ON THE ANALYTIC CONTINUATION
OF THE MINAKSHISUNDARAM-PLEIJEL ZETA FUNCTION
FOR COMPACT RIEMANN SURFACES

BY

BURTON RANDOL

ABSTRACT. A formula is derived for the Minakshisundaram-Pleijel zeta
function in the half-plane $\Re s < 0$.

Let $S$ be a compact Riemann surface, which we will regard as the quotient
of the upper half-plane $H$ by a discontinuous group $\Gamma$ of hyperbolic transfor-
mations. We will assume that $H$ is endowed with the metric $y^{-2}((dx)^2 + (dy)^2)$,
and we will denote the area of $S$ by $A$. Let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$
be the eigenvalues corresponding to the problem $\Delta f + \lambda f = 0$ on $S$, where $\Delta$
is the Laplace operator on $S$, derived from the metric induced on $S$ by that of $H$. In
the coordinates of $H$, the Laplacian is $y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$. Finally, let
$Z(s) = \sum_{n=1}^{\infty} \lambda_n^{-s}$. Since it is known [3] that $A(\tau) = 2\pi^2 \tau^2$ is asymptotic to $(A/4\pi)\tau$,
it follows that the series for $Z(s)$ converges absolutely in the half-plane $\Re s > 1$.

In this note we will use the Selberg trace formula to derive an expression for
the continuation of $Z(s)$ in the half-plane $\Re s < 0$. Accounts of the trace for-
mula can be found in [1], [2], and [4]. The formula, adjusted to the present situ-
tion, goes as follows.

Suppose $h(z)$ is an even function, holomorphic in a strip of the form
$|\Im z| < \frac{1}{2} + \epsilon$ ($\epsilon > 0$), and satisfying a growth condition of the form $|h(z)| = O((1 + |z|^2)^{-1-\epsilon})$ uniformly in the strip. Associate with the sequence $\lambda_0, \lambda_1, \lambda_2, \cdots$ of eigenvalues the set $R$ consisting of those numbers which satisfy an
equation of the form $\lambda_n = \frac{1}{4} + r^2$ ($n = 0, 1, 2, \cdots$). Apart from the possibility
$r = 0$, the elements of $R$ will then occur in pairs, of which each member is the
negative of the other, and it is always the case that every element of $R$ is either
real or pure imaginary, with imaginary part $\leq \frac{1}{2}$. If one of the $\lambda_n$'s happens to
be $\frac{1}{4}$, the corresponding $r = 0$ will be counted with double multiplicity in its
occurrence on the left side of the trace formula.

Now all the elements of $\Gamma$ except the identity are hyperbolic. I.e., each
$\gamma \in \Gamma$ is conjugate in $PSL(2, R)$ to a unique transformation of the form

Received by the editors December 10, 1973.

$z \rightarrow e^{t \gamma} z$, where $l_\gamma$ is real and positive. For geometric reasons, we will call the number $l_\gamma$ the length of the transformation $\gamma$ (cf. [2]). Clearly $l_\gamma$ is the same within a conjugacy class. We will denote by $\{\gamma\}$ the conjugacy class corresponding to $\gamma$ within $\Gamma$ itself. Also, we will call $\gamma \in \Gamma$ primitive, if it is not a positive integral power of any other element of $\Gamma$. Clearly we can speak of a conjugacy class in $\Gamma$ as being primitive. The trace formula then reads

$$\sum_{r_n \in R} h(r) = \frac{A}{2\pi} \int_{-\infty}^{\infty} h(r) r \tanh nr dr + \sum_{\{\gamma\}_p} \sum_{n=1}^{\infty} (l_\gamma \csch \frac{1}{2} nl_\gamma) h(nl_\gamma),$$

where

$$\hat{h}(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-ixr} h(r) dr,$$

and the outer sum is taken over all primitive conjugacy classes in $\Gamma$. Moreover, all series in the formula converge absolutely.

In order to study $Z(s)$, it is convenient to begin by studying a more general Dirichlet series. Namely, suppose $e > 0$, and define $Z_e(s) = \sum_{n=0}^{\infty} (\lambda_n + e)^{-s}$. As before, the series converges absolutely in the half-plane $\Re s > 1$. Next define $a > \frac{1}{2}$ by requiring that $\alpha^2 - \frac{1}{4} = e$. Letting $t$ be a positive number and setting $h(r) = e^{-(\alpha^2 + r^2)t}$ in the trace formula, we obtain

$$\sum_{n=0}^{\infty} e^{-(\lambda_n + e)t} = \frac{A}{4\pi} \int_{-\infty}^{\infty} e^{-(\alpha^2 + r^2)t} r \tanh nr dr$$

$$+ \frac{1}{2} (4\pi t)^{-1/2} \sum_{\{\gamma\}_p} \sum_{n=1}^{\infty} (l_\gamma \csch \frac{1}{2} nl_\gamma) e^{-(4\alpha^2 t^2 + n^2/4)}$$

with all series convergent.

Denote by $\theta_1^e(t)$ and $\theta_2^e(t)$, respectively, the first and second terms on the right side of (1). Then $\sum_{n=1}^{\infty} e^{-(\lambda_n + e)t} = \theta_1^e(t) + \theta_2^e(t) - e^{-et}$.

Throughout what follows, when we say that a result holds uniformly in $e$, we will mean that it holds uniformly for $e \in [0, 1]$, or what is the same thing, for $\alpha \in [1/2, \sqrt{5}/2]$.

The following two lemmas are obvious.

**Lemma 1.** $\sum_{n=1}^{\infty} e^{-(\lambda_n + e)t} = O(e^{-\lambda_1 t})$ uniformly in $e$, as $t \to \infty$.

**Lemma 2.** $\theta_1^e(t) = O(e^{-t/4})$ uniformly in $e$, as $t \to \infty$.

Combining Lemmas 1 and 2, we obtain the following lemma.

**Lemma 3.** $\theta_2^e(t) - e^{-et} = O(e^{-\min(\lambda_1, 1/4)t})$ uniformly in $e$, as $t \to \infty$.

**Lemma 4.** $\theta_2^e(t)$ is of rapid decrease as $t \to 0$, uniformly in $e$. I.e., for any negative integer $k$, $t^k \theta_2^e(t) \to 0$ as $t \to 0$, uniformly in $e$. 

Proof. For any \( k, t^k \theta_2^\varepsilon(t) \) is equal to a convergent series of positive terms. Moreover, for \( t \) less than some \( \eta \), which depends on \( k \) but can be taken independent of \( \varepsilon \), the terms all tend monotonically to zero as \( t \uparrow 0 \). The result then follows from Beppo Levi's theorem.

**Lemma 5.** Define \( \theta_2^\varepsilon(0) = 0 \). Then for each \( \varepsilon \geq 0 \), \( \theta_2^\varepsilon(t) \) is continuous, and the series for \( \theta_2^\varepsilon(t) \) converges uniformly to \( \theta_2^0(t) \) on compact subsets of \([0, \infty)\).

**Proof.** For any \( \varepsilon \geq 0 \), \( \theta_2^\varepsilon(t) \) is continuous at \( t = 0 \) by Lemma 4. For \( t > 0 \), \( \theta_2^\varepsilon(t) \) is clearly continuous, being a linear combination of three continuous functions. Moreover, since all the terms of the series that defines \( \theta_2^\varepsilon(t) \) are positive, it follows from Dini's theorem that the series converges uniformly on compact subsets of \([0, \infty)\).

**Lemma 6.** On any compact subset of \([0, \infty)\), \( \theta_2^\varepsilon(t) \to \theta_2^0(t) \) uniformly, as \( \varepsilon \to 0 \).

**Proof.** \( \theta_2^\varepsilon(t) \) increases monotonically to \( \theta_2^0(t) \) as \( \varepsilon \downarrow 0 \). The result thus follows from Dini's theorem.

Now suppose \( \Re s > 1 \). Taking the Mellin transform of \( \sum_{n=1}^{\infty} e^{-(\lambda_n + \varepsilon)t} \), and adding \( 1/s \) to the result, we obtain

\[
\Gamma(s)Z_\varepsilon(s) + \frac{1}{s} = \frac{A\Gamma(s)}{4\pi} \int_{-\infty}^{\infty} (\alpha^2 + r^2)^{-s} r \tanh \pi r \, dr + \int_0^{\infty} (\theta_2^\varepsilon(t) - e^{-\varepsilon t}) t^s \frac{dt}{t} + \frac{1}{s},
\]

or

\[
\Gamma(s)Z_\varepsilon(s) + \frac{1}{s} = \frac{A\Gamma(s)}{8(s - 1)} \int_{-\infty}^{\infty} (\alpha^2 + r^2)^{1-s} \text{sech}^2 \pi r \, dr
\]

\[
= \int_0^1 (\theta_2^\varepsilon(t) + 1 - e^{-\varepsilon t}) t^s \frac{dt}{t} + \int_1^{\infty} (\theta_2^\varepsilon(t) - e^{-\varepsilon t}) t^s \frac{dt}{t}.
\]

In view of Lemmas 3 and 4, the right side of the last equation gives, for any \( \varepsilon \geq 0 \), a meromorphic continuation of \( \Gamma(s)Z_\varepsilon(s) + 1/s \), and hence of \( \Gamma(s)Z_\varepsilon(s) \), into \( \Re s > -1 \), and indeed, into the whole plane if \( \varepsilon = 0 \) (since the first and third integrals are entire for any \( \varepsilon \geq 0 \), and the second integral is holomorphic in \( \Re s > -1 \), and entire if \( \varepsilon = 0 \)). If \( \varepsilon > 0 \), and we observe, using the power series for \( e^{-\varepsilon t} \), that \( \int_0^1 (1 - e^{-\varepsilon t}) t^s \, dt/t \) can be continued to the left of \( \Re s > -1 \), with simple poles at the negative integers, we obtain a meromorphic continuation of \( \Gamma(s)Z_\varepsilon(s) \) into the entire plane in this case as well. Since it is clear from this that the only possible poles of \( \Gamma(s)Z_\varepsilon(s) \) are simple poles at \( 1, 0, -1, -2, \cdots \), with the pole at \( s = 1 \) always present, we
conclude that for any \( \epsilon > 0 \), \( Z_{\epsilon}(s) \) can be meromorphically continued into the whole plane, with a single simple pole at \( s = 1 \), having residue \( A/4\pi \) (since \( \int_{-\infty}^{\infty} \text{sech}^2 \pi r \, dr = 2/\pi \)).

Suppose now \( \Re s > -1 \), and \( s \neq 0, 1 \). Then in view of Lemma 6, it is evident, by inspecting the right side of (2), bearing in mind Lemmas 3 and 4, that \( \Gamma(s)Z_{\epsilon}(s) + 1/s \to \Gamma(s)Z(s) + 1/s \), and hence \( \Gamma(s)Z_{\epsilon}(s) \to \Gamma(s)Z(s) \), as \( \epsilon \to 0 \).

On the other hand, if \( \Re s > 0 \),

\[
\Gamma(s)Z_{\epsilon}(s) = \frac{A\Gamma(s)}{8(s-1)} \int_{-\infty}^{\infty} (\alpha^2 + r^2)^{1-s} \text{sech}^2 \pi r \, dr
\]

and if \( \epsilon > 0 \), it is permissible to split the last integral into two integrals, since it follows from Lemma 3 that for positive \( \epsilon \), \( \theta_{2/4}(t) \) is of exponential decrease as \( t \to \infty \). We thus obtain, at first for \( \Re s > 0 \), and then for the whole plane by analytic continuation,

\[
\Gamma(s)Z_{\epsilon}(s) = \frac{A\Gamma(s)}{8(s-1)} \int_{-\infty}^{\infty} (\alpha^2 + r^2)^{1-s} \text{sech}^2 \pi r \, dr
\]

Now taking \( -1 < \Re s < 0 \), letting \( \epsilon \to 0 \), and bearing in mind that by Lemma 3, \( \theta_{2/4}(t) = O(1) \) uniformly in \( \epsilon \), as \( t \to \infty \), we find that

\[
\Gamma(s)Z(s) = \frac{A\Gamma(s)}{8(s-1)} \int_{-\infty}^{\infty} (\alpha^2 + r^2)^{1-s} \text{sech}^2 \pi r \, dr + \int_{0}^{\infty} \theta_{2/4}(t) t^{s} \, dt
\]

Since both integrals are holomorphic in \( \Re s < 0 \), we have obtained an expression for \( \Gamma(s)Z(s) \) in the left half-plane.

Let us examine \( \int_{0}^{\infty} \theta_{2/4}(t) t^{s} \, dt/s \), assuming \( \Re s < 0 \). Now

\[
\int_{0}^{\infty} e^{-(t^2 + (nl_{\gamma})^2)/4t} t^{s-1/2} \frac{dt}{t} = 2(nl_{\gamma})^{s-1/2} K_{1/2-s}(\sqrt{nl_{\gamma}}) \quad [5, \text{p. 183}],
\]

so we obtain the following result (the interchange of summation and integration being justified by Lemma 5):

**Theorem 1.** If \( \Re s < 0 \),

\[
\Gamma(s)Z(s) = \frac{A\Gamma(s)}{8(s-1)} \int_{-\infty}^{\infty} (\alpha^2 + r^2)^{1-s} \text{sech}^2 \pi r \, dr
\]

\[
+ (4\pi)^{-1/2} \sum_{\{\gamma\}p} \sum_{n=1}^{\infty} (l_{\gamma}/n)^{1/2} (\text{csch} \sqrt{2nl_{\gamma}}) (nl_{\gamma})^s K_{1/2-s}(\sqrt{2nl_{\gamma}}).
\]

Now if \( \Re s < \frac{1}{2} \),
(\frac{n!}{2\pi i}) = \pi^{1/2} \Gamma(1 - s)(nl^n)^{1/2} \int_0^\infty ((\frac{n}{2} nl^n)^2 + x^2)^{-(1-s)} \cos x \, dx

[5, p. 172],

so if we define \( \phi_s(A) \), for \( \text{Re } s > \frac{1}{2} \) and positive \( A \), by setting \( \phi_s(A) = (2\pi)^{-1} \int_0^\infty ((A/2)^2 + x^2)^{-s} \cos x \, dx \), we obtain the following reformulation of Theorem 1.

**Theorem 2.** If \( \text{Re } s < 0 \),

\[
\Gamma(s) Z(s) = \frac{A\Gamma(s)}{8(s - 1)} \int_{-\infty}^\infty \left( \frac{1}{4} + r^2 \right)^{-s} \text{sech}^2 \pi r \, dr
\]

\[
+ \Gamma(1 - s) \sum_{\{\gamma\}_p} \sum_{n=1}^\infty \frac{\ell_n (\text{csch } \frac{n}{2} nl^n) \phi_{1-s}(nl^n)}{n^s}.
\]

Suppose now \( \text{Re } s > 1 \). Then

\[
\frac{A\Gamma(s)}{8(s - 1)} \int_{-\infty}^\infty \left( \frac{1}{4} + r^2 \right)^{-s} \text{sech}^2 \pi r \, dr = \frac{A\Gamma(s)}{2\pi} \int_0^\infty \left( \frac{1}{4} + r^2 \right)^{-s} r \tanh \pi r \, dr.
\]

Now as we have pointed out, it is well known that \( \Sigma_{\lambda \leq T} 1 \sim AT/4\pi \), so it follows that \( \Sigma_{0 < r_n < T} 1 \sim AT^2/4\pi. \) But \((A/2\pi)\int_0^T r \tanh \pi r \, dr \sim AT^2/4\pi\), and in view of the trace formula, is the correct principal term in the asymptotic analysis of \( \Sigma_{0 < r_n < T} 1 \). This suggests defining a remainder term

\[
R(T) = \sum_{0 < r_n < T} 1 - \frac{A}{2\pi} \int_0^T r \tanh \pi r \, dr.
\]

Then if we denote the eigenvalues in \((0, \frac{1}{4})\) by \( \lambda_1, \cdots, \lambda_N \), and define \( \lambda(r) = \frac{1}{4} + r^2 \), Theorem 2 and the previous arguments tell us that \( \Gamma(s) \{ \sum_{n=1}^N \lambda_n^{-s} + \int_0^\infty \lambda^{-s} dR(r) \} \) can be meromorphically continued from \( \text{Re } s > 1 \) into the whole plane, and for \( \text{Re } s < 0 \), equals \( \Gamma(1 - s) \Phi(1 - s) \), where \( \Phi(s) \) is defined in the half-plane \( \text{Re } s > 1 \) by setting

\[
\Phi(s) = \sum_{\{\gamma\}_p} \sum_{n=1}^\infty \frac{\ell_n (\text{csch } \frac{n}{2} nl^n) \phi_{1-s}(nl^n)}{n^s}.
\]

If, now, we define \( R^*(T) = R(\sqrt{T - \frac{1}{4}}) \), integrate by parts, and make the change of variable \( \lambda = \lambda(r) \), the previous statement becomes the statement that \( \Gamma(s) \{ \sum_{n=1}^N \lambda_n^{-s} + s \int_{1/4}^{\infty} \lambda^{-s-1} R^*(\lambda) \, d\lambda \} \) can be meromorphically continued into the whole plane, and for \( \text{Re } s < 0 \), equals \( \Gamma(1 - s) \Phi(1 - s) \).

Thus, setting

\[
\Psi(s) = s \int_{1/4}^\infty \lambda^{-s-1} R^*(\lambda) \, d\lambda = s \int_{\log(1/4)}^\infty e^{-\lambda s} R^*(e^\lambda) \, d\lambda = \int_{\log(1/4)}^\infty e^{-\lambda s} dR^*(e^\lambda),
\]
we find that \( \Psi(s) \), the Laplace transform of the exponential form of the eigenvalue remainder measure, satisfies the following identity:

**Theorem 3.** If \( \text{Re} \, s < 0 \), \( \Gamma(s) \{ \sum_{n=1}^{N} \lambda_n^{-s} + \Psi(s) \} = \Gamma(1-s) \Phi(1-s) \).

**Corollary.** If there are no eigenvalues in \((0, \frac{1}{2})\) and \( \text{Re} \, s < 0 \), we have \( \Gamma(s) \Psi(s) = \Gamma(1-s) \Phi(1-s) \).

**REFERENCES**


