ABSTRACT. This paper studies pairs \((M, \xi)\) where \(M\) is a closed manifold and \(\xi\) is a \(k\)-dimensional subbundle of the tangent bundle of \(M\) in terms of cobordism.

1. Introduction. The purpose of this note is to analyze pairs \((M, \xi)\) where \(M\) is an \(n\)-dimensional manifold and \(\xi\) is a \(k\)-dimensional subbundle of the tangent bundle of \(M\), \(k \leq n\), in terms of cobordism.

In §2, the cobordism class of \(M\) is analyzed and the main result is

**Proposition.** A class \(\alpha \in \mathcal{N}_n\) is represented by a manifold \(M^n\) whose tangent bundle has a \(k\)-dimensional subbundle, \(k \leq n\), if and only if either

(a) \(k\) is even, or
(b) \(k\) is odd and \(w_n(\alpha) = 0\).

In section §3, the case \(k = 1\), i.e., \(\xi\) a line bundle, will be studied more closely. One defines a homomorphism \(\theta : \mathcal{N}_n(BO_1) \to \mathbb{Z}_2\) as follows. If \(\alpha \in \mathcal{N}_n(BO_1)\), choose a manifold \(M^n\) and map \(f : M^n \to BO_1\) representing \(\alpha\). Let \(i \in H^1(BO_1 ; \mathbb{Z}_2)\) be the nonzero class, and let \(\theta(\alpha)\) be the characteristic number

\[\{w_n(M) + w_{n-1}(M)f^*(i) + \cdots + w_{n-k}(M)(f^*(i))^r + \cdots + (f^*(i))^n\} [M].\]

Letting \(\gamma\) be the universal line bundle over \(BO_1\), the class \(\alpha\) is the class of the pair \((M, f^*(\gamma))\), and interpreting \(\mathcal{N}_n(BO_1)\) as the cobordism classes of \(n\)-manifolds with a line bundle, one has

**Proposition.** A class \(\alpha \in \mathcal{N}_n(BO_1)\) is represented by a pair \((M^n, \xi)\) where \(\xi\) is a sub-line-bundle of the tangent bundle of \(M\) if and only if \(\theta(\alpha) = 0\).

**Note.** In order to make this result seem plausible, one should note that the given characteristic number is the \(n\)th Stiefel-Whitney number of \(\tau_M - f^*(\gamma)\), which is an \((n - 1)\)-plane bundle if \(f^*(\gamma)\) is a subbundle of \(\tau_M\).

In §4, the problem is stabilized, and the main result is

**Proposition.** A class \(\alpha = [M, f] \in \mathcal{N}_n(BO_k)\) is represented by a pair \((M', \xi')\) with \(\tau_{M'} \oplus 1 \cong \xi' \oplus \eta' \oplus 1\) where \(\eta'\) is an \(n - k\) plane bundle if...
and only if every Stiefel-Whitney number of \( \alpha \) involving a class \( w_i(\tau - f^*(\gamma)) \) for \( i > n - k \) is zero.

In §5, the case \( k = 2 \) is studied more closely.

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2. The cobordism class of \( M \).

**Lemma 2.1.** If \( M^n \) is a closed \( n \)-manifold and \( \xi^k \) is a subbundle of the tangent bundle of \( M \) with \( k \) odd, then \( w_n[M] = 0 \); i.e., \( M \) has even Euler characteristic.

**Proof.** If \( n \) is odd, \( w_n[M] = 0 \), so one may assume \( n \) even. Let \( k = 2p + 1, n - k = 2q + 1 \) and let \( \eta \) be a complement of \( \xi \) in \( \tau \), the tangent bundle of \( M \), so that \( \xi \oplus \eta = \tau \). Then

\[
w_n[M] = w_n(\tau)[M] = w_{2p+1}(\xi) \cup w_{2q+1}(\eta)[M],
\]

\[
= (Sq^1 w_{2p}(\xi) + w_1(\xi) \cup w_{2p}(\xi)) \cup w_{2q+1}(\eta)[M]
\]

\[
= \{Sq^1 w_{2p}(\xi) \cup w_{2q+1}(\eta) + (w_1(\tau) + w_1(\eta)) \cup w_{2p}(\xi) \cup w_{2q+1}(\eta)\}[M]
\]

\[
= \{Sq^1 w_{2p}(\xi) \cup w_{2q+1}(\eta) + v_1(\tau) \cup w_{2p}(\xi) \cup w_{2q+1}(\eta)
\]

\[
+ w_{2p}(\xi) \cup Sq^1 w_{2q+1}(\eta)[M]
\]

\[
= \{v_1(\tau) \cup + Sq^1\} w_{2p}(\xi) w_{2q+1}(\eta)[M]
\]

but cup-product with the Wu class \( v_1(\tau) = w_1(\tau) \) gives \( Sq^1 \), and so this vanishes. \( \Box \)

In order to prove the converse, one needs some examples of manifolds. For this, one may use the result of [5, 3.4]:

**Lemma 2.2.** Let \( RP(n_1, n_2, \cdots, n_t), t > 1, \) be the bundle of lines in the fibers of \( \lambda_1 \oplus \cdots \oplus \lambda_t \) over \( RP(n_1) \times \cdots \times RP(n_t) \), where \( \lambda_i \) is the pull-back of the canonical bundle over \( RP(n_i) \). Then \( RP(n_1, \cdots, n_t) \) is a closed manifold of dimension \( n + t - 1 \) where \( n = n_1 + \cdots + n_t \), and is indecomposable in \( \mathcal{R}_* \) if and only if

\[
\binom{n + t - 2}{n_1} + \cdots + \binom{n + t - 2}{n_t}
\]

is odd.
One now defines manifolds $X^n$ of dimension $n$ for $n \neq 2s - 1$ and $n \neq 2$ as follows:

(a) if $n = 4s$, $s > 1$,  
$$X^n = \mathbb{R}P(1, \cdots, 1, 0),$$ 
(b) if $n = 4s + 2$, $s > 1$,  
$$X^n = \mathbb{R}P(1, \cdots, 1, 0, 0, 0),$$ 
(c) if $n = 2^p(2q + 1) - 1$, $p, q > 0$,  
$$X^n = \mathbb{R}P(2^p, 1, \cdots, 1, 0).$$

The above criterion immediately shows that these manifolds are indecomposable in $\mathcal{N}_{2s}$.

The manifolds $X^n$ have the additional property that, for each integer $k \leq n$, the tangent bundle of $X^n$ has a $k$-dimensional subbundle. In fact, for $n \leq 5$, the tangent bundle of $X^n$ is a Whitney sum of line bundles.

To see this, let $\lambda$ be the canonical line bundle over $\mathbb{R}P(n_1, \cdots, n_t)$ and $\pi: \mathbb{R}P(n_1, \cdots, n_t) \to \mathbb{R}P(n_1) \times \cdots \times \mathbb{R}P(n_t)$ the projection. Let $\lambda_i$ denote $\pi^*(\lambda_i)$ and $\tau_i$ the pullback of the tangent bundle of $\mathbb{R}P(n_i)$. Then

$$\tau_{\mathbb{R}P(n_1, \cdots, n_t)} \cong \pi^*\tau_{\mathbb{R}P(n_1)} \times \cdots \times \tau_{\mathbb{R}P(n_t)} \oplus \mu \cong \tau_1 \oplus \cdots \oplus \tau_t \oplus \mu$$

where $\mu$ is the bundle along the fibers. Then

$$\mu \oplus l \cong (\lambda \otimes \lambda_1) \oplus \cdots \oplus (\lambda \otimes \lambda_t)$$
and $\tau_i \oplus l = (n_i + 1)\lambda_i$

where $l$ is the trivial line bundle. If $n_i = 0$ or $1$, $\tau_i$ is trivial, since the tangent bundles of $\mathbb{R}P(1) = S^1$ and $\mathbb{R}P(0) = \text{point}$ are trivial. Adding the trivial $\tau_i$ with $n_i = 1$ to other $\tau_i$ or $\mu$ represents them as sums of line bundles.

For $n = 5$, $\mathbb{R}P(2, 1, 0)$ has tangent bundle $\tau_1 \oplus l \oplus \mu$ which is a line bundle and two 2-plane bundles, while in all other cases there are at least two $l$'s and the tangent bundle is a sum of line bundles.

One now has

**Proposition 2.3.** A class $\alpha \in \mathcal{N}_n$ is represented by a manifold $M^n$ whose tangent bundle has a $k$-dimensional subbundle, $k \leq n$, if either:

(a) $k$ is even, or
(b) $k$ is odd and $w_n(\alpha) = 0$.  

Proof. Every class $\alpha \in \mathcal{H}_n$ is represented by the disjoint union of manifolds

$$\underbrace{RP(2) \times \cdots \times RP(2)}_q \times X^{n_1} \times \cdots \times X^{n_s}$$

with $2q + n_1 + \cdots + n_s = n$. For any integer $k \leq n$ of the form $2u + v$ with $u \leq q$, $v \leq n_1 + \cdots + n_s$, this component has a subbundle of its tangent bundle of dimension $k$. In particular, every even integer can be put in this form, and every odd integer will be of this form except for the component $[RP(2)]^{n/2}$ which has $w_n \neq 0$. □

This completes the proof of the proposition given in the introduction.

Remark. If $\xi$ is the line bundle over $RP(1)$ and $\lambda$ is the line bundle over the Klein bottle $RP(\xi \oplus I)$, then the 5-manifold $RP(\xi \oplus 3)$ is indecomposable in $\mathcal{H}_5$ and has tangent bundle a sum of line bundles. This manifold could be used in place of $X^5$ and so five plays no special role.

3. Line bundles.

Lemma 3.1. If $M^n$ is a closed $n$-manifold, $\xi$ a sub-line-bundle of the tangent bundle of $M$ and $f: M \to BO_1$ classifies $\xi$, then $\theta([M, f]) = 0$.

Proof. Let $\eta$ be a complement in $\tau$ for $\xi$. Then $w(\eta) = w(\tau)/w(\xi)$, so since $\eta$ is an $(n-1)$-plane bundle

$$0 = w_n(\eta) = w_n(\tau) + w_{n-1}(\tau)w_1(\xi) + \cdots + (w_1(\xi))^n.$$ 

Since $w_1(\xi) = f^*(i)$ and $w_1(\tau) = w_1(M)$, this gives $\theta([M, f]) = 0$. □

In order to prove the converse, one needs to analyze the bordism of $BO_1$. Henceforth, classes of $\mathcal{H}_*(BO_1)$ will be denoted $[M, \xi]$ where $M$ is a closed manifold and $\xi$ is a line bundle over $M$. There is a homomorphism of $\mathcal{H}_*$ modules, called the Smith homomorphism,

$$\Delta: \mathcal{H}_*(BO_1) \to \mathcal{H}_*(BO_1)$$

of degree $-1$ assigning to $[M, \xi]$ the class $[N, \xi[N]$ where $N \subset M$ is the codimension one submanifold of $M$ dual to $\xi$. Specifically, if $f: M \to BO_1 = RP(\infty)$ classifies $\xi$, $f$ maps $M$ into some $RP(n)$ and may be homotoped in $RP(n)$ to be transverse regular on $RP(n-1)$, with $N$ then taken to be the inverse image of $RP(n-1)$.

Letting $1 = [\text{point}, I] \in \mathcal{H}_0(BO_1)$, there are unique classes $x_i = [M^i, \xi^i] \in \mathcal{H}_i(BO_1), i \geq 0$, with

1. $x_0 = 1$,
2. $\Delta x_i = x_{i-1}$, and
3. for $i > 0$, $M^i$ bounds.
These classes form a base for \( \mathfrak{N}_*(BO_1) \) as \( \mathfrak{N}_* \) module. (A proof of this statement, or more precisely, its complex analogue appears in \([2, (5.3)]\).)

**Lemma 3.2.** For \( i > 0 \), \( x_i \) is the class of the canonical bundle \( \lambda \) over \( RP(1, 0, \cdots, 0) \) \((i - 1 \ 0\text{'s})\).

**Proof.** In \([1, (2.2)]\), \( RP(1, 0, \cdots, 0) \) \((i - 1 \ 0\text{'s})\) is denoted \( RP(\xi \oplus (i - 1)) \), where \( \xi \) is the canonical line bundle over \( RP(1) \), and is shown to bound. In \([4, p. 160]\) it is shown that for any vector bundle \( \rho \) over \( M \), the submanifold dual to \( \lambda \) over \( RP(\rho \oplus 1) \) is \( RP(\rho) \), from which the behaviour of \( \Delta \) follows. □

For \( i > 1 \), the tangent bundle of \( RP(1, 0, \cdots, 0) \) \((i - 1 \ 0\text{'s})\) is \( 1 \oplus \mu = \lambda \oplus \pi^*(\xi) \oplus (i - 1)\lambda \), which contains a copy of \( \lambda \), so \( \theta(x_i) = 0 \) if \( i > 1 \).

Now if \( \xi \) is a line bundle over \( M \), and \( N \) is a closed manifold, \( \pi_M^*(\xi) \) is a line bundle over \( M \times N \), with \([N] \cdot [M, \xi] = [M \times N, \pi_M^*(\xi)]\) giving the module structure of \( \mathfrak{N}_*(BO_1) \). If \( N \) has dimension \( n \), it is immediate that \( \theta([N] \cdot [M, \xi]) = w_n[N] \cdot \theta([M, \xi]) \).

Since \( \theta(x_0) = \theta(x_1) = 1 \), one then has

**Lemma 3.3.** \( \theta(\Sigma [N^{n-i}] x_i) = w_n(N^n) + w_{n-1}(N^{n-1}) \).

**Proposition 3.4.** If \( \alpha \in \mathfrak{N}_*(BO_1) \) with \( \theta(\alpha) = 0 \), then \( \alpha = [M, \xi] \) where \( \xi \) is a sub-line-bundle of the tangent bundle of \( M \).

**Proof.** Let \( \alpha = \Sigma_{i=1}^n a_i x_i \) with \( a_i \in \mathfrak{g}_{n-i} \). Then \( w_n(a_0) = 0, w_{n-1}(a_1) = 0 \), for if \( n \) is odd \( w_n(a_0) = 0 \) for dimensional reasons while \( w_{n-1}(a_1) = \theta(\alpha) = 0 \) and if \( n \) is even \( w_{n-1}(a_1) = 0 \) for dimensional reasons while \( w_n(a_0) = \theta(\alpha) = 0 \). By \([1, (4.5)]\) there are manifolds \( N^n \) and \( N^{n-1} \) fibered over \( S^1 \), with \([N^{n-i}] = a_i, i = 0, 1 \). Choose manifolds \( N^{n-i} \) representing \( a_i \) for \( i > 1 \), and let

\[
M^n = N^n \cup (N^{n-1} \times RP(1)) \cup \bigcup_{i>1} (N^{n-i} \times RP(1, 0, \cdots, 0))
\]

and let \( \xi \) be the line bundle over \( M \) whose restriction to \( N^n \) is trivial, to \( N^{n-1} \times RP(1) \) is the pullback of the canonical bundle over \( RP(1) \) and to \( N^{n-i} \times RP(1, 0, \cdots, 0) \) is the pullback of \( \lambda \). Then \( \alpha = [M, \xi] \).

Since \( N^n \) fibers over \( S^1 \), the pullback of \( \tau_{S^1} \) is a trivial line bundle in \( \tau_{N^n} \). Since \( N^{n-1} \times RP(1) \) fibers over \( S^1 \times S^1 \), its tangent bundle contains a trivial 2 plane-bundle, but if \( \xi' \) is the canonical bundle over \( RP(1) \), \( 2\xi' = 2 \) so the tangent bundle contains two copies of the pullback of \( \xi' \). As noted, \( \lambda \) is a subbundle of the tangent bundle of \( RP(1, 0, \cdots, 0) \) \((i - 1 \ 0\text{'s})\) if \( i > 1 \).

Thus \( \xi \) is a subbundle of the tangent bundle of \( M \). □

Combining this with Lemma 3.1 gives the second proposition of the introduction.
Proposition 3.5. A class $\alpha \in \Omega_n$ is represented by an oriented manifold $M^n$ whose tangent bundle contains a line bundle if and only if the Stiefel-Whitney number $w_n(\alpha)$ is zero.

A class $\alpha \in \Omega_n(RP(\infty))$ is represented by a pair $[M^n, \xi]$ where $\xi$ is a sub-line-bundle of the tangent bundle of the oriented manifold $M$ if and only if the Stiefel-Whitney number $\theta(\alpha)$ is zero.

Proof. These conditions are clearly necessary. To see that they are sufficient, consider $\alpha \in \Omega_n$ for which $w_n(\alpha) = 0$ and choose a representative $M^n$ for $\alpha$. Using surgery, one may replace $M$ by the connected sum of its components; i.e., may assume $M$ connected. If $n$ is odd, the tangent bundle has a nonvanishing section, while if $n$ is even, such a section exists if and only if the Euler class of the tangent bundle $X(\tau)$ is zero. Since $M$ is connected, $X(\tau) = \chi(M)\sigma$, where $\chi(M)$ is the Euler characteristic of $M$ and $\sigma$ is a generator of $H^n(M; \mathbb{Z}) \cong \mathbb{Z}$. Mod 2, $\chi(M) = w_n(\alpha)$ so $\chi(M)$ is even, and by forming the connected sum of $M$ with copies of $S^p \times S^q$ for suitable $p, q > 0$, one obtains a new $M$ with $\chi(M) = 0$ also in $\alpha$. [Note. If $n = 2$, $\alpha = 0$ and $M$ may be taken empty or $S^1 \times S^1$ while if $n = 2k, k > 1$, the connected sum with $S^2 \times S^{n-2}$ increases $\chi$ by 2 while that with $S^1 \times S^{n-1}$ decreases it by 2.] Thus every $\alpha \in \Omega_n$ with $w_n(\alpha) = 0$ is represented by a manifold $M^n$ for which $\tau_M$ contains a trivial line bundle.

Now turning to $\Omega^*(RP(\infty))$, one has $\Omega^*(RP(\infty)) \cong \Omega^* \oplus \Omega^*_1(RP(\infty))$ and $\Omega^*_1(RP(\infty)) \cong \Omega^*_1$. A class in the $\Omega_n$ summand of $\Omega_n(RP(\infty))$ is represented by a manifold $M^n$ with trivial line bundle, and $\theta([M, 1]) = \langle w_n(\tau), [M] \rangle$ so that by the above, a class $\alpha$ in the $\Omega_*$ summand is represented by a subbundle if and only if $\theta(\alpha) = 0$. The summand $\Omega_{n-1}$ of $\Omega_n(RP(\infty))$ is realized as follows. If $\beta \in \Omega_{n-1}$, let $N^{n-1}$ be a manifold in $\beta$ and let $M^n$ be the real projective space bundle $RP(\xi \oplus 1)$ where $\xi$ is the determinant bundle of the tangent bundle of $N$ and let $\lambda$ be the canonical line bundle over $RP(\xi \oplus 1)$. Assigning to $\beta$ the class of $[M, \lambda]$ gives the isomorphism $\Omega_{n-1} \cong \Omega_n(RP(\infty))$. Now $\theta([M, \lambda]) = w_{n-1}(\beta)$, and if $\theta([M, \lambda]) = 0$, $\beta$ is represented by a manifold $N$ whose tangent bundle has a section and so $\lambda$ is a subbundle of the tangent bundle of $RP(\xi \oplus 1)$. Noting that $\theta$ vanishes on the $\Omega_*$ summand if $n$ is odd and on the $\Omega_1(RP(\infty))$ summand if $n$ is even, one sees that every class in the kernel of $\theta$ is realized by a subbundle of the tangent bundle.

4. Stabilization. One now considers stabilization of the subbundle problem. This permits the use of homotopy theoretic techniques.

One may consider a manifold $M^n$ together with an isomorphism $\tau_M \oplus$
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The manifold $M^n$ with this structure bounds if $M = \partial V$ where $\tau_V \oplus (j - 1) \cong \rho^k \oplus \sigma^{n-k+1} \oplus (j - 1)$ is a compatible decomposition; i.e., $\rho$ restricts to $\xi$ and $\sigma$ to $\eta \oplus 1$. Assuming $V$ has no closed components, $V$ has the homotopy type of an $n$-dimensional complex, so $\tau_V \cong \rho \oplus \sigma$, but this need not be compatible with the chosen isomorphism along $M$ unless $j > 1$.

Let $\phi^k_r: BO_k \times BO_r \to BO$ be a map classifying the complement of the Whitney sum $\gamma_k \oplus \gamma_r$ of the universal bundles (converted to a fibration). The structure on $M$ is precisely a lift of the normal map of $M$ to $BO_k \times BO_{n-k}$, while that of $V$ is a lift to $BO_k \times BO_{n-k+1}$.

The techniques of bordism of manifolds with normal structure [3] give that the bordism group of manifolds $M^n$ of the given type is the image of the stable homotopy homomorphism

$$\pi^S_n(T(BO_k \times BO_{n-k})) \to \pi^S_n(T(BO_k \times BO_{n-k+1}))$$

where $T(BO_k \times BO_r)$ is the Thom spectrum associated with the fibration $\phi^k_r$.

Specifically, if one takes the induced fibration

$$\begin{array}{ccc}
E & \to & BO_k \times BO_r \\
\pi & \downarrow & \downarrow \phi^k_r \\
BO_s & \to & BO
\end{array}$$

then $\pi^S_n(T(BO_k \times BO_r)) = \lim_{s \to \infty} \pi_{n+s}(T(\pi^*(\gamma_s)))$. One may also describe these groups as

$$\pi^S_n(T(BO_k \times BO_r)) = \lim_{s,t \to \infty} \pi_{n+s+t}(T(\gamma_s \oplus \gamma_t))$$

where $\gamma_s$, $\gamma_t$ are the universal $s$ and $t$ plane bundles over the Grassmann manifolds $G_{k,s}$ and $G_{r,t}$.

One may now consider the homomorphism

$$\pi^S_n(T(BO_k \times BO_{n-k})) \to \lim_{r \to \infty} \pi^S_n(T(BO_k \times BO_r))$$

One has $\pi_1 \times \Theta: BO_k \times BO \to BO_k \times BO$, which is a homotopy equivalence, and induces an equivalence $T(BO_k \times BO) \cong BO_k^+ \wedge MO$ and hence

$$\pi^S_n(T(BO_k \times BO)) \cong \mathcal{R}_n(BO_k).$$
This describes the forgetful homomorphism assigning to $M^n$ with its structure the bordism class of $(M, \xi)$.

One now embarks on a program of analyzing the stable homotopy groups involved.

**Lemma 4.1.** Let $\gamma_s$ be the universal $s$ plane bundle over $G_{r,s}$, $s > r$, and let $p$ be an odd prime. Then $\tilde{H}^i(T(\gamma_s); \mathbb{Z}_p) = 0$ for $i < r + s$.

**Proof.** One has the inclusion $G_{r,s} \subset G_{r+1,s}$ with $G_{r+1,s}$ obtained by attaching cells of dimension $(r + 1)$ and higher. This induces an inclusion of Thom spaces $T(\gamma_s|G_{r,s}) \subset T(\gamma_s|G_{r+1,s})$ and the cofiber has cells of dimension $r + 1 + s$ and higher. From the exact cohomology sequence

$$
\tilde{H}^i(T(\gamma_s|G_{r,s}); \mathbb{Z}_p) \cong \tilde{H}^i(T(\gamma_s|G_{r+1,s}); \mathbb{Z}_p) \quad \text{if} \quad i < r + s.
$$

Thus

$$
\tilde{H}^i(T(\gamma_s|G_{r,s}); \mathbb{Z}_p) \cong \tilde{H}^i(T(\gamma_s|G_{r+t,s}); \mathbb{Z}_p) \quad \text{if} \quad i < r + s, \ t \geq 0,
$$

but for $t$ large this is $\tilde{H}^i(MO_s; \mathbb{Z}_p)$ which is zero. □

**Lemma 4.2.** $\pi_n^S(T(BO_k \times BO_r))$ is a 2 group if $i < k + r$.

**Proof.** Let $\gamma_s, \gamma_t$ be the universal bundles over $G_{k,s}$ and $G_{r,t}$, $s$ and $t$ large. Then $T(\gamma_s \oplus \gamma_t) = T(\gamma_s) \wedge T(\gamma_t)$ and $\tilde{H}^i(T(\gamma_s \oplus \gamma_t); \mathbb{Z}_p) = 0$ if $i < k + r + s + t$ if $p$ is odd. By the mod $C$ Hurewicz theorem $\pi_i(T(\gamma_s \oplus \gamma_t))$ is a 2 group if $i < k + r + s + t$. □

Thus, for $r \geq n - k + 1$, $\pi_n^S(T(BO_k \times BO_r))$ is a 2 group, and the problem is entirely a 2 primary problem.

In order to begin the 2 primary analysis, one analyzes a cofibration of spectra

$$
T(BO_k \times BO_r) \to T(BO_k \times BO_{r+1}) \to X
$$

which one realizes by a cofibration $T(\gamma_s \oplus \gamma_t) \to T(\gamma_s \oplus \gamma'_t) \to X$ where $\gamma_s, \gamma_t$ are universal bundles over $G_{k,s}$, $G_{r,t}$, and $\gamma'_t$ is the universal bundle over $G_{r+1,t}$, with $s$ and $t$ being large.

First, consider $G_{r+1,t}$ as the space of $r + 1$ planes in $R^{r+1+t}$ with $\pi: D(\gamma_{r+1}) \to G_{r+1,t}$ the projection of the disc bundle. Letting $S(\gamma_{r+1})$ be the unit sphere bundle, one has a cofibration

$$
\begin{array}{ccc}
\frac{D(\pi^*(\gamma'_t)|S(\gamma_{r+1}))}{S(\pi^*(\gamma'_t)|S(\gamma_{r+1}))} & \to & \frac{D(\pi^*(\gamma'_t))}{S(\pi^*(\gamma'_t))} \\
\| & & \| \\
\frac{D(\pi^*(\gamma'_t)|S(\gamma_{r+1}))}{S(\pi^*(\gamma'_t)|S(\gamma_{r+1})) \cup S(\pi^*(\gamma'_t))} & \to & B \\
\| & & \|
\end{array}
$$

where $A = C$. 
Since $D(\pi^*(\gamma'_t))$ is identifiable with $D(\gamma_{r+1} \oplus \gamma'_t)$, $C$ is the Thom space of the trivial bundle $\gamma_{r+1} \oplus \gamma'_t$, and $C \cong \Sigma^{r+t+1}(G_{r+1, t}^*)$ is the $(r + t + 1)$-fold suspension of $G_{r+1, t}$ with a base point adjoined. Since $\pi$ is a homotopy equivalence, $B \cong T(\gamma'_t)$.

Finally, $S(\gamma_{r+1})$ may be considered as pairs $(\alpha, x)$ with $\alpha$ an $(r + 1)$-plane in $R^{r+1+t}$ and $x$ a unit vector in $\alpha$. Assigning to $(\alpha, x)$ the point $x \in S^{r+t}$ defines a fibration $p : S(\gamma_{r+1}) \to S^{r+t}$. The inverse image of $x \in S^{r+t}$ is the space of $r$ planes in $R^{r+1+t}$ orthogonal to $x$, i.e., $S(\gamma_{r+1})$ is the Grassmann bundle of $r$ planes in the fibers of the tangent bundle of $S^{r+t}$.

The inclusion $G_{r, t} \to G_{r+1, t}$ may then be considered as factoring via the inclusion as a fiber in $S(\gamma_{r+1})$. The inclusion of the fiber $G_{r, t} \to S(\gamma_{r+1})$ induces isomorphisms in homotopy and homology in dimensions less than $r + t - 1$, and so the inclusion $T(\gamma_t) \to A$ is a homotopy equivalence (for the prime 2) in dimensions less than $r + 2t - 1$. Since $t$ is large, one then obtains a cofibration

$$T(\gamma_t) \to T(\gamma'_t) \to \Sigma^{r+t+1}(G_{r+1, t}^*).$$

Smashing with $T(\gamma_s)$ gives a cofibration sequence

$$T(\gamma_{s} \oplus \gamma_t) \to T(\gamma_s \oplus \gamma'_t) \to T(\gamma_s) \wedge \Sigma^{r+t+1}(G_{r+1, t}^*)$$

(i.e., $X$ may be identified with $T(\gamma_s) \wedge \Sigma^{r+t+1}(G_{r+1, t}^*)$ for the prime 2, having isomorphic mod 2 cohomology up to dimension $s + r + 2t - 1$ induced by a map of spaces).

One now considers $T(\gamma_s) \wedge \Sigma^{r+t+1}(G_{r+1, t}^*)$ as $\Sigma^{r+t+1} T(\gamma_s) \wedge G_{r+1, t}^*$ and analyzes the maps

$$G_{k, s} \to G_{k, s+r+t+1} \to G_{m, s+r+t+1}$$

inducing

$$\Sigma^{r+t+1} T(\gamma_s) \to T(\gamma_{s+r+t+1}) \to MO_{s+r+t+1}$$

($m$ being large). The maps of Grassmannians induce isomorphisms in mod 2 cohomology in dimensions less than or equal to $k$ and hence the Thom spaces have isomorphic mod 2 cohomology in dimensions less than or equal to $k + s + r + t + 1$.

Thus $X$ may be identified with $MO_{s+r+t+1} \wedge (G_{r+1, t})^+$ in dimensions less than or equal to $k + s + r + t + 1$ (in mod 2 cohomology). In particular, in dimensions less than or equal to $k + s + r + t + 1$, $\widetilde{H}^*(X; \mathbb{Z}_2)$ is a free module over the Steenrod algebra and

$$\pi_{t+s+t}(X) \cong \pi_{t+s+t}(MO_{s+r+t+1} \wedge (G_{r+1, t})^+) \cong \mathcal{H}_{t-r-1}(G_{r+1, t})$$
if \( i + s + t \leq k + s + r + t, i \leq k + r \) (for 2 primary structure).

Being given a manifold \( M^i \) with \( \tau_M \oplus j \cong \xi^k \oplus \eta^{r+1} \oplus (i + j - k - r - 1) \) representing a class in \( \pi_i^S(T(BO_k \times BO_{r+1})) \), \( i \leq k + r \), the class in \( \pi_i^S(X) \cong \mathcal{N}_{i-r-1}(BO_{r+1}) \) obtained from the cofibration is represented by the submanifold of \( M^i \) dual to \( \eta^{r+1} \) with the \((r + 1)\)-plane bundle obtained by restricting \( \eta \). The map to \( X \) is induced by including \( T(\gamma^1_i) \) in \( T(\gamma^1_i \oplus \gamma_{r+1}) \) and making the maps transverse regular involves finding the submanifold dual to \( \gamma_{r+1} \), from which one has the given assertion.

On the other hand, a class in \( \pi_i^S(T(BO_k \times BO_{r+1})) \), \( i \leq k + r \), is in the image of \( \pi_i^S(T(BO_k \times BO_r)) \) if and only if it goes to zero in \( \pi_i^S(X) \). Since \( \tilde{H}^*(X; \mathbb{Z}_2) \) is a free module over the Steenrod algebra in dimensions up to \( k + s + r + t + 1 \), a homotopy element in \( \pi_{i+s+t}(X) \) is detected by mod 2 cohomology. Since \( T(\gamma_s \oplus \gamma^1_i) \rightarrow X \) maps \( \tilde{H}^*(X; \mathbb{Z}_2) \) isomorphically onto the multiples of \( \Phi(w_{r+1}) \), the Thom isomorphism image of \( w_{r+1} \), in the \( H^*(G_{k,s} \times G_{r+1,s}; \mathbb{Z}_2) \) module structure, this asserts that all characteristic numbers involving \( w_{r+1} \) should vanish. Thus, one has

**Lemma 4.3.** A manifold \( M^i \) with \( \tau_M \oplus j \cong \xi^k \oplus \eta^{r+1} \oplus (i + j - k - r - 1) \) representing a class in \( \pi_i^S(T(BO_k \times BO_{r+1})) \), \( i \leq k + r \), comes from \( \pi_i^S(T(BO_k \times BO_r)) \) if and only if all characteristic numbers involving \( w_{r+1} \) are zero.

For \( r \geq n - k \), this determines the image of

\[
\pi_n^S(T(BO_k \times BO_r)) \rightarrow \pi_n^S(T(BO_k \times BO_{r+1})).
\]

For \( r \geq n - k + 1 \), this homomorphism is monic, which may be seen as follows. Consider the homomorphism

\[
\pi_{n+1}^S(T(BO_k \times BO_{r+1})) \rightarrow \pi_{n+1}^S(X).
\]

Now \( \pi_{n+1}^S(X) \cong \mathcal{N}_{n-r}(BO_{r+1}) \) for \( n + 1 \leq k + r \), and \( \mathcal{N}_{n-r}(BO_{r+1}) \) is generated over \( \mathbb{Z}_2 \) by the manifolds

\[
P = M^m \times RP(\lambda_1 \oplus k_1) \times \cdots \times RP(\lambda_s \oplus k_s) \times \text{(point)}
\]

where \( \lambda_i \) is the nontrivial bundle over \( RP(1) \), \( k_i \geq 0 \), with \( m + (k_1 + 1) + \cdots + (k_s + 1) = n - r \) with bundle

\[
\lambda^{(1)} \oplus \cdots \oplus \lambda^{(s)} \oplus (r + 1 - s)
\]

where \( \lambda^{(i)} \) is the canonical bundle over \( RP(\lambda_i \oplus k_i) \). To see this, one notes that the \( RP(\lambda \oplus k) \), \( k > 0 \), and the point generate \( \mathcal{N}_*(BO_1) \), over \( \mathcal{N}_* \) and forming the products of \( r + 1 \) of these gives a \( \mathcal{N}_* \) generating set for \( \mathcal{N}_*(BO_{r+1}) \).
One then considers the manifold

$$Q = M^m \times RP(\lambda_1 \oplus k_1 \oplus 1) \times \cdots \times RP(\lambda_s \oplus k_s \oplus 1) \times RP(r + 1 - s)$$

of dimension $m + (k_1 + 2) + \cdots + (k_s + 2) + r + 1 - s = n - r + s + r + 1 - s = n + 1$ and letting $\lambda$ be the canonical line bundle over $RP(r + 1 - s)$, the submanifold dual to $\lambda^{(1)} \oplus \cdots \oplus \lambda^{(s)} \oplus (r + 1 - s)\lambda = \eta^{r+1}$ is the manifold $P$ given above, with $\eta$ restricting to the given bundle. Now the tangent bundle of $RP(\lambda_t \oplus k_t \oplus 1)$ is $\lambda^{(t)} \otimes \lambda_t \otimes (k_t + 1)\lambda^{(t)}$ so

$$\tau_Q \oplus 1 \cong [\tau_M \oplus (\lambda^{(1)} \oplus \lambda_t \oplus k_t\lambda^{(1)}) \oplus \cdots \oplus (\lambda^{(s)} \oplus \lambda_s \oplus k_s\lambda^{(s)}) \oplus \lambda] \oplus \eta$$

where $\xi'$ is an $m + (k_1 + 1) + \cdots + (k_s + 1) + 1 = n - r + 1 < k$ bundle. Thus $\tau_Q \oplus 1 \oplus (k + r - n - 1) \cong [\xi' \oplus (k + r - 1)] \oplus \eta = \xi k \oplus \eta$ giving a structure on $Q$ mapping to the class of $P$ in $\mathcal{N}_{n-r}(BO_{r+1})$.

This proves that the forgetful homomorphism $\pi_n^S(T(BO_k \times BO_r)) \to \mathcal{N}_n(BO_k)$ is monic for $r \geq n - k + 1$, and that

$$\text{im}\{\pi_n^S(T(BO_k \times BO_{n-k})) \to \pi_n^S(T(BO_k \times BO_{n-k+1}))\}$$

is mapped monomorphically into $\mathcal{N}_n(BO_k)$ with image precisely those classes for which all numbers involving $w_i(r - f^*(\gamma_k))$ for $i > n - k$ are zero, or one has

**Proposition 4.4.** A class $\alpha = [M, f] \in \mathcal{N}_n(BO_k)$ is represented by a manifold $M^n$ with $\tau_M \oplus 1 \cong f^*(\gamma_k) \oplus \eta^{n-k} \oplus 1$ if and only if all Stiefel-Whitney numbers of $\alpha$ involving $w_i(r - f^*(\gamma_k))$ for $i > n - k$ are zero.

**5. Two plane bundles.** The purpose of this section is to prove

**Proposition 5.1.** A class $\alpha = [M, f] \in \mathcal{N}_n(BO_2)$ is represented by a pair $[M, f]$ with $f^*(\gamma_2)$ a subbundle of the tangent bundle of $M$ if and only if all characteristic numbers of $\alpha$ involving $w_i(r - f^*(\gamma_2))$ with $i > n - 2$ are zero.

To begin the proof, one wants manifolds $M_{i,j}$ of dimension $i + 2j$ for each $(i, j)$ and 2 plane bundles $\lambda_{i,j}$ over $M_{i,j}$ for which

$$w^p_1(\lambda_{i,j})w^q_2(\lambda_{i,j})[M_{i,j}] = \begin{cases} 0 & \text{if } q > j, \quad p + 2q = i + 2j, \\ 1 & \text{if } q = j, \quad p = i. \end{cases}$$
Any collection of such manifolds form a base for $\mathcal{N}_*(BO_2)$ as $\mathcal{N}_*$ module. The representatives will be chosen so that $\lambda_{i,j}$ is a subbundle of the tangent bundle of $M_{i,j}$ except for $j = 0$ and $i \leq 3$.

For $j \geq 2$, one lets

$$M_{i,j} = \text{RP}(1, 0, \ldots, 0) \times \text{RP}(1, 0, \ldots, 0)_{j-1} \times \text{RP}(1, 0, \ldots, 0)_{i+j-1}$$

and lets $\lambda_{i,j} = \pi_1^*(\lambda) \oplus \pi_2^*(\lambda)$, where $\lambda$ is the canonical line bundle over $\text{RP}(1, 0, \ldots, 0)$.

For $j = 1, i \geq 3$, one lets $M_{i,j} = \text{RP}(1) \times \text{RP}(3, 0, \ldots, 0) (i - 2 \text{ 0's})$ and lets $\lambda_{i,j} = \pi_1^*(\xi) \oplus \pi_2^*(\lambda), \xi$ being the Hopf bundle over $\text{RP}(1)$. The tangent bundle of $\text{RP}(3)$ is trivial and so the tangent bundle of $M_{i,j}$ is $3 \oplus \pi_2^*(\lambda) \otimes (\pi_2^*(\xi') \oplus (i - 2)), where \xi'$ is the Hopf bundle over $\text{RP}(3)$. Since $2\xi = 2$ and $i \geq 3$, $\lambda_{i,j}$ is a subbundle of the tangent bundle.

For $j = 0, i \geq 4$, one lets $M_{i,j} = \text{RP}(3, 0, \ldots, 0) (i - 3 \text{ 0's})$ and $\lambda_{i,j} = 1 \oplus \lambda$.

For $j = 1, i = 0$, one lets $M_{i,j} = \text{RP}(2)$, and $\lambda_{i,j} = \tau$, the tangent bundle of $\text{RP}(2)$.

For $j = 1, i = 1$, one lets $M_{i,j} = \text{RP}(1) \times \text{RP}(2)$, the tangent bundle being $1 \oplus \pi_2^*(\tau) = 3\pi_2^*(\xi) = (2\pi_1^*(\xi) \otimes \pi_2^*(\xi)) + \pi_2^*(\xi)$ and lets $\lambda_{i,j} = [\pi_1^*(\xi) \otimes \pi_2^*(\xi)] \oplus \pi_2^*(\xi)$.

For $j = 1, i = 2$, let $M_{i,j}$ be the bundle of lines in the fibers of $\lambda \oplus 2$ over $\text{RP}(1, 0) = \text{RP}(\xi \oplus 1)$ where $\xi$ is the Hopf bundle over $\text{RP}(1)$, giving projections

$$\pi: M_{i,j} \to \text{RP}(1, 0), \quad p: \text{RP}(1, 0) \to \text{RP}(1).$$

Let $\theta$ be the bundle along the fibers of $p$, $\eta$ the bundle along the fibers of $\pi$, and $\lambda'$ the canonical line bundle over $M_{i,j}$. Then

$$\tau_{M_{i,j}} = \eta \oplus \pi^*(\tau_{\text{RP}(1, 0)}) = \eta \oplus \pi^*(\theta) \oplus 1 = (\lambda' \otimes \pi^*(\lambda + 2)) \oplus \pi^*(\theta)$$

which contains a copy of $\lambda_{i,j} = \lambda' \oplus \pi^*(\theta)$.

Finally, let $M_{0,0}$ be a point with $\lambda_{0,0}$ trivial, $M_{1,0} = \text{RP}(1)$ with $\lambda_{1,0} = \xi \oplus 1$, and $M_{2,0} = \text{RP}(1, 0), M_{3,0} = \text{RP}(1, 0, 0)$ with $\lambda_{i,0} = \lambda \oplus 1$.

Note that for $M_{i,0}, i \leq 3, \lambda_{i,0}$ is a subbundle of $\tau \oplus 2$. In particular, if $\alpha \in \mathcal{N}_p$ and $w_p(\alpha) = 0, \alpha$ is represented by a manifold $M^p$ fibered over $S^1 \times S^1$ [5, Proposition 6.1] and hence $\tau_M$ has 2 sections, so $\lambda_{i,0}$ is a subbundle of the tangent bundle of $M \times M_{i,0}$.

Every class in $\mathcal{N}_* (BO_2)$ is of the form $\Sigma \alpha_{(i,j)} [M_{i,j}, \lambda_{i,j}]$ with $\alpha_{(i,j)} \in \mathcal{N}_{n-i-2j}$ and every class $\alpha \in \mathcal{N}_p$ has the form $\beta + aR^2(p/2), a \in \mathbb{Z}_2, \beta \in \mathcal{N}_p$ with $w_p(\beta) = 0$. Thus if $I$ is the $\mathcal{N}_*$ submodule of classes in $\mathcal{N}_* (BO_2)$.
represented by $[M, f]$ with $f^*(\gamma_2)$ a subbundle of the tangent bundle of $M$, then $\mathcal{N}_*(BO_2)/I$ is a $\mathbb{Z}_2$ vector space generated by the classes

$$[\mathbb{R}P(2)^8] \cdot [M_{i,0}, \lambda_{i,0}] \quad \text{with} \quad i \leq 3.$$  

The characteristic numbers $w_n(r - f^*(\gamma_2))$ and $w_1(r - f^*(\gamma_2)) \cdot w_{n-1}(r - f^*(\gamma_2))$ (for $n \geq 2$) may be readily seen to give a homomorphism $\mathcal{N}_n(BO_2)/I \to \mathbb{Z}_2 \oplus \mathbb{Z}_2$ (or to $\mathbb{Z}_2$ if $n \leq 1$) sending the classes $[\mathbb{R}P(2)^8] \cdot [M_{i,0}; \lambda_{i,0}]$ of dimension $n$ to linearly independent elements.

This completes the proof of the proposition.

REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VIRGINIA, CHARLOTTESVILLE, VIRGINIA 22903