x_0) + \lambda \sum_{y > x_0} p(y) && \text{if} \quad |x| = x_0, \\ &= (1 - \lambda)p(x) && \text{if} \quad |x| > x_0. \end{aligned}$$

Then $\sum_{n=N_1+1}^{\infty} p_\lambda^{(n)}(0)$ is continuous in λ for $0 \leq \lambda < 1$, is less than $A/2$ for $\lambda = 0$ by (ii), and tends to infinity as λ tends to one since the random walk with transition probabilities $p_1(y - x)$ is recurrent. Therefore for some λ, $0 \leq \lambda < 1$, $\sum_{n=N_1+1}^{\infty} p_\lambda^{(n)}(0) = A$, and $W^1(n)$ can be taken to be the random walk with transition probabilities $p_\lambda(y - x)$.

In general, if N_{k-1} and $W^{k-1}(n)$ have already been chosen, choose N_k so large that

(i) $\beta_n \leq 2^{-k}$ for all $n \geq N_k$, and

(ii) $\sum_{n=N_{k-1}+1}^{N_k} P(W^{k-1}(n) = 0) > A/2$.

Then modify the random walk $W^{k-1}(n)$ to get $W^k(n)$ in such a way that

(iii) $A/2 < \sum_{n=N_{i-1}+1}^{N_i} P(W^k(n) = 0) < A$ for $1 \leq i \leq k$, and

(iv) $\sum_{n=N_k+1}^{\infty} P(W^k(n) = 0) = A$.

If the modifications are always made by symmetrically moving some of the mass in towards the origin, then the limit $W(n)$ of $W^k(n)$ exists in distribution as $k \to \infty$. Then $W(n)$ is a symmetric, irreducible random walk on Z^1 which satisfies

$$\frac{A}{2} \leq \sum_{n=N_{i-1}+1}^{N_i} P(W(n) = 0) \leq A$$

for all $i \geq 1$. Therefore

$$\sum_{n=1}^{\infty} P(W(n) = 0) = \sum_{i=1}^{\infty} \sum_{n=N_{i-1}+1}^{N_i} P(W(n) = 0) = \infty,$$

and

$$\sum_{n=1}^{\infty} \beta_n P(W(n) = 0) \leq \sum_{i=1}^{\infty} \frac{1}{2^{i-1}} \sum_{n=N_{i-1}+1}^{N_i} P(W(n) = 0) \leq 2A < \infty.$$

Construction of the example. Let $p(x,y)$ be the transition probabilities for a symmetric, irreducible, recurrent Markov chain on the countable set S which has the following additional properties:

(a) There is an $x_0 \in S$ so that $p(x_0, x_0) > 0$.

(b) For all $x \in S$, $x \neq x_0$, $p(x,x) = 0$.

(c) The Markov chain $(X(k), Y(k))$ on S^2 with transition operator U_2 (based on p) is transient.

Define stopping times T_l for the chain $(X(k), Y(k))$ by $T_0 = 0$ and

$$T_{l+1} = \min\{k > T_l \mid X(k) = x_0 \text{ or } Y(k) = x_0\}.$$

These are finite with probability one since p is recurrent. Let R_l be the number of $k \leq T_l$ for which $X(k) = X(k-1)$ and $Y(k) = Y(k-1)$, and put

$$\beta_n = \sum_{l=0}^{\infty} P^{(x_0,x_0)}(R_l = n, X(k) = Y(k) \neq x_0 \text{ for some } k, T_l < k < T_{l+1}).$$

Lemma 4.2. $\lim_{n\to\infty} \beta_n = 0$.

Proof. Let $g(x,y) = P^{(x,y)}(X(k) = Y(k) \neq x_0$ for some $k < T_1)$. Then by the strong Markov property, the lth summand in the definition of β_n is equal to

$$E^{(x_0,x_0)}[g(X(T_l), Y(T_l)), R_l = n].$$

Since $(X(k), Y(k))$ is transient and $\lim_{y\to\infty} g(x_0,y) = 0$,

$$g(X(T_l), Y(T_l)) \to 0 \tag{4.3}$$

with probability one. Since $R_{l+1} - R_l = 0$ or 1, an increasing sequence of integers N_n can be defined by $R_l = n$ if and only if $N_n \leq l < N_{n+1}$. Let $\gamma = 1 - \frac{1}{2}p(x_0, x_0) < 1$. Since $P^{(x_0,z)}(R_1 = 0) \leq \gamma$ for all $z \in S$, $P(N_{n+1} - N_n \geq k) \leq \gamma^{k-1}$ for $k \geq 0$. Therefore, $E[(N_{n+1} - N_n - m)^+] \leq \gamma^{m-1}/(1-\gamma)$ for all $n, m \geq 0$. Using (4.3) and $0 \leq g \leq 1$, we obtain

$$\overline{\lim_{n\to\infty}}\,\beta_n = \overline{\lim_{n\to\infty}}\,E^{(x_0,x_0)}\left[\sum_{l=N_n}^{N_{n+1}-1} g(X(T_l),Y(T_l))\right]$$
$$\leq \overline{\lim_{n\to\infty}}\,E^{(x_0,x_0)}\left[\sum_{l=N_n}^{N_n+m} g(X(T_l),Y(T_l))\right] + \frac{\gamma^{m-1}}{1-\gamma} = \frac{\gamma^{m-1}}{1-\gamma}.$$

Since this is true for each m, $\lim_{n\to\infty}\beta_n = 0$.

Returning to the construction, let $p(k) = P(W(1) = k)$, where $W(n)$ is the random walk constructed in Lemma 4.1 corresponding to this sequence $\{\beta_n\}$. Define a new Markov chain on $\tilde{S} = S \times Z^1$ with transition probabilities

$$\begin{aligned}\tilde{p}((x,i),(y,j)) &= p(x,y) && \text{if } i = j, \text{ and } x \neq x_0 \text{ or } y \neq x_0,\\ &= p(x_0,x_0)p_{j-i} && \text{if } x = y = x_0,\\ &= 0 && \text{otherwise.}\end{aligned}$$

We will show that this Markov chain has the required properties. The symmetry, irreducibility, and recurrence of the chain follow from the corresponding properties of $p(\cdot,\cdot)$ and $\{p_k\}$. It remains to show that $f_2 < 1$ for this chain. So, let $(\tilde{X}(k), \tilde{Y}(k))$ be the Markov chain on $\tilde{S}^2$ with transition operator $\tilde{U}_2$ (based on $\tilde{p}$) and initial state $((x_0,0),(x_0,0))$. Note that $(\tilde{X}_1(k), \tilde{Y}_1(k))$ has the same distribution as $(X(k), Y(k))$, and that $\tilde{X}_2(\tilde{T}_l) - \tilde{Y}_2(\tilde{T}_l)$ has the same distribution as $W(R_l)$, where the process $W(n)$ is independent of the sequence R_l. The stopping times $\tilde{T}_l$ are defined by $\tilde{T}_0 = 0$ and

$$\tilde{T}_{l+1} = \min\{k > \tilde{T}_l \mid \tilde{X}_1(k) = x_0 \text{ or } \tilde{Y}_1(k) = x_0\}.$$

Then, since $\tilde{X}_2(k) = \tilde{X}_2(T_l)$ and $\tilde{Y}_2(k) = \tilde{Y}_2(T_l)$ for $T_l < k < T_{l+1}$,

$$\sum_{l=0}^{\infty} P(\exists k, \tilde{T}_l < k < \tilde{T}_{l+1}, \tilde{X}(k) = \tilde{Y}(k), \tilde{X}_1(k) \neq x_0)$$
$$= \sum_{n=0}^{\infty} P(W(n) = 0)\beta_n < \infty.$$

So, by the Borel-Cantelli lemma,

$$P(\tilde{X}(k) = \tilde{Y}(k), \tilde{X}_1(k) \neq x_0 \text{ for } \infty \text{ many } k) = 0.$$

But since the process $(\tilde{X}_1(k), \tilde{Y}_1(k))$ is transient,

$$P(\tilde{X}(k) = \tilde{Y}(k), \tilde{X}_1(k) = x_0 \text{ for } \infty \text{ many } k) = 0$$

also, and thus $f_2 < 1$ by part (d) of Lemma 2.1.

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References

1. R. Holley, *A class of interactions in an infinite particle system*, Advances in Math. **5** (1970), 291–309. MR **42** #3857.

2. J. Kemeny, J. Snell, and A. Knapp, *Denumerable Markov chains*, Van Nostrand, Princeton, N. J., 1966. MR **34** #6858.

3. T. Liggett, *A characterization of the invariant measures for an infinite particle system with interactions*, Trans. Amer. Math. Soc. **179** (1973), 433–453.

4. ———, *Existence theorems for infinite particle systems*, Trans. Amer. Math. Soc. **165** (1972), 471–481.

5. F. Spitzer, *Interaction of Markov processes*, Advances in Math. **5** (1970), 246–290. MR **42** #3856.

6. ———, *Recurrent random walk of an infinite particle system*, Trans. Amer. Math. Soc. **198** (1974), 191–199.

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