THE $p$-CLASS IN A DUAL $B^*$-ALGEBRA

BY

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ABSTRACT. In this paper, we introduce and study the class $A_p$ ($0 < p < \infty$) in a dual $B^*$-algebra $A$. We show that, for $1 < p < \infty$, $A_p$ is a dual $A^*$-algebra which is a dense two-sided ideal of $A$. If $1 < p < \infty$, we obtain that $A_p$ is uniformly convex and hence reflexive. We also identify the conjugate space of $A_p$ ($1 < p < \infty$). This is a generalization of the class $C_p$ of compact operators on a Hilbert space.

1. Introduction. Let $H$ be a Hilbert space and $LC(H)$ the algebra of all compact operators on $H$. Then $LC(H)$ is a simple dual $B^*$-algebra and every simple dual $B^*$-algebra is of this form. The class $C_p$ of compact operators in $LC(H)$ has many interesting properties and has been studied in various articles (e.g., see [2], [3] and [4]). The present work is an attempt to introduce a similar class of spaces in an arbitrary dual $B^*$-algebra.

Let $A$ be a dual $B^*$-algebra. The class $A_p$ ($0 < p < \infty$) is introduced in §3. After establishing some crucial inequalities, we show that $A_p$ ($1 < p < \infty$) is a dual $A^*$-algebra which is a dense two-sided ideal of $A$. In §4, we study the algebras $A_1$ and $A_2$. We obtain that every proper $H^*$-algebra is of the form $A_2$ and $A_1 = \{xy: x, y \in A_2\}$. §5 is devoted to showing the uniform convexity in $A_p$ ($1 < p < \infty$). Finally we identify the conjugate space of $A_p$ ($1 < p < \infty$) in §6.

In this paper, our approach is elementary and the techniques are not new. In fact, they are borrowed from [3], [4], [10] and [11]. The author is grateful for these invaluable references.

2. Notation and preliminaries. Definitions not explicitly given are taken from Rickart's book [7].

For any set $E$ in a Banach algebra $A$, let $l(E)$ and $r(E)$ denote the left and right annihilators of $E$, respectively. Then $A$ is called a dual algebra.
if for every closed right ideal \( R \) and every closed left ideal \( I \), we have \( r(I(R)) = R \) and \( l(r(I)) = I \). See [5] and [7] for some of its properties.

An idempotent \( e \) in a Banach algebra \( A \) is said to be minimal if \( eAe \) is a division algebra. In case \( A \) is semisimple, this is equivalent to saying that \( Ae \) (\( eA \)) is a minimal left (right) ideal of \( A \).

Let \( A \) be a Banach algebra. A bounded linear operator \( S \) on \( A \) is called a right centralizer if \( S(xy) = (Sx)y \) for all \( x, y \) in \( A \). For each \( a \) in \( A \), the operator \( L_{a}: x \to ax \ (x \in A) \) is a right centralizer on \( A \).

Let \( H \) be a Hilbert space with an inner product \((\cdot, \cdot)\). If \( x \) and \( y \) are elements in \( H \), then \( x \otimes y \) will denote the operator on \( H \) defined by \((x \otimes y)(h) = (h, y)x \) for all \( h \) in \( H \).

In this paper, all algebras and linear spaces under consideration are over the field of complex numbers.

**NOTATION.** In this paper, \( A \) will denote a dual \( B^* \)-algebra with norm \( \| \cdot \| \).

We shall oftter use, without explicitly mentioning, the following fact: For any orthogonal family \( \{e_{\alpha}\} \) of hermitian idempotents of \( A \), \( \Sigma_{\alpha} e_{\alpha}x \) is summable in \( A \), and especially when \( \{e_{\alpha}\} \) is a maximal family, \( x = \Sigma_{\alpha} e_{\alpha}x \) for all \( x \) in \( A \) (see [5, p. 30, Theorem 16]).

Let \( B \) be a closed commutative \( \ast \)-subalgebra of \( A \) and \( e \) a minimal idempotent in \( B \). It follows easily from [7, p. 261, Lemma (4.10.1)] that \( e \) is hermitian. Also if \( f \) is any other minimal idempotent in \( B \), then \( fe = ef = 0 \). If \( B \) is maximal, then \( e \) is a minimal idempotent in \( A \).

**LEMMA 2.1.** Let \( e \) be a hermitian minimal idempotent in \( A \), \( a \in A \), and \( \{f_{\beta}\} \) a maximal orthogonal family of hermitian minimal idempotents in \( A \). Then \( \|ae\|^2 = \Sigma_{\beta} \|f_{\beta}ae\|^2 \).

**PROOF.** Since \( A \) is a dual \( B^* \)-algebra, it follows from [7, p. 259, Theorem (4.9.24)] and [7, p. 269, Corollary (4.10.20)] that \( A = (\Sigma_{\lambda} LC(H_{\lambda}))_{\ast} \), where \( LC(H_{\lambda}) \) is the algebra of all compact operators on a Hilbert space \( H_{\lambda} \). It is easy to see that \( e \in LC(H_{\lambda_0}) \) for some \( \lambda_0 \). Let \( \{f_{\gamma}\} = \{f_{\beta}\} \cap LC(H_{\lambda_0}) \). Then we can write \( f_{\gamma} = x_{\gamma} \otimes x_{\gamma} \) with \( x_{\gamma} \in H_{\lambda_0} \) and \( \|x_{\gamma}\| = 1 \). Similarly \( e = y \otimes y \) with \( y \in H_{\lambda_0} \) and \( \|y\| = 1 \). Since \( \{f_{\gamma}\} \) is a maximal orthogonal family of hermitian minimal idempotents in \( LC(H_{\lambda_0}) \), it follows easily that \( \{x_{\gamma}\} \) is a complete orthonormal set in \( H_{\lambda_0} \). Put \( b = ae \). Then \( b \in LC(H_{\lambda_0}) \) and \( be = ae \). Hence

\[
\|ae\|^2 = \|eb^\ast be\| = \|(y \otimes y)b^\ast b(y \otimes y)\| = \|by\|^2.
\]

Similarly \( \|f_{\gamma}be\| = \|(by, x_{\gamma})\| \). Since \( f_{\beta}ae = 0 \) if \( f_{\beta} \not\in \{f_{\gamma}\} \), by Parseval's identity we have
\[ \sum_{\beta} ||f_{\beta}ae||^2 = \sum_{\gamma} ||f_{\gamma}be||^2 = \sum_{\gamma} ||b(y, x_{\gamma})||^2 = ||by||^2 = ||ae||^2. \]

This completes the proof.

The following lemma is useful in this paper and it is similar to [10, p. 29, Lemma 1].

**Lemma 2.2.** Let \( a \in A \) and \( \{e_{\alpha}\}, \{f_{\beta}\} \) any two maximal orthogonal families of hermitian minimal idempotents in \( A \). Then

\[ \sum_{\alpha} ||ae_{\alpha}||^2 = \sum_{\beta} ||af_{\beta}||^2 = \sum_{\beta} ||af_{\beta}||^2. \]

**Proof.** We note first that \( ||f_{\beta}ae_{\alpha}|| = ||e_{\alpha}a^*f_{\beta}|| \). If \( \Sigma_{\alpha} ||ae_{\alpha}||^2 \) is summable, then by Lemma 2.1, we have

\[ \Sigma_{\alpha} ||ae_{\alpha}||^2 = \Sigma_{\beta} ||af_{\beta}||^2 = \Sigma_{\beta} ||af_{\beta}||^2. \]

Hence, in particular, \( \Sigma_{\beta} ||af_{\beta}||^2 = \Sigma_{\beta} ||af_{\beta}||^2 \). The lemma now follows from (2.1).

Suppose \( b \) is a normal element in \( A \). Let \( B \) (resp. \( B' \)) be a maximal commutative \( * \)-subalgebra of \( A \) containing \( b \) and \( \{e_{\alpha}\} \) (resp. \( \{e_{\omega}\} \)) the maximal orthogonal family of hermitian minimal idempotents in \( B \) (resp. \( B' \)). Then \( be_{\alpha} = e_{\alpha}be_{\alpha} = k_{\alpha}e_{\alpha} \) for some constant \( k_{\alpha} \). Similarly \( be_{\omega} = k_{\omega}e_{\omega} \) for some constant \( k_{\omega} \). Let \( K \) (resp. \( K' \)) be the set of all nonzero \( k_{\alpha} \) (resp. \( k_{\omega} \)).

We note that \( k_{\alpha_1} \) may be equal to \( k_{\alpha_2} \) for some \( \alpha_1 \neq \alpha_2 \). However we consider them as different elements in \( K \).

**Lemma 2.3.** The set \( K \) is either finite or countable and \( K = K' \). The set of all distinct constants in \( K \) is precisely the set of all nonzero constants in the spectrum of \( b \).

**Proof.** Let \( B_0 \) be the intersection of all maximal commutative \( * \)-subalgebras of \( A \) containing \( b \). Let \( \{f_{\beta}\} \) be the maximal orthogonal family of hermitian minimal idempotents in \( B_0 \). Since \( B_0 \) is a dual \( B^* \)-algebra, \( b = \Sigma_{\beta} b f_{\beta} = \Sigma_{\beta} \lambda_{\beta} f_{\beta} \), where \( \lambda_{\beta} \) are constants. Therefore there exists only a countable number of \( f_{\beta} \) for which \( b f_{\beta} \neq 0 \). Also, for each nonzero \( \lambda_{\beta_0} \), the set \( \{\lambda_{\beta} : \lambda_{\beta} = \lambda_{\beta_0}\} \) is finite. It is now easy for us to write \( b = \Sigma_{n=1}^{\infty} \lambda_n f_n \), where \( \lambda_n \) are distinct nonzero constants and \( \{f_n\} \) is an orthogonal family of hermitian idempotents in \( B_0 \) such that \( \lambda_n f_n = b f_n \). Note that \( f_n \) is not necessarily minimal. Since \( B \) is dual and \( f_n \in B \), it is well known that

\[ f_n = e_{\alpha_{n_1}} + \cdots + e_{\alpha_{n_p}}, \text{ where } e_{\alpha_{n_i}} \in \{e_{\alpha}\} (i = 1, 2, \cdots, p). \]
Considering the right ideal \( f_n A \) of \( A \), by \([1, \text{p. 497, Theorem 2.2}]\), the number \( n_p \) is independent of the choice of \( B \). Since \( bf_n = \lambda_n f_n \), we see easily that

\[
be_{\alpha n_i} = \lambda_n e_{\alpha n_i} \quad (i = 1, 2, \cdots, p).
\]

If \( f_m = e_{\alpha m_1} + \cdots + e_{\alpha m_q} \) \((m \neq n)\), then it follows from (2.2) that

\[
\{e_{\alpha m_1}, \cdots, e_{\alpha m_q}\} \cap \{e_{\alpha n_1}, \cdots, e_{\alpha n_p}\} = \emptyset,
\]

because \( \lambda_n \neq \lambda_m \). Also

\[
b = \sum_n \lambda_n f_n = \sum \lambda_n (e_{\alpha n_1} + \cdots + e_{\alpha n_p}).
\]

Let \( E \) be the set of all such \( e_{\alpha n_p} \). Then \( E \) is countable. For simplicity, we write \( E = \{e_1, e_2, \cdots\} \) and \( b = \sum_{n=1}^{\infty} k_n e_n \), where \( \sum k_n e_n = be_n \) and \( k_n \neq 0 \) (because \( \lambda_n \neq 0 \)). Let \( \{e_{\gamma}\} = \{e_{\alpha}\} - E \). We show that \( b e_{\gamma} = 0 \) for all \( \gamma \).

In fact, since \( b = \sum_{\alpha} b_{e_{\alpha}} = \sum_{\alpha} b_{e_{\alpha}} \), it follows that \( \sum_{\gamma} b e_{\gamma} = 0 \). Let \( F_{\alpha} \) be the multiplicative linear functional on \( B \) corresponding to the maximal modular ideal \( B(1 - e_{\alpha}) \) of \( B \). For any fixed \( \gamma_0 \), we have \( b e_{\gamma_0} = k_{\gamma_0} e_{\gamma_0} \), for some constant \( k_{\gamma_0} \). Then

\[
k_{\gamma_0} = F_{\gamma_0}(be_{\gamma_0}) = \sum_{\gamma} F_{\gamma_0}(be_{\gamma}) = F_{\gamma_0} \left( \sum_{\gamma} be_{\gamma} \right) = 0.
\]

Hence it follows that \( b e_{\gamma} = 0 \) for all \( \gamma \). Consequently \( K = \{k_n\} \). Similarly we can show that \( K' = \{k_n\} \). Therefore \( K = K' \). Now the last part of the lemma follows easily from \([7, \text{p. 111, Theorem (3.1.6)}]\). This completes the proof.

Let \( b, \{e_{\alpha}\} \) and \( \{e_n\} \) be as in the proof of Lemma 2.3. Then

\[
b = \sum_{\alpha} k_{\alpha} e_{\alpha} = \sum_{n} k_n e_n \quad \text{is called a spectral representation of} \ b. \ By \ Lemma 2.3, \ \{k_n\} \ is independent of \ \{e_n\}. \ Also \ if \ k_{\alpha} \neq k_n \ for \ all \ n, \ then \ k_{\alpha} = 0.
\]

Suppose \( a \) is a nonzero element in \( A \). Let \( a^*a = \sum_n r_n e_n \) be a spectral representation of \( a^*a \). We claim that

\[
a = \sum_n a e_n.
\]

In fact, since \( \sum_n ae_n \) is summable and \( a^*a = \sum_n a^*ae_n = \sum_n e_n a^*ae_n = \sum_n r_n e_n \), it follows that \( (a - \sum_n ae_n)^*(a - \sum_n ae_n) = 0 \). Hence \( a = \sum_n ae_n \).

We note that \( ae_n \neq 0 \); for otherwise \( r_n e_n = a^*ae_n = 0 \).

Since \( a^*a \) is a positive element, \( r_n > 0 \) for all \( n \). Put \( k_n = \sqrt{r_n} > 0 \). We show that \( \sum_n k_n e_n \) is summable in \( A \). In fact, for any two positive integers \( m, n \) \((m < n)\), \( \| \sum_{i=m}^{n} k_i e_i \|^2 = \| \sum_{i=m}^{n} r_i e_i \| \). Since \( \sum_n r_n e_n \) is summable, so
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is \( \sum_n k_n e_n \). Put

\[(a) = \sum_n k_n e_n.\]

Then \( (a)^* = (a) \) and \( (a)^2 = a^* a \). Hence \( (a) = (a^* a)^{\frac{1}{2}} \). For each \( x \) in \( A \),

\[
\left\| \sum_{i=m}^{n} k_i^{-1} a e_i x \right\|^2 = \left\| \left( \sum_{i=m}^{n} k_i^{-1} a e_i x \right)^* \left( \sum_{i=m}^{n} k_i^{-1} a e_i x \right) \right\|
\]

\[
= \left\| \sum_{i=m}^{n} x^* e_i x \right\|^2 = \left\| \sum_{i=m}^{n} e_i x \right\|^2 \leq \|x\|^2.
\]

Since \( \sum_n e_n x \) is summable in \( A \), so is \( \sum_n k_n^{-2} a e_n x \). Define a mapping \( W \) on \( A \) by

\[
Wx = \sum_n k_n^{-2} a e_n x \quad (x \in A).
\]

Then it follows from (2.3), (2.4) and (2.5) that \( W(a) = a \) and \( \|W\| = 1 \).

We note that \( a e_n a^* \neq 0 \); for otherwise \( r_n^2 e_n = a^* a e_n a = 0 \). Put \( f_n = k_n^{-2} a e_n a^* \). Since \( (0) \neq f_n A \subset a e_n A \) and \( a e_n \neq 0 \), it follows from [7, p. 45, Lemma (2.1.8)] that \( f_n A = a e_n A \) is a minimal right ideal of \( A \). Hence we see that \( \{f_n\} \) is an orthogonal family of hermitian minimal idempotents in \( A \).

By (2.3), \( aa^* = \sum_n a e_n a^* = \sum_n k_n^2 f_n \) and so it is a spectral representation of \( aa^* \) by the proof of Lemma 2.3. For each \( x \) in \( A \), by a similar argument in (2.5), we have

\[
\left\| \sum_{i=m}^{n} k_i^{-1} e_n a^* x \right\|^2 = \left\| \sum_{i=m}^{n} f_i x \right\|^2 \leq \|x\|^2.
\]

Since \( \sum_n f_n x \) is summable, so is \( \sum_n k_n^{-1} e_n a^* x \). Therefore we can define a mapping \( W^* \) on \( A \) by

\[
W^* x = \sum_n k_n^{-1} e_n a^* x \quad (x \in A).
\]

It follows easily from (2.4) and (2.7) that \( W^* a = [a] \) and \( \|W^*\| = 1 \). Also both \( W \) and \( W^* \) are right centralizers on \( A \). We shall refer to the operator \( W \) as the partial isometry associated with \( a \).

We remark that similar concepts were introduced in [9].

3. The \( p \)-class in \( A \). As before, \( A \) will be a dual \( B^* \)-algebra with norm \( \|\cdot\| \). Suppose \( a \) is a nonzero element in \( A \). Let \( a^* a = \Sigma_n r e_n \) be a spectral representation of \( a^* a \) and \( k_n = \sqrt{r_n} \). Since \( a^* a \) is a positive element in \( A \), \( r_n > 0 \) and so \( k_n > 0 \). We define
(3.1) $$|a|_p = \left( \sum_n k_n^p \right)^{1/p} \quad (0 < p < \infty),$$

$$|a|_\infty = \max \{k_n: n = 1, 2, \cdots \}.$$ 

For $a = 0$, we define $|a|_p = 0 \quad (0 < p \leq \infty)$.

**Remark.** By Lemma 2.3, $|a|_p$ is well defined.

**Definition.** For $0 < p < \infty$, let $A_p = \{a \in A: |a|_p < \infty\}$.

**Remark.** For $0 < p < \infty$, $|a|_p > 0$ and $|a|_p = 0$ if and only if $a = 0$. Also $|ka|_p = |k| |a|_p$ for any constant $k$.

We now have some elementary properties of $|a|_p$.

**Lemma 3.1.** Let $a$ be an element in $A$ and $0 < p \leq \infty$. Then

(i) $||a|| = |a|_\infty \leq |a|_p$. Thus $A_\infty = A$.

(ii) $|a|_p = ||a||_p$. Hence $a \in A_p$ if and only if $[a] \in A_p$.

(iii) If $p \leq q$, then $|a|_p \geq |a|_q$ and so $A_p \subset A_q$.

(iv) If $e$ is a hermitian minimal idempotent in $A$, then $|e|_p = 1$ and so $e \in A_p$.

(v) $|a|_p = |a^*|_p$. Hence $a \in A_p$ if and only if $a^* \in A_p$.

**Proof.** Let $a^*a = \sum_n r_n e_n$ be a spectral representation of $a^*a$ and $[a] = \sum_n k_n e_n$ with $k_n = \sqrt{r_n}$.

(i) This follows from $||a||^2 = ||a^*a||$ and [7, p. 112, Corollary (3.1.7)].

(ii) This follows from $[a] = [[a]] = \sum_n k_ne_n$.

(iii) and (iv). This is clear.

(v) We can assume that $a \neq 0$. Put $f_n = k_n^{-2} a^*e_n a^*$. Then $a a^* = \sum_n k_n f_n$ is a spectral representation of $a a^*$ (see §2). Therefore it follows that $|a^*|_p = |a|_p$. This completes the proof of the lemma.

Let $a$ be a positive element in $A$ and $B_0$ the intersection of all maximal commutative *-subalgebras of $A$ containing $a$. If $\{f_\beta\}$ is the maximal orthogonal family of hermitian minimal idempotents in $B_0$, then $a = \sum_\beta f_\beta = \sum_\beta \lambda_\beta f_\beta$, where $\lambda_\beta$ are nonnegative constants.

**Definition.** For $0 < p < \infty$, we define $a^p = \sum_\beta \lambda_\beta^p f_\beta$.

**Remark.** Let $a = \sum_\alpha k_\alpha e_\alpha = \sum_n k_ne_n$ be a spectral representation of $a$. If $a^p$ exists, then by the proof of Lemma 2.3 $a^p = \sum_\alpha k_\alpha^p e_\alpha = \sum_n k_n^p e_n$ is a spectral representation of $a^p$.

**Lemma 3.2.** Let $a$ be a positive element in $A$ and $0 < p, q < \infty$. If $a^q$ exists, then $|a^q|_{p/q} = |a|_p^q$.

**Proof.** This is clear.
Lemma 3.3. Let $a \in A$ and $0 < p < \infty$. Then the following statements are equivalent:

(i) $a \in A_p$.
(ii) $[a]^p \in A_1$.
(iii) $[a]^{p/2} \in A_2$.

If any of these conditions holds, then $|a|^p = \sum f_{\beta} |a|^{p/2} f_{\beta}$, where $\{f_{\beta}\}$ is a maximal orthogonal family of hermitian minimal idempotents in $A$.

Proof. Let $[a] = \sum k_{\alpha} e_{\alpha} = \sum k_{n} e_{n}$ be a spectral representation of $[a]$.

(i) $\iff$ (ii) This follows from the equality $|a|^p = \sum k_{n}^p = \|a\|^p_1$.
(ii) $\iff$ (iii) This follows from the equality $\|a\|^p_1 = \sum k_{n}^p = \|a|^{p/2}\|_2^2$.

If any of these conditions holds, then by Lemma 2.2, we have

$$|a|^p = \sum k_{\alpha}^p = \sum \|a|^{p/2} e_{\alpha}\|^2$$

$$= \sum \|a|^{p/2} f_{\beta}\|^2 = \sum \|f_{\beta} [a]|^{p/2} f_{\beta}\|.$$ 

This completes the proof.

Lemma 3.4. Let $a$ be a positive element in $A$ and $f$ a hermitian minimal idempotent in $A$. Then

(i) $\|fa^p f\| > \|fa^p f\|^p$ ($1 < p < \infty$).
(ii) $\|fa^p f\| < \|fa^p f\|^p$ ($0 < p < 1$).

Proof. Let $a = \sum k_{\alpha} e_{\alpha}$ be a spectral representation of $a$.

(i) Clearly we can assume that $1 < p < \infty$. Then by Hölder's inequality and Lemma 2.1, we have

$$\|fa^p f\| = \sum |e_{\alpha} a^{p/2} f|^2 = \sum k_{\alpha} \|e_{\alpha} f\|^2$$

$$\leq \left( \sum k_{\alpha}^p \|e_{\alpha} f\|^2 \right)^{1/p} \left( \sum \|e_{\alpha} f\|^2 \right)^{(p-1)/p}$$

$$= \left( \sum \|e_{\alpha} a^{p/2} f\|^2 \right)^{1/p} (\|f\|^2)^{(p-1)/p} = \|fa^p f\|^{1/p}.$$ 

(ii) Replacing $a$ by $a^p$ and $p$ by $1/p$ in (i), we get (ii).

Lemma 3.5. Let $a \in A_p$ and $\{f_{\beta}\}$ be a maximal orthogonal family of hermitian minimal idempotents in $A$. Then

(i) $|a|^p \leq \sum f_{\beta} |a| f_{\beta}\|^p$ ($1 < p < 2$).
(ii) $|a|^p > \sum f_{\beta} |a| f_{\beta}\|^p$ ($2 < p < \infty$).

If $[a] = \sum k_{n} e_{n}$ is a spectral representation of $[a]$, then $|a|^p = \sum k_{n} \|ae_{n}\|^p$ ($0 < p < \infty$).
PROOF. (i) If $1 < p < 2$, then by Lemma 3.4(ii), we have $\|f_\beta [a]^p f_\beta\| \leq \|f_\beta[a]^2 f_\beta\|^p/2 = \|af_\beta\|^p$. Therefore (i) follows now from Lemma 3.3.

(ii) This can be proved similarly.

If $[a] = \Sigma_n k_n e_n$, then $\|ae_n\| = |e_n a^* a e_n|^{1/2} = k_n$. Therefore $|a|^p = \Sigma_n |ae_n|^p$ $(0 < p < \infty)$. This completes the proof.

**Lemma 3.6.** Suppose $a, b \in A$ and $1 \leq p \leq \infty$, then the following statements hold:

(i) If $a \in A_p$ and $S$ is a right centralizer on $A$, then $Sa \in A_p$ and $|Sa|^p \leq \|S\| |a|^p$.

(ii) If $a \in A_p$ and $b \in A$, then $|ab|^p \leq \|b\| |a|^p$ and $|ba|^p \leq \|b\| |a|^p$.

Hence $ab$ and $ba$ are in $A_p$.

(iii) If $a, b$ are in $A_p$, then $|ab|^p < |a|^p |b|^p$.

**Proof.** Clearly we can assume that $1 \leq p < \infty$.

(i) Suppose $1 < p < 2$. Let $[a] = \Sigma k e_a$ be a spectral representation of $[a]$. Then by Lemma 3.5, we have

$$|Sa|^p \leq \sum_\alpha \|(Sa) e_\alpha\|^p \leq \|S\|^p \sum_\alpha |ae_\alpha|^p = \|S\|^p |a|^p.$$ 

If $2 < p < \infty$, let $[Sa] = \Sigma k e_\alpha$ be a spectral representation of $[SA]$. Then by a similar argument, we have $|Sa|^p \leq \|S\| |a|^p$. This proves (i).

(ii) This follows easily from (i) and Lemma 3.1(v).

(iii) This follows from (ii) and Lemma 3.1(i).

**Lemma 3.7.** Let $a \in A_p$ and $\{f_\beta\}$ a maximal orthogonal family of hermitian minimal idempotents in $A$. Then

(3.2) $\sum_\beta \|f_\beta a f_\beta\|^p \leq |a|^p$ $(1 \leq p < \infty)$.

**Proof.** Let $W$ be the partial isometry associated with $a$ and $b = W[a]^{1/2}$. Then $a = W[a] = b[a]^{1/2}$. It follows from Cauchy’s inequality that

(3.3) $\sum_\beta \|f_\beta a f_\beta\|^p \leq \left(\sum_\beta \|f_\beta b\|^2\right)^{1/2} \left(\sum_\beta \|a\|^{1/2} |f_\beta|^2\right)^{1/2}.$

By Lemma 3.3 and Lemma 3.4, we have

(3.4) $\sum_\beta \|a\|^{1/2} |f_\beta|^2 = \sum_\beta \|f_\beta a f_\beta\| \leq \sum_\beta \|f_\beta a f_\beta\|^p = |a|^p.$

By Lemma 3.2, Lemma 3.3 and Lemma 3.4, we have

(3.5) $\sum_\beta \|f_\beta b\|^2 \leq \sum_\beta \|f_\beta (b^* b) f_\beta\| = |b^* b|^2 = |b|^2,$

$$\leq \|a\|^{2p} = |a|^p.$$
Substituting (3.4) and (3.5) into (3.3), we get (3.2). This completes the proof.

In order that \( |\cdot|_p \) be a norm on \( A_p \) \( (1 \leq p \leq \infty) \), it is sufficient now to show the triangle inequality.

**Lemma 3.8.** Let \( a, b \in A_p \), then \( |a + b|_p \leq |a|_p + |b|_p \) \( (1 \leq p \leq \infty) \). Hence \( a + b \in A_p \).

**Proof.** We can assume \( 1 \leq p < \infty \). Write \( [a + b] = \sum \alpha k_\alpha e_\alpha \) and \( [a + b] = W^*(a + b) \) (see (2.8)). Then by Lemma 3.5, Lemma 3.7 and Minkowski's inequality, we have

\[
|a + b|^p = \left( \sum |e_\alpha W^*a e_\alpha|^p \right)^{1/p} + \left( \sum |e_\alpha W^*b e_\alpha|^p \right)^{1/p} < |W^*a|^p + |W^*b|^p < |a|^p + |b|^p.
\]

This completes the proof.

Now we have the main result of this section.

**Theorem 3.9.** For \( 1 \leq p \leq \infty \), \( A_p \) is a dual \( A^* \)-algebra which is a dense two-sided ideal of \( A \).

**Proof.** By a similar argument in the proof of [4, p. 265, Corollary 3.2], we can show that \( A_p \) is complete. (We use maximal orthogonal families of hermitian minimal idempotents instead of orthonormal bases.) Hence \( A_p \) is an \( A^* \)-algebra which is a two-sided ideal of \( A \). It follows from Lemma 3.1(iv) that \( A_p \) contains the socle \( S \) of \( A \). Since \( S \) is dense in \( A \), so is \( A_p \). We claim that, for each \( a \) in \( A_p \), \( a \) belongs to the closure of \( aA_p \) in \( A_p \). In fact, let \( [a] = \sum_{i=1}^\infty k_i e_i \) be a spectral representation of \( [a] \) and \( W \) the partial isometry associated with \( a \). Put \( f_n = \sum_{i=1}^n e_i \) \( (n = 1, 2, \ldots) \). Then

\[
|a - af_n|^p \leq ||[a] - [a]f_n||_p = \left( \sum_{i=n+1}^\infty k_i e_i \right)^{1/p} = \left( \sum_{i=n+1}^\infty k_i^p \right)^{1/p}.
\]

Since \( a \in A_p \), it follows that \( |a - af_n|^p \to 0 \) as \( n \to \infty \). Hence by [5, p. 29, Lemma 8 (3)], \( A_p \) is a dual algebra. This completes the proof.

We shall need the following result.

**Corollary 3.10.** Let \( \{e_\gamma\} \) be any orthogonal family of hermitian minimal idempotents of \( A \) and \( x \in A_p \) \( (1 \leq p \leq \infty) \), then \( \Sigma e_\gamma x \) is summable in \( ||\cdot||_p \) and especially when \( \{e_\gamma\} \) is a maximal family \( x = \Sigma e_\gamma x \) in \( A_p \).

**Proof.** This follows from Theorem 3.9 and Theorem 5.2 in [12].

Finally we remark that many statements and proofs in this section are similar to those given in [4] and [11].
4. The algebras $A_1$ and $A_2$. We have a characterization of a proper $H^*$-algebra.

**Theorem 4.1.** The algebra $A_2$ is a proper $H^*$-algebra. Conversely, every proper $H^*$-algebra is of the form $A_2$ for some dual $B^*$-algebra $A$.

**Proof.** Let $a, b \in A_2$ and $\{f_\beta\}$ a maximal orthogonal family of hermitian minimal idempotents in $A$. Then $f_\beta b^*a f_\beta = \lambda_\beta f_\beta$ for some constant $\lambda_\beta$. We claim that $\sum_\beta \lambda_\beta$ is absolutely summable and independent of the choice of $\{f_\beta\}$. In fact, let $x, y \in Af_\beta$. Then $y^*x = \langle x, y \rangle_\beta f_\beta$ for some constant $\langle x, y \rangle_\beta$. It follows from [7, p. 261, Theorem (4.10.3)] and [7, p. 263, Theorem (4.10.6)] that $\langle x, y \rangle_\beta$ defines a complete inner product on $Af_\beta$ such that $\langle x, x \rangle_\beta = ||x||^2$. Now by Lemma 2.2 and the proof of [10, p. 30, Lemma 4], we can show that $\sum_\beta \lambda_\beta$ is absolutely summable and independent of $\{f_\beta\}$. Define

$$ (a, b) = \sum_\beta \lambda_\beta \quad (a, b \in A_2). \tag{4.1} $$

Then by the proof of [10, p. 31, Lemma 5], $( , )$ is an inner product on $A_2$ such that $(xa, b) = (a, x^*b)$ and $(ax, b) = (a, bx^*)$ for all $x$ in $A$. Also $|a|^2 = (a, a)$. Therefore $A_2$ is a proper $H^*$-algebra.

Conversely, let $B$ be a proper $H^*$-algebra. Then $B$ is a dense two-sided ideal of some dual $B^*$-algebra $A$. We can show that $B = A_2$ and this completes the proof.

**Lemma 4.2.** Let $1/p + 1/q = 1$, where $1 < p, q < \infty$. If $a \in A_p$ and $b \in A_q$, then $ab \in A_1$ and $|ab|_1 \leq |a|_p |b|_q$.

**Proof.** Suppose first that $2 < p < \infty$, $1 < q < 2$. Let $[b] = \sum_\alpha k_\alpha e_\alpha$ be a spectral representation of $[b]$. Also write $[ab] = W^*ab$. Then by Lemma 3.3, Lemma 3.5 and Hölder's inequality, we have

$$ |ab|_1 = \sum_\alpha ||e_\alpha [ab] e_\alpha|| = \sum_\alpha ||e_\alpha W^*a e_\alpha|| \leq |W^*a|_p |b|_q \leq |a|_p |b|_q. \tag{4.2} $$

By a similar argument, we can show that (4.2) holds for $1 < p \leq 2$, $2 \leq q < \infty$.

We now identify $A_1$.

**Theorem 4.3.** $A_1 = \{xy: x, y \in A_2\}$.

**Proof.** If $a \in A_1$, then by Lemma 3.3, $[a]^\frac{1}{2} \in A_2$. Let $W$ be the
partial isometry associated with \( a \). Then \( a = W[a] = (W[a]^{1/2})([a]^{1/2}) \in \{ xy: x, y \in A_2 \} \). The converse follows from Lemma 4.2 and this completes the proof.

Let \( a \in A_1 \). Then by Theorem 4.3, \( a = c^*b \) for some \( b, c \) in \( A_2 \).

Define

\[
\text{tr } a = (b, c) \quad (a \in A_1),
\]

where \((b, c)\) is given by (4.1).

**Lemma 4.4.** Let \( a \in A_1 \), \( \{f_\beta\} \) a maximal orthogonal family of hermitian minimal idempotents in \( A \) and \( \lambda_\beta f_\beta = f_\beta a f_\beta \). Then \( \text{tr } a \) is well defined, \( \text{tr } a = \Sigma_\beta \lambda_\beta = \Sigma_\beta (a f_\beta, f_\beta) \) and \( |\text{tr } a| \leq |a|_1 \).

**Proof.** By the proof of Theorem 4.1, \( \Sigma_\beta \lambda_\beta \) is absolutely summable and independent of \( \{f_\beta\} \). It is clear that \( \text{tr } a = \Sigma_\beta \lambda_\beta = \Sigma_\beta (a f_\beta, f_\beta) \). Therefore \( \text{tr } a \) is well defined. By Lemma 3.7, \( |\text{tr } a| \leq \Sigma_\beta ||f_\beta a f_\beta|| \leq \langle a \rangle \).

5. The uniform convexity of \( A_p \) (\( 1 < p < \infty \)). For each \( a \) in \( A \), we define a linear operator \( L_a \) on \( A_2 \) by

\[
L_a(x) = ax \quad (x \in A_2).
\]

Since \( ||ax||_2 \leq ||a|| |x|_2 \), it follows that \( L_a \) is bounded on \( A_2 \). Let \( (, ) \) be the given inner product on \( A_2 \).

**Lemma 5.1.** Let \( a \) be a positive element in \( A \). Then \( L_a \) is positive and \( L_a^r = (L_a)^r \) (\( 0 < r < \infty \)).

**Proof.** This is clear.

We now establish [4, p. 260, Lemma 2.6] for \( A_p \).

**Lemma 5.2.** Let \( a \) and \( b \) be two positive elements in \( A \) and \( 0 < r < \infty \). If \( (a + b)^r, a^r \) and \( b^r \) are in \( A_1 \), then

(i) \( \text{tr } (a + b)^r \leq \text{tr } a^r + \text{tr } b^r \) (\( 0 < r \leq 1 \)).

(ii) \( \text{tr } (a + b)^r \geq \text{tr } a^r + \text{tr } b^r \) (\( 1 \leq r < \infty \)).

**Proof.** We assume first that \( 0 < r \leq 1 \). Let \( S = L_a, T = L_b \) and \( U = L_{a+b} \). Then by the proof of [4, p. 260, Lemma 2.6], there exist operators \( C \) and \( D \) on \( A_2 \) such that

\[
||C|| \leq 1, \quad ||D|| \leq 1, \quad CU^{1/2} = S^{1/2}, \quad DU^{1/2} = T^{1/2},
\]

\[
U^r = U^{r/2}C^*C^{1/2}U^{1/2} + U^{r/2}D^*D^{1/2}U^{1/2}.
\]

Let \( \{f_\beta\} \) be a maximal orthogonal family of hermitian minimal idempotents in \( A \). Then by Lemma 5.1, we have
\[
\text{tr} (a + b)^r = \sum_{\beta} ((a + b)^r f_{\beta}, f_{\beta}) = \sum_{\beta} (U^r f_{\beta}, f_{\beta})
\]
(5.2)
\[
= \sum_{\beta} (CU^{r/2} f_{\beta}, CU^{r/2} f_{\beta}) + \sum_{\beta} (DU^{r/2} f_{\beta}, DU^{r/2} f_{\beta}).
\]

Since \(C(a + b)^{r/2} \in A_2\) and \(CU^{r/2} f_{\beta} = C(a + b)^{r/2} f_{\beta}\), it follows from (5.2) that
\[
\text{tr} (a + b)^r = |C(a + b)^{r/2}|_2^2 + |D(a + b)^{r/2}|_2^2
\]
(5.3)
\[
= |(C(a + b)^{r/2})^*|_2^2 + |(D(a + b)^{r/2})^*|_2^2.
\]

Let \(a = \Sigma_{\alpha} k_{\alpha} e_{\alpha}\) be a spectral representation of \(a\). Since \((C(a + b)^{r/2})^* e_{\alpha} = (CU^{r/2})^* e_{\alpha} = (a + b)^{r/2} e_{\alpha}\), it follows from [4, p. 252, Lemma 2.1] that
\[
((C(a + b)^{r/2})^* e_{\alpha}, (C(a + b)^{r/2})^* e_{\alpha}) = ((a + b)^{r/2} e_{\alpha}, C^* e_{\alpha})
\]
\[
\leq (a + b)^{r/2} e_{\alpha}, C^* e_{\alpha})^* = (a e_{\alpha}, e_{\alpha})^r = k_{\alpha}^r = (a e_{\alpha}, e_{\alpha}).
\]

Therefore \(|(C(a + b)^{r/2})^*|_2^2 \leq \text{tr} a^r\). Similarly \(|(D(a + b)^{r/2})^*|_2^2 \leq \text{tr} b^r\). Hence by (5.3), we have \(\text{tr} (a + b)^r \leq \text{tr} a^r + \text{tr} b^r\). The case \(1 \leq r < \infty\) can be proved in a similar way and the proof is complete.

By using maximal orthogonal families of hermitian minimal idempotents and a similar argument in the proof of [4, p. 259, Lemma 2.5], we have:

**Lemma 5.3.** Let \(a\) be a positive element in \(A\) and \(b\) a hermitian element in \(A\) such that \(a + b\) and \(a - b\) are positive. Suppose \((a + b)^r\), \((a - b)^r\) and \(a^*\) are in \(A_1\). Then

(i) \(\text{tr} (a + b)^r + \text{tr} (a - b)^r \leq \text{tr} a^r (0 < r \leq 1)\).

(ii) \(\text{tr} (a + b)^r + \text{tr} (a - b)^r \geq \text{tr} a^r (1 \leq r < \infty)\).

Now we have the following result.

**Theorem 5.4.** Let \(a\) and \(b\) be two elements in \(A_p\) and \(1/p + 1/q = 1\). Then

(i) \(2^{p-1}(|a|_p^p + |b|_p^p) \leq |a + b|_p^p + |a - b|_p^p \leq 2(|a|_p^p + |b|_p^p) (0 < p \leq 2)\)

(ii) \(|a + b|_p^q + |a - b|_p^q \leq 2(|a|_p^p + |b|_p^p)^{q/p} (1 < p \leq 2)\)

(iii) \(2(|a|_p^p + |b|_p^p) \leq |a + b|_p^p + |a - b|_p^p \leq 2^{p-1}(|a|_p^p + |b|_p^p) (2 \leq p < \infty)\)

(iv) \(2(|a|_p^p + |b|_p^p)^{q/p} \leq |a + b|_p^q + |a - b|_p^q (2 \leq p < \infty)\).

**Proof.** This can be proved by using Lemma 5.2, Lemma 5.3 and the proof of [4, p. 261, Theorem 2.7]. We omit the details.
As observed in [4], we have:

**Corollary 5.5.** For $1 < p < \infty$, $A_p$ is uniformly convex and reflexive.

6. The conjugate space of $A_p$. In this section, we always assume that $1 \leq p < \infty$ and $1/p + 1/q = 1$. Let $A_p^*$ be the conjugate space of $A_p$. We shall show that $A_p = A_q^*$ in a natural way.

For each $a$ in $A_p$ ($1 \leq p < \infty$), we define

$$F_a(x) = \text{tr} ax \quad (x \in A_q).$$

**Theorem 6.1.** For each $a$ in $A_p$ ($1 < p < \infty$), $F_a \in A_q^*$ and $\|F_a\| = |a|_p$.

**Proof.** By Lemma 4.2, $F_a$ is well defined. It is clear that $F_a \in A_q^*$ and $\|F_a\| \leq |a|_p$. By a similar argument in the proof of [11, p. 786, Proposition 3.26], we can show that $\|F_a\| \geq |a|_p$. This completes the proof.

We now establish a converse of Theorem 6.1.

**Theorem 6.2.** For $1 < p < \infty$, every continuous linear functional $F$ on $A_q$ is of the form $F_a$ for some $a$ in $A_p$, where $F_a$ is defined in (6.1).

**Proof.** We assume first that $p = 1$ and $F \in A_\infty^* = A^*$. Then it is clear that $F \in A_2^*$. Since $A_2$ is a Hilbert space, by the Riesz representation theorem, there exists some $a$ in $A_2$ such that $F(x) = (x, a^*) = \text{tr} ax$ for all $x$ in $A_2$. By the proof of Theorem 6.1, we can show that $a \in A_1$ and so $F = F_a$.

Now we consider the case $1 < p < \infty$ and assume $F \in A_q^*$. Then $F \in A_1^*$. Hence by the proof of [8, p. 103, Theorem 2], there exists a right centralizer $S$ on $A_2$ such that

$$F(y) = \text{tr} Sy \quad (y \in A_1).$$

Let $\{e_\alpha\}$ be a maximal orthogonal family of hermitian minimal idempotents in $A$ and $\{E_\gamma\}$ the direct set of all finite sums $e_{\alpha_1} + e_{\alpha_2} + \cdots + e_{\alpha_n}$. Define $F_\gamma$ on $A_q$ by

$$F_\gamma(x) = F(E_\gamma x) \quad (x \in A_q).$$

Since $S(E_\gamma x) = (SE_\gamma)(E_\gamma x) = ((SE_\gamma)E_\gamma)x = (SE_\gamma)x$ and $E_\gamma x \in A_1$, by (6.2) and (6.3) we have

$$F_\gamma(x) = \text{tr} S(E_\gamma x) = \text{tr} (SE_\gamma)x \quad (x \in A_q).$$

Since $SE_\gamma = (SE_\gamma)E_\gamma \in A_p$, by (6.4) and Theorem 6.1, $|SE_\gamma|_p = \|F_\gamma\| \leq \|F\|$. Therefore $\{SE_\gamma\}$ is a bounded set in $A_p$. Since $A_p$ is reflexive (Corollary 5.5), we can assume that $SE_\gamma \to a$ weakly for some $a$ in $A_p$. Hence $ae_\alpha = Se_\alpha$ for all $\alpha$. Therefore by (6.2), $F(e_\alpha x) = \text{tr} ae_\alpha x$. For each $x$ in $A_q$, by
Corollary 3.10, $x = \sum_{\alpha} e_{\alpha} x$ in $| \cdot |_q$. Hence it follows that $F(x) = \text{tr} \, x \, (x \in A_q)$. This completes the proof.

**Remark.** Some arguments in the above proof are similar to those in the proof of [3, p. 130, Theorem III. 12.2].

**REFERENCES**


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