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PERTURBED SEMIGROUP LIMIT THEOREMS WITH APPLICATIONS TO DISCONTINUOUS RANDOM EVOLUTIONS

BY

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ABSTRACT. For $\epsilon > 0$ small, let $U^{\epsilon}(t)$ and $S(t)$ be strongly continuous semigroups of linear contractions on a Banach space L with infinitesimal operators $A(\epsilon)$ and B respectively, where $A(\epsilon) = A^{(1)} + \epsilon A^{(2)} + o(\epsilon)$ as $\epsilon \to 0$. Let $\{B(u); u \geq 0\}$ be a family of linear operators on L satisfying $B(\epsilon) = B + \epsilon\Pi^{(1)} + \epsilon^2\Pi^{(2)} + o(\epsilon^2)$ as $\epsilon \to 0$. Assume that $A(\epsilon) + \epsilon^{-1}B(\epsilon)$ is the infinitesimal operator of a strongly continuous contraction semigroup $T_\epsilon(t)$ on L and that for each $f \in L$, $\lim_{\lambda\to 0}\lambda\int_0^\infty e^{-\lambda t}S(t)f\,dt \equiv Pf$ exists. We give conditions under which $T_\epsilon(t)$ converges as $\epsilon \to 0$ to the semigroup generated by the closure of $P(A^{(1)} + \Pi^{(1)})$ on $\overline{R(P) \cap \mathcal{D}(A^{(1)}) \cap \mathcal{D}(\Pi^{(1)})}$. If $P(A^{(1)} + \Pi^{(1)})f = 0$, $Bh = -(A^{(1)} + \Pi^{(1)})f$, and we let $\hat{V}f = P(A^{(1)} + \Pi^{(1)})h$, then we show that $T_\epsilon(t/\epsilon)f$ converges as $\epsilon \to 0$ to the strongly continuous contraction semigroup generated by the closure of $V^{(2)} + \hat{V}$.

From these results we obtain new limit theorems for discontinuous random evolutions and new characterizations of the limiting infinitesimal operators of the discontinuous random evolutions. We apply these results in a model for the approximation of physical Brownian motion and in a model of the content of an infinite capacity dam.

1. Introduction. The perturbed semigroup limit theorems in this paper are motivated by results on discontinuous random evolutions. Let $X(t)$, $t \geq 0$, be a finite-state, continuous-time Markov chain with values in $\{1, 2, \cdots, N\}$; τ_1, $\tau_2, \cdots, \tau_\nu$ and ν denote the transition epochs and total number of transitions before time t/ϵ for the process $X(t)$. For each $1 \leq j \leq N$, let $T_j(t)$ be a semigroup of linear contractions on a Banach space L; for each $1 \leq j \neq k \leq N$, let $\Pi_{jk}(u)$, $u \geq 0$, be a family of linear contractions on L satisfying $\Pi_{jk}(\epsilon)f = f + \epsilon\Pi_{jk}f + o(\epsilon)$ as $\epsilon \to 0$ for $f \in \mathcal{D}(\Pi_{jk})$. We define the discontinuous random evolution by

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$$M_\epsilon(t) = T_{X(0)}(\epsilon\tau_1)\Pi_{X(0)X(\tau_1)}(\epsilon)T_{X(\tau_1)}(\epsilon(\tau_2 - \tau_1)) \cdots \Pi_{X(\tau_{\nu-1})X(\tau_\nu)}(\epsilon)T_{X(\tau_\nu)}(t - \epsilon\tau_\nu).$$

Investigations of $\lim_{\epsilon\to 0} E_j\,[M_\epsilon(t) f_{X(t/\epsilon)}]$ and $\lim_{\epsilon\to 0} E_j\,[M_\epsilon(t/\epsilon) f_{X(t/\epsilon^2)}]$ are motivations for and are shown to be applications of the perturbed semigroup limit theorems presented in §§2 and 3.

In §2 an application of the limit theorems for discontinuous random evolutions is given to the approximation of physical Brownian motion by the motion of a macroscopic particle within a medium of microscopic particles. Another application is made to the approximation of the content of an infinite capacity dam as the random epochs of rainfall become more frequent and random quantity of rainfall per occurence diminishes. Limit theorems for discontinuous random evolutions in which the "controlling" Markov process is a regular step process rather than a finite-state Markov chain constitute §4. Instead of the norm convergence used in §§2 and 3 we use buc-convergence, i.e., convergence of bounded families, uniformly on compact sets, in §4. In all applications of the limit theorems to discontinuous random evolutions we give new characterizations of the limiting infinitesimal generator.

In [6] Griego-Hersh introduced "continuous" random evolutions, i.e., random evolutions without the presence of the "jump operators" Π_{jk}, and used this concept to prove singular perturbation theorems. Perturbed semigroup limit theorems motivated by continuous random evolutions were proved by Thomas G. Kurtz [14]. Pinsky introduced discontinuous random evolutions as a representation for multiplicative operator functionals of a Markov chain in [15] and showed in [16] that $M_\epsilon(t)$ is the unique solution to the linear operator equation

$$M_\epsilon(t) = I + \int_0^t M_\epsilon(u) A_{X(u/\epsilon)}\, du + \sum_{0<\tau_k\leqslant t/\epsilon} M_\epsilon(\epsilon\tau_k)\{\Pi_{X(\tau_{k-1})X(\tau_k)}(\epsilon) - I\},$$

where, for $1 \leqslant j \leqslant N$, A_j is the infinitesimal operator of $T_j(t)$. The author has proved limit theorems for discontinuous random evolutions using other techniques and has applied these results to singular perturbation theorems and to central limit theorems for Markov processes on N lines [10], [11]. Surveys of the literature on random evolutions are given in the papers of Pinsky [16] and Cogburn-Hersh [3].

2. **"Weak-law-of-large numbers" type perturbation results with norm convergence.** Let L be a Banach space. Suppose $\{U(t); t \geqslant 0\}$ and $\{S(t); t \geqslant 0\}$ are strongly continuous semigroups of linear contractions on L with infinitesimal operators A and B respectively. Suppose that $\{B(t), t \geqslant 0\}$ is a family of linear operators on L and Π is a linear operator satisfying

$$B(\epsilon)f = Bf + \epsilon \Pi f + o(\epsilon) \tag{2.1}$$

for $f \in \mathcal{D}(B) \cap \mathcal{D}(\Pi)$ and $\epsilon \downarrow 0$. Take B to be the closure of B restricted to $\mathcal{D}(A) \cap \mathcal{D}(B) \cap \mathcal{D}(\Pi)$. Suppose for each $\epsilon > 0$ small that the closure of $A + \epsilon^{-1}B(\epsilon)$ is the infinitesimal operator of a strongly continuous contraction semigroup $T_\epsilon(t)$ on L.

Notation is that of [12]. A possibly multivalued operator A is written as a set of ordered pairs $A = \{(x, y); Ax = y\}$, with $\mathcal{D}(A) = \{x; (x, y) \in A\}$ and $\mathcal{R}(A) = \{y; (x, y) \in A\}$. We use $\lim_{n\to\infty}(x_n, y_n) = (x, y)$ to mean $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Limits here and below are taken to be strong limits. Proofs use techniques found in [14].

THEOREM 2.1. *Suppose* $U(t)$, $S(t)$, $B(t)$, Π, *and* $T_\epsilon(t)$ *are given as above. Assume that*

$$\lim_{\lambda\to 0} \lambda \int_0^\infty e^{-\lambda t} S(t) f\, dt \equiv Pf \tag{2.2}$$

exists for every $f \in L$. *Let*

$$D = \{f \in \mathcal{R}(P); f \in \mathcal{D}(A) \cap \mathcal{D}(\Pi)\} \tag{2.3}$$

and define for $f \in D$

$$Vf = P(A + \Pi)f. \tag{2.4}$$

Assume that $\overline{\mathcal{R}(\lambda - V)} \supset D$ *for some* $\lambda > 0$. *Then there is a strongly continuous contraction semigroup* $\{T(t); t \geqslant 0\}$ *defined on* $\bar{D}$ *with* $\lim_{\epsilon\to 0} T_\epsilon(t) f = T(t)f$ *for all* $f \in \bar{D}$. *The infinitesimal operator of* $T(t)$ *is the closure of* V *restricted so that* $Vf \in \bar{D}$.

PROOF. Let V_ϵ = closure of $A + \epsilon^{-1}B(\epsilon)$ = infinitesimal operator of $T_\epsilon(t)$. From Theorem 1.10 of [14] and Theorem 2.1 of [12, p. 357], it suffices to show

$$\{(f, Vf); f \in D\} \subset \left\{(f, g);\ \exists (f_\epsilon, g_\epsilon) \in V_\epsilon \text{ with } \lim_{\epsilon\to 0}(f_\epsilon, g_\epsilon) = (f, g)\right\},$$

i.e., given $f \in D$, we must find $f_\epsilon \in \mathcal{D}(V_\epsilon)$, $g_\epsilon = V_\epsilon f_\epsilon \in \mathcal{R}(V_\epsilon)$ such that $\lim_{\epsilon\to 0} f_\epsilon = f$ and $\lim_{\epsilon\to 0} g_\epsilon = Vf$. For then, using $\overline{\mathcal{R}(\lambda - V)} \supset D$, we have that there exists a strongly continuous contraction semigroup $\{T(t); t \geqslant 0\}$ on $\bar{D}$ such that $\lim_{\epsilon\to 0} T_\epsilon(t) f = T(t)f$ for each $f \in \bar{D}$. From this theorem it also follows that the infinitesimal operator of $T(t)$ is the closure of $\{(f, g); (f, g) \in V$ and $f, g \in \bar{D}\}$, i.e., the closure of V restricted so that $Vf \in \bar{D}$.

Recall that we are considering B as the closure of B restricted to $\mathcal{D}(A) \cap \mathcal{D}(B) \cap \mathcal{D}(\Pi)$. Hence for any $g \in \overline{\mathcal{R}(B)}$, there exist $h_\epsilon \in \mathcal{D}(A) \cap \mathcal{D}(B) \cap \mathcal{D}(\Pi)$ such that $\lim_{\epsilon\to 0} Bh_\epsilon = g$, and, if necessary by relabeling the index set, such that, in addition, $\|(A + \Pi)h_\epsilon\| = o(1/\epsilon)$ and $\|h_\epsilon\| = o(1/\epsilon)$ (see §A.4 of [10]).

Hence $\lim_{\epsilon\to 0}\epsilon V_\epsilon h_\epsilon = g$ since

$$\|\epsilon V_\epsilon h_\epsilon - g\| = \|\epsilon A h_\epsilon + B(\epsilon)h_\epsilon - g\| \leqslant \|\epsilon A h_\epsilon + B h_\epsilon + \epsilon \Pi h_\epsilon - g\| + o(\epsilon)$$

$$\leqslant \epsilon\|A h_\epsilon + \Pi h_\epsilon\| + \|B h_\epsilon - g\| + o(\epsilon) = \|B h_\epsilon - g\| + o(\epsilon).$$

From Theorem 18.6.2 of [8, p. 516], we have that P is a projection and $\mathcal{R}(B)$ is dense in $n(P)$, the null space of P. Hence, if $f \in D$, $P(A + \Pi)f - (A + \Pi)f$ is in $\overline{\mathcal{R}(B)}$, and we can choose $h_\epsilon \in \mathcal{D}(A) \cap \mathcal{D}(B) \cap \mathcal{D}(\Pi)$ such that $\lim_{\epsilon\to 0}\epsilon V_\epsilon h_\epsilon = P(A + \Pi)f - (A + \Pi)f$, with $\|h_\epsilon\| = o(1/\epsilon)$. Also from this theorem, $\mathcal{R}(P) = n(B)$; hence, since $f \in D$ we have $V_\epsilon f = (A + \epsilon^{-1}B(\epsilon))f = Af + \Pi f + o(\epsilon)/\epsilon$. If we set $f_\epsilon = f + \epsilon h_\epsilon$ then $\lim_{\epsilon\to 0} f_\epsilon = f$ and $\lim_{\epsilon\to 0} V_\epsilon f_\epsilon = Vf$, where we use the inequality

$$\|V_\epsilon f_\epsilon - Vf\| = \|V_\epsilon f + \epsilon V_\epsilon h_\epsilon - P(A + \Pi)f\|$$

$$\leqslant \|(A + \Pi)f + \epsilon V_\epsilon h_\epsilon - P(A + \Pi)f\| + o(\epsilon)/\epsilon = o(1).$$

Thus, given $f \in D$, there are $f_\epsilon \in \mathcal{D}(V_\epsilon)$ for $\epsilon \downarrow 0$ satisfying $\lim_{\epsilon\to 0} f_\epsilon = f$ and $\lim_{\epsilon\to 0} V_\epsilon f_\epsilon = Vf$. Q.E.D.

REMARK. Theorem 2.1 remains valid if we replace $U(t)$ by $U^{(\epsilon)}(t)$ in Theorem 2.1, where $U^{(\epsilon)}(t)$, for each $\epsilon > 0$, is a strongly continuous semigroup of linear contractions on L with infinitesimal operator $A(\epsilon)$ satisfying $A(\epsilon) = A + o(1)$ as $\epsilon \longrightarrow 0$, and if we then assume that $A(\epsilon) + \epsilon^{-1}B(\epsilon)$ is the infinitesimal operator of a strongly continuous contraction semigroup $T_\epsilon(t)$ on L.

EXAMPLE 2.1. Let $\{\xi(t); t \geqslant 0\}$ be a time-homogeneous, irreducible Markov chain with values in $E = \{1, 2, \cdots, N\}$. We assume $\xi(t)$ has generator $Q = (q_{\alpha\beta})$, $1 \leqslant \alpha, \beta \leqslant N$, stationary distribution (p_j), $1 \leqslant j \leqslant N$, and transition probabilities $\{p_{jk}(t); t \geqslant 0\}$, $1 \leqslant j, k \leqslant N$.

Suppose for each $1 \leqslant j \leqslant N$, that $T_j(t)$ is a strongly continuous, linear, contraction semigroup on a Banach space L with infinitesimal operator A_j. Let $\mathcal{L}$ be the Banach space of functions f: $E = \{1, 2, \cdots, N\} \longrightarrow L$, with $\|f\| = \max_{1\leqslant j\leqslant N}\|f_j\|_L$. The operators $\{U(t); t \geqslant 0\}$ and $\{S(t); t \geqslant 0\}$, defined by

$$(U(t)f)_j = T_j(t)f_j, \quad (S(t)f)_j = \sum_{1\leqslant k\leqslant N} p_{jk}(t)f_k \quad \text{for } j = 1, \cdots, N$$

are strongly continuous linear contraction semigroups on $\mathcal{L}$. $U(t)$ and $S(t)$ have infinitesimal operators A and B respectively, given by $(Af)_j = A_j f_j$ and $(Bf)_j = \Sigma_{1\leqslant k\leqslant N} q_{jk} f_k$ for $j = 1, \cdots, N$.

Suppose for each $1 \leqslant j \neq k \leqslant N$, that $\{\Pi_{jk}(t); t \geqslant 0\}$ is a family of linear contractions on L and Π_{jk} is a linear operator satisfying

$$\Pi_{jk}(\epsilon)f = f + \epsilon\Pi_{jk}f + o(\epsilon) \tag{2.5}$$

as $\epsilon \downarrow 0$ for $f \in \mathcal{D}(\Pi_{jk}) \subset L$. We denote by $\{B(t); t \geqslant 0\}$ the family of linear operators on $\mathcal{L}$ given by

$$(B(t)f)_j = \sum_{k=1,k\neq j}^{N} q_{jk}\Pi_{jk}(t)f_k + q_{jj}f_j$$

for $j = 1, \cdots, N$. We define the operator Π by

$$(\Pi f)_j = \sum_{k=1,k\neq j}^{N} q_{jk}\Pi_{jk}f_k$$

for $j = 1, \cdots, N$ and for $f \in \{f \in L; f_k \in \mathcal{D}(\Pi_{jk})$ for $j = 1, \cdots, N, j \neq k\}$. Then it follows that

$$(B(\epsilon)f)_j = (Bf)_j + \epsilon(\Pi f)_j + o(\epsilon) \tag{2.6}$$

as $\epsilon \downarrow 0$ for $f \in \mathcal{D}(\Pi) \cap \mathcal{D}(B) = \mathcal{D}(\Pi)$.

The operator $A + \epsilon^{-1}B(\epsilon)$ is given by

$$[(A + \epsilon^{-1}B(\epsilon))f]_j = A_j f_j + \epsilon^{-1} \sum_{k=1,k\neq j}^{N} q_{jk}\Pi_{jk}(\epsilon)f_k + \epsilon^{-1}q_{jj}f_j \tag{2.7}$$

for $j = 1, \cdots, N$, $\epsilon > 0$. $A + \epsilon^{-1}B(\epsilon)$ is the infinitesimal operator of the strongly continuous contraction semigroup $T_\epsilon(t)$ on L, defined by

$$\begin{aligned}(T_\epsilon(t)f)_j = E_j[&T_{\xi(0)}(\epsilon t_1^*)\Pi_{\xi(0)\xi(t_1^*)}(\epsilon)T_{\xi(t_1^*)}(\epsilon(t_2^* - t_1^*)) \\ &\cdots T_{\xi(t_\nu^*)}(t - \epsilon t_\nu^*)f_{\xi(t/\epsilon)}]\end{aligned} \tag{2.8}$$

for $j = 1, \cdots, N$, $\epsilon > 0$, where $t_1^*, t_2^*, \cdots, t_\nu^*$ and ν are the jump times and number of jumps for the process $\xi(u)$ in the time interval $[0, t/\epsilon]$.

We assume that B is the closure of B restricted to $\mathcal{D}(A) \cap \mathcal{D}(\Pi)$. In checking that the conditions of Theorem 2.1 are met, we note that $\lim_{\lambda\to 0}\lambda\int_0^\infty e^{-\lambda t}p_{jk}(t)dt = p_k$ implies that $\lim_{\lambda\to 0}\lambda\int_0^\infty e^{-\lambda t}S(t)f\,dt \equiv Pf$ exists for each $f \in L$, where P is given here by

$$(Pf)_j = \sum_{k=1}^{N} p_k f_k \tag{2.9}$$

for $j = 1, 2, \cdots, N$. In this setting

$$D = \left\{f \in L; f_j = w \text{ for } j = 1, \cdots, N, \; w \in \bigcap_{1\leq\alpha,j\neq k\leq N} (\text{domains of } A_\alpha, \Pi_{jk})\right\} \tag{2.10}$$

and

$$Vf = P(A + \Pi)f = \left(\sum_{j=1}^{N} p_j A_j + \sum_{1\leq j\neq k\leq N} p_j q_{jk}\Pi_{jk}\right)w \cdot (1) \tag{2.11}$$

for $f = w \cdot (1)$ in D. The notation $f = w \cdot (1)$ means $f = (f_j)$ and $f_j = w$ for all $j = 1, \cdots, N$. Finally, we assume that $\overline{R(\lambda - V)} \supset D$ for some $\lambda > 0$.

Then, by Theorem 2.1, there is a strongly continuous contraction semigroup $\{T(t); t \geqslant 0\}$ defined on $\bar{D}$ with $\lim_{\epsilon\to 0} T_\epsilon(t)f = T(t)f$ for all $f \in \bar{D}$. The infinitesimal operator of $T(t)$ is the closure of V restricted so that $Vf \in \bar{D}$.

APPLICATION 2.1. We consider the motion of a particle a_m, of mass m, moving in a one-dimensional medium. We suppose there are several position-dependent fields of force which act in the medium. We assume that the medium also contains homogeneous particles a_μ, of mass μ and with several possible velocity distributions given, independent of the motion of a_m. The motion of a_m is to be determined by one of the force fields between collisions of a_m with particles a_μ and at collisions is to be given by the "law of elastic impact." We assume that collisions occur "randomly" (see [9, p. 421]).

Specifically, functions $F_\alpha: R \longrightarrow R$, $1 \leqslant \alpha \leqslant N$, represent different force fields, and are assumed to be Lipschitz, twice continuously differentiable, and bounded. We let $\{\xi(t); t \geqslant 0\}$ denote a Markov chain taking values in $\{1, \cdots, N\}$ and with generator $Q = (q_{jk})$. We assume the hypotheses and notation with respect to Q and $\xi(\cdot)$ which are given in Example 2.1. For each $1 \leqslant j \neq k \leqslant N$, the family $\{\eta_i(j, k)\}_{i\geqslant 1}$ of independent, identically distributed random variables represents one of $N(N-1)$ possible velocities of the particles a_μ. These families of random variables are independent of each other and of the chain $\xi(\cdot)$, with family $\{\eta_i(j, k)\}_{i \geqslant 1}$ having distribution function $R_{jk}(y)$. We define the position-velocity process $\{Z^\mu(t) = (X^\mu(t), Y^\mu(t)); t \geqslant 0\}$ starting at (x, y) by $(X^\mu(t), Y^\mu(t)) = (x_\alpha(t), y_\alpha(t))$ for $t_j^* \leqslant t < t_{j+1}^*$, $j \geqslant 0$, where $\xi(t_j^*) = \alpha$ and $(x_\alpha(t), y_\alpha(t))$ is the solution at time t of the system

$$\begin{gathered} \frac{dx_\alpha}{dt} = y_\alpha, \qquad \frac{dy_\alpha}{dt} = F_\alpha(x_\alpha); \\ x_\alpha(t_j^*) = X^\mu(t_j^*), \quad y_\alpha(t_j^*) = Y^\mu(t_j^*); \end{gathered} \tag{2.12}$$

and at the times $t = t_j^*$, $X^\mu(t)$ remains continuous and equals $X^\mu(t-) = X^\mu(t+)$, $Y^\mu(t)$ is to be right-continuous, with $Y^\mu(t) = Y^\mu(t_j^*-) + \nu(\eta_j(\xi(t_{j-1}^*), \xi(t_j^*)) - Y^\mu(t_j^*-))$ where $\nu = 2\mu/(m+\mu)$, according to the law of elastic impact (see [17]). Note that $\{(X^\mu(t), Y^\mu(t)); t \geqslant 0\}$ is a Markov process.

We prove a limit theorem for this process in the following setting. We let $\xi(t) = \xi_\epsilon(t)$ depend upon $\epsilon > 0$ through its infinitesimal generator $Q_\epsilon = \epsilon^{-1}Q$ and for each $1 \leqslant j \neq k \leqslant N$ let $R_{jk}(dz) = R_{jk}^\epsilon(dz)$ depend upon ϵ and satisfy $\int zR_{jk}^\epsilon(dz) = 0$, $\mu_\epsilon \int z^2 R_{jk}^\epsilon(dz) = T_{jk}$ = constant, and $\lim_{\epsilon\to 0}\mu_\epsilon^2 \int |z|^3 R_{jk}^\epsilon(dz) = 0$, for mass $\mu = \mu_\epsilon = \epsilon$. For each ϵ, the gas is at rest $(E\{\eta_i^\epsilon(j, k)\} = 0)$ and the kinetic energy of the system is constant while the process remains in any given state, but may vary from one state to another $(E\{\mu_\epsilon(\eta_i^\epsilon(j, k))^2\} = T_{jk})$. In the limiting operation, we are letting mass $\mu_\epsilon \longrightarrow 0$, average velocities of $a_{\mu_\epsilon} \longrightarrow +\infty$

(but require constant temperatures), and collision rates $-q_{ij}^{\epsilon} = -q_{jj}/\epsilon \longrightarrow +\infty$ (but require constant viscosities proportional to $\alpha_j = -2q_{jj} = 2q_{jj}^{\epsilon}\mu_{\epsilon}$). Using Theorem 2.1, we show that under these conditions $Z^{\mu_\epsilon}(t)$ converges in distribution to physical Brownian motion; we do this by proving convergence of the semigroups of the processes.

Let L = space of bounded, continuous functions on R^2 with supremum norm; C^n = space of n-times continuously differentiable functions on R^2 with bounded support; and D^n = space of bounded n-times differentiable functions on R^2.

For each $1 \leqslant j \leqslant N$, $\{T_j(t); t \geqslant 0\}$ represents the linear contraction semigroup on L defined by $T_j(t)f(x, y) = f(x_j(t, x, y))$, with infinitesimal operator A_j given by $A_j f(x, y) = y\partial f/\partial x + F_j(x)\partial f/\partial y$ on D^1. For each $1 \leqslant j \neq k \leqslant N$, $\{\Pi_{jk}(\nu); \nu > 0\}$ represent linear contractions on L defined by

$$\Pi_{jk}(\nu)f(x, y) = \int_{-\infty}^{\infty} f(x, y + \nu(z - y))R_{jk}(dz).$$

From [9, p. 422] and [16, §1.3], we have the representation

$$\begin{aligned} w_j(t, x, y) &= E[f_{\xi^\epsilon(t)}(Z^{\mu_\epsilon}(t)) \mid Z^{\mu_\epsilon}(0) = (x, y),\ \xi^\epsilon(0) = j] = (T_\epsilon(t)f)_j \\ &= E\{T_{\xi(0)}(\epsilon t_1^*)\Pi_{\xi(0)\xi(t_1^*)}(\nu)T_{\xi(t_1^*)}(\epsilon(t_2^* - t_1^*)) \\ &\qquad \cdots T_{\xi(t^*_{N(t/\epsilon)})}(t - \epsilon t^*_{N(t/\epsilon)})f_{\xi(t/\epsilon)} \mid Z^{\mu_\epsilon}(0) = (x, y),\ \xi(0) = j\} \end{aligned}$$

for the semigroup of the $(Z^{\mu_\epsilon}(t), \xi^\epsilon(t))$ process and for the solution to the initial value problem

$$\text{(2.13)}\qquad \begin{aligned} \frac{\partial w_j^\epsilon}{\partial t} &= A_j w_j^\epsilon + \frac{1}{\epsilon} \sum_{k;k\neq j} q_{jk}\Pi_{jk}(\nu)w_k^\epsilon + q_{jj}w_j^\epsilon, \\ w_j^\epsilon(0) &= f_j, \qquad 1 \leqslant j \leqslant N,\ t > 0,\ f_j \text{ in } C^1. \end{aligned}$$

For each $1 \leqslant j \neq k \leqslant N$, from the assumptions on $R_{jk}^\epsilon(dz)$, we obtain that $\Pi_{jk}(\nu) = I + \epsilon\Pi_{jk} + o(\epsilon)$ as $\epsilon \longrightarrow 0$ on D^3, where Π_{jk} is defined by

$$\Pi_{jk}f(x, y) = \frac{2}{m}\left[-y\frac{\partial f}{\partial y} + (T_{jk}/m)\frac{\partial^2 f}{\partial y^2}\right].$$

On the set $\widetilde{D} = \{f \in L; f_j = w \text{ in } D^3, 1 \leqslant j \leqslant N\}$ we define V by

$$\begin{aligned} (Vf)_n &= \sum_{j=1}^N p_j A_j w + \sum_{1\leqslant j\neq k\leqslant N} p_j q_{jk}\Pi_{jk}w \\ &= y\frac{\partial w}{\partial x} + F(x)\frac{\partial w}{\partial y} + \frac{\alpha}{m}\left[-y\frac{\partial w}{\partial y} + (T/m)\frac{\partial^2 w}{\partial y^2}\right] \end{aligned}$$

for $1 \leqslant n \leqslant N$ with $F(x) = \Sigma_{j=1}^N p_j F_j(x)$, $\alpha = \Sigma_{j=1}^N p_j \alpha_j$, and $T = (2/\alpha)\Sigma_{1<j\neq k<N} p_j q_{jk} T_{jk}$. Since V satisfies the conditions of Theorem 2.1, there is a strongly continuous contraction semigroup $\{T(t); t \geqslant 0\}$ defined on $\Omega = \{f \in L;\ f_j = w$ is in L, $1 \leqslant j \leqslant N\}$ satisfying $\lim_{\epsilon\to 0} T_\epsilon(t)f = T(t)f$ for all $f \in \Omega$. We let $S(t)\colon L \longrightarrow L$ represent the strongly continuous contraction semigroup defined by $S(t)w = (T(t)f)_j$ for $f = w \cdot (1)$ in Ω and extend the set on which convergence holds (see [7, Corollary to Theorem 1]). We obtain $\lim_{\epsilon\to 0}(T_\epsilon(t)f)_j = S(t)\Sigma_{\alpha=1}^N p_\alpha f_\alpha$ for f in L. The infinitesimal operator of $T(t)$ is the closure of V. In particular, if $w(t, x, y)$ is the bounded solution of

$$(2.14)\qquad \begin{aligned} &\frac{\partial w}{\partial t} = y\frac{\partial w}{\partial x} + F(x)\frac{\partial w}{\partial y} + \frac{\alpha}{m}\left[-y\frac{\partial w}{\partial y} + (T/m)\frac{\partial^2 w}{\partial y^2}\right], \\ &w(0) = \sum_{j=1}^N p_j f_j, \qquad t \geqslant 0,\ f_j \text{ in } C^3, \end{aligned}$$

then $w(t, x, y) = S(t)(\Sigma_{j=1}^N p_j f_j)(x, y)$ and $\lim_{\epsilon\to 0} w_j^\epsilon(t) = w(t)$.

REMARKS. (a) Note that the Gaussian distribution with mean zero and variance $\epsilon^{-2}T_{jk} = \mu_\epsilon^{-1}T_{jk}$ satisfies the conditions imposed on $R_{jk}^\epsilon(dz)$.

(b) Khas′minskiĭ and Il′in have shown that there corresponds a Markov process $\{(X(t), Y(t)); t \geqslant 0\}$ whose transition density $p(x, y, t, x_1, y_1)$ is the Green's function for the equation in (21) (see [9, p. 437]). The above analysis gives that $(X^\mu(t), Y^\mu(t))$ converges to $(X(t), Y(t))$ in distribution as $\mu_\epsilon \longrightarrow 0$ in the prescribed manner.

APPLICATION 2.2. Suppose L is a Banach space, and notation is as given in Example 2.1 with $N = 2$, $q_{12} = q_{21} = a > 0$; $A_j = \Psi$ and $T_j(t) = T(t)$ for $j = 1, 2$; and $\Pi_{jk}(\epsilon) = \Pi^\epsilon = I + \epsilon\Pi + o(\epsilon)$ as $\epsilon \longrightarrow 0$ for $1 \leqslant j \neq k \leqslant 2$. The semigroup $T_\epsilon(t)$ is now given by

$$(T_\epsilon(t)\widetilde{f})_j = E_j[T(\epsilon t_1^*)\Pi^\epsilon T(\epsilon(t_2^* - t_1^*)) \cdots T(t - \epsilon t_{N(t/\epsilon)}^*)f] \qquad \text{for } \widetilde{f} = (f, f),\ f \text{ in } L.$$

Under the assumptions of Example 2.1 there exists a strongly continuous contraction semigroup $S(t)$ defined on $\overline{D(\Psi) \cap D(\Pi)}$ with $\lim_{\epsilon\to 0}(T_\epsilon(t)\widetilde{f})_j = S(t)f$ for f in $\overline{D(\Psi) \cap D(\Pi)}$. For B a Banach space of sufficiently smooth functions in L, we have $w_j^\epsilon(t) = (T_\epsilon(t)\widetilde{f})_j$ is a bounded solution of

$$(2.15)\qquad \begin{aligned} \partial w^\epsilon/\partial t &= \Psi w^\epsilon + (a/\epsilon)(\Pi^\epsilon w^\epsilon - w^\epsilon), \\ w^\epsilon(0) &= f, \qquad \epsilon > 0,\ t > 0,\ f \text{ in } B. \end{aligned}$$

As in the previous application we can obtain $w(t) = \lim_{\epsilon\to 0} w_1^\epsilon(t)$ exists and equals the bounded solution of

$$\begin{aligned} &\partial w/\partial t = (\Psi + a\Pi)w, \\ (2.16) \qquad &w(0) = f, \qquad t > 0, \ f \text{ in } B. \end{aligned}$$

As an application of Example 2.1 in the above form, we prove a limit theorem in the following storage theory model.

Suppose we are given a process $\{\xi(t); t \geqslant 0\}$ with independent, nonnegative increments, having jump rate $0 < b < \infty$, jump times given by $\tau_1, \tau_2, \cdots$, and with $\gamma(y)$ the distribution of the magnitude of a jump having two finite moments. (We assume that the linear part of $\xi(\cdot)$ is zero; the case where this part is nonzero is treated similarly.) We are also given a Lipschitz, strictly-increasing function r: $[0, \infty] \longrightarrow [0, \infty]$ satisfying $r(0) = 0$. The equation

$$(2.17) \qquad X_t = X_0 + \xi_t - \int_0^t r(X_u)du, \qquad t \geqslant 0, \ X_0 \geqslant 0,$$

has been analyzed in [2]. Here X_0 represents the initial content of a dam; ξ_t, the total input during time $[0, t]$; X_t, the content at time t; and $r(x)$, the releasing function. The equation (2.17) says that $Z_t = \int_0^t r(X_u)du$ is the total output during time $[0, t]$ and that the rate of output at time u is $r(X_u)$. In [2], $\{X(t); t \geqslant 0\}$, the unique solution to (2.17) is explicitly written down and shown to be a normal, standard Markov process.

We prove a limit theorem for the content process in the following setting. We let $\xi(t) = \xi^\epsilon(t)$ depend upon $\epsilon > 0$ by having jump rate $b_\epsilon = b/\epsilon$ and jump-size distribution $\gamma_\epsilon(y) = \gamma(y/\epsilon)$. We show that $X^\epsilon(t)$ converges to a deterministic process $x(t)$ as $\epsilon \longrightarrow 0$; we do this by showing convergence of the semigroups of the processes.

We let L be the Banach space of continuous functions on $[0, \infty)$ vanishing at infinity, with supremum norm. We define the group $T(t)$, $t \geqslant 0$, on L by $T(t)f(x) = f(q(x, t))$ where $q(x, t)$ is the unique solution to $\partial q/\partial t = -r(x)\partial q/\partial x$, $q_0 = x$. The infinitesimal operator of $T(t)$ is $\Psi = -r(x)\partial/\partial x$. For $\epsilon > 0$, we define the convolution operators $\Pi(\epsilon)$ on L by $\Pi(\epsilon)f(x) = \int_0^\infty f(x + z)\gamma_\epsilon(dz)$ where $\gamma_\epsilon(c) = \gamma(c/\epsilon)$. Then the transition semigroup of the content process $X^\epsilon(t)$ has the representation

$$\begin{aligned} w^\epsilon(t, x) &= P_t^\epsilon f(x) = E[f(X^\epsilon(t))] \\ &= E[T(\epsilon\tau_1)\Pi(\epsilon)T(\epsilon(\tau_2 - \tau_1)) \cdots T(t - \epsilon\tau_{N(t/\epsilon)})f(x)] \end{aligned}$$

with infinitesimal generator A^ϵ given by $A^\epsilon f(x) = -r(x)\partial f/\partial x + b\epsilon^{-1}[\Pi(\epsilon) - I]f$ (see [2]).

From these assumptions on $\gamma_\epsilon(y)$ we obtain $\Pi(\epsilon) = I + \epsilon\Pi + o(\epsilon)$ as $\epsilon \longrightarrow 0$ where $\Pi f = \mu\partial f/\partial x$, with $\mu = \int_0^\infty y\gamma(dy)$, on $F = \{f; f, f', f''$ are bounded$\}$. We define V on $\widetilde{F} = \{f = (f, f); f$ in $F\}$ by

$(V\widetilde{f})_j = \Psi f + b\Pi f = (-r(x) + m)\partial f/\partial x$, $j = 1, 2$, with $m = b\mu$ = input rate. From Theorem 2.1 and through the introductory remarks to this application, there is a strongly continuous contraction semigroup $S(t)$ defined on L with $\lim_{\epsilon\to 0} P_t^\epsilon f = S(t)f$ for f in L. In particular, if $w(t)$ is the bounded solution of

$$\frac{\partial w}{\partial t} = (\Psi + b\Pi)w = (-r(x) + m)\frac{\partial w}{\partial x},$$
$$w(0) = f, \qquad t \geq 0,\ f \text{ in } C_0^2, \tag{2.18}$$

then $w(t) = S(t)f$ and $\lim_{\epsilon\to 0} w^\epsilon(t) = w(t)$. From this convergence of semigroups we obtain that given $X^\epsilon(0) = x$, $X^\epsilon(t)$ converges in distribution as $\epsilon \longrightarrow 0$ to the solution $q(t)$ of $\partial q/\partial t = (-r(x) + m)\partial q/\partial x$, $q(0, x) = x$. For another physical interpretation of this model and a generalization to the level of Application 2.1, see [10].

3. "Central-limit-theorem" type perturbation results with norm convergence. Let L be a Banach space. Suppose $\{U^{(\epsilon)}(t); t \geq 0\}$ and $\{S(t); t \geq 0\}$ are strongly continuous semigroups of linear contractions on L with infinitesimal operators $A(\epsilon) = A^{(1)} + \epsilon A^{(2)} + o(\epsilon)$ and B respectively. Suppose that $\{B(t); t \geq 0\}$ is a family of linear operators on L and $\Pi^{(1)}$ and $\Pi^{(2)}$ are linear operators satisfying

$$B(\epsilon)f = Bf + \epsilon\Pi^{(1)}f + \epsilon^2\Pi^{(2)}f + o(\epsilon^2) \tag{3.1}$$

for $f \in \bigcap\{\text{domains of } B, \Pi^{(1)} \text{ and } \Pi^{(2)}\}$ and $\epsilon \downarrow 0$. Assume that B is the closure of B restricted to $\bigcap\{\text{domains of } B, A^{(1)}, A^{(2)}, \Pi^{(1)}, \text{and } \Pi^{(2)}\}$. Suppose for each ϵ small, that the closure of $A(\epsilon) + \epsilon^{-1}B(\epsilon)$ is the infinitesimal operator of a strongly continuous contraction semigroup $T_\epsilon(t)$ on L. Other notation is that of § 2.

THEOREM 3.1. *Suppose* $U(t)$, $S(t)$, $B(t)$, $\Pi^{(1)}$, $\Pi^{(2)}$, *and* $T_\epsilon(t)$ *are given as above. Assume that* $\lim_{\lambda\to 0}\lambda\int_0^\infty e^{-\lambda t}S(t)f\,dt \equiv Pf$ *exists for every* $f \in L$. *Define*

$$D_j = \{f \in R(P); f \in \mathcal{D}(A^{(j)}) \cap \mathcal{D}(\Pi^{(j)})\} \quad \textit{for } j = 1, 2, \tag{3.2}$$

$$D_0 = \{f \in D_1;\ \exists h \in \mathcal{D}(B) \cap \mathcal{D}(A^{(1)}) \cap \mathcal{D}(\Pi^{(1)}) \quad \textit{with } Bh = -(A^{(1)} + \Pi^{(1)})f\}, \tag{3.3}$$

$$V^{(j)}f = P(A^{(j)} + \Pi^{(j)})f \quad \textit{for } f \in D_j, j = 1, 2, \tag{3.4}$$

$$\hat{V}f = P(A^{(1)} + \Pi^{(1)})h \quad \textit{for } f \in D_0. \tag{3.5}$$

Suppose that $V^{(1)}f = 0$ *for all* $f \in D_1$. *Assume* $\overline{R(\lambda - (V^{(2)} + \hat{V}))} \supset D_0 \cap D_2$ *for some* $\lambda > 0$

Then there is a strongly continuous contraction semigroup $\{T(t); t \geqslant 0\}$ *defined on* $\overline{D_0 \cap D_2}$ *with* $\lim_{\epsilon\to 0} T_\epsilon(t/\epsilon)f = T(t)f$ *for all* $f \in \overline{D_0 \cap D_2}$.

The infinitesimal operator of $T(t)$ *is the closure of* $V^{(2)} + \hat{V}$ *restricted so that* $(V^{(2)} + \hat{V})f \in \overline{D_0 \cap D_2}$.

PROOF. Let V_ϵ denote the closure of $A(\epsilon) + \epsilon^{-1}B(\epsilon)$, i.e., the infinitesimal operator of $T_\epsilon(t)$; hence $\epsilon^{-1}V_\epsilon$ denotes the closure of $\epsilon^{-1}A(\epsilon) + \epsilon^{-2}B(\epsilon)$, i.e., the infinitesimal operator of $T_\epsilon(t/\epsilon)$.

By Theorem 1.10 of [14] and Theorem 2.1 of [12, p. 357] it suffices to show

$$\{(f, (V^{(2)} + \hat{V})f); f \in D_0 \cap D_2\}$$
$$\subset \left\{(f, g); \exists (f_\epsilon, g_\epsilon) \in V_\epsilon \text{ with } \lim_{\epsilon\to 0} (f_\epsilon, \epsilon^{-1}g_\epsilon) = (f, g)\right\}.$$

That is, given $f \in D_0 \cap D_2$, we must find $f_\epsilon \in \mathcal{D}(V_\epsilon)$ and $g_\epsilon = V_\epsilon f_\epsilon \in \mathcal{R}(V_\epsilon)$ such that $\lim_{\epsilon\to 0} f_\epsilon = f$ and $\lim_{\epsilon\to 0}\epsilon^{-1}g_\epsilon = (V^{(2)} + \hat{V})f$.

Let $f \in D_0 \cap D_2$ and $h \in \mathcal{D}(B) \cap \mathcal{D}(A^{(1)}) \cap \mathcal{D}(\Pi^{(1)})$ such that $Bh = -(A^{(1)} + \Pi^{(1)})f$. As in the proof of Theorem 2.1, we can find $h_\epsilon \in \bigcap\{\text{domains of } B, A^{(1)}, A^{(2)}, \Pi^{(1)}, \text{and } \Pi^{(2)}\}$ such that

$$\lim_{\epsilon\to 0} \epsilon V_\epsilon h_\epsilon = P(A^{(1)} + \Pi^{(1)})h - (A^{(1)} + \Pi^{(1)})h$$
$$+ P(A^{(2)} + \Pi^{(2)})f - (A^{(2)} + \Pi^{(2)})f,$$

$$\|(A^{(1)} + \Pi^{(1)})h_\epsilon\| = o(1/\epsilon), \quad \|(A^{(2)} + \Pi^{(2)})h_\epsilon\| = o(1/\epsilon^2),$$

and

$$\|h_\epsilon\| = o(1/\epsilon^2), \quad \text{as } \epsilon \downarrow 0.$$

Let $f_\epsilon = f + \epsilon h + \epsilon^2 h_\epsilon$. Then

$$\begin{aligned}
\epsilon^{-1}V_\epsilon f_\epsilon &= \epsilon^{-1}V_\epsilon f + V_\epsilon h + \epsilon V_\epsilon h_\epsilon \\
&= \epsilon^{-1}A(\epsilon)f + \epsilon^{-2}B(\epsilon)f + A(\epsilon)h + \epsilon^{-1}B(\epsilon)h + \epsilon V_\epsilon h_\epsilon \\
&= \epsilon^{-1}A^{(1)}f + A^{(2)}f + \epsilon^{-2}Bf + \epsilon^{-1}\Pi^{(1)}f + \Pi^{(2)}f + A^{(1)}h + \epsilon A^{(2)}h \\
&\quad + \epsilon^{-1}Bh + \Pi^{(1)}h + \epsilon\Pi^{(2)}h + \epsilon V_\epsilon h_\epsilon + o(1) \\
&= (A^{(2)} + \Pi^{(2)})f + (A^{(1)} + \Pi^{(1)})h + \epsilon V_\epsilon h_\epsilon + o(1) \quad (\text{as } \epsilon \downarrow 0).
\end{aligned}$$

Thus $\lim_{\epsilon\to 0}\epsilon^{-1}V_\epsilon f_\epsilon = P(A^{(2)} + \Pi^{(2)})f + P(A^{(1)} + \Pi^{(1)})h$. Given $f \in D_0 \cap D_2$, there are $f_\epsilon \in \mathcal{D}(V_\epsilon)$ for $\epsilon > 0$ such that $\lim_{\epsilon\to 0} f_\epsilon = f$ and $\lim_{\epsilon\to 0}\epsilon^{-1}V_\epsilon f_\epsilon = (V^{(2)} + \hat{V})f$. Q.E.D.

REMARK. If $\int_0^\infty \|(S(t) - P)f\|\, dt < \infty$ for all $f \in L$ and $Pg = 0$, then the solution of $Bh = -g$ is given by $h = \int_0^\infty (S(t) - P)g\, dt$ (see [5, p. 26]). This indicates how to solve $Bh = -(A + \Pi)f$ in the definition of D_0, although, in addition, the condition $h \in \mathcal{D}(A)$ must be satisfied. See Example 3.1.

EXAMPLE 3.1. Let $\xi(t)$, $Q = (q_{\alpha\beta})$, $p = (p_\alpha)$, and Banach spaces L and L be given as in Example 2.1.

For each $j = 1, \cdots, N$, $\epsilon > 0$, suppose that $\{T_j^{(\epsilon)}(t); t \geq 0\}$ is a strongly continuous linear contraction semigroup on L with infinitesimal operator $A_j(\epsilon) = A_j^{(1)} + \epsilon A_j^{(2)}$. Note that $U^{(\epsilon)}(t)$ and $S(t)$, defined by

$$(U^{(\epsilon)}(t)f)_j = T_j^{(\epsilon)}(t)f_j, \qquad (S(t)f)_j = \sum_{k=1}^{N} p_{jk}(t)f_k$$

for $1 \leq j \leq N$ are strongly continuous contraction semigroups on L. $U^{(\epsilon)}(t)$ and $S(t)$ have infinitesimal operators $A(\epsilon) = A^{(1)} + \epsilon A^{(2)}$ and B respectively, given by

$$(A(\epsilon)f)_j \equiv (A^{(1)}f)_j + \epsilon(A^{(2)}f)_j \equiv A_j^{(1)}f_j + \epsilon A_j^{(2)}f_j \qquad (= A_j(\epsilon)f_j),$$

$$(Bf)_j = \sum_{k=1}^{N} q_{jk}f_k \quad \text{for } j = 1, \cdots, N.$$

For each $1 \leq j \neq k \leq N$, suppose that $\{\Pi_{jk}(u); u \geq 0\}$ is a family of linear contractions on L and $\Pi_{jk}^{(1)}$ and $\Pi_{jk}^{(2)}$ are linear operators satisfying

$$\Pi_{jk}(\epsilon)f = f + \epsilon\Pi_{jk}^{(1)}f + \epsilon^2\Pi_{jk}^{(2)}f + o(\epsilon^2) \tag{3.6}$$

as $\epsilon \downarrow 0$ for $f \in \mathcal{D}(\Pi_{jk}^{(1)}) \cap \mathcal{D}(\Pi_{jk}^{(2)}) \subset L$. Denote by $\{B(t); t \geq 0\}$ the family of linear operators on L given by

$$(B(t)f)_j = \sum_{k=1;\, k\neq j}^{N} q_{jk}\Pi_{jk}(t)f_k + q_{jj}f_j$$

for $j = 1, 2, \cdots, N$. For $i = 1, 2$, we define $\Pi^{(i)}$ by

$$(\Pi^{(i)}f)_j = \sum_{k=1;\, k\neq j}^{N} q_{jk}\Pi_{jk}^{(i)}f_k$$

for $j = 1, 2, \cdots, N$ and $f \in \{f \in \mathsf{L}; f_k \in \mathcal{D}(\Pi_{jk}^{(i)}), 1 \leq j \leq N, j \neq k\}$. It follows that

$$(B(\epsilon)f)_j = (Bf)_j + \epsilon(\Pi^{(1)}f)_j + \epsilon^2(\Pi^{(2)}f)_j + o(\epsilon^2) \tag{3.7}$$

as $\epsilon \downarrow 0$ for $f \in \mathcal{D}(\Pi^{(1)}) \cap \mathcal{D}(\Pi^{(2)}) \cap \mathcal{D}(B) = \mathcal{D}(\Pi^{(1)}) \cap \mathcal{D}(\Pi^{(2)})$.

Now, $\epsilon^{-1}A(\epsilon) + \epsilon^{-2}B(\epsilon)$ is given by

$$\begin{aligned}[(\epsilon^{-1}A(\epsilon) + \epsilon^{-2}B(\epsilon))f]_j &= \epsilon^{-1}A_j^{(1)}f_j + A_j^{(2)}f_j \\ &\quad + \epsilon^{-2}\sum_{k=1;k\neq j}^{N} q_{jk}\Pi_{jk}(\epsilon)f_k + \epsilon^{-2}q_{jj}f_j\end{aligned} \tag{3.8}$$

for $j = 1, \cdots, N$. $\epsilon^{-1}A(\epsilon) + \epsilon^{-2}B(\epsilon)$ is the infinitesimal operator of the strongly continuous semigroup $T_\epsilon(t)$ on L, defined by

$$(3.9)\quad \begin{aligned}(T_\epsilon(t/\epsilon)f)_j = E_j\{&T^{(\epsilon)}_{\xi(0)}(\epsilon t_1^*)\Pi_{\xi(0)\xi(t_1^*)}(\epsilon)T^{(\epsilon)}_{\xi(t_1^*)}(\epsilon(t_2^* - t_1^*))\\ &\cdots T^{(\epsilon)}_{\xi(t_\nu^*)}(t/\epsilon - \epsilon t_\nu^*)f_{\xi(t/\epsilon^2)}\}\end{aligned}$$

for $j = 1, \cdots, N$, with $t_1^*, t_2^*, \cdots, t_\nu^*$ and ν the jump times and number of jumps respectively for the process $\xi(u)$ in the time interval $[0, t/\epsilon^2]$.

We assume that B is the closure of B restricted to $\bigcap$\{domains of $A^{(1)}, A^{(2)}$, $\Pi^{(1)}$, and $\Pi^{(2)}$\}. As in Example 2.1, we note that $\lim_{\lambda\to 0}\lambda\int_0^\infty e^{-\lambda t}S(t)f\,dt \equiv Pf$ exists for each $f \in L$, where Pf is given by

$$(2.9)\quad (Pf)_j = \sum_{k=1}^N p_k f_k$$

for $j = 1, \cdots, N$. In this setting

$$(3.10)\quad \begin{aligned}D_m = \Big\{&f = (f_j) \in L; f_j = w \text{ for } j = 1, \cdots, N,\\ &w \in \bigcap_{1\leqslant \alpha, j \neq k \leqslant N} (\text{domains of } A_\alpha^{(m)}, \Pi_{jk}^{(m)})\Big\} \quad \text{for } m = 1, 2\end{aligned}$$

and

$$(3.11)\quad \begin{aligned}V^{(m)}f &= P(A^{(m)} + \Pi^{(m)})f\\ &= \left(\sum_{j=1}^N p_j A_j^{(m)} + \sum_{1\leqslant j\neq k\leqslant N} p_j q_{jk}\Pi_{jk}^{(m)}\right) w \cdot (1)\\ &\qquad\qquad \text{for } f = w \cdot (1) \in D_m.\end{aligned}$$

Recall that the notation $f = w \cdot (1)$ means $f = (f_j)$ and $f_j = w$ for all $j = 1, \cdots, N$. We make the assumption that $V^{(1)}f = 0$ for all $f \in D_1$.

We let

$$(3.12)\quad \begin{aligned}D_0 = \{f \in D_1;\ \exists h \in \mathcal{D}(B) \cap \mathcal{D}(A^{(1)}) \cap \mathcal{D}(\Pi^{(1)}) = \mathcal{D}(A^{(1)}) \cap \mathcal{D}(\Pi^{(1)})\\ \text{with } Bh = -(A^{(1)} + \Pi^{(1)})f\}.\end{aligned}$$

If we assume that $f = w \cdot (1) \in D_0$, and we note that it is true that

$$(3.13)\quad \int_0^\infty |p_{jk}(t) - p_k|\,dt < \infty$$

for $1 \leqslant j, k \leqslant N$ (see [4, p. 236]), then the function h satisfying $Bh =$

$-(A^{(1)}+\Pi^{(1)})f$ has the form

$$h_j=\int_0^\infty \{(S(t)-P)(A^{(1)}+\Pi^{(1)})f\}_j\,dt=\sum_{k=1}^N v_{jk}\{(A^{(1)}+\Pi^{(1)})f\}_k$$

$$=\sum_{k=1}^N v_{jk}\left(A_k^{(1)}+\sum_{l=1;\,l\neq k}^N q_{kl}\Pi_{kl}^{(1)}\right)w$$

where for each $1\leqslant j,k\leqslant N$,

$$v_{jk}=\int_0^\infty (p_{jk}(t)-p_k)\,dt. \tag{3.14}$$

We have also assumed here that $h\in\mathcal{D}(A^{(1)})\cap\mathcal{D}(\Pi^{(1)})$. That is, we have assumed that each coordinate $f_k=w$ of $f=(f_k)$ is in

$$\bigcap_{1\leqslant\alpha,\,\beta,\,j\neq k,m\neq n\leqslant N}\{\text{domains of } A_\alpha^{(1)}A_\beta^{(1)}, A_\alpha^{(1)}\Pi_{jk}^{(1)}, \Pi_{jk}^{(1)}A_\beta^{(1)} \text{ and } \Pi_{jk}^{(1)}\Pi_{mn}^{(1)}\}.$$

We define

$$\hat{V}f=P(A^{(1)}+\Pi^{(1)})h \tag{3.15}$$

for $f=(f_k)=w\cdot(1)\in D_0$. Under condition (3.13) and with $h\in\mathcal{D}(A^{(1)})\cap\mathcal{D}(\Pi^{(1)})$ as required, we have

$$(\hat{V}f)_j=\sum_{j=1}^N p_jA_j^{(1)}h_j+\sum_{1\leqslant j\neq k\leqslant N}p_jq_{jk}\Pi_{jk}^{(1)}h_k$$

$$=\sum_{j=1}^N p_jA_j^{(1)}\left[\sum_{k=1}^N v_{jk}A_k^{(1)}+\sum_{1\leqslant k\neq l\leqslant N}v_{jk}q_{kl}\Pi_{kl}^{(1)}\right]w$$

$$+\sum_{1\leqslant j\neq k\leqslant N}p_jq_{jk}\Pi_{jk}^{(1)}\left[\sum_{m=1}^N v_{km}A_m^{(1)}+\sum_{1\leqslant m\neq n\leqslant N}v_{km}q_{mn}\Pi_{mn}^{(1)}\right]w.$$

Hence

$$\begin{aligned}(\hat{V}f)_j&=\sum_{1\leqslant j,k\leqslant N}p_jv_{jk}A_j^{(1)}A_k^{(1)}w+\sum_{1\leqslant j,k\neq l\leqslant N}p_jv_{jk}q_{kl}A_j^{(1)}\Pi_{kl}^{(1)}w\\&\quad+\sum_{1\leqslant j\neq k,m\leqslant N}p_jq_{jk}v_{km}\Pi_{jk}^{(1)}A_m^{(1)}w\\&\quad+\sum_{1\leqslant j\neq k,m\neq n\leqslant N}p_jq_{jk}v_{km}q_{mn}\Pi_{jk}^{(1)}\Pi_{mn}^{(1)}w.\end{aligned} \tag{3.16}$$

We also assume that $\overline{R(\lambda-(V^{(2)}+\hat{V}))}\supset D_0\cap D_2$ for some $\lambda>0$.

Then from Theorem 3.1 there is a strongly continuous contraction semigroup $\{T(t); t \geq 0\}$ defined on $\overline{D_0 \cap D_2}$ with $\lim_{\epsilon \to 0} T_\epsilon(t/\epsilon)f = T(t)f$ for all $f \in \overline{D_0 \cap D_2}$. The infinitesimal operator is the closure of $V^{(2)} + \hat{V}$ restricted so that $V^{(2)} + \hat{V} \in \overline{D_0 \cap D_2}$.

4. **Perturbation results with "buc-limits".** Let $\{X(t); t \geq 0\}$ be a regular step process with locally compact measurable state space $(E, \mathcal{B})$. $X(t)$ has Markov kernel $Q(x, A)$ on $E \times \mathcal{B}$, "holding function" $\lambda(x)$ measurable on $\mathcal{B}$ satisfying $0 < \lambda(x) < M < \infty$, and transition function $P(t, x, \Gamma)$. (For regular step processes, see [1, p. 63].)

Let L be a Banach space. For each $x \in E$, let $\{T_x(t); t \geq 0\}$ be a strongly continuous contraction semigroup on L with infinitesimal operator A_x. For each $x, y \in E$ with $x \neq y$, let $\{\Pi_{xy}(t); t \geq 0\}$ be a family of linear contractions on L and Π_{xy} be a linear operator satisfying $\Pi_{xy}(\delta)f = f + \delta \Pi_{xy} f + o(\delta)$ (as $\delta \downarrow 0$) for $f \in \mathcal{D}(\Pi_{xy}) \subset L$.

Let L be the Banach space of bounded, strongly measurable functions $f: E \longrightarrow L$ with $\|f\| = \sup_{x \in E} \|f(x)\|_L$. We say that $\text{buc-lim}_{\lambda \to 0^+} g_\lambda$ exists and equals g for $g_\lambda, g \in \mathsf{L}$ if

(i) $\sup_{0<\lambda<\delta} \|g_\lambda\| < \infty$ for some $\delta > 0$, and

(ii) $\lim_{\lambda \to 0^+} g_\lambda(x) = g(x)$ uniformly on compact subsets on E.

Define contraction semigroups $\{U(t); t \geq 0\}$ and $\{S(t); t \geq 0\}$ on L by

$$(U(t)f)(x) = T_x(t)f(x), \qquad (S(t)f)(x) = \int f(y)P(t, x, dy).$$

Let the subspace L_0 of L be given and satisfy

$$\mathsf{L}_0 \subseteq \Big\{ f \in \mathsf{L} \,|\, U(t)f \text{ and } S(t)f \text{ are buc-continuous}; \tag{4.1}$$
$$\int_0^\infty e^{-\lambda t} S(t)f\, dt \in \mathsf{L}_0, \text{ and } \int_0^\infty e^{-\lambda t} U(t)f\, dt \in \mathsf{L}_0 \Big\}.$$

We define operators A and B with domains $\mathcal{D}(A)$ and $\mathcal{D}(B)$ respectively by

$$Af = \underset{t \to 0}{\text{buc-lim}} \frac{U(t)f - f}{t}, \quad \mathcal{D}(A) = \{f \in \mathsf{L}_0; \text{limit exists and } Af \in \mathsf{L}_0\},$$

$$Bf = \underset{t \to 0}{\text{buc-lim}} \frac{S(t)f - f}{t}, \quad \mathcal{D}(B) = \{f \in \mathsf{L}_0; \text{limit exists and } Bf \in \mathsf{L}_0\}.$$

Note that A and B are restrictions of operators defined respectively by

$$(Af)(x) = A_x f(x)$$

for $f \in \{f \in \mathsf{L};\ f(x) \in \mathcal{D}(A_x) \text{ for } x \in E, \sup_{x \in E} \|A_x f(x)\| < \infty\}$ and

$$(Bf)(x) = \lambda(x) \int_{E-\{x\}} Q(x, dy) f(y) - \lambda(x) f(x)$$

for $f \in \mathsf{L}$.

Define bounded, linear operators $\{B(t); t \geqslant 0\}$ by

$$(B(t)f)(x) = \lambda(x)\int_{E-\{x\}} Q(x, dy)\Pi_{xy}(t)f(y) - \lambda(x)f(x)$$

for $f \in \mathcal{D}(B(t)) = \{f \in L_0 | B(t)f \in L_0\}$. The linear operator Π is defined by

$$(\Pi f)(x) = \lambda(x) \int_{E-\{x\}} Q(x, dy)\Pi_{xy} f(y)$$

for $f \in \mathcal{D}(\Pi)$, given by

$$\mathcal{D}(\Pi) = \left\{ f \in L_0 | \Pi f \in L_0; f(y) \in \mathcal{D}(\Pi_{xy}) \text{ for } x \neq y \in E; \right.$$
$$\left. \text{and } \sup_{x \neq y} \|\Pi_{xy} f(y)\| < \infty \right\}.$$

Note that for $f \in \bigcap\{\text{domains of } B, \Pi, \text{ and } B(\epsilon)\}$

$$\begin{aligned}(B(\epsilon)f)(x) &= \lambda(x) \int_{E-\{x\}} Q(x, dy)\Pi_{xy}(\epsilon)f(y) - \lambda(x)f(x) \\ &= \lambda(x) \int_{E-\{x\}} Q(x, dy)f(y) \\ &\quad + \epsilon\lambda(x) \int_{E-\{x\}} Q(x, dy)\Pi_{xy} f(y) - \lambda(x)f(x) + o(\epsilon).\end{aligned}$$

Hence,

$$(B(\epsilon)f)(x) = (Bf)(x) + \epsilon(\Pi f)(x) + o(\epsilon) \quad (\text{as } \epsilon \downarrow 0). \tag{4.2}$$

We assume that for each $\epsilon > 0$, there is a buc-continuous contraction semigroup $\{T_\epsilon(t); t \geqslant 0\}$ defined on L_0 such that $(A + \epsilon^{-1}B(\epsilon))f =$ buc-$\lim_{t\to 0}((T_\epsilon(t)f - f)/t) \equiv V_\epsilon f$. Assume also that B is the buc-closure of B restricted to $\mathcal{D}(A) \cap \mathcal{D}(B) \cap \mathcal{D}(\Pi)$. Notice that $A + \epsilon^{-1}B(\epsilon)$ is a restriction of the operator defined by

$$\begin{aligned}&((A + \epsilon^{-1}B(\epsilon))f)(x) \\ &\quad = A_x f(x) + \epsilon^{-1}\lambda(x)\int_{E-\{x\}} Q(x, dy)(\Pi_{xy}(\epsilon)f(y) - f(x))\end{aligned} \tag{4.3}$$

for $f \in \{f \in L | f(y) \in \mathcal{D}(A_y)$ for $y \in E$; and $\sup_{y\in E}\|A_y f(y)\| < \infty\}$. Also $\{T_\epsilon(t); t \geqslant 0\}$ is a restriction of the operator defined on L by

$$\begin{aligned}(T_\epsilon(t)f)(x) = E_x\{&T_{X(0)}(\epsilon t_1^*)\Pi_{X(0)X(t_1^*)}(\epsilon)T_{X(t_1^*)}(\epsilon(t_2^* - t_1^*)) \\ &\cdots T_{X(t_\nu^*)}(t - \epsilon t_\nu^*)f_{X(t/\epsilon)}\}\end{aligned} \tag{4.4}$$

where $t_1^*, t_2^*, \cdots, t_\nu^*$ and ν are the jump times and number of jumps for the process $X(s)$ during the time interval $[0, t/\epsilon]$.

We will need the following theorems. Theorem 4.1 is an application of

[13, p. 27]. The proof of Theorem 4.2 is similar to that of Theorem 18.6.2 of [8, pp. 512–517].

THEOREM 4.1. *Suppose* $\{W_n(t); t \geqslant 0\}$, $n = 1, 2, \cdots$, *are buc-continuous contraction semigroups on* L_0 *with operators* $C_n f = \text{buc-lim}_{t \downarrow 0}((W_n(t)f - f)/t)$ *having domain of those functions in* L_0 *for which this limit exists and* $C_n f \in L_0$. *Define*

$$C = \left\{(f, g);\ \exists f_n \in \mathcal{D}(C_n) \text{ with } g_n = C_n f_n \text{ satisfying } \underset{n\to\infty}{\text{buc-lim}}\, f_n = f \right.$$

$$\left. \text{and } \underset{n\to\infty}{\text{buc-lim}}\, g_n = g \right\}. \tag{4.5}$$

Then there exists a strongly continuous contraction semigroup $W(t)$ *on* $\overline{\mathcal{D}(C)}$ *such that* $W(t)f = \text{buc-lim}_{n\to\infty} W_n(t)f$ *for each* $f \in \overline{\mathcal{D}(C)}$ *if and only if* $\mathcal{R}(\lambda - C) \supset \mathcal{D}(C)$.

THEOREM 4.2. *Let* $S(t)$ *be a buc-continuous semigroup on* L_0 *with operator* B *defined by* $Bf = \text{buc-lim}_{t\to 0}((S(t)f - f)/t)$ *and with domain of* B *as those functions for which this limit exists and* $Bf \in L_0$. *Suppose the following conditions hold*:

(i) *for each compact set* $K \subset E$, *each* $\epsilon > 0$, *and each* $t > 0$, *there is a compact set* $K_\epsilon = K(\epsilon, t, K)$ *such that* $\sup_{x\in K} P(t, x, K_\epsilon^c) < \epsilon$; *and*

(ii) *for all* $f \in L_0$, $\text{buc-lim}_{\lambda\to 0} \lambda \int_0^\infty e^{-\lambda t} S(t) f\, dt \equiv Pf$ *exists.*

Then we have

$$P \text{ is a bounded, linear projection}; \tag{4.6}$$

$$S(t)P = PS(t) = P \text{ for all } t > 0; \tag{4.7}$$

$$\mathcal{R}(P) = n(B), \text{ the null space of } B; \tag{4.8}$$

$$\mathcal{R}(B) \text{ is buc-dense in } n(P); \tag{4.9}$$

$$BPf = 0 \text{ for all } f \in L_0, \qquad PBf = 0 \text{ for all } f \in \mathcal{D}(B). \tag{4.10}$$

THEOREM 4.3. *Let* E, $U(t)$, $S(t)$, $B(t)$, $T_\epsilon(t)$, Π, A, B, *and* V_ϵ *be as above. We assume that*

(i) *for each compact set* $K \subset E$, *each* $\epsilon > 0$, *and each* $t > 0$, *there is a compact set* $K_\epsilon = K(\epsilon, t, K)$ *such that* $\sup_{x\in K} P(t, x, K_\epsilon^c) < \epsilon$; *and*

(ii) for *all* $f \in L_0$, $\text{buc-lim}_{\lambda\to 0} \lambda \int_0^\infty e^{-\lambda t} S(t) f\, dt \equiv Pf$ *exists.*

We denote by D *the set given by*

$$D = \{f \in \mathcal{R}(P); f \in \mathcal{D}(A) \cap \mathcal{D}(\Pi)\} \tag{4.11}$$

and define the operator V *for* $f \in D$ *by*

$$Vf = P(A + \Pi)f. \tag{4.12}$$

We suppose that $\mathcal{R}(\lambda - V) \supset D$ *for some* $\lambda > 0$.

Then there is a strongly continuous contraction semigroup $\{T(t); t \geqslant 0\}$ *defined on* $\bar{D}$ *satisfying* $\text{buc-lim}_{n\to\infty} T_\epsilon(t)f = T(t)f$ *for all* $f \in \bar{D}$. *The infinitesimal operator of* $\{T(t); t \geqslant 0\}$ *is the closure of* V *restricted so that* $Vf \in \bar{D}$.

PROOF. The proof is similar to that of Theorem 2.1. From Theorem 4.1 it suffices to show

$$\{(f, Vf); f \in D\} \subset \left\{(f, g);\ \exists f_\epsilon \in \mathcal{D}(V_\epsilon) \text{ with } g_\epsilon = V_\epsilon f_\epsilon \text{ satisfying } \underset{\epsilon\to 0}{\text{buc-lim}}\, f_\epsilon = f \text{ and } \underset{\epsilon\to 0}{\text{buc-lim}}\, g_\epsilon = g\right\},$$

i.e., given $f \in D$, we must find $f_\epsilon \in \mathcal{D}(V_\epsilon)$, $g_\epsilon = V_\epsilon f_\epsilon \in \mathcal{R}(V_\epsilon)$ such that $\text{buc-lim}_{\epsilon\to 0} f_\epsilon = f$ and $\text{buc-lim}_{\epsilon\to 0} V_\epsilon f_\epsilon = Vf$.

Then, using $\mathcal{R}(\lambda - V) \supset D$, we have that there exists a strongly continuous contraction semigroup $\{T(t); t \geqslant 0\}$ on $\bar{D}$ such that $\text{buc-lim}_{\epsilon\to 0} T_\epsilon(t)f = T(t)f$ for each $f \in \bar{D}$. From Theorem 4.1 it also follows that the infinitesimal operator of $T(t)$ is the closure of $\{(f, g); f \in \mathcal{D}(V), g = Vf, \text{ and } f, g \in \bar{D}\}$, i.e., the closure of V restricted so that $Vf \in \bar{D}$.

Recall that B is the buc-closure of B restricted to $\mathcal{D}(A) \cap \mathcal{D}(\Pi) \cap \mathcal{D}(B)$. Hence for g in the buc-closure of $\mathcal{R}(B)$, there exist $h_\epsilon \in \mathcal{D}(A) \cap \mathcal{D}(B) \cap \mathcal{D}(\Pi)$ such that $\text{buc-lim}_{\epsilon\to 0} Bh_\epsilon = g$, and, if necessary by relabeling the index set, such that $\|(A + \Pi)h_\epsilon\| = o(1/\epsilon)$ and $\|h_\epsilon\| = o(1/\epsilon)$. Hence

$$\underset{\epsilon\to 0}{\text{buc-lim}}\, \epsilon V_\epsilon h_\epsilon = \underset{\epsilon\to 0}{\text{buc-lim}}\, (\epsilon A h_\epsilon + B(\epsilon)h_\epsilon)$$

$$= \underset{\epsilon\to 0}{\text{buc-lim}}\, (\epsilon(A + \Pi)h_\epsilon + Bh_\epsilon + o(\epsilon)) = g.$$

From Theorem 4.2 we obtain for $f \in D$ that $P(A + \Pi)f - (A + \Pi)f$ is in the buc-closure of $\mathcal{R}(B)$; and hence we can choose $\{h_\epsilon\} \subset \mathcal{D}(A) \cap \mathcal{D}(\Pi) \cap \mathcal{D}(B)$ such that $\text{buc-lim}_{\epsilon\to 0} \epsilon V_\epsilon h_\epsilon = P(A + \Pi)f - (A + \Pi)f$, with $\|h_\epsilon\| = o(1/\epsilon)$ and $\|Ah_\epsilon\| = o(1/\epsilon)$. Also from Theorem 4.2 we have for $f \in D$ that $V_\epsilon f = (A + \epsilon^{-1}B(\epsilon))f = Af + \Pi f + o(\epsilon)/\epsilon$.

If we set $f_\epsilon = f + \epsilon h_\epsilon$ then $\text{buc-lim}_{\epsilon\to 0} f_\epsilon = f$ and

$$\underset{\epsilon\to 0}{\text{buc-lim}}\, V_\epsilon f_\epsilon = \underset{\epsilon\to 0}{\text{buc-lim}}\, (V_\epsilon f + \epsilon V_\epsilon h_\epsilon)$$

$$= \underset{\epsilon\to 0}{\text{buc-lim}}\, ((A + \Pi)f + \epsilon V_\epsilon h_\epsilon + o(1)) = P(A + \Pi)f = Vf.$$

Thus, given $f \in D$, there are $f_\epsilon \in \mathcal{D}(V_\epsilon)$ for $\epsilon \downarrow 0$ satisfying $\text{buc-lim}_{\epsilon\downarrow 0} f_\epsilon = f$ and $\text{buc-lim}_{\epsilon\downarrow 0} V_\epsilon f_\epsilon = Vf$. Q.E.D.

Let $X(t)$, L, and $\mathcal{L}$ be given as before. For each $x \in E$, $\epsilon > 0$, let $\{T_x^{(\epsilon)}(t); t \geqslant 0\}$ be a strongly continuous contraction semigroup on $\mathcal{L}$ with infinitesimal operator satisfying $A_x(\epsilon)f = A_x^{(1)}f + \epsilon A_x^{(2)}f + o(\epsilon)$ as $\epsilon \downarrow 0$ for $f \in \bigcap\{\text{domains of } A_x^{(1)}, A_x^{(2)}, \text{and } A_x(\epsilon)\}$. Suppose that for each $x, y \in E$, $x \neq y$, $\{\Pi_{xy}(u); u \geq 0\}$ is a family of linear contractions on L and $\Pi_{xy}^{(i)}$ is a linear operator satisfying $\Pi_{xy}(\epsilon)f = f + \epsilon\Pi_{xy}^{(1)}f + \epsilon^2\Pi_{xy}^{(2)}f + o(\epsilon^2)$ as $\epsilon \downarrow 0$ for $f \in \mathcal{D}(\Pi_{xy}^{(1)}) \cap \mathcal{D}(\Pi_{xy}^{(2)})$.

Define contraction semigroups $\{S(t); t \geq 0\}$ and $\{U^{(\epsilon)}(t); t \geq 0\}$, $\epsilon > 0$, on L by

$$(U^{(\epsilon)}(t)f)(x) = T_x^{(\epsilon)}(t)f(x), \qquad (S(t)f)(x) = \int f(y)P(t, x, dy).$$

Let the subspace L_0 of L be given and satisfy

$$L_0 \subseteq \left\{f \in L \mid U^{(\epsilon)}(t)f \text{ and } S(t)f \text{ are buc-continuous}; \int_0^\infty e^{-\lambda t}S(t)f\,dt \in L_0; \text{ and} \int_0^\infty e^{-\lambda t}U^{(\epsilon)}(t)f\,dt \in L_0 \text{ for } \epsilon > 0\right\}. \tag{4.13}$$

We define operators $A(\epsilon)$ and B with domains $\mathcal{D}(A(\epsilon))$ and $\mathcal{D}(B)$ respectively by

$$A(\epsilon)f = \operatorname*{buc\text{-}lim}_{t\to 0} \frac{U^{(\epsilon)}(t)f - f}{t}, \quad \mathcal{D}(A(\epsilon)) = \{f \in L_0; \text{limit exists and } A(\epsilon)f \in L_0\},$$

$$Bf = \operatorname*{buc\text{-}lim}_{t\to 0} \frac{S(t)f - f}{t}, \qquad \mathcal{D}(B) = \{f \in L_0; \text{limit exists and } Bf \in L_0\}.$$

Note that $A(\epsilon)$ and B are restrictions of operators defined respectively by

$$(A(\epsilon)f)(x) = A_x(\epsilon)f(x) = A_x^{(1)}f(x) + \epsilon A_x^{(2)}f(x) + o(\epsilon) \quad (\text{as } \epsilon \downarrow 0)$$

for $f \in \{f \in L \mid f(x) \in \mathcal{D}(A_x^{(j)}) \cap \mathcal{D}(A_x(\epsilon))$ for $x \in E$, $j = 1, 2$; and $\sup_{x\in E; j=1,2}\|A_x^{(j)}f(x)\| < \infty\}$ and

$$(Bf)(x) = \lambda(x)\int_{E-\{x\}} Q(x, dy)f(y) - \lambda(x)f(x)$$

for $f \in L$.

Define bounded linear operators $\{B(t); t \geq 0\}$ by

$$(B(t)f)(x) = \lambda(x)\int_{E-\{x\}} Q(x, dy)\Pi_{xy}(t)f(y) - \lambda(x)f(x)$$

for $f \in \mathcal{D}(B(t)) = \{f \in L_0; B(t)f \in L_0\}$. The linear operator $\Pi^{(j)}$ for $j = 1, 2$ is defined by

$$(\Pi^{(j)}f)(x) = \lambda(x)\int_{E-\{x\}} Q(x, dy)\Pi_{xy}^{(j)}f(y)$$

for $f \in \mathcal{D}(\Pi^{(j)})$, given by $\mathcal{D}(\Pi^{(j)}) = \{f \in L_0 \mid \Pi^{(j)}f \in L_0;\ f(y) \in \mathcal{D}(\Pi_{xy}^{(j)})$ for $x \neq y \in E$; and $\sup_{x\neq y}\|\Pi_{xy}^{(j)}f(y)\| < \infty\}$. Note that for $f \in \bigcap\{\text{domains of } B, \Pi^{(1)}, \Pi^{(2)}, \text{and } B(\epsilon)\}$

$$(B(\epsilon)f)(x) = \lambda(x)\int_{E-\{x\}} Q(x, dy)\Pi_{xy}(\epsilon)f(y) - \lambda(x)f(x)$$

$$= \lambda(x)\int_{E-\{x\}} Q(x, dy)f(y) + \epsilon\lambda(x)\int_{E-\{x\}} Q(x, dy)\Pi^{(1)}_{xy}f(y)$$

$$+ \epsilon^2\lambda(x)\int_{E-\{x\}} Q(x, dy)\Pi^{(2)}_{xy}f(y) - \lambda(x)f(x) + o(\epsilon).$$

Hence,

$$(4.14)\qquad \begin{aligned}(B(\epsilon)f)(x) &= (Bf)(x) + \epsilon(\Pi^{(1)}f)(x)\\ &\quad + \epsilon^2(\Pi^{(2)}f)(x) + o(\epsilon^2) \quad (\text{as } \epsilon \downarrow 0).\end{aligned}$$

We assume that for each $\epsilon > 0$, there is a buc-continuous contraction semigroup $\{T_\epsilon(t); t \geqslant 0\}$ defined on L_0 such that $(A(\epsilon) + \epsilon^{-1}B(\epsilon))f = \text{buc-lim}_{t\to 0}((T_\epsilon(t)f - f)/t) \equiv V_\epsilon f$. Assume also that B is the buc-closure of B restricted to $\bigcap\{\text{domains of } A^{(j)}, \Pi^{(j)}, \text{and } B, j = 1, 2\}$. Notice that $A(\epsilon) + \epsilon^{-1}B(\epsilon)$ is a restriction of the operator defined by

$$(4.15)\qquad \begin{aligned}&((A(\epsilon) + \epsilon^{-1}B(\epsilon))f)(x)\\ &\quad = A_x(\epsilon)f(x) + \epsilon^{-1}\lambda(x)\int_{E-\{x\}} Q(x, dy)(\Pi_{xy}(\epsilon)f(y) - f(x))\end{aligned}$$

for $f \in \{f \in L \mid f(y) \in \mathcal{D}(A_y(\epsilon))$ for $y \in E$; and $\sup_{y\in E}\|A_y(\epsilon)f(y)\| < \infty\}$. Also $\{T_\epsilon(t); t \geqslant 0\}$ is a restriction of the operator defined on L by

$$(4.16)\qquad \begin{aligned}(T_\epsilon(t)f)(x) &= E_x\{T^{(\epsilon)}_{X(0)}(\epsilon t_1^*)\Pi_{X(0)X(t_1^*)}(\epsilon)T^{(\epsilon)}_{X(t_1^*)}(\epsilon(t_2^* - t_1^*))\\ &\qquad \cdots T^{(\epsilon)}_{X(t_\nu^*)}(t - \epsilon t_\nu^*)f_{X(t/\epsilon)}\}.\end{aligned}$$

THEOREM 4.4. *Assume in addition to the above that*

(i) *for each compact set* $K \subset E$, *each* $\epsilon > 0$, *and each* $t > 0$, *there is a compact set* $K_\epsilon = K(\epsilon, t, K)$ *such that* $\sup_{x\in K} P(t, x, K_\epsilon^c) < \epsilon$; *and*

(ii) *for all* $f \in L_0$, $\text{buc-lim}_{\lambda\to 0}\lambda\int_0^\infty e^{-\lambda t}S(t)f\,dt \equiv Pf$ *exists. Define*

$$(4.17)\qquad D_j = \{f \in R(P); f \in \mathcal{D}(A^{(j)}) \cap \mathcal{D}(\Pi^{(j)})\} \quad (\textit{for } j = 1, 2),$$

$$(4.18)\qquad \begin{aligned}D_0 = \{f \in D_1;\ \exists h \in \mathcal{D}(B) \cap \mathcal{D}(A^{(1)}) \cap \mathcal{D}(\Pi^{(1)})\\ \textit{with } Bh = -(A^{(1)} + \Pi^{(1)})f\},\end{aligned}$$

$$(4.19)\qquad V^{(j)}f = P(A^{(j)} + \Pi^{(j)})f \quad (\textit{for } f \in D_j),$$

$$(4.20)\qquad \hat{V}f = P(A^{(1)} + \Pi^{(1)})h \quad (\textit{for } f \in D_0).$$

Assume that $V^{(1)}f = 0$ *for all* $f \in D_1$ *and that* $\overline{R(\lambda - (V^{(2)} + \hat{V}))} \supset D_0 \cap D_2$ *for some* $\lambda > 0$.

Then there is a strongly continuous contraction semigroup $\{T(t); t \geqslant 0\}$ *defined on* $\overline{D_0 \cap D_2}$ *with* $\text{buc-lim}_{\epsilon\to 0} T_\epsilon(t/\epsilon)f = T(t)f$ *for all* $f \in \overline{D_0 \cap D_2}$. *The infinitesimal operator of* $T(t)$ *is the closure of* $V^{(2)} + \hat{V}$ *restricted so that* $(V^{(2)} + \hat{V})f \in \overline{D_0 \cap D_2}$.

The proof is like that of Theorem 3.1 with limit changes introduced as in the proof of Theorem 4.3. Note that the remark given after the proof of Theorem 3.1 holds here also.

EXAMPLE 4.1. Let $\{X(t); t \geqslant 0\}$ be a temporally homogeneous, positive recurrent Markov chain with state space $E = \{1, 2, 3, \cdots\}$. $\{X(t); t \geqslant 0\}$ has generator $Q = (q_{jk})$, $j, k \in E$, $0 \leqslant q_{jk} < \infty$ for $j \neq k$, $\Sigma_{k=1;k\neq j}^{\infty} q_{jk} = -q_{jj} < \infty$; transition probabilities $(p_{jk}(t); t \geqslant 0)$, $j, k \in E$; and stationary probability distribution (p_k), $k \in E$, where $\lim_{t\to\infty} p_{jk}(t) = p_k$. Assume that $\sup_{j\in E}|q_{jj}| < \infty$.

Let the spaces and operators in the hypothesis of Theorem 4.3 be given. These operators now take on the following form

(i)
$$(U(t)f)_j = T_j(t)f_j \text{ and } (S(t)f)_j = \Sigma_{k=1}^{\infty} f_k p_{jk}(t)$$
$$\text{with } (Af)_j = A_j f_j \text{ and } (Bf)_j = \Sigma_{k=1}^{\infty} q_{jk} f_k,$$

(ii)
$$(B(t)f)_j = \Sigma_{k=1,k\neq j}^{\infty} q_{jk}\Pi_{jk}(t)f_k + q_{jj}f_j \text{ and } (\Pi f)_j = \Sigma_{k=1,k\neq j}^{\infty} q_{jk}\Pi_{jk}f_k$$
$$\text{with } (B(\epsilon)f)_j = (Bf)_j + \epsilon(\Pi f)_j + o(\epsilon),$$

(iii)
$$((A + \epsilon^{-1}B(\epsilon))f)_j = A_j f_j + \epsilon^{-1}\Sigma_{k=1,k\neq j}^{\infty} q_{jk}\Pi_{jk}(\epsilon)f_k + \epsilon^{-1}q_{jj}f_j \text{ and}$$
$$(T_\epsilon(t)f)_j = E_j[T_{X(0)}(\epsilon t_1^*)\,\Pi_{X(0)X(t_1^*)}(\epsilon)T_{X(t_1^*)}(\epsilon(t_2^* - t_1^*)) \cdots T_{X(t_\nu^*)}(t - \epsilon t_\nu^*)f_{X(t/\epsilon)}].$$

Theorems 2.1 and 3.1 do not apply in this situation. Condition (2.2) does not hold; for $f \in L$, $\lambda\int_0^\infty e^{-\lambda t}S(t)f\,dt$ does not converge in the strong topology as $\lambda \to 0$. But $\text{buc-lim}_{\lambda\to 0}\lambda\int_0^\infty e^{-\lambda t}S(t)f\,dt \equiv Pf$ exists and $(Pf)_j = \Sigma_{j=1}^{\infty} p_j f_j$. Also for each compact set (finite set) K, each $\epsilon > 0$ and each $t > 0$, we can use $\Sigma_{k=1}^{\infty} p_{jk}(t) = 1$ to obtain a compact set $K_\epsilon = K(\epsilon, t, K)$ such that $\sup_{x\in K} P(t, x, K_\epsilon^c) < \epsilon$. Recall that B is assumed to be the buc-closure of B restricted to $\mathcal{D}(A) \cap \mathcal{D}(\Pi)$.

In this setting

$$D = \left\{ f \in L_0 \mid f_j = h \text{ for } j \in E; h \in \bigcap_{\alpha, j\neq k\in E} (\text{domains of } A_\alpha, \Pi_{jk}); \right.$$
$$\left. \sup_j \|A_j h\| < \infty; \text{ and } \sup_{j\neq k} \|\Pi_{jk}h\| < \infty \right\}$$

and for $f = (f_j) = (h) \in D$,

$$(Vf)_m = (P(A + \Pi)f)_m = \sum_{j=1}^{\infty} p_j A_j h + \sum_{1 \leq j \neq k < \infty} p_j q_{jk} \Pi_{jk} h.$$

Finally, assume $\overline{R(\lambda - V)} \supset D$ for some $\lambda > 0$.

Then by Theorem 4.3 there is a strongly continuous contraction semigroup $\{T(t); t \geq 0\}$ defined on $\bar{D}$ with buc-$\lim_{\epsilon\to 0} T_\epsilon(t)f = T(t)f$ for all $f \in \bar{D}$. The infinitesimal operator of $T(t)$ is the closure of V restricted so that $Vf \in \bar{D}$.

EXAMPLE 4.2. Let $\{X(t); t \geq 0\}$ be the Markov chain with state space $E = \{1, 2, \cdots\}$ given in Example 4.1. Let the spaces and operators in the hypotheses of Theorem 4.4 be given. These operators now take on the following form

(i) $(U^{(\epsilon)}(t)f)_j = T_j^{(\epsilon)}(t)f_j$ and $(S(t)f)_j = \Sigma_{k=1}^{\infty} f_k p_{jk}(t)$

with $(A(\epsilon)f)_j = A_j(\epsilon)f_j$ and $(Bf)_j = \Sigma_{k=1}^{\infty} q_{jk} f_k$.

(ii) $(B(t)f)_j = \Sigma_{k=1,k\neq j}^{\infty} q_{jk}\Pi_{jk}(t)\, f_k + q_{jj} f_j$ and $(\Pi^{(i)}f)_j = \Sigma_{k=1,k\neq j}^{\infty} q_{jk}\Pi_{jk}^{(i)} f_k$, $i = 1, 2$,

with $(B(\epsilon)f)_j = (Bf)_j + \epsilon(\Pi^{(1)}f)_j + \epsilon^2(\Pi^{(2)}f)_j + o(\epsilon^2)$, as $\epsilon \downarrow 0$.

(iii) $((A(\epsilon) + \epsilon^{-1}B(\epsilon))f)_j = A_j(\epsilon)f_j + \epsilon^{-1}\Sigma_{k=1,k\neq j}^{\infty} q_{jk}\Pi_{jk}(\epsilon) f_k + \epsilon^{-1} q_{jj} f_j$ and

$$(T_\epsilon(t)f)_j = E_j\{T_{X(0)}^{(\epsilon)}(\epsilon t_1^*)\Pi_{X(0)X(t_1^*)}(\epsilon)T_{X(t_1^*)}^{(\epsilon)}(\epsilon(t_2^* - t_1^*)) \cdots T_{X(t_\nu^*)}^{(\epsilon)}(t - \epsilon t_\nu^*) f_{X(t/\epsilon)}\}.$$

(iv)
$$(Pf)_j = \sum_{j=1}^{\infty} p_j f_j.$$

$D_j = \{f \in L_0 \mid f_k = w$ for $k \in E$;

$w \in \bigcap_{\alpha, m\neq n\in E}$(domains of $A_\alpha^{(j)}, \Pi_{mn}^{(j)}$);

and $\sup_{\alpha,m\neq n\in E}(\|A_\alpha^{(j)}w\| \vee \|\Pi_{mn}^{(j)}\, w\|) < \infty\}$ $(j = 1, 2)$.

(v) $D_0 = \{f \in D_1 \mid \exists h \in \bigcap_{\alpha,m\neq n}$(domains of $A_\alpha^{(1)}, \Pi_{mn}^{(1)}$) with

$Bh = (A^{(1)} + \Pi^{(1)})f\}$.

$(V^{(j)}f)_k = \Sigma_{\alpha=1}^{\infty} p_\alpha A_\alpha^{(j)} w + \Sigma_{1\leq m\neq n<\infty} p_m q_{mn} \Pi_{mn}^{(j)} w$

(for $k \in E, j = 1, 2, f \in D_j$).

We assume that B is the buc-closure of B restricted to $\bigcap$(domains of $B, A^{(1)}, A^{(2)}, \Pi^{(1)}$, and $\Pi^{(2)}$). We also assume that $(V^{(1)}f)_j = \Sigma_{\alpha=1}^{\infty} p_\alpha A_\alpha w + \Sigma_{1\leq\alpha\neq\beta<\infty} p_\alpha q_{\alpha\beta}\Pi_{\alpha\beta} w \equiv 0$ for all $f = w \cdot (1) \in D_1$. Note that conditions (i) and (ii) of Theorem 4.4 hold, as in Example 4.1.

We assume that the following condition holds:

$$\int_0^\infty |p_{jk}(t) - p_k|\, dt < \infty \quad \text{for all} \quad 1 \leq j,\, k < \infty. \tag{4.21}$$

For each $1 \leq j,\, k < \infty$, we denote $v_{jk} = \int_0^\infty (p_{jk}(t) - p_k)\, dt$. If we assume that $f = (f_j) = w \cdot (1) \in D_0$, that $\sup_{j\in E}(\sum_{k=1}^\infty \|v_{jk}((A^{(1)} + \Pi^{(1)})f)_k\|) < \infty$, and that h is the function satisfying $h \in \mathcal{D}(A^{(1)}) \cap \mathcal{D}(\Pi^{(1)})$ with $Bh = -(A^{(1)} + \Pi^{(1)})f$, then we have that h has form

$$h_j = \int_0^\infty \{(S(t) - P)(A^{(1)} + \Pi^{(1)})f\}_j\, dt = \sum_{k=1}^\infty v_{jk}((A^{(1)} + \Pi^{(1)})f)_k$$

$$= \sum_{k=1}^\infty v_{jk}A_k^{(1)}w + \sum_{1\leq l\neq k<\infty} v_{jk}q_{kl}\Pi_{kl}^{(1)}w.$$

Note that the condition $h \in \mathcal{D}(A^{(1)}) \cap \mathcal{D}(\Pi^{(1)})$ implies that

(i) $$h_j = \sum_{k=1}^\infty v_{jk}A_k^{(1)}w + \sum_{1\leq k\neq l<\infty} v_{jk}q_{kl}\Pi_{kl}^{(1)}w \in \mathcal{D}(A_j^{(1)}),$$

(ii) $$\sup_j \|A_j^{(1)}h_j\| = \sup_j \left\| A_j^{(1)}\left\{ \sum_{k=1}^\infty v_{jk}A_k^{(1)}w + \sum_{1\leq k\neq l<\infty} v_{jk}q_{kl}\Pi_{kl}^{(1)}w \right\}\right\| < \infty,$$

(iii) For each $1 \leq i < \infty$,

$$h_j = \sum_{k=1}^\infty v_{jk}A_k^{(1)}w + \sum_{1\leq k\neq l<\infty} v_{jk}q_{kl}\Pi_{kl}^{(1)}w \in \mathcal{D}(\Pi_{ij}^{(1)}), \text{ and}$$

(iv) $$\sup_{i\neq j} \|\Pi_{ij}^{(1)}h_j\| = \sup_{i\neq j}\left\| \Pi_{ij}^{(1)}\left\{ \sum_{k=1}^\infty v_{jk}A_k^{(1)}w + \sum_{1\leq k\neq l<\infty} v_{jk}q_{kl}\Pi_{kl}^{(1)}w \right\}\right\| < \infty.$$

Define $\hat{V}f = P(A^{(1)} + \Pi^{(1)})h$ for $f = (f_k) = w \cdot (1) \in D_0$. Under condition (4.21), with $h \in \mathcal{D}(A^{(1)}) \cap \mathcal{D}(\Pi^{(1)})$ as required, and with

$$\sup_j \left(\sum_{k=1}^\infty \|v_{jk}((A^{(1)} + \Pi^{(1)})f)_k\| \right) < \infty$$

we have

$$(\hat{V}f)_j = \sum_{j=1}^\infty p_jA_j^{(1)}h_j + \sum_{1\leq j\neq k<\infty} p_jq_{jk}\Pi_{jk}^{(1)}h_k$$

$$= \sum_{j=1}^\infty p_jA_j^{(1)} \left\{ \sum_{k=1}^\infty v_{jk}A_k^{(1)} + \sum_{1\leq k\neq l<\infty} v_{jk}q_{kl}\Pi_{kl}^{(1)} \right\} w$$

$$+ \sum_{1\leq j\neq k<\infty} p_jq_{jk}\Pi_{jk}^{(1)} \left\{ \sum_{m=1}^\infty v_{km}A_m^{(1)} + \sum_{1\leq m\neq n<\infty} v_{km}q_{mn}\Pi_{mn}^{(1)} \right\} w.$$

Hence,

$$
\begin{aligned}
(\hat{V}f)_j = {} & \sum_{j=1}^{\infty} p_j A_j^{(1)} \left\{ \sum_{k=1}^{\infty} v_{jk} A_k^{(1)} \right\} w \\
& + \sum_{j=1}^{\infty} p_j A_j^{(1)} \left\{ \sum_{1 \leqslant k \neq l < \infty} v_{jk} q_{kl} \Pi_{kl}^{(1)} \right\} w \\
& + \sum_{1 \leqslant j \neq k < \infty} p_j q_{jk} \Pi_{jk}^{(1)} \left\{ \sum_{m=1}^{\infty} v_{km} A_m^{(1)} \right\} w \\
& + \sum_{1 \leqslant j \neq k < \infty} p_j q_{jk} \Pi_{jk}^{(1)} \left\{ \sum_{1 \leqslant m \neq n < \infty} v_{km} q_{mn} \Pi_{mn}^{(1)} \right\} w.
\end{aligned}
\tag{4.22}
$$

Then from Theorem 4.4 there is a strongly continuous contraction semigroup $\{T(t); t \geqslant 0\}$ defined on $\overline{D_0 \cap D_2}$ with $\text{buc-lim}_{\epsilon \to 0} T_\epsilon(t/\epsilon)f = T(t)f$ for all $f \in \overline{D_0 \cap D_2}$. The infinitesimal operator of $T(t)$ is the closure of $V^{(2)} + \hat{V}$ restricted so that $(V_2 + \hat{V}) \in \overline{D_0 \cap D_2}$.

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