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THE SPACE OF CONJUGACY CLASSES OF A TOPOLOGICAL GROUP

BY

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ABSTRACT. The space $G^{\#}$ of conjugacy classes of a topological group G is the orbit space of the action of G on itself by inner automorphisms. For a class of connected and locally connected groups which includes all analytic $[Z]$-groups, the universal covering space of $G^{\#}$ may be obtained as the space of conjugacy classes of a group which is locally isomorphic with G, and the Poincaré group of $G^{\#}$ is found to be isomorphic with that of G/G', the commutator quotient group. In particular, it is shown that the space $G^{\#}$ of a compact analytic group G is simply connected if and only if G is semisimple. The proof of this fact has not appeared in the literature, even though more specialized methods are available for this case.

I. Definitions and elementary properties. Two elements x, y of a topological group G are called *conjugate*, and we write $x \approx y$, if there is an element $t \in G$ such that $y = txt^{-1}$. The equivalence class of a point x under this relation is called the *conjugacy class* of x, denoted I_x. A subset of G which is a union of conjugacy classes is invariant under inner automorphisms and will be said to be *invariant*.

If G acts on itself by inner automorphisms, the inner automorphisms determined by the center $Z(G)$ of G are trivial and $G/Z(G)$ acts effectively on G. The orbit space under the action of G or $G/Z(G)$ is called the *space of conjugacy classes* of G, denoted $G^{\#}$. If G is the direct product of groups G_i, then $G^{\#}$ is homeomorphic with the Cartesian product of the spaces $G_i^{\#}$ (see [5, p. 130]).

The space $G^{\#}$ of a compact analytic group G is homeomorphic with the orbit space T/W of the action of the Weyl group W on a maximal toroid T of G [1, p. 95]. If G is semisimple, $G^{\#}$ may be obtained by identifying certain boundary points of a compact convex polyhedron in the Lie algebra of T (see [2, Example 6]). Some elementary proofs and [11, p. 231] give the following:

LEMMA 1. *If G is a compact analytic group, then $G^{\#}$ is compact, Hausdorff, second countable, and locally arcwise simply connected.*

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The natural map $p\colon G \to G^{\#}$ may not be closed if G is not discrete, and $G/Z(G)$ is not compact. From [4, p. 303], we have the following:

PROPOSITION 2. *If G is a connected, locally compact group, then the following are equivalent*:

(i) *the natural map $p\colon G \to G^{\#}$ is closed*;

(ii) *each neighborhood of e contains an invariant neighborhood of e* (*the* [SIN] *property*);

(iii) *G is the direct product of a compact group and a vector group.*

EXAMPLE 1. Let H be the subgroup of $SL(3, R)$ consisting of matrices of the form

$$M(r, s, t) = \begin{pmatrix} 1 & r & t \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}.$$

It is easily checked that

$$M(a, b, c)M(r, s, t)(M(a, b, c))^{-1} = M(r, s, t + as - br)$$

and that

$$M(a, b, c)M(r, s, t)(M(a, b, c))^{-1}(M(r, s, t))^{-1} = M(0, 0, as - br).$$

An element of the form $M(0, 0, t)$ is central, and $D = \{M(0, 0, n)\colon n \in Z\}$ is a discrete central subgroup. The conjugacy class of a noncentral element $M(r, s, t)$ is $\{M(r, s, w)\colon w \in R\}$. In particular, $\{M(1/n, 0, w)\colon w \in R\}$ is a conjugacy class for each $n \in Z^{+}$. Hence, H is not an [SIN] group. The quotient group H/D has compact conjugacy classes and is not an [SIN] group.

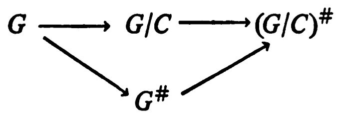
EXAMPLE 2. In the group H of Example 1, consider the subgroup $G = \{M(m, n, t); m, n \in Z; t \in R\}$. The conjugacy class of a noncentral element $M(m, n, t)$ is $\{M(m, n, t + kd)\colon k \in Z;\ d$ the greatest common divisor of m and $n\}$. The space $G^{\#}$ is normal, because each component of $G^{\#}$ is homeomorphic with R/dZ for some $d \in Z$. The component of e is exactly the center, so that G is an [SIN] group which is not the direct product of a vector group and a compact group.

For connectedness, we have

PROPOSITION 3. *Suppose that p is a closed map or that G is locally connected or that each conjugacy class is connected. Then $G^{\#}$ is connected if and only if G is connected.*

PROOF (of the nontrivial implication). Let C be the (invariant) component subgroup of G. If $C \neq G$, there is an open and closed set E which does not meet C. If p is closed, then $p(E)$ is an open and closed set which does not meet $p(C)$.

In the other two cases, consider the space $(G/C)^{\#}$ and the diagram:



If G is locally connected, then G/C and $(G/C)^{\#}$ are discrete. If each conjugacy class is connected, then $G/C = (G/C)^{\#}$ is totally disconnected [8, p. 60]. But in either case, $(G/C)^{\#}$ is connected, hence, trivial. Thus, $G = C$.

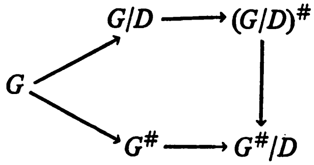
Clearly, if $x, y \in G$ and $z \in Z(G)$, then $x \approx y$ if and only if $zx \approx zy$. This suggests that we define an action of $Z(G)$ on $G^{\#}$ by

(*) $$zI_x = I_{zx}.$$

This action of $Z(G)$ on $G^{\#}$ constitutes a transformation group, in the sense of [11], except that $G^{\#}$ may not be a Hausdorff space.

LEMMA 4. *If D is a closed subgroup of $Z(G)$, then the orbit space $G^{\#}/D$ is homeomorphic with $(G/D)^{\#}$.*

PROOF. Consider the diagram:



II. Stability subgroups for the action of $Z(G)$ on $G^{\#}$. The stability subgroups for the action (*) are conveniently described in terms of the sets $I_x I_x^{-1}$. For each $x \in G$, the set $I_x I_x^{-1}$ is invariant and inversion-invariant and $e \in I_x I_x^{-1} \subset (G, G)$, the algebraic commutator subgroup. The following theorem, which was proved by Gotô in [6], will be used to show that the main result of this paper (Theorem 16) holds for analytic [Z]-groups:

THEOREM 5 (GOTÔ). *If G is a compact semisimple analytic group, then there is an element $x \in G$ such that $I_x I_x^{-1} = G$.*

In a more general situation, we have the following relationship between the algebraic and topological structure of the conjugacy classes:

PROPOSITION 6. *Suppose that the set $I_x I_x^{-1}$ is locally compact in its relative topology. Then $\bar{I}_x \subset I_x I_x^{-1} I_x \subset x(G, G)$. Moreover, if the set $I_x I_x^{-1}$ is* *closed under the group operation, then it is a closed invariant subgroup of* G *contained in* (G, G).

PROOF. The second part follows from [8, p. 35], for then $I_x I_x^{-1}$ is a locally compact subgroup.

For the first part, let U, V be neighborhoods of e with $\bar{U} \cap I_x I_x^{-1}$ compact and $V^2 \subset U$. Let $w \in \bar{I}_x$, $z \in I_x^{-1} \cap w^{-1}V$, and let $\{w_i\}$ be a net in I_x converging to w. Then, eventually,

$$w_i z \in Vww^{-1}V \cap I_x I_x^{-1} = V^2 \cap I_x I_x^{-1} \subset U \cap I_x I_x^{-1}.$$

Thus, wz is in the closed set $\bar{U} \cap I_x I_x^{-1}$ and $w = wzz^{-1} \in I_x I_x^{-1} I_x$.

EXAMPLE 3. If G is the affine group, $\{\left(\begin{smallmatrix} r & s \\ 0 & 1 \end{smallmatrix}\right): r \in R^+, s \in R\}$, and $x = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, then $e \in \bar{I}_x$, and $I_x I_x^{-1} = I_x \cup I_x^{-1} \cup \{e\} = (G, G)$ (see [8, p. 350]).

We now identify the stability subgroups for the action (*).

LEMMA 7. *If* D *is a closed subgroup of* $Z(G)$, *then the set* $D_x = D \cap I_x I_x^{-1} = D \cap x I_x^{-1}$ *is the stability subgroup in* D *of* $I_x \in G^\#$.

PROOF. An element $d \in D$ is in the stability subgroup if and only if it translates some (and hence, every) conjugate of x to another conjugate of x. Then, for some $s, t \in G$, $d = sxs^{-1}tx^{-1}t^{-1} = s^{-1}ds = xs^{-1}tx^{-1}t^{-1}s$.

These stability subgroups are related to the zeros of characters of finite-dimensional irreducible representations:

COROLLARY 8. *Let* π *be a finite-dimensional irreducible representation of* G *and let* $x \in G$. *If* $\operatorname{trace}(\pi(x)) \neq 0$, *then* $Z(G) \cap I_x I_x^{-1} \subset \operatorname{kernel}(\pi)$. *If moreover,* π *is faithful, then the stability subgroup of* $I_x \in G^\#$ *under* (*) *is trivial.*

PROOF. Let $z \in Z(G) \cap I_x I_x^{-1}$, then Schur's lemma shows that $\operatorname{trace}(\pi(x)) = \operatorname{trace}(\pi(zx)) = \operatorname{trace}(\pi(x))\operatorname{trace}(\pi(z))/\operatorname{trace}(\pi(e))$.

COROLLARY 9. *If* G *is a compact semisimple analytic group and* $x \in G$ *is a regular point, that is, a point whose centralizer has minimum dimension, then* D_x *is isomorphic with a subgroup of the Weyl group* W *of* G.

PROOF. There is a maximal toroid T which contains x (and $Z(G)$) (see [1]) and for each $d \in D_x$ there is exactly one $nT \in W$ such that $dx = nxn^{-1}$. This correspondence effects an isomorphism between D_x and an Abelian subgroup of W.

EXAMPLE 4. *If* $G = SU(2)$, there is only one conjugacy class with a nontrivial stability subgroup, that of $\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$. The Weyl group is of order two.

III. The Poincaré group of $G^{\#}$. The relationship between the structures $G^{\#}$ and $G^a = G/G'$, where G' is the closed commutator subgroup of G, is a consequence of the fact that the natural map from G to G^a factors through $G^{\#}$ (see [8, p. 358]).

LEMMA 10. *The map* $q: G^{\#} \to G^a$ *defined by* $q(I_x) = xG'$ *is continuous, open and surjective.*

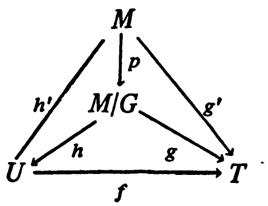
A connected and locally connected space S will be said to be *simply connected* if, for each covering space (U, f) of a space T and continuous map $g: S \to T$, there is a unique continuous map $h: S \to U$ such that $f \circ h = g$ and $h(s) = u$, where s and u are prescribed points such that $f(u) = g(s)$.

This is the definition used by Hochschild [10], and is equivalent to that used by Chevalley [3], except that we do not require the Hausdorff property. The stability of this lifting property under two types of maps which appear leads to sufficient conditions for the spaces $G^{\#}$ and G^a to be simply connected. For analytic $[Z]$-groups, we show that these spaces are *arcwise* simply connected if they are simply connected.

A space is said to be locally simply connected if each point has a simply connected neighborhood. A connected space has a simply connected covering space if and only if it is locally simply connected (see [10] and [3]).

LEMMA 11. *Let* G *be a group which acts on a simply connected space* M *with a fixed point* m. *Then the orbit space* M/G *is simply connected.*

PROOF. Consider the commutative diagram



where f is a covering, g is continuous, $g' = g \circ p$ and h' is a specified lift of g'. To show that there is a map h as indicated, we show that h' is constant on G-orbits. Since h' is the unique map taking m to $h'(m)$ and satisfying $f \circ h' = g'$, precomposition of h' with an action of G does not alter h', that is h' is constant on G-orbits.

We can now give some sufficient conditions for the spaces $G^{\#}$ and G^a to be simply connected:

PROPOSITION 12. *If* G *is locally connected and* $G^{\#}$ *is simply connected, then* $G^a = G/G'$ *is simply connected.*

PROOF. By Proposition 3, G is connected. Hence, the G'-cosets are connected [9, p. 142]. Use [10, p. 56], and Lemma 10.

PROPOSITION 13. *If G is simply connected, then $G^{\#}$ is simply connected.*

PROOF. The stability subgroup of e for the action of G on itself by inner automorphisms is G. Use Lemma 11.

PROPOSITION 14. *Let D be a discrete subgroup of $Z(G)$ and let D_x be the stability subgroup of $I_x \in G^{\#}$ under $(*)$. If $G^{\#}$ is simply connected, then $(G/D_x)^{\#}$ is simply connected.*

PROOF. The subgroup D_x is closed (we have not assumed any separation properties for $G^{\#}$). Use Lemmas 11 and 4.

PROPOSITION 15. *Let D be a discrete subgroup of $Z(G)$ which is generated by the stability subgroups D_x under $(*)$. If $G^{\#}$ is simply connected, then $(G/D)^{\#}$ is simply connected.*

PROOF. Partially order by inclusion the collection of subgroups D^* of D such that the orbit space $G^{\#}/D^*$ is simply connected, and use Zorn's lemma. The uniqueness of lifts in the definition of "simply connected" implies that the union of the elements of a chain is an upper bound for the chain, and D is the only possible maximal element because of Proposition 14.

EXAMPLE 5. In Example 1, the group H is the universal covering group of H/D and $(H/D)^{\#}$ is simply connected.

We are now ready to prove the main result.

THEOREM 16. *Let G be a connected and locally simply connected group with universal covering group $\widetilde{G}$. If D is a discrete subgroup of $Z(\widetilde{G})$ such that $G \cong \widetilde{G}/D$ and $D \cap (\widetilde{G})'$ is generated by the stability groups D_x under $(*)$, and $D(\widetilde{G})'/(\widetilde{G})'$ is discrete in $(\widetilde{G})^a = \widetilde{G}/(\widetilde{G})'$, then the Poincaré groups of $G^{\#}$ and G^a are isomorphic with $D/(D \cap (\widetilde{G})')$.*

PROOF. Let $D_1 = D \cap (\widetilde{G})'$ and let $f_1\colon G \to G/D_1$ and $f_2\colon G/D_1 \to G/D$ be the natural covering maps. Since $D(\widetilde{G})'/(\widetilde{G})'$ is closed in $(\widetilde{G})^a$, $f_1((\widetilde{G})') = (\widetilde{G}/D_1)'$ and $f_2(f_1(D(\widetilde{G})')) = (\widetilde{G}/D)'$. Thus, we have the diagram

$$\begin{array}{ccccc}
\widetilde{G} & \longrightarrow & (\widetilde{G})^{\#} & \longrightarrow & (\widetilde{G})^a \\
{\scriptstyle f_1}\downarrow & & \downarrow & & \downarrow{\scriptstyle f_1^a} \\
\widetilde{G}/D_1 & \longrightarrow & (G/D_1)^{\#} & \longrightarrow & (\widetilde{G}/D_1)^a \\
{\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle f_2^{\#}} & & \downarrow{\scriptstyle f_2^a} \\
\widetilde{G}/D & \longrightarrow & (\widetilde{G}/D)^{\#} & \longrightarrow & (\widetilde{G}/D)^a
\end{array}$$

where f_1^a is the topological isomorphism induced by f_1, $f_2^{\#}$ and f_2^a are induced by f_2 and other maps are as in Lemmas 4 and 10.

Propositions 13, 15, and 12 show that the spaces $(\widetilde{G}/D_1)^{\#}$ and $(\widetilde{G}/D_1)^a$ are simply connected. It remains to show that D/D_1 is a properly discontinuous group of homeomorphisms of these spaces [12, p. 87]. In $(\widetilde{G}/D_1)^a$, $D/D_1 \cong (D(\widetilde{G})'/D_1)/((\widetilde{G})'/D_1)$, a discrete subgroup. The action (*) of D/D_1 on $(\widetilde{G}/D_1)^{\#}$ is also properly discontinuous, because the elements of a D/D_1 orbit are conjugacy classes lying in distinct $(\widetilde{G}/D_1)'$-cosets (in $\widetilde{G}$, we have $(dI_x)I_x^{-1} \subset (\widetilde{G})'$ only if $d \in I_x I_x^{-1}(\widetilde{G})' = (\widetilde{G})'$).

Corollary 17. *If G is an analytic $[Z]$-group, the spaces $G^{\#}$ and G^a are locally arcwise simply connected and have isomorphic fundamental groups.*

Proof. First of all, the group G is the direct product of a vector group and a compact group (Proposition 2), so we may assume that G is compact. Then $\widetilde{G}$ is the direct product of a vector group and a simply connected compact semisimple analytic group H (see [10] and [13]). If D is a discrete subgroup of $Z(\widetilde{G})$ such that $\widetilde{G}/D = G$, then $DH/H = D(\widetilde{G})'/(\widetilde{G})'$ is discrete [10, p. 6] and $D \cap H = D \cap (\widetilde{G})' = D_x$ for some $x \in H$ (Theorem 5). The result follows from [12, p. 88], because $H^{\#}$ is arcwise simply connected (see Proposition 13, Lemma 1, and [10]).

Corollary 18. *A compact analytic group is semisimple if and only if $G^{\#}$ is simply connected.*

Proof. A compact analytic group G is semisimple if and only if the toroid G^a is trivial; use Corollary 17.

Example 6. One maximal toroid of $G = SO(4)$ consists of matrices

$$M(\theta, \varphi) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\varphi & -\sin\varphi \\ 0 & 0 & \sin\varphi & \cos\varphi \end{pmatrix}$$

and the nontrivial elements of the Weyl group are represented by the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

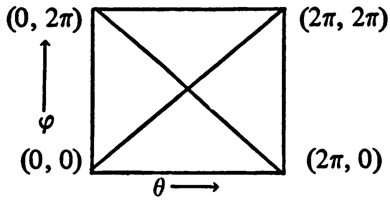
and

$$C = AB = BA = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

One checks easily that

$$\begin{aligned} A(M(\theta, \varphi))A^{-1} &= M(2\pi - \theta, 2\pi - \varphi), \\ B(M(\theta, \varphi))B^{-1} &= M(\varphi, \theta), \\ C(M(\theta, \varphi))C^{-1} &= M(2\pi - \varphi, 2\pi - \theta), \end{aligned}$$

so that each conjugacy class is represented by a matrix $M(\theta, \varphi)$ with $0 \leqslant \theta \leqslant \pi$ and $\theta \leqslant \varphi \leqslant 2\pi - \theta$. The space $G^{\#}$ may be realized as the small triangle on the left in the square



where pairs $M(0, \varphi)$, $M(0, 2\pi - \varphi)$ on the left-hand boundary must be identified. The space $G^{\#}$ is simply connected, as indicated by Corollary 18.

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