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SMOOTH COMPLEX PROJECTIVE SPACE BUNDLES AND $B\widetilde{U}(n)$

BY

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ABSTRACT. Smooth fiberings with complex projective and Dold manifold fibers are studied and a bordism classification for even complex projective space bundles is given. The Z_2-cohomology of $B\widetilde{U}(n)$ is computed with its Steenrod algebra action.

1. **Introduction.** Let H^* be a Z_2-Poincaré algebra [3], d a formal class of degree 2 and b in H^1. In [2] it was shown that if $H^*[d]$ is given the Steenrod algebra structure determined by $Sq^1 d = bd$ and if $\Sigma_{i=0}^{n}(1+b)^{n-i}a_i$ is an "sw-class" in H^*, where a_i is in H^{2i}, then

$$K^* = H^*[d]/\langle d^n + a_i d^{n-1} + \cdots + a_n\rangle$$

is a Poincaré algebra. It was also shown that if K^*, as above, is a Poincaré algebra where d is in K^2, H^* is a Poincaré algebra, a_i are in H^{2i} and $Sq^1 d = bd$ for some b in H^1, then $\Sigma_{i=0}^{n}(1+b)^{n-i}a_i$ is an sw-class in H^*.

In this paper, we will use the above result to characterize those unoriented bordism classes which have a representative which fibers smoothly (over another manifold) with fiber an "even" complex projective space, $CP(2k)$. See [6] for the case $k = 1$ as well as for fiberings with real projective fibers. We discuss $P(n, m)$ fiberings (where $P(n, m)$ denotes the Dold manifold $S^n \times_{Z_2} CP(m)$) and show that "most" (unoriented) bordism classes contain a representative which fibers with $P(1, 2)$ as fiber. To get our results, we need to consider $B\widetilde{U}(n)$, the classifying space of $\widetilde{U}(n)$, see [5], which is to sw-pairs as $BO(n)$ is to sw-classes.

All algebras will be over Z_2 and cohomology will be singular theory with Z_2 coefficients. If η is a bundle, then $E(\eta)$ and $B(\eta)$ denote the total and base spaces of η. $RP(\eta)$ will denote the real projective space bundle associated with η and $CP(\eta)$ the complex projective space bundle (if η is complex). Γ and Λ will denote the canonical line bundles (real and complex, respectively)

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over $RP(\eta)$ and $CP(\eta)$. γ_n and λ_n will denote the universal bundles over $BO(n)$ and $BU(n)$ and for $n = 1$, the "universal" line bundles over $RP(m)$ and $CP(m)$.

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2. $B\widetilde{U}(n)$. Let $\widetilde{U}(n)$ be the subgroup of $O(2n)$ generated by $U(n)$ and conjugation. The inclusion of $\widetilde{U}(n)$ in $O(2n)$ gives a map on classifying spaces, which we will call j. There is the homomorphism from $\widetilde{U}(n)$ to Z_2 given by dividing out $U(n)$ which yields a fibration of $B\widetilde{U}(n)$ over BZ_2 with fiber $BU(n)$. Call the projection π and the fiber inclusion i. According to Stong [5],

$$H^*(B\widetilde{U}(n)) \cong Z_2[\pi^*(\iota), j^*(w_2), \cdots, j^*(w_{2n})],$$

where ι is nonzero in $H^1(BZ_2)$ and $w_{2k} = w_{2k}(\gamma_{2n})$. Denote $j^*(\gamma_{2n})$ by η.

Let δ be the $CP(n-1)$ bundle associated with the universal principal $\widetilde{U}(n)$ bundle over $B\widetilde{U}(n)$ and $E(\hat{\delta})$ be the total space of the associated $E(\lambda_1)$ bundle over $B\widetilde{U}(n)$. Then $\hat{\delta}$ is a fibering of $E(\hat{\delta})$ over $E(\delta)$, which pulls back, via the inclusion of $CP(n-1)$ in $E(\delta)$, to λ. Hence

$$H^*(E(\delta)) \cong H^*(B\widetilde{U}(n))[d]/\langle d^n + \alpha_1 d^{n-1} + \cdots + \alpha_n\rangle,$$

where $d = w_2(\hat{\delta})$ and α_i is in $H^{2i}(B\widetilde{U}(n))$. (To see α_i in terms of $w_{2i}(\eta)$, see [1].)

Similarly, if ψ denotes the $RP(2n-1)$ bundle associated with the universal principal $\widetilde{U}(n)$ bundle over $B\widetilde{U}(n)$ and $E(\hat{\psi})$ denotes the corresponding $E(\gamma_1)$ bundle, then there is a fibering $\hat{\psi}$ of $E(\hat{\psi})$ over $E(\psi)$ which pulls back to γ_1 over the fiber of ψ. Moreover ψ classifies naturally into $RP(\gamma_{2n})$, which gives that

$$H^*(E(\psi)) \cong H^*(B\widetilde{U}(n))[c]/\langle c^{2n} + w_1(\eta)c^{2n-1} + \cdots + w_{2n}(\eta)\rangle;$$

where $c = w_1(\hat{\psi})$.

LEMMA 2.1. *$H^*(E(\psi)) \cong H^*(E(\delta))[c]/\langle c^2 + \beta' C + d\rangle$, where $\beta' = w_1(\hat{\delta})$.*

PROOF. The sphere of $\hat{\delta}$ is the S^{2n-1} bundle associated with the universal principal $\widetilde{U}(n)$ bundle and hence $RP(\hat{\delta})$ is $E(\psi)$ which fibers over $E(\delta)$ with $RP(1)$ as fiber. Note that $\hat{\psi}$ pulls back to γ_1 over this fiber. The asserted relation is the usual one for the projective bundle associated with a vector bundle. □

Since β' is in $H^1(E(\delta))$, $\beta' = \pi_\delta^*(\beta)$ for some β in $H^1(B\widetilde{U}(n))$.

LEMMA 2.2. $\beta = \pi^*(\iota)$

PROOF. It suffices to show that $\beta \neq 0$. Let $\widetilde{U}(n)$ act on $S^m \times U(n)$ via the antipodal map and conjugation, which is a principal action. Classifying it and pulling back δ, we get the usual fibration of $P(m, n-1)$, the Dold manifold, over $RP(m)$ and a map of this bundle into δ.

It is known that (see [8]):

(i) $H^*(P(m, n-1)) \cong Z_2[c, d]/\langle c^{m+1}, d^n\rangle$, where the degree of c is one and that of d is two.

(ii) The Stiefel-Whitney class of $S^m \times_{Z_2} E(\lambda_1) = S^m \times U(n) \times_{\widetilde{U}(n)} E(\lambda_1)$, as a two plane bundle over $P(m, n-1)$, is $1 + c + d$.

Therefore β' pulls back to c in $H^1(P(m, n-1))$ and is nonzero. □

If $x = 1 + x_1 + x_2 + \cdots + x_n$, where x_i is a $2i$-dimensional class and y is one dimensional, we will call (x, y) an "sw-pair" if $\Sigma_{j=0}^n x_j(1 + y)^{n-j}$ is an sw-class.

THEOREM 2.3. (i) $H^*(B\widetilde{U}(n)) \cong Z_2[\beta, \alpha, \cdots, \alpha_n]$;

(ii) (α, β) *is an sw-pair; where* $\alpha = 1 + \alpha_1 + \cdots + \alpha_n$;

(iii) *If* (a, b) *is an sw-pair in a left unstable* $A(2)$ *algebra* X^*, *then there is an* $A(2)$-*homomorphism* σ: $H^*(B\widetilde{U}(n)) \to X^*$ *with* $\sigma(\alpha) = a$ *and* $\sigma(\beta) = b$.

PROOF. On $H^*(E(\psi))$, there are the relations $\Sigma_{j=0}^{2n} c^{2n-j}w_j(\eta) = 0$, $\Sigma_{k=0}^n d^{n-k}\alpha_k = 0$ and $d = c^2 + c\beta$. Hence

$$\sum_{i=0}^{2n} c^i w_{2n-i} = \sum_{j=0}^{n} \alpha_j(c^2 + \beta c)^{n-j},$$

identically in c. Hence $w(\eta) = \Sigma\ \alpha_j(1 + \beta)^{n-j}$, which gives part (ii). Moreover,

$$w_{2k} = \sum_{j=0}^{k} \binom{n-k+j}{2j} \alpha_{k-j}\beta^{2j}.$$

Therefore w_{2k} and α_k are equally acceptable polynomial generators and Stong's result cited above gives (i).

To finish, it is enough to show that the epimorphism from the cohomology of the product

$$K(Z_2, 2) \times K(Z_2, 4) \times \cdots \times K(Z_2, 2n) \times K(Z_2, 1)$$

of Eilenberg-Mac Lane spaces to $H^*(B\widetilde{U}(n))$, defined by α and β, has for kernel precisely those relations imposed by (α, β) being an sw-pair. There are unique polynomials $p_{ij}(x, y_1, \cdots, y_n)$ with $Sq^i\alpha_j + p_{ij}(\beta, \alpha_1, \cdots, \alpha_n) = 0$ for all

i and j. Suppose the cohomology of the above product is generated (as an $A(2)$ algebra) by $\iota_1, \iota_2, \cdots, \iota_{2n}$. Let the ideal J be generated by the elements $Sq^i\iota_{2j} + p_{ij}(\iota_1, \cdots, \iota_{2n})$. Let $K^* = Z_2[\iota_1, \cdots, \iota_{2n}]$. Let L^* be the cohomology of the above product and denote the epimorphism to $H^*(B\widetilde{U}(n))$ by τ. Note that τ restricted to K^* is a ring isomorphism and that the projection of L^* to L^*/J is an epimorphism when restricted to K^*. Since J is in the kernel of τ, τ factors through L^*/J and must give an isomorphism between L^*/J and $H^*(B\widetilde{U}(n))$. □

3. ***CP*(2*k*)-fibrations and bordism.** We wish now to connect Theorem 2.3 with the result cited in the introduction. Our main result is:

THEOREM 3.1. *The ideal in N_*, the unoriented bordism ring, of classes having representatives which fiber over closed smooth manifolds with fiber $CP(2k)$ is the image of $N_*(B\widetilde{U}(2k+1))$ in N_* of the homomorphism which sends the class of the $\widetilde{U}(2k+1)$ bundle over M to the class of the total space of its associated $CP(2k)$ bundle.*

(Compare, in [6], Proposition 8.5 and the remarks following 8.6.)

PROOF. Suppose $\pi: M \longrightarrow P$ is a smooth fibration of closed manifolds with $CP(2k)$ as fiber. Since $w_2(M)$ must pull back nontrivially to the generator of the fiber, $H^*(M)$ is freely generated, as an $H^*(P)$ module, by classes $1, e, \cdots, e^{2k}$, where $e = w_2(M)$. Moreover, there will be a relation $\Sigma_i e^i f_{2k+1-i} = 0$ which will give the product, where f_j is in $H^j(P)$.

If $Sq^1 e = be + g$, then $Sq^1 f_1 = bf_1 + g$ (applying Sq^1 to the above relation). Setting $d = e + f_1$ and defining the class $a = 1 + a_1 + \cdots + a_{2k+1}$ by the relation

$$\sum_{i=0}^{2k+1} e^i f_{2k+1-i} = \sum_{i=0}^{2k+1} d^i a_{2k+1-i},$$

we conclude that (a, b) must be an sw-pair (see introduction).

Hence there is a homomorphism $\sigma: H^*(B\widetilde{U}(2k+1)) \longrightarrow H^*(P)$ taking β to b and α to a. The results of [3] imply that there is a manifold Q and a map $f: Q \longrightarrow B\widetilde{U}(2k+1)$ such that f^* and σ are bordant in the algebraic bordism of $H^*(B\widetilde{U}(2k+1))$. Let $M' = f^*(E(\delta))$. We claim that M' is bordant to M.

LEMMA 3.2. *The correspondence*

$$(H^*, (a, b)) \longrightarrow H^*[d]/\langle d^n + a_1 d^{n-1} + \cdots + a_n\rangle,$$

where (a, b) is an sw-class, H^* is a Poincaré algebra and d is a formal two-

dimensional class, defines a homomorphism from the mth algebraic bordism group of $H^*(B\widetilde{U}(n))$ *to* N_{m+2n-2}.

PROOF. Suppose $(H^*, (a, b))$ bounds. Then there is a self-annihilating, homogeneous subalgebra J^* in H^* which is closed under the left and right $A(2) \otimes H^*(B\widetilde{U}(n))$ action. J^* is the image of the bounding Lefschetz algebra. See [7].

Let $R^* = J^*[d]/\langle d^n + a_1 d^{n-1} + \cdots + a_n\rangle$ which is a homogeneous subalgebra of $K^* = H^*[d]/\langle d^n + \cdots + a_n\rangle$ and is closed under the left $A(2)$ action. One shows, by straightforward arguments, that R^* is self-annihilating and closed under the right $A(2)$ action. Hence K^* bounds. Since the correspondence is clearly additive, the result follows. □

The theorem now follows by the equivalence of N_* with the algebraic bordism of $H^*(pt)$. □

4. $N_*(B\widetilde{U}(n))$ **and** $P(n, m)$ **fibrations.** In this section, we find generators for $N_*(B\widetilde{U}(n))$ and the indecomposables in the image of $N_*(B\widetilde{U}(n)) \to N_*$, the homomorphism of the previous section. We also collect several related results on $P(n, m)$ fibrations.

There is an involution of $\widetilde{U}(n)$ (which on the included $U(n)$ is conjugation) whose fixed subgroup is $Z_2 \times O(n)$. The composition

$$\theta\colon Z_2 \times (O(1) \times \cdots \times O(1)) \to Z_2 \times O(n) \to \widetilde{U}(n)$$

is clearly the inclusion of a maximal torus and the induced homorphism

$$\theta^*\colon H^*(B\widetilde{U}(n)) \to H^*(BZ_2) \otimes \left\{\bigotimes_{i=1}^{n} H^*(BO(1))\right.$$

is a monomorphism.

LEMMA 4.1. $\theta^*(\beta) = y$ *and* $\theta^*(\alpha_j) = \sigma_j(x_1(y + x_1), \cdots, x_n(y + x_n))$, *where* σ_j *denotes the jth elementary symmetric function,* y *generates the cohomology of* BZ_2 *and* x_i *generates the cohomology of the ith factor* $BO(1)$.

PROOF. Since the composition, $Z_2 \times O(n) \to \widetilde{U}(n) \to Z_2$, is projection on the first factor, $\theta^*(\beta) = y$. We claim that η, the bundle over $B\widetilde{U}(n)$ given by the inclusion of $\widetilde{U}(n)$ in $O(2n)$, pulls back over $BZ_2 \times BO(n)$ to $(\gamma_1 \hat{\otimes} \gamma_n) + (1 \hat{\otimes} \gamma_n)$, where $\hat{\otimes}$ denotes the exterior tensor product of vector bundles. Clearly, this will complete the proof.

If $f\colon BO(n) \to BU(n)$ denotes the usual complexification (induced by the inclusion of $O(n)$ in $U(n)$), then the inclusion of $BZ_2 \times BO(n)$ in $B\widetilde{U}(n)$

classifies $EZ_2 \times_{Z_2} f^*(\lambda_n)$, where λ_n is a Z_2-space via conjugation. This space is

$$\{(s, y, a, b) \in S^\infty \times BO(n) \times R^\infty \times R^\infty : a \in y, b \in y\}$$

modulo the relation $(s, y, a, b) \sim (-s, y, a, -b)$, where we are thinking of $BO(n)$ as n-planes in R^∞. This bundle is $(1 \hat{\otimes} \gamma_n) + (S^\infty \times_{Z_2} \gamma_n)$, where

$$S^\infty \times_{Z_2} \gamma_n = \{(s, a, b) \in S^\infty \times BO(n) \times R^\infty : b \in a\}/(s,a,b) \sim (-s,a,-b).$$

Moreover

$$\gamma_1 \hat{\otimes} \gamma_n = \{(x, t, u, v) \in BO(1) \times R^\infty \times BO(n) \times R^\infty : t \in x, v \in u\},$$

modulo $(x, rt, u, v) \sim (x, t, u, rv)$ for any r in R^1. The correspondence $(s, a, b) \longrightarrow ([s], s, a, b)$ induces the isomorphism. □

Let $M(q, j_1, \cdots, j_n)$ be the product manifold

$$RP(q) \times RP(2j_1) \times RP(2j_1 + 2j_2) \times \cdots \times RP(2j_1 + \cdots + 2j_n).$$

Then there is the map

$$M(q, j_1, \cdots, j_n) \longrightarrow BZ_2 \times \prod_{i=1}^{n} BO(1) \longrightarrow B\widetilde{U}(n),$$

which we will denote by $f_{q,j_1,\cdots,j_n}$. Ordering the $(n+1)$-tuples $(q, j_1, \cdots, j_n)$ lexiographically, one easily shows, using the previous lemma, that $(q, j_1, \cdots, j_n) < (p, k_1, \cdots, k_n)$ implies that

$$f^*_{q,j_1,\cdots,j_n}(\beta^p \alpha_1^{k_1} \cdots \alpha_n^{k_n}) = 0.$$

It follows that the classes $[M(q, j_1, \cdots, j_n), f_{q,j_1,\cdots,j_n}]$ are an N_* basis for $N_*(B\widetilde{U}(n))$.

Our main result is:

THEOREM 4.2. *The image of the class of* $[M(q, j_1, \cdots, j_n), f_{q,j_1,\cdots,j_n}]$ *is decomposable in* N_* *if and only if the term of degree* p *in the expansion of*

$$\frac{\Sigma_{i=1}^{n} \{(1 + y + x_i)^{p+2n-2} + (1 + x_i)^{p+2n-2}\}}{\Pi_{i=1}^{n} (1 + y + x_i)(1 + x_i)}$$

is zero, where p *is the dimension of* $M(q, j_1, \cdots, j_n)$.

To demonstrate this, we need a preliminary result.

LEMMA 4.3. *If* $f\colon M \to B\widetilde{U}(n)$ *is a map, then*

$$w(f^*(\delta)) = \pi^*(w(M))(1+b)^{-1}\left\{\sum_{i=0}^{n}(1+b+d)^{n-i}a_i\right\},$$

where $a_i = f^*(\alpha_i)$ *and* $1+b+d = f^*(w(\hat{\delta}))$.

PROOF. Since $f^*(\delta)$ fibers smoothly over M (with fiber $CP(n-1)$), $w(f^*(\delta)) = \pi^*(w(M))w(\theta)$, where θ is the bundle along the fibers of $f^*(\delta) \to M$. We claim that θ is the pull back of a "universal" bundle $\widetilde{\theta}$, over δ, such that, when pulled back over the diagram

$$\begin{array}{ccc}
S^\infty \times_{Z_2} CP\left(\bigoplus_{i=1}^{n}(\gamma_1 \otimes C)\right) & \longrightarrow & E(\delta) \\
\downarrow & & \downarrow{\scriptstyle\delta} \\
BO(1) \times \prod_{i=0}^{n} BO(1) \to BO(1) \times BO(n) & \to & B\widetilde{U}(n),
\end{array}$$

satisfies the relation:

$$(*) \qquad \hat{\delta} \otimes_R \pi^*\left(\bigoplus_{i=1}^{n}\gamma_1\right) \cong \widetilde{\theta} \oplus \det\hat{\delta} \oplus 1.$$

If $(*)$ holds, then $w(\widetilde{\theta}) = (1+\beta)^{-1}\,\Sigma_{i=0}^{n}(1+\beta+d)^{n-i}\alpha_i$, since $BO(1) \times \Pi_{i=1}^{n} BO(1) \to B\widetilde{U}(n)$ is monic on cohomology.

To prove $(*)$, we work over the double covers of δ and $B\widetilde{U}(n)$ defined by β. Pulling back η and $\hat{\delta}$, we receive the complex bundles $\hat{\lambda}_n$ and $\hat{\Lambda}$, of complex dimension n and 1 respectively. It is then standard that $\Lambda \otimes_C \pi^*(\hat{\lambda}_n) \cong S^\infty \times (1 \oplus \theta)$, where $\theta = \{(x, y) \in S(\lambda_n) \times E(\lambda_n)\colon x \perp y\}$ modulo the usual S^1 action (here $\perp$ is as complex vectors). Pulling back to $BO(1) \times \Pi_{i=1}^{n} BO(1)$ and dividing out the Z_2 action gives $(*)$. $\square$

PROOF OF THEOREM 4.2. Let

$$f^*_{q,j_1,\dots,j_n}(\delta) = X, \quad RP(f^*_{q}, {}_{j_1,\dots,j_n}(\hat{\delta})) = Y \text{ and } k_l = j_1 + \cdots + j_l.$$

Then

$$w(X) = (1+y)^q \prod_{i=1}^{n}(1+x_i)^{2k_i+1}\{(1+y+d)^n + \cdots + a_n\},$$

where y and x_i generate the cohomology of $RP(q)$ and $RP(2k_i)$ respectively. Moreover,

$$w(Y) = w(X)\{(1 + c)^2 + (1 + c)y + d\},$$

where $c = w_1(f^*_{q,j_1,\ldots,j_n}(\hat{\psi}))$. Therefore

$$w(Y) = (1 + y)^{q+1} \prod_{i=1}^{n} (1 + x_i)^{2k_i+1} \left\{ \sum_{j=0}^{n} a_{n-j}(1 + y + d)^j \right\}.$$

Let m be the dimension of X, so that $m = p + 2(n - 1)$, and denote the mth s-class of Y by $s_m(Y)$; we have

$$s_m(Y) = (q + 1)y^m + \sum_{i=1}^{n} x_i^m + s_m\left\{ \sum_{j=0}^{n} a_{n-j}(1 + y + d)^j \right\}.$$

Since $m > q$ and $m > 2k_i$,

$$\begin{aligned} s_m(Y) &= s_m\left\{ \sum_{j=0}^{n} a_{n-j}(1 + y + d)^j \right\} \\ &= \sum_{i=1}^{n} \{(y + x_i + c)^m + (c + x_i)^m\} = \sum_{j=0}^{m} \binom{m}{j} c^{m-j} \left\{ \sum_{i=1}^{n} \{(y + x_i)^j + x_i^j\} \right\}. \end{aligned}$$

Hence

$$s_m(Y) = \binom{m}{1} c^{m-1} s_1(w) + \binom{m}{2} c^{m-2} s_2(w) + \cdots + \binom{m}{2n-2} c^{2n-2} s_{m-2n+2}(w),$$

where $w = w(f^*_{q,j_1,\ldots,j_n}(\eta))$. Since $c^{2n-1+i} \equiv \bar{w}_i c^{2n-1}$ modulo lower degree terms in c, $cs_m(Y)$ evaluates on the fundamental class of Y as does the expression

$$c^{2n-1} \left\{ \sum_{j=1}^{m-(2n-2)} \bar{w}_{m-(2n-2)-j} s_j(w) \right\}.$$

But for any $x \in H^*(M; Z_2)$, $c^{2n-1}x$ evaluates on the fundamental class of Y as x does on the fundamental class of M. Since $\bar{w} = \Pi_{i=1}^{n}(1 + y + x_i)^{-1}(1 + x_i)^{-1}$, the result follows. □

We will finish this section with several related results on smooth fiberings with Dold manifolds as fibers.

LEMMA 4.4. *If* $\pi: X \to M$ *is a smooth fibration with* $i: P(m, n) \to X$ *the inclusion of a fiber, then* $\pi_1(M)$ *acts trivially on* $H^*(P(m, n))$ *if either* $m \neq 2$ *or* n *is even.*

PROOF. Let $\theta: [0, 1] \to M$ with $\theta(0) = \theta(1) = x$. Then there is a diagram

$$\begin{tikzcd} P(m, n) \times [0, 1] \arrow[r, "\tilde{\theta}"] \arrow[d] & X \arrow[d, "\pi"] \\ x \times [0, 1] \arrow[r, "\theta"] & M \end{tikzcd}$$

giving $\theta: P(m, n) \to P(m, n)$, defined by $\theta(p) = \tilde{\theta}(p, 1)$. Hence θ^*: $H^*(P(m, n)) \to H^*(P(m, n))$ is a ring automorphism and a homomorphism of $A(2)$ algebras.

Since $H^*(P(m, n) \cong Z_2[c, d]/\langle c^{m+1}, d^{n+1}\rangle$, where the degrees of c and d are 1 and 2 respectively, $\theta^*(c) = c$. If $\theta^*(d) = d + c^2$, then

$$\theta^*(cd) = \theta^*(Sq^1 d) = Sq^1\theta^*(d) = Sq^1(d + c^2) = cd.$$

But $\theta^*(cd) = \theta^*(c)\theta^*(d) = c(d + c^2)$. Hence m is not greater than 2. If $m = 1$, then clearly $\theta^*(d) = d$.

According to [4], $\theta^*(w_i) = w_i$, where w_i is the ith Stiefel-Whitney class of $P(m, n)$. If $n = 2k$, $w = (1 + c)^m(1 + c + d)^{2k+1}$ (see [8]), then

$$\begin{aligned} w &= \left\{1 + \binom{m}{1}c + \binom{m}{2}c^2 + \cdots\right\}(1 + c + d)(1 + c^2 + d^2)^k \\ &= \left\{1 + c\left(\binom{m}{1} + 1\right) + c^2\left(\binom{m}{2} + \binom{m}{1} + \binom{k}{1}\right) + d + \cdots\right\}. \end{aligned}$$

Hence,

$$\left(\binom{m}{2} + \binom{m}{1} + k\right)c^2 + \theta^*(d) = \left(\binom{m}{2} + \binom{m}{1} + k\right)c^2 + d,$$

and $\theta^*(d) = d$. □

LEMMA 4.5. *If* $\pi: X \to M$ *and* $i: P(m, n) \to X$ *are as in the previous lemma and* n *is even, then* π *is totally nonhomologous to zero.*

PROOF. Let $n = 2k$ and $a = \binom{m}{2} + \binom{m}{1} + k$. Then, as above, $w_2(P(m, 2k)) = ac^2 + d$. Since π is locally trivial, $i^*(w_2(X)) = w_2(P(m, 2k))$ and $ac^2 + d$ is in the image of i^*. Hence $ac^2 + d$ is in the kernel of every differential d_i of the (cohomology) spectral sequence of π.

Since $d_2(ac^2 + d) = 0$, $d_2(d) = 0$. But $Sq^1(ac^2 + d) = Sq^1 d = cd$, which is therefore in the image of i^*. Hence, $0 = d_2(cd) = dd_2(c) + cd_2(d) = dd_2(c)$. Since $d_2(c)$ is in $H^*(M)$, $d_2(c) = 0$. But $d_3(c^2) = 0$ also and hence, $0 = d_3(ac^2 + d) = d_3(d)$.

Therefore, the spectral sequence is trivial and the result follows. □

THEOREM 4.6. *If* i: $P(1, 2k) \rightarrow X$ *is the inclusion of a fiber in the smooth fibration* π: $X \rightarrow M$, *X and M closed, then X is bordant to a manifold which fibers smoothly with fiber* $CP(2k)$.

PROOF. By Lemma 4.5, $H^*(X) \cong H^*(M)[1, c, d, cd, \cdots, d^i, cd^i, \cdots, cd^{2k}]$, as $H^*(M)$ modules. There are the relations $c^2 = \gamma c + \delta$ and $d^{2k+1} = \Sigma_{j=1}^{2k+1} a_j d^{2k-j+1}$, where γ is in $H^1(M)$, δ is in $H^2(M)$ and a_j is a $2j$-degree class in $K^* = H^*(M)[c]/\langle c^2 + \gamma c + \delta\rangle$.

Since $Sq^1\delta = \gamma\delta$, K^* is a Poincaré algebra (see [2]), $H^*(X)$ is a Poincaré algebra and $H^*(X) \cong K^*[d]/\langle d^{2k+1} + a_1 d^{2k} + \cdots + a_{2k+1}\rangle$. Since $2k$ is even, we can change generators, if necessary, to get $Sq^1 d = cd$. Hence the pair (a, c), where $a = 1 + a_1 + \cdots + a_{2k+1}$, is an sw-pair.

It follows that there is a homomorphism θ: $H^*(B\widetilde{U}(2k + 1)) \rightarrow K^*$ with $\theta(\beta) = c$ and $\theta(\alpha) = a$, and, as before, a pair (N, f) with f: $N \rightarrow B\widetilde{U}(2k + 1)$ bordant to (K^*, θ). Pulling δ back along f, we get a manifold fibering over N with fiber $CP(2k)$ which is bordant to X. (This uses Theorem 3.2.) □

THEOREM 4.7. *There are indecomposable manifolds which fiber over closed manifolds with fiber* $P(1, 2)$ *in all dimensions* m *of the form* $4k + 2$ *for* $k = 1, 2, \cdots$ *or* $2^p(2q + 1) - 1$ *for* $p > 0$ *and* $q > 0$ (*i.e., all odd dimensions, not of the form* $2^i - 1$).

PROOF. First note that for $q = 1$, the rational function of Theorem 4.2 becomes

$$my\left(\sum_{i=1}^{n}(1 + x_i)^{m-1}\right)\Big/\prod_{i=1}^{n}(1 + y + x_i)(1 + x_i).$$

Hence, we need the coefficient of $x_1^{2k_1} \cdots x_n^{2k_n}$ in

$$\frac{(1 + x_i)^{m-2}\{1 + (y + x_i) + (y + x_i)^2 + \cdots\}}{\Pi_{j=1;j\neq i}^{n}(1 + y + x_j)(1 + x_j)}.$$

Therefore, we want the coefficient of $x_1^{2k_1}\cdots x_n^{2k_n}$, deleting $x_i^{2k_i}$, in

$$\begin{aligned}\{1 &+ (x_1^2 + y(1 + x_1)) + (x_1^2 + y(1 + x_1))^2 + \cdots\}\\ &\cdots(1 + x_n^2 + \cdots)\cdots\\ &= (1 + x_1^2 + x_1^4 + \cdots + y\psi_1)(1 + x_2^2 + x_2^4 + \cdots + y\psi_2)\cdots.\end{aligned}$$

Therefore, the required coefficient is

$$m\left\{\binom{m-2}{2k_i} + \binom{m-2}{2k_i - 1} + \cdots + \binom{m-2}{1} + \binom{m-2}{0}\right\}$$

which equals $m\binom{m-3}{2k_i}$. Hence, in this case,

$$s_m(X) = \sum_{i=1}^{n}\binom{m-3}{2k_i} = \sum_{i=1}^{n}\binom{k_1 + \cdots + k_n + n - 2}{k_i}.$$

Hence if $n = 3$, $s_m(X) = \Sigma_{j=1}^{3}\binom{k_1+k_2+k_3+1}{k_j}$.

If $m = 2^p(2q + 1) - 1$, $k_1 + k_2 + k_3 = 2^pq + 2^{p-1} - 2$. If $k_1 + k_2 + k_3 = l + 1$ is odd, then $\binom{l+1}{l} + \binom{l+1}{0} + \binom{l+1}{0}$ is odd and the manifold $S^1 \times_{Z_2} CP((\gamma_1 \otimes C) \hat{\oplus} 1_C \hat{\oplus} 1_C$ will do.

If $l + 1$ is even, we claim that

$$\binom{l+1}{2^{p-1} - 2} + \binom{l+1}{2^{p-1}q} + \binom{l+1}{2^{p-1}q - 1}$$

is odd. To see this, set $q = \Sigma_{i=0}^{r} a_i 2^i$ and note that

$$\binom{l+1}{2^{p-1} - 2} \equiv \binom{a_r}{0}\cdots\binom{a_0}{0}\binom{0}{0}\binom{1}{1}\cdots\binom{1}{1} \quad \text{(modulo 2)},$$

which is odd. Moreover,

$$\begin{aligned}\binom{l+1}{2^{p-1}q} + \binom{l+1}{2^{p-1}q - 1} &= \binom{l+2}{2^{p-1}q} = \binom{2^pq + 2^{p-1} - 1}{2^{p-1}q}\\ &= \binom{a_r}{0}\cdots\binom{a_i}{a_{i+1}}\cdots\binom{a_0}{a_1}\binom{0}{a_0}\binom{1}{0}\cdots\binom{1}{0},\end{aligned}$$

which is odd only if $a_0 = a_1 = \cdots = a_r = 0$. But $a_r = 1$. Hence $k_1 = 2^{p-1} - 2$, $k_2 = 2^{p-1}q$ and $k_3 = 2^{p-1}q - 1$ define a manifold which works if $m = 2^p(2q+1) - 1$ and $p > 1$.

To get the even dimensions, we want to consider the manifolds Q^n defined as follows. Let v be the smooth involution of $P(1, 2)$ defined by $v[(t_1, t_2), x] = [(-t_1, t_2), x]$, where t_i are in R^1, $t_1^2 + t_2^2 = 1$, x is in $CP(2)$ and $[\ ,\ ]$ denotes the usual equivalence class. Let $Q^n = S^n \times_{Z_2} P(1, 2)$, where the action takes (s, y) to $(-s, vy)$. Then

$$H^*(Q) \cong H^*(RP(n))[1, c, d, cd, d^2, cd^2],$$

as $H^*(RP(n))$ modules.

Now there is a diagram:

$$\begin{array}{ccc} CP(2) & \xrightarrow{i} & Q \\ \downarrow & & \downarrow p \\ RP(1) & \xrightarrow{j} & S^n \times_{Z_2} RP(1) \\ & & \downarrow q \\ & & RP(n) \end{array}$$

where p and q are fibrations with inclusions of fibers i and j respectively. Clearly $S^n \times_{Z_2} RP(1) = RP(\gamma_1 \oplus 1)$.

Letting T denote a tangent bundle and θ a bundle along the fibers, we have

$$T(Q) \oplus 2 \cong p^*(\Gamma \otimes q^*(\gamma_1 \oplus 1)) \oplus \theta_p \oplus p^*q^*((n+1)\gamma_1),$$

where Γ denotes the canonical line bundle over $RP(\gamma_1 \oplus 1)$. Let p' denote the usual fibration of $P(1, 2)$ over $RP(1)$, then $\theta_p = S^n \times_{Z_2} \theta_{p'}$. If σ' and ρ' denote the usual line and 2-plane bundle over $P(1, 2)$, $\sigma = S^n \times_{Z_2} \sigma'$ and $\rho = S^n \times_{Z_2} \rho'$, then $\theta_p \oplus \sigma \oplus 1 = 3\rho$. Let $c = w_1(\rho)$, $d = w_2(\rho)$ and x generate the cohomology of $RP(n)$. Then $c^2 = cx$ in $H^*(Q)$. Since Γ pulls back to σ via p, $w_1(\sigma) = c$. Hence $d^3 = 0$ in $H^*(Q)$.

It follows that

$$H^*(Q) \cong \frac{H^*(RP(n))[c]}{\langle c^2 + cx \rangle} \otimes \frac{Z_2[d]}{\langle d^3 \rangle},$$

as rings. Note that, since $Sq^1 d = cd$, the splitting is not as $A(2)$ algebras.

We have that

$$T(Q) \oplus 3 \oplus \sigma \cong p^*\{q^*((n+1)\gamma_1) \oplus (\Gamma \otimes q^*(\gamma_1 \oplus 1))\} \oplus 3\rho;$$

hence

$$\begin{aligned} w(Q) &= (1+x)^{n+1}(1+c+d)^3\{(1+c)^2 + x(1+c)\}(1+c)^{-1} \\ &= (1+x)^{n+1}(1+c+d)^3(1+c+x). \end{aligned}$$

It follows that

$$\begin{aligned} s_{n+5}(Q) &= \sum_{i=0}^{n+5} \binom{n+5}{i} c^i x^{n+5-i} + \sum_{p+2q=n+5} \binom{p+q-1}{q} c^p d^q \\ &= \binom{n+2}{2} c x^n d^2, \end{aligned}$$

since $c^i x^{n+5-i} = cx^{n+4} = 0$ for $i \geqslant 1$ and $c^p d^q = cx^{p-1} d^q = 0$, unless $q = 2$, $p - 1 = n$. But $\binom{n+2}{2}$ is odd precisely when $n \equiv 0$ or $1 \bmod 4$.

Hence Q^n is indecomposable in dimensions $4k + 1$ and $4k + 2$, which gives the result. □

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