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CLOVERLEAF REPRESENTATIONS OF SIMPLY CONNECTED 3-MANIFOLDS

BY

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ABSTRACT. Let M be a triangulated 3-manifold satisfying the hypothesis of the Poincaré Conjecture. In the present paper it is shown that there is a finite linear graph K_1 in the 3-sphere, with exactly two components, and a finite linear graph K_2 in M, such that when the components of the graphs K_i are regarded as points, the resulting hyperspaces are homeomorphic. K_2 satisfies certain conditions which imply that each component of K_2 is contractible in M. Thus the conclusion of the theorem proved here is equivalent to the hypothesis of the Poincaré Conjecture.

1. **Statement of results.** All manifolds discussed in this paper will be piecewise linear. By a linear graph we mean either a 1-dimensional complex or a 1-dimensional polyhedron, according to convenience in the context. By a *loop-graph* we mean a linear graph L, containing a point P_0 such that L is the union of a finite collection of 1-spheres $\{J_1, J_2, \cdots, J_n\}$ such that the intersection of every two of them is P_0. These 1-spheres are called the *loops* of L, and P_0 is called the *center* of L. Let L be a loop-graph in a triangulated 3-manifold M. Suppose that there is a collection $\{D_1, D_2, \cdots, D_n\}$ of polyhedral disks, such that $\operatorname{Bd} D_i = J_i$ for each i, and such that the intersection of every two of the disks D_i is P_0. Then L is called a *cloverleaf.* (Here, as usual, if D is an m-manifold with boundary, then $\operatorname{Bd} D$ is the boundary of D. The interior $D - \operatorname{Bd} D$ of D will be denoted by $\operatorname{Int} D$.) The set $\{D_1, D_2, \cdots, D_n\}$ is called *a set of spanning disks for* L.

Now let $K = \bigcup_{i=1}^n I_i$ and $L = \bigcup_{i=1}^n J_i$ be cloverleaves, with the same number of loops, in the same triangulated 3-manifold M. Suppose that we can choose the order of the loops in K and L, and choose spanning disks $\{D_1, D_2, \cdots, D_n\}$ for K, and spanning disks $\{E_1, E_2, \cdots, E_n\}$ for L, in such a way that (1) D_i intersects E_j only if $i = j$, and (2) D_i and E_i intersect in the same way, topologically, as two linked circular regions in Euclidean 3-space $\mathbf{R}^3$. Then K and L are *simply linked.*

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Let K be a finite linear graph in a triangulated 3-manifold M. By a slight abuse of language, M/K denotes the space whose points are the components of K and the points of $M-K$ (with the usual topology).

THEOREM 1.1 (THE CLOVERLEAF THEOREM). *Let M be a compact, connected, simply connected triangulated 3-manifold. Then there is a linear graph K_1 in the 3-sphere S^3, and a linear graph K_2 in M, such that* (1) *either K_2 is empty or K_2 is the union of two simply linked cloverleaves, and* (2) *S^3/K_1 and M/K_2 are homeomorphic.*

The following theorems throw some light on the cloverleaf theorem. Their proofs are easy, and are omitted.

THEOREM 1.2. *Let K be a finite linear graph in a connected triangulated 3-manifold M. If M is simply connected, then so also is M/K.*

THEOREM 1.3. *Let M and K be as in Theorem 1.2, and suppose that the components of K are cloverleaves. If M/K is simply connected, then so also is M.*

THEOREM 1.4. *Let M and K be as in Theorems 1.2 and 1.3. Suppose that there is a linear graph H in S^3 such that S^3/H and M/K are homeomorphic. Then M is simply connected.*

Theorem 1.4 is a corollary of the two preceding theorems. By Theorem 1.2, S^3/H is simply connected. Therefore so also is M/K. By Theorem 1.3, M is simply connected.

Theorem 1.4 indicates that it is not impossible a priori for the conclusion of Theorem 1.1 to be used as a working hypothesis for M in a proof of the Poincaré Conjecture.

Without the hypothesis that K is a finite linear graph, Theorem 1.2 becomes false. See Bing's announcement [**B**] that if K is a solenoid in S^3, then S^3/K is not simply connected.

Without the hypothesis that the components of K are cloverleaves, Theorems 1.3 and 1.4 become false. To see this, let M be a triangulated 3-manifold which is compact, connected, and orientable, but not simply connected. Let K be the 1-skeleton of M, and let N be a regular neighborhood of K. Then the closure $\mathrm{Cl}(M-N)$ is a tubular set, that is, it is homeomorphic to a regular neighborhood of a linear graph, with a certain 1-dimensional Betti number $p^1 = p^1(\mathrm{Cl}(M-N))$ (with integers modulo 2 as coefficients). Since $\mathrm{Cl}(M-N)$ cannot contain a "solid Klein bottle", it follows that the topology of $\mathrm{Cl}(M-N)$ is completely described by p^1. Evidently $M-K$ is homeomorphic to $M-N$, and M/K is homeomorphic to the one-point compactification of $M-K$.

But all this can be copied in S^3. Let H be a linear graph in S^3, lying in a polyhedral 2-sphere, with $p^1(H)$ chosen so that $S^3 - H$ is homeomorphic to $M - K$. Then S^3/H is homeomorphic to the one-point compactification of $S^3 - H$. Therefore, by Theorem 1.2, S^3/H is simply connected. Therefore, so also is M/K. But, at the outset, M was not simply connected.

Thus the distinctive topology of a compact connected orientable triangulated 3-manifold is destroyed when we map its 1-skeleton onto a point. This observation is of course the first step in the proofs of deeper results of Lickorish [L] and Hempel [H].

2. **Special monotonic mappings.** The following was proved in [M].

THEOREM 2.1. *Let M be a triangulated 3-manifold, and suppose that M is compact, connected, and simply connected. Then there is a subcomplex K of a triangulation of the 3-sphere S^3, and a mapping $f\colon S^3 \longrightarrow M$ of S^3 onto M, such that*

(1) $\dim K \leqslant 2$,

(2) $f|K$ *is simplicial (relative to K and a subdivision of M),*

(3) $f|(S^3 - K)$ *is one-to-one,*

(4) $f(K) \cap f(S^3 - K) = 0$,

(5) f *is monotonic, and*

(6) *each set $f^{-1}(x)$ is either a point or a linear graph.*

We shall now modify the mapping f, strengthening some of its properties and weakening others; (6) will be lost.

First, we may suppose that the given triangulation X of S^3 (in which K forms a subcomplex) has the property that

(7) If $\sigma \in X$, then $\sigma \cap K$ either is empty or is a single face of σ.

(If the given X does not have this property, then we form a subdivision of X, by introducing exactly one new vertex in the interior of each simplex σ^i of X $(i > 0)$ that does not lie in K. The new triangulation satisfies (7).)

Second, we may suppose that

(8) K is the closure of the union of the nondegenerate sets $f^{-1}(x)$ $(x \in M)$.

(If (8) does not hold, then $f|\mathrm{Int}\,\sigma$ is one-to-one, for some $\sigma \in K$. We can then delete $\mathrm{Int}\,\sigma$ from K, preserving the stated properties of K.)

DEFINITION 2.1. By a *cell-complex* we mean a locally finite collection $\mathbf{C}$ of topological cells, such that (1) if $C \in \mathbf{C}$, then $\mathrm{Bd}\, C$ is a union of elements of $\mathbf{C}$, (2) different sets $\mathrm{Int}\, C$ are disjoint, and (3) if two elements of $\mathbf{C}$ intersect, then their intersection is the union of the elements of a subcollection of $\mathbf{C}$. If $C_1 \subset C_2$, then C_1 is a *face* of C_2 and a *face* of $\mathbf{C}$. If $\dim C_1 = i$, then C_1 is an *i-face* of C_2 (and of $\mathbf{C}$). $\mathbf{C}$ is called a *cell-decomposition* of the union $\mathbf{C}^*$ of the elements of $\mathbf{C}$.

The images $f(\sigma)$ of the simplices of X now form a cell-decomposition $f(X)$ of M, whose faces in various dimensions are the images of the faces of X. Evidently $f(X)$ forms a subcomplex of a triangulation Y of M, and so the elements of $f(X)$ are polyhedral in M, relative to Y. Suppose that K is not connected, and let B be a polygonal arc in the 1-skeleton X^1, joining two vertices v, v' of different components of K, and intersecting K only in v and v'. Then $f(B)$ is a polygonal arc in M. Therefore there is a mapping $\phi: M \longrightarrow M$, such that (1) $\phi(f(B))$ is a point, (2) ϕ is a homeomorphism everywhere else, and (3) $\phi | f(K)$ is piecewise linear. Let $f' = \phi(f)$, and let $K' = K \cup B$. Then f' and K' satisfy all the conditions for f and K, except for (2); the images $f'(\sigma)$ $(\sigma \in K')$ are homeomorphic to simplices, but they are not necessarily simplices; in general, we have

(2′) For each $\sigma \in K'$, $f'|\sigma$ is a simplicial mapping $f|\sigma: \sigma \longrightarrow \tau$, followed by a piecewise linear homeomorphism $\tau \longrightarrow M$.

Geometrically, in $X = S^3$, the sets $\sigma^2 \cap f'^{-1}(f'(x))$ $(\sigma^2 \in K)$ are the same as the sets $\sigma^2 \cap f^{-1}(f(x))$. Note that, to preserve (7), we may need to subdivide X, by the same method as before.

In a finite number of such steps, we get f' and K' satisfying (1), (2), (3)–(8), and

(9) K' is connected.

Now let $L = f'(K')$. Then L forms a linear graph, in the strict sense of a complex, in which the edges and vertices are known; these are the images of the simplices of K, and L forms a polyhedron in M. We want to produce a situation in which L becomes the union L' of a finite number of polyhedral arcs, every two of which have the same endpoints. (See condition (2″) below.) Note that L is already connected, because K' is.

Step 1. Let V be a subgraph of L, such that (a) V is connected, (b) V is acyclic, and (c) V is maximal with respect to properties (a) and (b). Let W be the closure of $L - V$. Then every edge of W joins two vertices of V. Thus if V is mapped onto a point, every edge of W is mapped onto a 1-sphere.

Step 2. We now subdivide X in the following way. Given $\sigma^2 = v_0 v_1 v_2 \in K'$, such that $f'(\sigma^2)$ is an edge $f'(v_0 v_1) = f'(v_0 v_2)$ of W, we subdivide σ^2, using the midpoints of $v_0 v_1$ and $v_0 v_2$ as the only new vertices. This gives a subdivision K'' of K'. In each σ^3 that contains such a σ^2, we introduce a new vertex v in the interior, and subdivide $\sigma^{3\prime}$ by forming the join of v with the subdivison of Bd σ^3 already defined. This gives a subdivision X' of X, in which K'' forms a subcomplex. As before, $f'(X')$ forms a cell decomposition of M, and forms a subcomplex of a triangulation Y' of M.

Now $S^3 - K''$ is connected, because its homeomorphic image $f'(S^3 - K'')$ is the complement of the linear graph L in M. Let $N(K'')$ be the union of the simplices of X' that intersect K'', and let T be the closure of $S^3 - N(K'')$, so that T forms a subcomplex of X'. Then T is connected. To see this, let P and Q be points of T, and let PQ be a broken line in $S^3 - K''$, joining P to Q. Then PQ can be forced off the simplices of $N(K'')$, one at a time; and this gives a path from P to Q in T. It follows that the 1-skeleton T^1 of T is connected. Note also that T^1 contains all new vertices of X' in interiors of 3-simplices of X. It follows that there is a connected acyclic linear graph J_1 in T^1, containing all such new vertices. For each edge $e = f(\sigma^2)$ of W, we choose an edge σ^1 of K' such that $e = f'(\sigma^1)$; and we let v_e be the mid-point of σ^1, so that v_e is a vertex of X'. For each such v_e, we add to J_1 an edge of X' that joins v_e to a point of J_1. This gives a connected acyclic linear graph J_2.

Finally, let X'' be a subdivision of X', such that every simplex of X'' intersects $K'' \cup J_2$ in a simplex of $K'' \cup J_2$ (or in the empty set). To do this, we use the same join-construction as before. Let $K''' = K \cup J_2$. As before, $f'(X'')$ forms a cell decomposition of M, and there is a triangulation Y'' of M in which $f'(X'')$ forms a subcomplex.

Step 3. Now V and $f'(J_2)$ form subcomplexes of Y'', and each is a connected acyclic linear graph. Let $\psi: M \longrightarrow M$ be a mapping such that $\psi(V)$ and $\psi(f'(J_2))$ are points, such that ψ is a homeomorphism elsewhere, and such that each set $\psi(f'(\sigma))$ $(\sigma \in X'')$ is a polyhedron. Thus $\psi(f'(X''))$ is a polyhedral cell decomposition of M, relative to the triangulation Y''. Let

$$f'' = \psi(f'): S^3 \longrightarrow M.$$

Then $f''(J_2)$ is a point, $f''(f'^{-1}(V))$ is a point, and K''' is the closure of the union of the nondegenerate sets $f''^{-1}(P)$ $(P \in M)$. Thus f'', K''', and X'' satisfy all the conditions for f', K', and X', and also the following:

(2″) There are points v, v' of M such that for each $\sigma^2 \in K'''$, either $f''(\sigma^2)$ is a point or $f''|\sigma^2$ is a simplicial mapping $\sigma^2 \longrightarrow \tau^1$ followed by a piecewise linear homeomorphism $\tau^1 \leftrightarrow B_i$, where B_i is a broken line joining v to v'. Different sets B_i intersect only at v and v'.

We also have:

(10) $f''(X'')$ forms a polyhedral cell-decomposition of M (relative to the triangulation Y'').

If f'' and K''' satisfy (1), (2″), (3)–(5), and (8)–(10), then f'' is called a *special monotonic mapping*. Note that (6) was lost, under the construction which gave us (2″). These conditions can be reformulated more simply as follows. In the following definition, the given triangulation is understood to be the triangulation Y'' used at the end of the preceding discussion. K''', f'', and X'' are replaced by K, f, and X respectively.

DEFINITION 2.2. Let M be a triangulated 3-manifold, let X be a triangulation of S^3, let f be a mapping of S^3 onto M, and let K be the closure of the union of the nondegenerate sets $f^{-1}(P)$ $(P \in M)$. Suppose that

(1) K forms a subcomplex of X;

(2) $\dim K \leqslant 2$;

(3) Every simplex of X intersects K in a simplex (or in the empty set);

(4) There are points v, v' of M such that for each $\sigma^2 \in K$, either $f(\sigma^2)$ is a point or $f|\sigma^2$ is a simplicial mapping $\sigma^2 \longrightarrow \tau^1$ followed by a piecewise linear homeomorphism $\tau^1 \leftrightarrow B_i$, where B_i is a broken line joining v to v'. Different sets B_i intersect only at v and v';

(5) f is monotonic; and

(6) the images $f(\sigma)$ of the simplices of X form a polyhedral cell decomposition of M.

Then f is a *special monotonic mapping* (relative to X).

The preceding discussion has proved the following:

THEOREM 2.2. *Let M be a triangulated 3-manifold, and suppose that M is compact, connected, and simply connected. Then S^3 and M can be triangulated in such a way that M is the image of S^3 under a special monotonic mapping.*

Hereafter we shall have no occasion to use Theorem 2.1.

The loss of condition (6) of Theorem 2.1 is not a serious matter; we shall get it back, at the end of the argument, by the unknotting process in §12 of [M].

3. **An outline of the rest of the proof.** The logical apparatus used in the proof of the cloverleaf theorem is tedious, and the technical definitions in later sections may be easier to understand if we first give a heuristic sketch, indicating the problems that they are designed to deal with.

Given a special monotonic mapping $f\colon S^3 \longrightarrow M$, let $G = \{f^{-1}(y) | y \in M\}$, let S^3/G be the resulting hyperspace, and let π be the projection $S^3 \longrightarrow S^3/G$. Let $G' = M$, and let π' be the projection $M \longrightarrow M/G'$. Let y be a point of a set $\operatorname{Int} B_i$. Then $f^{-1}(y)$ is a linear graph. Such a set $f^{-1}(y)$ will be called a *generic* element of G. Each edge e_i of $f^{-1}(y)$ lies in a 2-simplex of the "singularity complex" K. Suppose, for the moment, that we are dealing with the simplest of the nontrivial cases, namely, Bing's example, described in [M], so that $f^{-1}(y)$ is a figure 8, and $f(K)$ is a linear segment B. Let $\sigma^2 = v_0v_1v_2$ be a 2-simplex of K, such that $f(\sigma^2) = f(v_0v_1) = f(v_0v_2)$, an edge of $f(K)$. In the figure on the left below, the sets $\sigma^2 \cap f^{-1}(y)$ $(y \in \operatorname{Int} B)$ are horizontal segments. B is shown on the right.

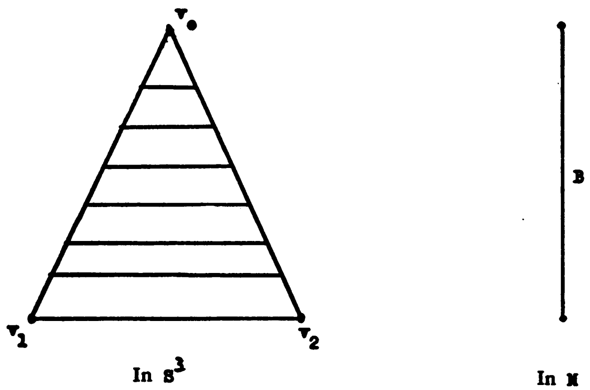


FIGURE 1

If the sets $f^{-1}(y)$ $(y \in \operatorname{Int} B)$ are acyclic, then the cloverleaf theorem follows easily, and it also follows that M is a 3-sphere. (See Finney [F].) If the sets $f^{-1}(y)$ are not acyclic, then σ^2 can be chosen so that each set $f^{-1}(y) - f^{-1}(y) \cap \sigma^2$ $(y \in \operatorname{Int} B)$ is connected. (In Bing's example, every 2-simplex of K has this property.) Hereafter, we shall assume that σ^2 satisfies this condition.

Step 1. First we split $v_1 v_2$ into two arcs e_1, e_2, each of which projects onto a point, in a new hyperspace S^3/G_1 of S^3, as shown in Figure 2.

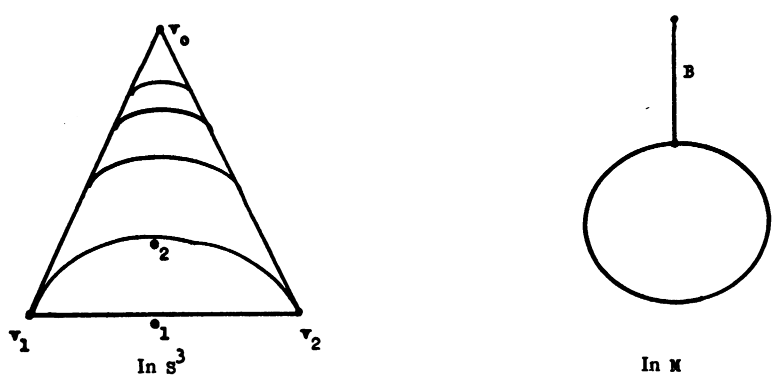


FIGURE 2

To copy this operation in M, so as to preserve the homeomorphism between the hyperspaces, we introduce a polyhedral 1-sphere J in M, spanning a polyhedral disk, and map J onto a point, getting a new hyperspace M/G_1' of M. J intersects B in a point, as shown in the figure on the right above.

In later steps, π_i and π_i' will denote the projections $S^3 \longrightarrow S^3/G_i$ and

$M \longrightarrow M/G'_i$. K_i and K'_i will denote the "singularity complexes" in S^3 and M, that is, the closures of the unions of the nondegenerate elements of G_i and G'_i. From now on, K'_i will be nonempty.

Step 2, Next, in S^3, we introduce a polyhedral disk D_1 mapped onto the same linear interval as σ^2, in a new hyperspace, so that $D_1 \cup \sigma^2$ looks like a portion of the configuration used in Bing's example. (Here, and hereafter in this section, when we "change" geometric configurations, we do not assign new names to the new sets thus obtained.) We now have a new hyperspace S^3/G_2 of S^3, as shown in Figure 3.

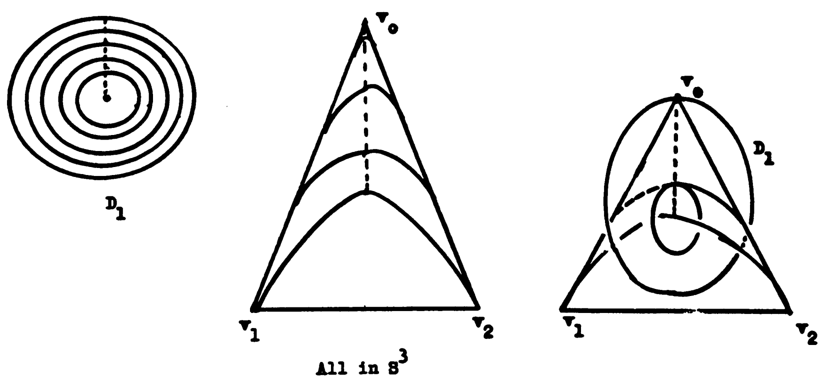


FIGURE 3

We copy this operation in M, by introducing a disk D_2 linked with J as in the figure below; and we define a new M/G'_2 in which $\pi'_2(D_2) = \pi'_2(B)$. In Figure 4, $I = \text{Bd}\, D_2$.

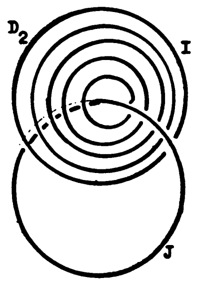


FIGURE 4

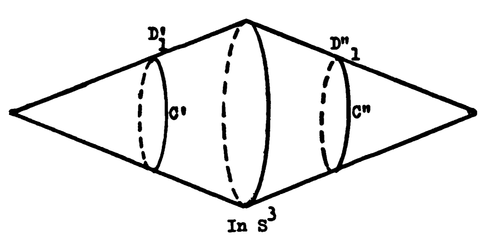


FIGURE 5

Step 3. Next we simplify S^3/G_2 (or rather G_2) by the operation used in the author's proof (presented in [M]) that Bing's hyperspace is homeomorphic to S^3. That is, we split D_1 apart, along Int D_1, into two polyhedral 2-cells

D_1', D_1'', bounding a polyhedral 3-cell C. In the new hyperspace, we regard Int C as empty. See Figure 5. This operation induces a homeomorphism of S^3/G_2; the point is that not only in Bing's example, but also in general, it does no harm to split a circle c in D_1 into two circles c' and c'', because c' and c'' always lie in the same element of the resulting upper semicontinuous decomposition G_3 of $S^3 - \operatorname{Int} C$. In the general case, this is a consequence of the initial hypothesis that for each generic set $f^{-1}(y)$, the set $f^{-1}(y) - f^{-1}(y) \cap \sigma^2$ is connected. Thus $(S^3 - \operatorname{Int} C)/G_3$ is homeomorphic to S^3/G_2, and hence to M/G_2'. Let $G_3' = G_2'$.

Step 4. We now map C onto the union of two linear intervals, by a mapping which is a homeomorphism on $S^3 - C$, in such a way that the latitudinal circles c' and c'' are mashed onto points. We now have hyperspaces S^3/G_4 and $M/G_4' = M/G_3'$. The local situation in S^3 is shown in Figure 6. The situation in M is still as in Figure 4.

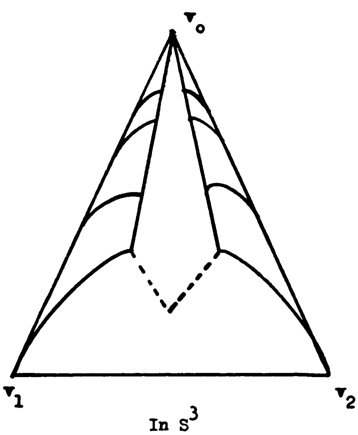


FIGURE 6

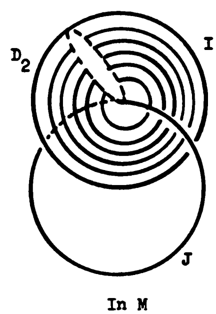


FIGURE 7

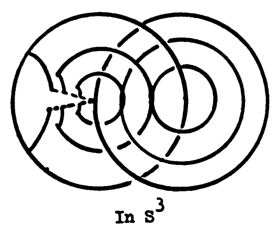


FIGURE 8

Here the quadrilateral in the middle of the figure shows the place where Figure 5 used to be.

Step 5. In M/G_4', $\pi_4'(D_2)$ is a linear interval, and separates space locally, in its interior. That is, $\pi_4'(D_2)$ is the intersection of two 3-cells, each of which contains the interior of the edge in its interior. Since S^3/G_4 and M/G_4' are homeomorphic, $\pi_4(v_0v_1)$ must have the same property. This property of M/G_4' can be abolished by splitting D_2 along the interior of a radial edge, as shown in Figure 7. It is a fact that this operation can be copied in S^3, by splitting K_4 apart in some way. In Bing's example, this is immediately plausible. The situation in S^3 is as in Figure 8. The disk on the left has already been split. To copy in

S^3 the splitting of M, we split the disk on the right, along any radial edge. Thus we have homeomorphic hyperspaces S^3/G_5 and M/G'_5.

Step 6. In Bing's example, the nondegenerate elements of both G_5 and G'_5 now consist of two simply linked circles (that is, two simply linked cloverleaves with one loop apiece), plus a continuous family of generic sets which are acyclic linear graphs. The latter can be shrunk to points by operations preserving the topologies of the hyperspaces. (See, for example, the operation α', defined on p. 463 of [M].) This gives homeomorphic hyperspaces, each of which looks like Figure 9.

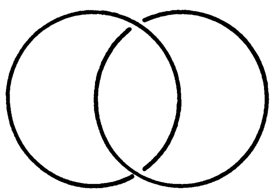


FIGURE 9

In the general case, we can still get two simply linked circles in M, by shrinking the generic sets; and by the same operations in S^3, we can get rid of the "free endpoints" of the corresponding generic sets. But we normally expect that there will still be 2-simplices in the resulting singularity complex K_6 which are mapped onto B by the homeomorphism between the hyperspaces.

Step 7. The next step, then, is to deform B so as to isolate it from the two 1-spheres in G'_6 and from the disks that they bound (except at the endpoints). See Figure 10.

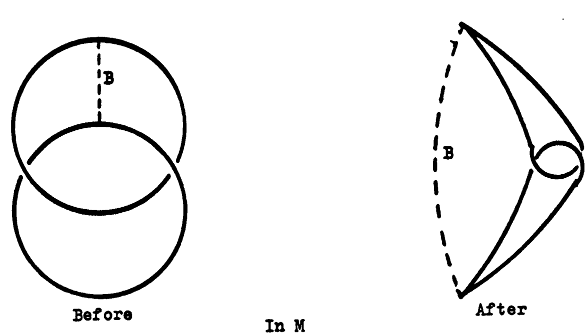


FIGURE 10

We now repeat Steps 1 through 6, keeping clear of the disks on the right. The main stages in M are shown in Figure 11. The corresponding figures for the operations in S^3 would be exactly the same as before.

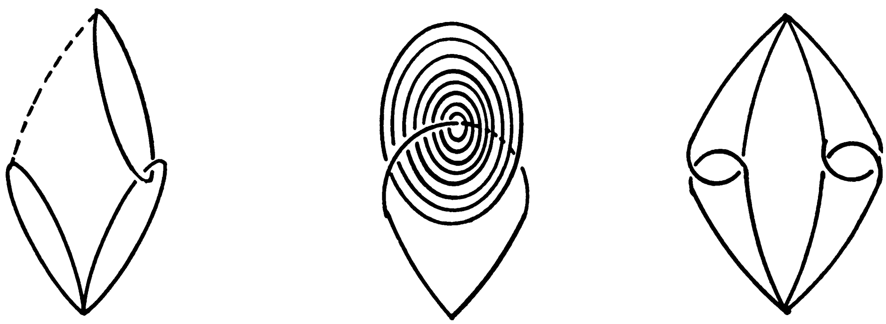


FIGURE 11

And, of course, in the general case, we have more than one B to deal with. But the process must terminate, because it reduces the 1-dimensional Betti numbers of the generic sets $\pi_i^{-1}(y)$.

The general character of the technical problems should now be clear.

(I) The sets and mappings that we use are "built with the hands," and so, in the last resort, there cannot be problems arising from wild imbedding. But in Step 2, we need for the disk D_2 to be the image of D_1, and we need for D_2 to be a polyhedron in M. Since a special monotonic mapping is never piecewise linear, even at the outset, except in trivial cases, we cannot simply claim that the image of a polyhedron in S^3 is a polyhedron in M. Therefore we need a "sufficiently dense" family of 2-dimensional polyhedra in S^3 whose images in M are polyhedra; and we need for this relationship to be defined in such a way that it is preserved under iterations of Steps 1 through 7.

(II) In the general case, the splitting operation in S^3, described in Step 5, does not take the simple form that it takes in Bing's example; we may need not only to split an edge of K_4 into two edges, but also to split a 2-simplex of K_4 into two 2-simplices. (See §4 below.) These duplicate 2-simplices are not always eliminated in Step 6, and so, in general, K_5 and K_6 are not complexes. Therefore, to use a recursion argument, we need an apparatus in which K is not required to be a complex.

(III) We need to assign a meaning to Step 7. Topologically speaking, the situations shown on the left and right in Figure 10 are indistinguishable. Much the same problem of logical definition already arises in Step 1, where we need to assign a meaning to the relationship suggested by Figure 2.

Hence the apparatus described in the following sections. A word of warning: in the deductive form of the proof, the order of the steps will be changed, for technical reasons; Step 7 will be postponed until all iterations of the preceding steps have been carried out. Thus Problem (III) will be avoided rather than solved.

4. **Shrink-equivalences of pseudo-simplicial hyperspaces.** If $f\colon S^3 \longrightarrow M$ is a special monotonic mapping, and $G = \{f^{-1}(x) | x \in M\}$, then the pair $[S^3, G]$ will be called a *pseudo-simplicial* hyperspace of S^3. We shall need this idea in a more general form, in which (1) S^3 is allowed to be any compact connected triangulated 3-manifold M, (2) the hyperspace M/G is not required to be a manifold, and (3) the mapping $\pi|K$ (where π is the projection $M \longrightarrow M/G$ and K is the closure of the union of the nondegenerate sets $\pi^{-1}(x)$) is almost simplicial, but not quite.

DEFINITION 4.1. Let $\mathbf{C}$ be a cell complex, in a triangulated 3-manifold, such that the elements of $\mathbf{C}$ are polyhedra. Then $\mathbf{C}$ is a *pseudo-simplicial* cell complex, or a PS *cell complex*, and is a PS *decomposition* of the union $\mathbf{C}^*$ of its elements. Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be PS cell complexes, and let ϕ be a homeomorphism $\mathbf{C}_1^* \leftrightarrow \mathbf{C}_2^*$. If $\phi(\mathbf{C}_1) = \mathbf{C}_2$ (that is, if ϕ maps elements of $\mathbf{C}_1$ onto elements of $\mathbf{C}_2$), then ϕ is *pseudo-simplicial.*

Note that while a PS cell decomposition is always a collection of polyhedra, the system $\mathbf{C}$ has not been assigned a linear structure, and so a pseudo-simplicial homeomorphism need not be piecewise linear.

Let K be a finite cell-complex, in a finite complex M, such that the elements of K are polyhedra in M. Let K^* be the union of the elements of K. Let ρ be a mapping $K^* \longrightarrow L^*$, where L is a finite complex, such that, for each $\sigma \in K$, $\rho|\sigma$ is a piecewise linear homeomorphism of σ onto a simplex τ of L. Then the pair $[K, \rho]$ is called a *skew-complex.* If $[K_1, \rho_1]$ and $[K_2, \rho_2]$ are skew-complexes, then a mapping $f\colon K_1^* \longrightarrow K_2^*$ is *skew-simplicial* if (1) if $\sigma_1 \in K_1$, then $f(\sigma_1) = \sigma_2 \in K_2$, and (2) for each $\sigma_1 \in K_1$, $\sigma_2 = f(\sigma_1)$, the mapping

$$\rho_2 f(\rho_1|\sigma_1)^{-1}\colon \tau_1 \longrightarrow \tau_2 = \rho_2(\sigma_2)$$

is simplicial.

Thus a skew-complex resembles a complex, in that (1) its simplices look, combinatorially, like simplices, and (2) each simplex σ has a linear structure, induced by $(\rho|\sigma)^{-1}$, relative to which skew-simplicial mappings are defined. Note, however, that a simplex is not determined when its vertices are named. Thus the set $\mathbf{S}$ of all skew-complexes is closed under the operation of "splitting the complex into two parts, combinatorially, along the interior of a simplex." For example, suppose that $[K, \rho]$ is a trivial skew-complex, in which K is a single 2-simplex σ^2 (plus its faces) and ρ is the identity. We can get a new skew-complex $[K', \rho']$ by replacing σ^2 by two PL 2-cells σ_1^2, σ_2^2, with $\sigma_1^2 \cap \sigma_2^2 = \operatorname{Bd} \sigma_1^2 \cap \operatorname{Bd} \sigma_2^2 = \operatorname{Bd} \sigma^2$. We then define ρ' so that, for $i = 1, 2$, $\rho'|\sigma_i^2$ is a piecewise linear homeomorphism (PLH) $\sigma_i^2 \leftrightarrow \sigma^2$ and $\rho'|\operatorname{Bd} \sigma_i^2$ is the identity.

More generally, suppose that the elements of K are PL cells in a PL 3-

manifold M, and let ρ and L be arbitrary. Let D_1^3 and D_2^3 be PL 3-cells in M, with

$$D_1^3 \cap D_2^3 = \mathrm{Bd}\, D_1^3 \cap \mathrm{Bd}\, D_2^3 = D^2,$$

where D^2 is a 2-cell, and $D^2 \cap K^*$ is a union of elements of K. For $i = 1, 2$ let $E_i = \mathrm{Cl}(\mathrm{Bd}\, D_i^3 - D^2)$. Then for $i = 1, 2$ there is a PLH

$$f_i\colon D_i^3 \longrightarrow D_i^3 - \mathrm{Int}\, D^2,$$

such that $f_i|E_i$ is the identity. The operation which replaces M by the polyhedron

$$M' = [M - (D_1^3 \cup D_2^3)] \cup f_1(D_1^3) \cup f_2(D_2^3)$$

will be called a *splitting of* M *at* D^2. Evidently there is a PL mapping $g\colon M' \longrightarrow M$, such that $g|f_i(D_i^3) = f_i^{-1}$ and such that g is the identity elsewhere. Let K' be the set of all cells τ such that $g(\tau) \in K$. Now define $\rho'\colon K'^* \longrightarrow L$ by the condition $\rho'|\tau = \rho(g|\tau)$. Then $[K', \rho']$ is a skew-complex. The operation which replaces $[K, \rho]$ by $[K', \rho']$ is called a *splitting* of $[K, \rho]$ at $K^* \cap \mathrm{Int}\, D^2$.

Similarly, consider a collection $K = \{v, v', B_1, B_2, \cdots, B_n\}$, where v and v' are points, and the B_i's ,are broken lines from v to v', not intersecting elsewhere. Then K forms a skew-complex, relative to an obvious mapping ρ: $K^* \longrightarrow \sigma^1$, where σ^1 is fixed and each mapping $\rho|B_i$ is a PLH.

By abuse of language, we may speak of a PS cell-complex as a skew-complex. if it is clear what sort of mapping ρ is intended.

We can now generalize our preliminary definition of a pseudo-simplicial hyperspace.

DEFINITION 4.2. Let M be a compact connected triangulated 3-manifold, let $\mathbf{C}$ be a pseudo-simplicial cell-decomposition of M, let G be an upper semicontinuous decomposition of M, and let π be the projection $M \longrightarrow M/G$. Let K be the closure of the union of the nondegenerate elements of G. Suppose that

(1) K forms a skew-complex in M;

(2) $\dim K \leqslant 2$;

(3) if $C \in \mathbf{C}$, and C intersects K, then $C \cap K$ forms a subcomplex both of $\mathbf{C}$ and of K;

(4) each $C \in \mathbf{C}$ intersects each element of G in a contractible set;

(5) there is a triangulation Y of M/G such that $\pi(\mathbf{C})$ is a PS cell-decomposition $\mathbf{C}'$ of M/G;

(6) $\pi(K)$ forms a skew-complex in Y (with at most two vertices);

(7) $\pi|K$ is skew-simplicial; and

(8) π is monotonic.

Then the pair $[M, G]$ is a PS *hyperspace* of M (with respect to $\mathbf{C}$ and Y).

Note that we do not call M/G a PS hyperspace, because in the applications we shall be dealing with the structure of G, and this is not determined when M and M/G are known. For example, in Bing's example, S^3/G is homeomorphic to S^3, and therefore homeomorphic to S^3/S^3; but G and S^3 are different.

Note that $\mathbf{C}$ and Y are not part of the structure of the pseudo-simplicial hyperspace $[M, G]$; we are merely requiring that such objects exist. The projection $M \longrightarrow M/G$ will always be denoted by π, and $\mathbf{C}'$ will always denote $\pi(\mathbf{C})$.

Evidently we have been generalizing the definition of a special monotonic mapping, and so we have immediately:

THEOREM 4.1. *If $f\colon S^3 \longrightarrow M$ is a special monotonic mapping, and $G = \{f^{-1}(x) | x \in M\}$, then $[S^3, G]$ is a PS hyperspace of S^3.*

For $i = 1, 2$, let $[M_i, G_i]$ be a PS hyperspace of M_i, with respect to $\mathbf{C}_i$, and let ϕ be a homeomorphism $M_1/G_1 \leftrightarrow M_2/G_2$. If $\mathbf{C}_1$ and $\mathbf{C}_2$ can be chosen so that $\phi(\mathbf{C}_1') = \mathbf{C}_2'$, then ϕ is a *shrink-equivalence* between $[M_1, G_1]$ and $[M_2, G_2]$. If such a ϕ exists, then $[M_1, G_1]$ and $[M_2, G_2]$ are *shrink-equivalent.*

THEOREM 4.2. *If $f\colon S^3 \longrightarrow M$ is a special monotonic mapping, then $[S^3, G]$ and $[M, M]$ are shrink-equivalent.*

PROOF. Let π be the projection $S^3 \longrightarrow S^3/G$, and let $\phi(\pi(x)) = f(x)$.

THEOREM 4.3. *Let ϕ be a shrink-equivalence between $[M_1, G_1]$ and $[M_2, G_2]$, with respect to $\mathbf{C}_1$ and $\mathbf{C}_2$. Let B_2 be an edge of $\mathbf{C}_2$, such that $K_2 \cap \operatorname{Int} B_2 = 0$. Then there is a shrink-equivalence ϕ'', between $[M_1, G_1]$ and $[M_2, G_2]$, with respect to subdivisions $\mathbf{C}_{1,1}$ and $\mathbf{C}_{2,1}$ of $\mathbf{C}_1$ and $\mathbf{C}_2$ respectively, such that if X_2 is the union of the 3-cells of $\mathbf{C}_{2,1}$ that have B_2 as an edge, then X_2 is a 3-cell and $\pi_2 | X_2$ is a homeomorphism.*

PROOF. Let $C^2_{2,i}$ be a 2-cell of $\mathbf{C}_2$, having B_2 as an edge. Let P_2 and Q_2 be the endpoints of B_2. Let $B_{2,i}$ be a polyhedral arc from P_2 to Q_2 in $C^2_{2,i}$, intersecting $\operatorname{Bd} C^2_{2,i}$ only in P_2 and Q_2. As always, let P_2' and Q_2' be $\pi_2(P_2)$ and $\pi_2(Q_2)$. Let $C^2_{1,i}$. be the element of $\mathbf{C}_1$ such that

$$\phi(\pi_1(C^2_{1,i})) = \pi_2(C^2_{2,i}).$$

Then there are vertices $P_{1,i}$ and $Q_{1,i}$ such that

$$\phi(P'_{1,i}) = P_2' \quad \text{and} \quad \phi(Q'_{1,i}) = Q_2'.$$

(Note the double subscripts: $P_{1,i}$ depends, in general, on i as well as on P_2, and similarly for $Q_{1,i}$.) Let $B_{1,i}$ be a polygonal arc in $C^2_{1,i}$, from $P_{1,i}$ to $Q_{1,i}$, intersecting $\mathrm{Bd}\, C^2_{1,i}$ only in $P_{1,i}$ and $Q_{1,i}$.

There is now a shrink-equivalence

$$\begin{aligned}\phi'\colon M_1/G_1 &\leftrightarrow M_2/G_2,\\ : \mathbf{C}'_1 &\longrightarrow \mathbf{C}'_2,\end{aligned}$$

such that

$$\phi'(P'_{1,i}) = P'_2, \quad \phi'(Q'_{1,i}) = Q'_2, \quad \text{and} \quad \phi'(B'_{1,i}) = B'_{2,i}.$$

To construct such a ϕ', first define $\phi' = \phi$ on every 2-cell of $\mathbf{C}'_2$ other than the cells $\pi_1(C^2_{1,i})$, on $\phi^{-1}(B'_2)$, and in the interior of every 3-cell of $\mathbf{C}'_1$ that does not contain a cell $\pi_1(C^2_{1,i})$. Then extend ϕ' so that $\phi'(B'_{1,i}) = B'_{2,i}$ for each i. Now $B'_{1,i}$ separates $C^2_{1,i}$ into 2-cells, on whose boundaries ϕ' is already defined. Similarly for the sets $B'_{2,i} \subset \pi_2(C^2_{2,i})$. Now extend ϕ' so that

$$\phi'(\pi_1(C^2_{1,i})) = \pi_2(C^2_{2,i}).$$

Finally, extend ϕ' to the interiors of the 3-cells $C'_{1,i}$ such that $\phi(C'_{1,i})$ is a 3-cell containing the cell $\pi_2(C^2_{2,i})$.

Now consider a 3-cell $C_{1,i}$ of $\mathbf{C}_1$, such that $C'_{1,i}$ is of this last type, and let $C_{2,i} \in \mathbf{C}_2$ be such that $\phi'(C'_{1,i}) = C'_{2,i}$. Let $C^2_{2,i}$ and $C^2_{2,i+1}$ be the 2-faces of $C_{2,i}$ that contain B_2. Then $B_{2,i} \cup B_{2,i+1}$ is a (simple closed) polygon $J_{2,i}$ in $\mathrm{Bd}\, C_{2,i}$, and bounds a polyhedral disk $D_{2,i}$ in $C_{2,i}$ such that

$$D_{2,i} \cap \mathrm{Bd}\, C_{2,i} = \mathrm{Bd}\, D_{2,i}.$$

Under our hypothesis, $\pi_2 | D_{2,i}$ is a homeomorphism. Let $\sigma_{2,i}$ be the closure of the component of $C_{2,i} - D_{2,i}$ whose boundary contains B_2. Then $\pi_2 | \sigma_{2,i}$ is also a homeomorphism.

We need to copy this pattern in M_1. In $C_{1,i}$, the endpoints $P_{1,i}$ and $P_{1,i+1}$ are not necessarily the same. But $P'_{1,i} = P'_{1,i+1}$; and since each element of G_1 intersects $\mathrm{Bd}\, C_1$ in a contractible set, it follows that $\pi_1^{-1}(P'_{1,i}) \cap \mathrm{Bd}\, C_1$ is connected. Since $P_{1,i}$ and $P_{1,i+1}$ are vertices, it also follows that $P_{1,i}$ can be joined to $P_{1,i+1}$ by a polygonal arc $P_{1,i}P_{1,i+1}$ lying in $K^1 \cap \mathrm{Bd}\, C_1$. Similarly for $Q_{1,i}$ and $Q_{1,i+1}$. Now let

$$J_{1,i} = B_{1,i} \cup B_{1,i+1} \cup P_{1,i}P_{1,i+1} \cup Q_{1,i}Q_{1,i+1}.$$

Now $J_{1,i}$ is the boundary of a polyhedral 2-cell $D_{1,i}$ such that $\mathrm{Int}\, D_{1,i} \subset \mathrm{Int}\, C_{1,i}$, and $D'_{1,i}$ is a 2-cell. (The projection π_1 merely shrinks to points each of two disjoint arcs in $\mathrm{Bd}\, D_{1,i}$, and is a homeomorphism elsewhere.)

Finally, we define a new shrink-equivalence ϕ'', such that $\phi''(D'_{1,i}) = D'_{2,i}$ for each i. First we define $\phi'' = \phi$ on the set $M - \bigcup_i \operatorname{Int} C_{1,i}$. Then we extend the given $\phi'' | \operatorname{Bd} D'_{1,i}$ to $D'_{1,i}$, so that $\phi''(D'_{1,i}) = D'_{2,i}$. Finally we observe that $D'_{1,i}$ decomposes $C'_{1,i}$ into two 3-cells, intersecting in $D'_{1,i}$; and similarly for $D'_{2,i}$ in $C'_{2,i}$. Now extend ϕ'' to the interiors of these 3-cells, in such a way that $\phi''(\sigma'_{1,i}) = \sigma'_{2,i}$, where $\sigma_{1,i}$ is one of the 3-cells into which $D_{1,i}$ decomposes $C_{1,i}$.

Now we have subdivisions $\mathbf{C}_{1,1}$ of $\mathbf{C}_1$ and $\mathbf{C}_{2,1}$ of $\mathbf{C}_2$, such that $\phi''(\mathbf{C}'_{1,1}) = \mathbf{C}'_{2,1}$. The subdivision process preserves condition (4) of Definition 4.2. Thus all the conditions of the conclusion of the theorem are satisfied.

I believe that if M_1/G_1 and M_2/G_2 are homeomorphic, then $[M_1, G_1]$ and $[M_2, G_2]$ are shrink-equivalent. An easy proof of this would simplify the present paper, but I have not been able to find one.

5. **The shrink-operation** α. Given two shrink-equivalent hyperspaces $[M_1, G_1]$ and $[M_2, G_2]$, a *shrink-operation* is an operation performed on one or both of them, preserving the relation of shrink-equivalence between them. Such operations will be defined in the following three sections.

Let ϕ be a shrink-equivalence between $[M_1, G_1]$ and $[M_2, G_2]$, with respect to $\mathbf{C}_1$ and $\mathbf{C}_2$. Now K_1 is a skew-complex, $\pi_1 | K_1$ is skew-simplicial, and $\pi_1(K_1)$ is a skew-complex with at most two vertices. Generalizing slightly a definition given in §3, we say that an element $\pi_1^{-1}(x)$ of G_1 is *generic* if x lies in the interior of an edge of $\pi_1(K_1)$. Let $\sigma^2 \in K_1$, suppose that $\pi_1(\sigma^2)$ is an edge of $\pi_1(K_1)$, and suppose that if $\pi_1^{-1}(x)$ is a generic set intersecting σ^2, then $\operatorname{Int} \sigma^2 \cap \pi_1^{-1}(x)$ does not separate $\pi_1^{-1}(x)$. Let B be a broken line in M_2, such that $B' = \phi(\pi_1(\sigma^2))$, and suppose that B intersects K_2 in at most one or both of the endpoints of B. By Theorem 4.3, we may suppose that ϕ, $\mathbf{C}_1$, and $\mathbf{C}_2$ are chosen so that if X is the union of the cells of $\mathbf{C}_2$ that contain B, then X is a 3-cell and $\pi_2 | X$ is a homeomorphism. In the operation α, presently to be defined, changes will be made only in X, with no changes on the boundary, except for subdivisions. Thus, locally, we are working in a manifold.

Let $C_{1,1}$ and $C_{1,2} \in \mathbf{C}_1$, such that $C_{1,1} \cap C_{1,2} \supset \sigma^2$; and let $\sigma^2 = v_0 v_1 v_2$, as in Figure 1 in §3. (Note that here and hereafter the notation of simplices is used for skew-simplices, although a skew-simplex is not necessarily determined when its vertices are named.) Let $C_{2,j}$ $(j = 1, 2)$ be the elements of $\mathbf{C}_2$ such that

$$C'_{2,j} = \pi_2(C_{2,j}) = \phi(C'_{1,j}) = \phi(\pi_1(C_{1,j})).$$

Now split $v_1 v_2$ into two edges e_1, e_2, as in Figure 2. The edge e_2 now cuts σ^2 into 2-cells. We now subdivide again, like this:

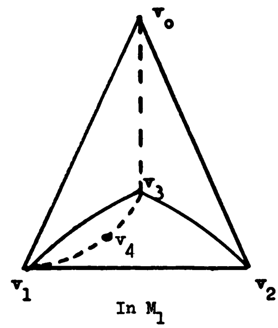


FIGURE 12

Here v_3 and v_4 are new vertices, and the three dotted edges are new edges. Thus we have subdivided each set Bd $C_{1,j}$. We introduce a new vertex $v_{1,j}$ in Int $C_{1,j}$, and subdivide $C_{1,j}$ by forming the join. This gives a cell-decomposition $\mathbf{C}_{1,1}$ of M_1, in which the new singularity complex $K_{1,1}$ forms a subcomplex. Note the purpose of the two new vertices, and the new dotted edges, in Bd $C_{1,j}$: we need to subdivide finely enough so that the elements of $\pi_1(\mathbf{C}_{1,1})$ are cells.

We now need to copy all this in M_2, so as to get a shrink-equivalence between two new hyperspaces. Let B be as in the beginning of this section. We express B as the union of two broken lines B_1 and B_2, end to end. We know that $C_{2,1} \cap C_{2,2} = B$. (See Theorem 4.3; we are not distinguishing between sets in X and their projections under π_2.) We now split $C_{2,1} \cup C_{2,2}$ apart, along Int B_2, getting broken lines B_2', B_2'', whose union is a polygon J. As before, the dotted edges belong to the subdivision.

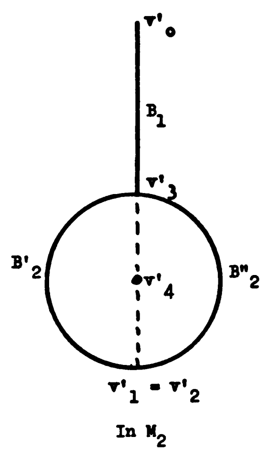


FIGURE 13

We form a new hyperspace $[M_2, G_2']$ in which $\pi_2'(J)$ is a point.

The sets $C_{2,j}$ have now been replaced by 3-cells $E_{2,j}$, intersecting in the union of B_1 and a 2-cell spanning J. In the interior of $E_{2,j}$ we introduce a new vertex $v_{2,j}$, and form the join with the given cell-decomposition of $\mathrm{Bd}\, E_{2,j}$, with identifications along B_1. This gives a cell-decomposition $\mathbf{C}_{2,1}$ of M_2, in which the new singularity set $K_{2,1}$ forms a subcomplex. In fact, there is a homeomorphism ϕ: $M_1/G_1' \leftrightarrow M_2/G_2'$ which is a shrink-equivalence with respect to $\mathbf{C}_{1,1}$ and $\mathbf{C}_{2,1}$. To construct such a ϕ_1, we use the old ϕ on the old sets $C_1' = \pi_1(C_1)$; and on the new sets $C_1' = \pi_1'(C_1)$ $(C_1 \in \mathbf{C}_{1,1})$ we follow the directions given by the primes in Figure 13. That is,

$$\pi_1'(v_0v_1v_3) \leftrightarrow \pi_2'(v_0'v_3') = \pi_2'(B_1);$$

$$\pi_1'(v_0v_3v_2) \leftrightarrow \pi_2'(B_1);$$

the upper edge $\pi_1'(v_1v_3)$ is mapped onto $\pi_2'(v_3')$, and similarly for the other new edges. The extension of ϕ_1 to the new 2-faces shown in the figure is obvious. Finally, we define

$$\phi_1(\pi_1'(v_{1,j})) = \pi_2'(v_{2,j}),$$

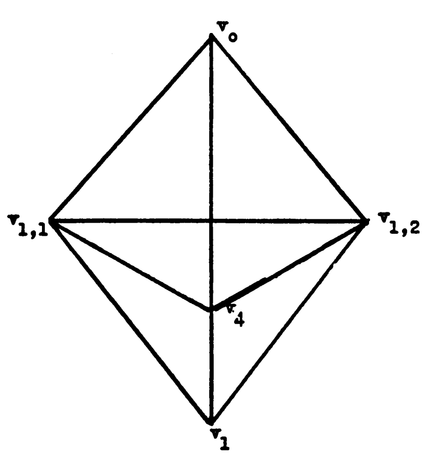
and extend ϕ_1 so as to preserve joins.

Thus ϕ_1 is a shrink-equivalence, with respect to $\mathbf{C}_{1,1}$ and $\mathbf{C}_{2,1}$. The operation $[M_i, G_i] \longrightarrow [M_i, G_i']$, defined in the above discussion, will be called the operation α_1. Thus we have proved the following.

THEOREM 5.1. *α_1 is a shrink-operation.*

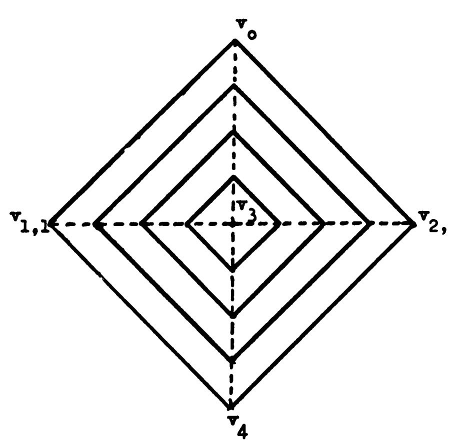
This theorem does not supersede the preceding discussion; the rest of the section uses the latter.

Consider now the 2-faces $v_{1,j}v_0v_3$, $v_{1,j}v_3v_4$, $v_{1,j}v_4v_1$ of $\mathbf{C}_{1,1}$ $(j = 1, 2)$. Their union is a polyhedral disk; see Figure 14.



In M1

FIGURE 14



In M1

FIGURE 15

Let $D_1 = v_{1,1}v_0v_3 \cup v_{1,1}v_3v_4 \cup v_{2,1}v_0v_3 \cup v_{2,1}v_3v_4$, so that D_1 is a polyhedral disk. Adjoining these "simplices," plus their faces, to $K_{1,1}$, we get a new pseudo-simplicial complex $K_{1,2}$. We now define a new hyperspace G_1'', in which D_1 is mapped onto an edge, as indicated in Figure 15. There is now a subdivision $\mathbf{C}_{1,2}$ of $\mathbf{C}_{1,1}$, in which D_1 forms a subcomplex.

All this can be copied in M_2; the figures would be the same, except for primes on the vertices. Thus we get D_2 in M_2, $K_{2,2}$, G_2'', and $\mathbf{C}_{2,2}$; and there is a shrink-equivalence between $[M_1, G_1'']$ and $[M_2, G_2'']$.

Steps 3 and 4 of §3 are now mathematically legitimate: first we split M_1 along $\operatorname{Int} D_1$, and then we map the resulting 3-cell onto the union of two edges. (See Figures 5 and 6 of §3.) This gives $[M_1, G_1''']$. We supposed at the outset that if $\pi_1^{-1}(x)$ was a generic set intersecting σ^2, then $\pi_1^{-1}(x) \cap \operatorname{Int} \sigma^2$ does not separate $\pi_1^{-1}(x)$. It follows that Steps 3 and 4 preserve the topology of the hyperspace. And in fact they also preserve the relation of shrink-equivalence. Under the splitting, where D_1 is replaced by a 3-cell C, cells of $\mathbf{C}_{1,2}$ go onto cells of a cell-decomposition $\mathbf{C}_{1,3}$ of $M - \operatorname{Int} C$; and under the "mashing" operation of Step 4, cells of $\mathbf{C}_{1,3}$ go onto elements of a cell-decomposition $\mathbf{C}_{1,4}$ of M_1. It is thus a straightforward matter to define a homeomorphism

$$\begin{aligned} \phi_2 &: M_1/G_1''' \leftrightarrow M_2/G_2'', \\ &: \mathbf{C}_{1,4}' \leftrightarrow \mathbf{C}_{2,2}'. \end{aligned}$$

The total operation described in this section is called the operation α. We have proved:

THEOREM 5.2. *α is a shrink-operation.*

Recall that the configuration that we start with, in the operation α, is described in Figure 1 of §3. The final result is described by Figures 4 and 6.

Let $[M, G]$ be a pseudo-simplicial hyperspace of M, with projection π. In the interior of each edge e_i of the skew-complex $\pi(K)$, we choose a point x_i. Then $\pi^{-1}(x_i)$ is a generic element of G. The *index* $\operatorname{Ind} G$ of G is the sum of the 1-dimensional Betti numbers $p^1(\pi^{-1}(x_i))$, with integers modulo 2 as coefficients. Since $\pi|K$ is skew-simplicial, it is evident that $\operatorname{Ind} G$ is independent of the choice of the points x_i. Note also that if $\operatorname{Ind} G = 0$, then all generic sets are acyclic.

THEOREM 5.3. *If* $[M_1, G_1] \longrightarrow [M_1, G_1']$ *and* $[M_2, G_2] \longrightarrow [M_2, G_2']$, *under the operation* α, *then* $\operatorname{Ind} G_1' = \operatorname{Ind} G_1 - 1$.

PROOF. Compare Figures 1 and 6, and recall the initial hypothesis for σ^2.

6. **The shrink-operation** β.

THEOREM 6.1. *Let* ϕ *be a shrink-equivalence between* $[M_1, G_1]$ *and* $[M_2, G_2]$, *with respect to* $\mathbf{C}_1$ *and* $\mathbf{C}_2$. *Let* $\sigma^2 \in K_2$, *with two edges* σ_1^1 *and* σ_2^1, *such that* $\pi_2(\sigma^2)$ *is an edge of* $\pi_2(K_2)$ *and* $\pi_2(\sigma_1^1) = \pi_2(\sigma_2^1) = \pi_2(\sigma^2)$. *Let* G_2' *be the decomposition of* M_2 *obtained by splitting* σ^2 *away from the rest of* K_2 *along* $\operatorname{Int}\sigma_1^1$. *Then there is a pseudo-simplicial hyperspace* $[M_1, G_1']$ *such that* (1) $[M_1, G_1']$ *and* $[M_2, G_2']$ *are shrink-equivalent, and* (2) $\operatorname{Ind} G_1' \leqslant \operatorname{Ind} G_1$.

(For an explicit definition of what we mean by "splitting" operations, see the remarks preceding Definition 4.2.)

DEFINITION. The operation described in Theorem 6.1 will be called the operation β.

Evidently Theorem 6.1 is designed to justify Step 5 of §3.

PROOF. Let C_2 and D_2 be the two 3-faces of $\mathbf{C}_2$ that have σ^2 as a face, and let c_2 and d_2 be the 2-faces of C_2 and D_2 (other than σ^2) that have σ_1^1 as an edge. Let c_1 and d_1 be the 2-faces of $\mathbf{C}_1$ such that

$$\pi_2(c_2) = \phi(\pi_1(c_1)), \qquad \pi_2(d_2) = \phi(\pi_1(d_1)).$$

Since ϕ is a homeomorphism, it follows that $\operatorname{Int}\pi_1(c_1 \cup d_1)$ separates M_1/G_1 locally if and only if $\operatorname{Int}\pi_2(c_2 \cup d_2)$ separates M_2/G_2 locally.

Now perform the splitting of K_2 along $\operatorname{Int}\sigma_1^1$ by splitting M_2 apart along $\operatorname{Int}(c_2 \cup d_2)$, in the sense explained just before Definition 4.2. Thus $c_2 \cup d_2$ is replaced by the union of two 3-cells $E_{2,1}$ and $E_{2,2}$, intersecting in a common 2-face, where σ_1^1 used to be. Thus we have a new pseudo-simplicial decomposition $\mathbf{C}_{2,1}$ of M_2.

We need to copy this splitting operation, in some way, in M_1. Let e be the edge of c_1 such that $\phi(\pi_1(e)) = \pi_2(\sigma_1^1)$, let y be a point of $\operatorname{Int}\pi_1(e)$, and let $x_1 = c_1 \cap \pi_1^{-1}(y)$. Now $\pi_1^{-1}(y)$ is a connected linear graph, and so $\pi_1^{-1}(y)$ contains a broken line B joining x_1 to a point x of d_1. Since $\pi_1|K_1$ is skew-simplicial, it follows that B intersects every $\tau^2 \in K_1$ in a broken line, a point, or the empty set. Let $B_1, B_2, \cdots, B_n$ be the broken lines $\tau^2 \cap B$, in the order of their appearance on B, starting at x_1. (The B_j's are not necessarily linear intervals, because K_1 is not in general a complex.) Let the endpoints of the B_j's be $x_1, x_2, \cdots, x_{n+1}$ (in the order from x_1), so that $x = x_{n+1}$. For each j, let τ_j^2 be the skew-2-simplex of K that contains the broken line $B_j = x_j x_{j+1}$. For each j, let τ_j^1 be the edge of K_1 that contains x_j. Then τ_j^1 is an edge of τ_j^2 for each j; and if $j > 1$, then τ_j^1 is also an edge of τ_{j-1}^2. Since K_1 is merely a skew-complex, $\tau_{j-1}^2 \cap \tau_j^2$ may contain more than τ_j^1. Note also that nonconsecutive simplices τ_j^2 may intersect in unpredictable ways, on edges which are mapped onto points by π_1. Thus the set

$$R = \bigcup_{j=1}^{n} \operatorname{Int} \tau_j^2 \cup \bigcup_{j=1}^{n+1} \operatorname{Int} \tau_j^i$$

is homeomorphic to the union of an open rectangular region and a pair of opposite faces, but $\bigcup_{j=1}^{n} \tau_j^2$ is a "singular 2-cell with singularities on its boundary."

We are now ready to "split M_1 apart along R." First we split M_1 apart along $\operatorname{Int}(c_1 \cup \tau_1^2)$. Then we split along $\operatorname{Int} \tau_2^2 \cup \operatorname{Int} \tau_2^1$. Continuing in this fashion, we finally split along $\operatorname{Int}(d_1 \cup \tau_n^2)$. Thus the union of c_1, d_1, and the closure of R is replaced by a 3-cell C, with singularities on its boundary, such that the boundary of C is the union of the sets obtained by splitting c_1, d_1, and the simplices τ_j^2. This gives a new G_1', and a new K_1', on which the new π_1' is skew-simplicial. Obviously it does not give a pseudo-simplicial cell-decomposition of M_1, because C is not a cell. But we can subdivide C into a chain $C_0, C_1, \cdots, C_n, C_{n+1}$ of polyhedral 3-cells, such that the following are true.

(1) For each $i < n + 1$, $C_j \cap C_{j+1}$ is a polyhedral 2-cell, bounded by the 1-sphere obtained by splitting τ_{j+1}^1, and intersecting K_1' nowhere else.

(2) Bd C_0 is the union of $C_0 \cap C_1$ and the 2-cells obtained by splitting c_1.

(3) Bd C_{n+1} is the union of $C_n \cap C_{n+1}$ and the 2-cells obtained by splitting d_1.

(4) For $0 < j < n + 1$, Bd C_j is the union of $C_j \cap C_{j-1}$, $C_j \cap C_{j+1}$ and the 2-cells obtained by splitting τ_j^2.

This gives a pseudo-simplicial cell-decomposition $\mathbf{C}_{1,1}$ of M_1.

We recall that in M_2, $c_2 \cup d_2$ was replaced by two 3-cells $E_{2,1}$ and $E_{2,2}$ with a common 2-face. To copy the pattern of the chain $C_0, C_1, \cdots, C_n, C_{n+1}$ in M_2, it is sufficient to split apart the common 2-face, n times. This gives a pseudo-simplicial cell-decomposition $\mathbf{C}_{2,2}$ of M_2.

Obviously $\operatorname{Ind} G' \leqslant \operatorname{Ind} G_1$; the splitting operation performed on K_1 cannot increase the 1-dimensional Betti numbers of the generic sets.

It remains to construct a shrink-equivalence ϕ' between $[M_1, G_1']$ and $[M_2, G_2']$. We have a natural correspondence between the faces of $\mathbf{C}_{1,1}' = \pi_1'(\mathbf{C}_{1,1})$ and those of $\mathbf{C}_{2,2}' = \pi_2'(\mathbf{C}_{2,2})$. On each 3-cell C_1' in $\mathbf{C}_{1,1}'$, we can realize this correspondence by a homeomorphism $\phi'|C_1'\colon C_1' \leftrightarrow C_2' \in \mathbf{C}_{2,2}'$, mapping faces onto faces, in such a way that the mappings agree on common faces of different 3-cells. To do this, we proceed as in the proof of Theorem 4.3, making repeated applications of the theorem which states that every homeomorphism between the boundaries of two cells can be extended to give a homeomorphism between the cells.

So far, the notation $\phi'|C_1'$ is an abuse of language, because we have not yet defined a global homeomorphism ϕ' whose restriction to C'_1 is $\phi'|C'_1$. But we noted at the outset that $\operatorname{Int}(c_1 \cup d_1)$ separates M_1/G_1 locally if and only if $\operatorname{Int}(c_2 \cup d_2)$ separates M_2/G_2 locally. As in the preceding discussion, let e be the edge of c_1 such that $\phi(\pi_1(e)) = \pi_2(\sigma_1^1)$. Then the edges of $\mathbf{C}_{1,1}$, obtained by splitting e, are identified by π'_1 if and only if the edges of $\mathbf{C}'_{2,2}$, obtained by splitting σ_1^1, are identified by π'_2. Thus the mappings $\phi'|C'_1$ fit together to give a well-defined global mapping ϕ'; and by the same reasoning, in reverse, ϕ' is one-to-one.

7. **The shrink-operation** γ. Suppose that ϕ is a shrink-equivalence between $[M_1, G_1]$ and $[M_2, G_2]$, with respect to $\mathbf{C}_1$ and $\mathbf{C}_2$. Let $\sigma^2 = v_0v_1v_2 \in K_1$, such that $\pi_1(\sigma^2)$ is an edge of $\pi_1(K_1)$; and suppose that σ^2 is *free* in K_1, in the sense that

(1) $\pi_1(v_0v_1) = \pi_1(v_0v_2) = \pi_1(\sigma^2)$, and

(2) v_0v_2. is an edge of no 2-face of K_1 other than σ^2.

The operation γ deletes $\operatorname{Int}\sigma^2 \cup \operatorname{Int} v_0v_2$ from K_1. (Similarly if $\sigma^2 \in K_2$, and analogous conditions are satisfied.)

THEOREM 7.1. *γ is a shrink-operation.*

PROOF. Let D be a polyhedral 3-cell in M_1, such that

(1) $\operatorname{Int}\sigma^2 \cup \operatorname{Int} v_0v_2$ lies in $\operatorname{Int} D$;

(2) $v_0v_1 \cup v_1v_2$ lies in $\operatorname{Bd} D$;

(3) If c is a 2-face of $\mathbf{C}_1$, other than σ^2, with v_0v_2 as an edge, then $c \cap D$ is a polyhedral 2-cell d, $d \cap \operatorname{Bd} D$ is a broken line b from v_0 to v_2, and $\operatorname{Int} b \subset \operatorname{Int} c$; and

(4) If c is a 2-face of $\mathbf{C}_1$, and v_0v_2 is not an edge of c, then c intersects D at most in a subset of $v_0v_1 \cup v_1v_2$.

Now the 2-cells d intersect one another only in v_0v_2. We replace each of them, in the 2-skeleton $\mathbf{C}_1^2$ of $\mathbf{C}_1$, by a polyhedral disk d' in D, such that $\operatorname{Bd} d' = b \cup v_0v_1 \cup v_1v_2$, in such a way that different sets d' intersect one another only where they must, in $v_0v_1 \cup v_1v_2$. Thus each $c \in \mathbf{C}_1^2$, with v_0v_2 as an edge, is replaced by a 2-cell c'; and these form the 2-skeleton of a cell-decomposition $\mathbf{D}_1$ of M. In fact, if $[M_1, G'_1]$ is a new hyperspace, then there is a homeomorphism

$$\psi: M_1/G'_1 \leftrightarrow M_1/G_1,$$
$$: \mathbf{D}_1 \leftrightarrow \mathbf{C}_1.$$

(The obvious method works: first we map the projections $\pi'_1(c')$ onto the corresponding sets $\pi_1(c)$, preserving edges; and then we extend the mapping to the interiors of the 3-cells.) Now $\phi(\psi)$ is a shrink-equivalence $M_1/G'_1 \leftrightarrow M_2/G_2$, $\mathbf{D}_1 \leftrightarrow \mathbf{C}_2$, which is what we wanted.

8. **Proof of the cloverleaf theorem: weak form.** In this section we shall show that under the hypothesis of Theorem 1.1, there are hyperspaces S^3/K_1 and M/K_2 which satisfy all the conditions of Theorem 1.1, except that K_1 is not necessarily a linear graph. Here the notation of quotient spaces is that of Theorem 1.1. In the following discussion, we resume the notation of the preceding sections.

Let f be a special monotonic mapping $S^3 \longrightarrow M$, let

$$\begin{aligned} G_1 &= \{f^{-1}(x) | x \in M\} \\ &= \{\pi_1^{-1}(y) | y \in S^3/G_1\}, \end{aligned}$$

and let K_1 be the singularity complex of $[S_3, G_1]$. The corresponding G_2 is M, and K_2 is empty. By Theorem 4.2, $[S^3, G_1]$ and $[M, G_2]$ are shrink-equivalent, under a homeomorphism ϕ, with respect to cell-decompositions $\mathbf{C}_1$ and $\mathbf{C}_2$. Hereafter, the conventional notations G_i, K_i, π_i, and $\mathbf{C}_i$ will be used without comment.

In the proof of the weak form of the cloverleaf theorem, we first need to take care of a special case.

Case 1. Suppose that all generic sets $\pi_1^{-1}(y)$ are acyclic. Then they all have endpoints, and so they can be eliminated by the operation γ. This gives $G_{1,1}$ and $K_{1,1}$; and $K_{1,1}$ is the union of two disjoint connected skew-complexes, each of which is mapped onto a point by $\pi_{1,1}$. Thus the weak form of the cloverleaf theorem holds, with $K_2 = 0$.

Case 2. Suppose that some generic set $\pi_1^{-1}(y)$ is not acyclic. Then the operation α can be used.

Step A. Perform the operation α. This gives hyperspaces $[S^3, G_{1,1}]$ and $[M, G_{2,1}]$ which are shrink-equivalent, under a homeomorphism ϕ_1, with respect to cell-decompositions $\mathbf{C}_{1,1}$ and $\mathbf{C}_{2,1}$. In M, we now have two simply linked polygons I_1, J_1, and spanning disks D_1' and E_1, as in Figure 4 of §3. The new notation is shown in the figure below. $D_1' \cap E_1$ is the broken line B'.

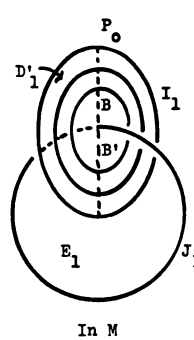


FIGURE 16

D_1' and E_1 form subcomplexes of $\mathbf{C}_{2,1}$.

Step B. Let σ^2 be a 2-simplex of D_1' which contains B. Split σ^2 away from $\operatorname{Int} B$ by the operation β, leaving B fixed. This gives new hyperspaces $[S^3, G_{1,2}]$ and $[M, G_{2,2}]$, which are shrink-equivalent with respect to cell-decompositions $\mathbf{C}_{1,2}$ of S^3 and $\mathbf{C}_{2,2}$ of M.

Step C. By the operation γ, reduce the generic elements of $G_{2,2}$ to the points of B that contain them, leaving B fixed throughout the process. This gives the situation shown on the left in Figure 10, §3. We now have hyperspaces $[S^3, G_{1,3}] = [S^3, G_{1,2}]$ and $[M, G_{1,3}]$, and these are shrink-equivalent with respect to $\mathbf{C}_{1,3}$ and $\mathbf{C}_{2,3}$.

Step D. The disks D_1' and E_1 are not necessarily subcomplexes of $\mathbf{C}_{2,2}^2$, but they still exist; we have

(1) $\operatorname{Bd} D_1' = I_1$;

(2) $\operatorname{Bd} E_1 = J_1$;

(3) $D_1' \cap E_1$ is the broken line B';

(4) $D_1' \cap \pi_{2,3}^{-1}\phi_3\pi_{1,3}(K_{1,3}) = I_1 \cup B$; and

(5) $E_1 \cap \pi_{2,3}^{-1}\phi_3\pi_{1,3}(K_{1,3})$ is the union of J_1 and the lower endpoint of B'.

Therefore there is a disk D_1 such that

(1′) $\operatorname{Bd} D_1 = I_1$;

(2′) D_1 and E_1 are as in the definition of two simply linked cloverleaves; and

(4′) $D_1 \cap \pi_{2,3}^{-1}\phi_3\pi_{1,3}(K_{1,3})$ is the union of I_1 and a single point of J_1.

(To get such a D_1, we merely move $\operatorname{Int} D_1'$ slightly off to one side, preserving the intersection properties of D_1' and E_1.) We now have the following configuration.

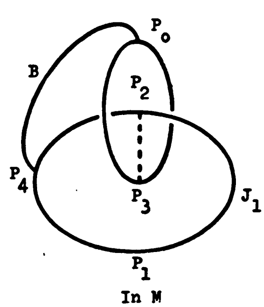


FIGURE 17

In the figure, $D_1 \cap E_1$ is the broken line P_2P_3, and the figure has been distorted so as to convey something of condition (4′).

Let $G_{1,4} = G_{1,3}$, and let $G_{2,4} = G_{2,3}$. At this stage, the singularity complex in M is $K_{2,4} = I_1 \cup J_1$, and B intersects this set only at its endpoints.

By Theorem 4.3 it follows that there are subdivisions $\mathbf{C}_{1,4}$ and $\mathbf{C}_{2,4}$, of $\mathbf{C}_{1,3}$ and $\mathbf{C}_{2,3}$, such that $[S^3, G_{1,4}]$ and $[M, G_{2,4}]$ are shrink-equivalent with respect to the new subdivisions, and such that if X is the union of the 3-cells of $\mathbf{C}_{2,4}$ that contain B, then $\pi_{2,4}|X$ is a homeomorphism and X is a 3-cell. Therefore Steps A through D can be iterated if need be.

Step E. If all generic sets in S^3 are acyclic, proceed immediately to Step F. If not, repeat Steps A through D until this condition holds. (The process must terminate, because the operation α reduces the index. See Theorem 5.3.) Then eliminate the generic sets from the singularity complex in S^3, by iterations of the operation γ. We then have the following situation.

(1) $S^3/G_{1,5}$ and $M/G_{2,5}$ are homeomorphic. (We no longer need shrink-equivalence.)

(2) $K_{1,5}$ contains no generic sets; it has exactly two components A_1, A_2, each of which is mapped onto a point by $\pi_{1,5}$.

(3) We have $I_1, I_2, \cdots, I_n$ and $J_1, J_2, \cdots, J_n$ in M, with spanning disks D_j and E_j. These satisfy all the conditions for two simply linked cloverleaves, except that $\bigcup I_j$ and $\bigcup J_j$ may be not quite cloverleaves: we have not shown that there is a single point P which is the intersection of every two polygons I_j; and similarly for the sets J_j. But $\bigcup I_j \cap \bigcup J_j = 0$; different sets I_j (and J_j) intersect only in a single point; D_j intersects D_k only in $I_j \cap I_k$ (and similarly for the sets J_j and E_j); and D_j intersects E_k in the way required in the definition of two simply linked cloverleaves. In the figure, we show the case $n = 2$.

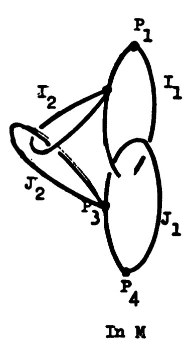


FIGURE 18

To convert such a configuration to the desired final form, we merely map the short arcs P_1P_2 and P_3P_4 onto the points P_1 and P_4, by a mapping which is a homeomorphism everywhere else, and which maps the D_j's and E_j's onto polyhedra. (To be precise, the short arcs P_1P_2 (and P_3P_4) are the arcs in I_1 and J_1 respectively which do not intersect E_1 (and D_1).) In the general case, iterate this operation $n - 1$ times.

This completes the proof of the weak form of the cloverleaf theorem, with $K_1 = A_1 \cup A_2$ and $K_2 = \bigcup I_j \cup \bigcup J_j$.

9. **Proof of the cloverleaf theorem: conclusion.** Let $[S^3, K_1]$ and $[M, K_2]$ be as in the weak form of the cloverleaf theorem. We need to reduce K_1 to a linear graph. For this purpose, we use the unknotting process described in §§11, 12, and 13 of [M]. The following is a sketch. For details, see [M], which gives a complete proof in a more difficult case.

Step 1. First "enlarge" each of the two components A_1 and A_2 of K_1, so that each of them becomes a polyhedral 3-manifold B_i with boundary. Let $B = B_1 \cup B_2$. This process can be carried out in such a way that S^3/B is homeomorphic to S^3/K_1. (Here quotient spaces are being described in the notation of Theorem 1.1.)

Step 2. Let W be the closure of $S^3 - B$. By a theorem of R. H. Fox [F], W is homeomorphic to a polyhedron X in S^3, such that the closure T of $S^3 - X$ is a tubular set; that is, T is homeomorphic to a regular neighborhood of a linear graph. Thus S^3/B and S^3/T are homeomorphic.

Step 3. There is now a mapping

$$\begin{aligned} g\colon S^3 &\longrightarrow S^3 \\ \colon T &\longrightarrow K_{1,1}, \end{aligned}$$

where $K_{1,1}$ is a linear graph and $g|(S^3 - T)$ is a homeomorphism. Thus S^3/T and $S^3/K_{1,1}$ are homeomorphic.

Therefore $S^3/K_{1,1}$ and M/K_2 are homeomorphic, which was to be proved.

10. **A strong form of the cloverleaf theorem.**

THEOREM 10.1. *Under the conditions of the cloverleaf theorem, there are regular neighborhoods T_1 and T_2, of K_1 in S^3 and of K_2 in M respectively, such that there is a homeomorphism $\phi\colon S^3/K_1 \leftrightarrow M/K_2$, such that ϕ is piecewise linear on every finite polyhedron in $S^3 - K_1$, and $\phi(T_1) = T_2$.*

In the proof, we shall use the following.

LEMMA 1. *There is a homeomorphism $g\colon S^3/K_1 \leftrightarrow M/K_2$ such that g is piecewise linear on every finite polyhedron in $S^3 - K_1$.*

Here finite polyhedra and piecewise linearity are defined relative to the given triangulations of S^3 and M, in which K_1 and K_2 form subcomplexes.

PROOF OF LEMMA. Let $U_1 = S^3 - K_1$, $U_2 = M - K_2$. Let $f\colon S^3/K_1 \leftrightarrow M/K_2$ be a homeomorphism. Let $f_1 = f|U_1$. Then f_1 is a homeomorphism $U_1 \leftrightarrow U_2$. For each point P of U_1, let $\rho(P)$ be half the minimum of (a) the distance from P to K_1 and (b) the distance from $f(P)$ to K_2. Then $\rho(P) > 0$

for every P; ρ is continuous; and $\rho(P) \longrightarrow 0$ as $P \longrightarrow K_1$. By Theorem 2 of [M_5], there is a homeomorphism $g_1: U_1 \longrightarrow U_2$ of U_1 into U_2, such that g_1 is piecewise linear on every finite polyhedron in U_1 and such that g_1 is a ρ-approximation of f_1, in the sense that $d(g_1(P), f_1(P)) < \rho(P)$ for every P. (Here d is the distance function in M.)

We now extend g_1 to get a mapping g: $S^3/K_1 \longrightarrow M/K_2$. To do this, we define $g(C) = f(C)$, for each of the two components C of K_1. The resulting function is continuous:

$$P \approx C \Rightarrow f(P) \approx f(C) \Rightarrow g_1(P) \approx f(C) \Rightarrow g(P) \approx g(C).$$

Now $g_1(U_1)$ is locally Euclidean, and so $g(U_1)$ is open in M/K_2. If $g(U_1)$ is not all of U_2, then $g(S^3/K_1)$ is not compact, which is impossible. Therefore g is surjective. Therefore g is a homeomorphism $S^3/K_1 \leftrightarrow M/K_2$, which was to be proved.

We proceed to the proof of the theorem. Let L_2 be a component of K_2. Then the spanning disks of K_2 that contain the loops of L_2 intersect only at the intersection point of the loops of L_2. Therefore L_2 lies in the interior of a polyhedral disk D_2 in M. Let T_2 be a regular neighborhood of K_2, and let U_2 be the component of T_2 that contains L_2. We choose T_2 sufficiently small so that the set $A_2 = D_2 \cap U_2$ forms a regular neighborhood of L_2 in D_2. Thus A_2 is a "2-cell with holes," and decomposes U_2 into two homeomorphic compact pieces U_2^+ and U_2^-, such that $U_2^+ \cap U_2^- = A_2$.

There is now a collection $\{d_1, d_2, \cdots, d_n\}$ of disjoint polyhedral 2-cells in U_2, such that $d_i \cap \operatorname{Bd} U_2 = \operatorname{Bd} d_i$ for each i, and such that the d_i's decompose U_2 into polyhedral 3-cells C_j which intersect only in sets d_i, and such that each d_i intersects A_2 in a linear interval. (The d_i's are "orthogonal to the edges of L_2".)

Now let T_1 be a regular neighborhood of K_1, chosen sufficiently small so that $g(\pi_1(T_1)) \subset \pi_2(\operatorname{Int} T_2)$. Let U_1 be the component of T_1 such that $g(\pi_1(U_1)) \subset \pi_2(U_2)$. Let $S = g(\pi_1(\operatorname{Bd} U_1))$. We adjust the homeomorphism g, if need be, so that S is in general position relative to $A_2 \cup \bigcup d_i$, in the sense that neither of these sets contains a vertex of the other. It follows that:

(1) for each i, $S \cap d_i$ is the union of a finite collection of disjoint (simple closed) polygons;

(2) $S \cap A_2$ has the same property; and

(3) for each i, the intersections of $S \cap d_i$ with A_2 are "true crossing points" of $S \cap d_i$ with A_2.

LEMMA 2. *Let J be a polygon in S, and suppose that J is contractible in $g(\pi_1(U_1)) - \pi_2(K_2)$. Then J bounds a 2-cell in S.*

PROOF OF LEMMA. Since S is a retract of $g(\pi_1(U_1)) - \pi_2(K_2)$, it follows that J is contractible in S; and the lemma follows.

Now let p be the total number of points in all sets of the form $S \cap d_i \cap A_2$. Let q be the total number of components of all sets of the form $S \cap d_i \cap U_2^+$, $S \cap d_i \cap U_2^-$, and $S \cap A_2$. Hereafter we assume that g is chosen in such a way as to minimize p and q, in the stated order.

LEMMA 3. *No set* $S \cap d_i \cap U_2^+$ *or* $S \cap d_i \cap U_2^-$ *contains a polygon.*

PROOF OF LEMMA. Let J be such a polygon. Then J bounds a 2-cell D_J in d_i. We may suppose that J is *inmost* in d_i, in the sense that D_J contains no point of $S - J$. Obviously J is contractible in $g(\pi_1(U_1)) - \pi_2(K_2)$. By Lemma 2 it follows that J bounds a 2-cell d_J' in S. There is therefore a piecewise linear homeomorphism $h\colon M \leftrightarrow M$, $d_J' \leftrightarrow d_J$, such that $h|(d_i - d_J)$ is the identity, and such that h differs from the identity only in a small neighborhood of the 3-cell bounded by $d_J \cup d_J'$. We can now move d_J slightly off d_i (into the interior of one of the sets C_j that contains d_i), by a piecewise linear homeomorphism which differs from the identity only in a small neighborhood of d_J.

All this is impossible, because it reduces q without increasing p.

LEMMA 4. *Let* B *be a broken line with endpoints* P *and* Q, *forming a component of a set* $S \cap d_i \cap U_2^+$ *or* $S \cap d_i \cap U_2^-$. *Let* PQ *be the broken line from* P *to* Q *in* $d_i \cap A_2$. *Then* PQ *contains the point* $d_i \cap K_2$.

PROOF OF LEMMA. If not, B can be moved into $d_i \cap U_2^-$ (or $d_i \cap U_2^+$) by a piecewise linear homeomorphism $M \leftrightarrow M$, $d_i \leftrightarrow d_i$ which differs from the identity only in a small neighborhood of the 2-cell in d_i bounded by $B \cup PQ$. This is impossible, because it reduces p.

LEMMA 5. *Let* B *be a component of a set* $S \cap C_i \cap A_2$. *Then* B *is a broken line, and the endpoints of* B *lie in different 2-cells* d_j *and* d_k *which lie in* Bd C_i.

PROOF OF LEMMA. Obviously B is either a broken line or a polygon. If B were a polygon, then q could be reduced, without increasing p; the proof is exactly the same as for Lemma 3. Therefore B is a broken line. Suppose that the endpoints P, Q of B lie in the same set d_j. Let PQ be the broken line from P to Q in $d_j \cap A_2$. Then $B \cup PQ$ bounds a 2-cell D_B in $C_i \cap A_2$. Since $B \cap K_2 = 0$, it follows that $PQ \cap K_2 = 0$ and $D_B \cap K_2 = 0$. There is therefore a piecewise linear homeomorphism $\phi\colon M \leftrightarrow M$, $A_2 \leftrightarrow A_2$, such that ϕ differs from the identity only in a small neighborhood of D_B, and such that ϕ moves B into $A_2 \cap \operatorname{Int} C_k$, where C_k is the other 3-cell that contains

d_j. This is impossible, because it reduces p.

LEMMA 6. *Let E be a component of a set $S \cap C_i \cap U_2^+$ (or $S \cap C_i \cap U_2^-$), and let J be a component of* $\mathrm{Bd}\, E$. *Then* (1) *J intersects each component of $(C_i \cap A_2) - L_2$ in a broken line, joining points of different sets d_j, d_k, and* (2) *if $d_j \subset C_i$, then $J \cap d_j$ is a broken line, joining points of different components of $(C_i \cap A_2) - L_2$.*

PROOF OF LEMMA. Consider the 2-cell D^2 which is the union of (1) the 2-cell $C_i \cap A_2$ and (2) the nonempty intersections $C_i \cap U_2^+ \cap d_j$ (each of which is a 2-cell). Let $Z = D^2 - L_2$. Since $D^2 \cap L_2$ is an acyclic linear graph lying in $\mathrm{Int}\, D^2$, it follows that Z is homeomorphic to a half-open plane annulus (i.e., a plane annulus bounded by two concentric circles, minus one component of its boundary). The components of $(C_i \cap A_2) - L_2$, together with the nonempty sets of the form

$$(C_i \cap U_2^+ \cap d_j) - L_2,$$

form a decomposition of Z, and these sets appear in a cyclic order around $C_i \cap L_2$, such that any two sets intersect if and only if they are consecutive. Let the sequence be $e_1, e_2, \cdots, e_m$. By the preceding two lemmas, we know that J "runs straight through" the cyclic sequence $e_1, e_2, \cdots, e_m$; J can never reverse its direction. Therefore J does not bound a 2-cell in Z. It follows that $J \cup \mathrm{Bd}\, Z$ is the boundary of a (closed) annulus in Z. Therefore any 1-cycle that generates the 1-dimensional homology group $H^1(J)$ (with integer coefficients) is homologous on Z to a generator of $H^1(\mathrm{Bd}\, Z)$. It follows that the signed sum of the crossing numbers of J with each set $e_j \cap e_{j+1}$ is ± 1. Since J cannot reverse its direction, the intersections $J \cap e_i$ must be connected, and the lemma follows.

LEMMA 7. *Let E be as in Lemma* 6. *Then* $\mathrm{Bd}\, E$ *is connected.*

PROOF OF LEMMA. Since each component J of $\mathrm{Bd}\, E$ is contractible in $g(U_1) - L_2$, it follows by Lemma 2 that J bounds a disk D_J in S. Either $D_J \supset E$ or $D_J \cap E = 0$. If $D_J = E$, then there is nothing to prove. Suppose, then, that either (a) $D_J \supset E$ and $D_J \neq E$ or (b) $D_J \cap E = 0$. Since every component of $\mathrm{Bd}\, E$ contains a broken line lying in a set $d_j \cap U_2^+$, it follows that D_J does not lie in C_i. Therefore D_J contains a polygon J', lying in a set d_j. Suppose that J' is inmost in D_J, in the sense that J' bounds a 2-cell $D_{J'}$ whose interior intersects no set d_k. Then $D_{J'}$ lies in a single set C_m, and J' bounds a 2-cell $d_j(J')$ in d_j. Since $D_J \cap L_2 = 0$, it follows that $d_j(J') \cap L_2 = 0$. If $J' \subset U_2^+$ (or $J' \subset U_2^-$) this contradicts Lemma 3. If J' intersects A_2, then J' contains a broken line of the type ruled out by Lemma 4.

The proof of the theorem can now be completed by standard "pushing and pulling" methods. Let W be the 3-manifold with boundary such that

$$\mathrm{Bd}\, W = S \cup \mathrm{Bd}\, U_2.$$

From the preceding lemmas it follows by a straightforward construction that W is the image of the Cartesian product $S \times [1/3, 2/3]$ under a piecewise linear homeomorphism. It follows that there is a closed neighborhood N of W which is the image of $S \times [0, 1]$, under a piecewise linear homeomorphism

$$h\colon S \times [0, 1] \leftrightarrow N,$$

such that

$$h(S \times 1/3) = \mathrm{Bd}\, U_2, \qquad h(S \times 2/3) = S.$$

Therefore there is a piecewise linear homeomorphism

$$H\colon M \leftrightarrow M$$

$$\colon S \leftrightarrow \mathrm{Bd}\, U_2.$$

Let $\phi = H(g)$. Then $\phi(U_1) = U_2$.

We go through exactly the same procedure with the other component of T_1 and the other component of T_2. This completes the proof of the theorem.

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