

ON THE ANALYTIC CONTINUATION
OF THE MINAKSHISUNDARAM-PLEIJEL ZETA FUNCTION
FOR COMPACT RIEMANN SURFACES

BY

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ABSTRACT. A formula is derived for the Minakshisundaram-Pleijel zeta function in the half-plane $\operatorname{Re} s < 0$.

Let S be a compact Riemann surface, which we will regard as the quotient of the upper half-plane H by a discontinuous group Γ of hyperbolic transformations. We will assume that H is endowed with the metric $y^{-2}((dx)^2 + (dy)^2)$, and we will denote the area of S by A . Let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ be the eigenvalues corresponding to the problem $\Delta f + \lambda f = 0$ on S , where Δ is the Laplace operator on S , derived from the metric induced on S by that of H . In the coordinates of H , the Laplacian is $y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$. Finally, let $Z(s) = \sum_{n=1}^{\infty} \lambda_n^{-s}$. Since it is known [3] that $N(T) = \sum_{\lambda_n < T} 1$ is asymptotic to $(A/4\pi)T$, it follows that the series for $Z(s)$ converges absolutely in the half-plane $\operatorname{Re} s > 1$.

In this note we will use the Selberg trace formula to derive an expression for the continuation of $Z(s)$ in the half-plane $\operatorname{Re} s < 0$. Accounts of the trace formula can be found in [1], [2], and [4]. The formula, adjusted to the present situation, goes as follows.

Suppose $h(z)$ is an even function, holomorphic in a strip of the form $|\operatorname{Im} z| < \frac{1}{2} + \epsilon$ ($\epsilon > 0$), and satisfying a growth condition of the form $|h(z)| = O((1 + |z|^2)^{-1-\epsilon})$ uniformly in the strip. Associate with the sequence $\lambda_0, \lambda_1, \lambda_2, \dots$ of eigenvalues the set R consisting of those numbers which satisfy an equation of the form $\lambda_n = \frac{1}{4} + r^2$ ($n = 0, 1, 2, \dots$). Apart from the possibility $r = 0$, the elements of R will then occur in pairs, of which each member is the negative of the other, and it is always the case that every element of R is either real or pure imaginary, with imaginary part $\leq \frac{1}{2}$. If one of the λ_n 's happens to be $\frac{1}{4}$, the corresponding $r = 0$ will be counted with double multiplicity in its occurrence on the left side of the trace formula.

Now all the elements of Γ except the identity are hyperbolic. I. e., each $\gamma \in \Gamma$ is conjugate in $PSL(2, R)$ to a unique transformation of the form

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$z \rightarrow e^{l_\gamma z}$, where l_γ is real and positive. For geometric reasons, we will call the number l_γ the length of the transformation γ (cf. [2]). Clearly l_γ is the same within a conjugacy class. We will denote by $\{\gamma\}$ the conjugacy class corresponding to γ within Γ itself. Also, we will call $\gamma \in \Gamma$ primitive, if it is not a positive integral power of any other element of Γ . Clearly we can speak of a conjugacy class in Γ as being primitive. The trace formula then reads

$$\sum_{n \in R} h(r_n) = \frac{A}{2\pi} \int_{-\infty}^{\infty} h(r) r \tanh \pi r \, dr + \sum_{\{\gamma\}_p} \sum_{n=1}^{\infty} (l_\gamma \operatorname{csch} \frac{1}{2} n l_\gamma) \hat{h}(n l_\gamma),$$

where

$$\hat{h}(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-ixr} h(r) \, dr,$$

and the outer sum is taken over all primitive conjugacy classes in Γ . Moreover, all series in the formula converge absolutely.

In order to study $Z(s)$, it is convenient to begin by studying a more general Dirichlet series. Namely, suppose $\epsilon \geq 0$, and define $Z_\epsilon(s) = \sum_{n=0}^{\infty} (\lambda_n + \epsilon)^{-s}$. As before, the series converges absolutely in the half-plane $\operatorname{Re} s > 1$. Next define $\alpha \geq \frac{1}{2}$ by requiring that $\alpha^2 - \frac{1}{4} = \epsilon$. Letting t be a positive number and setting $h(r) = e^{-(\alpha^2 + r^2)t}$ in the trace formula, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} e^{-(\lambda_n + \epsilon)t} &= \frac{A}{4\pi} \int_{-\infty}^{\infty} e^{-(\alpha^2 + r^2)t} r \tanh \pi r \, dr \\ (1) \quad &+ \frac{1}{2} (4\pi t)^{-1/2} \sum_{\{\gamma\}_p} \sum_{n=1}^{\infty} (l_\gamma \operatorname{csch} \frac{1}{2} n l_\gamma) e^{-(4\alpha^2 t^2 + n^2 l_\gamma^2)/4t} \end{aligned}$$

with all series convergent.

Denote by $\theta_1^\epsilon(t)$ and $\theta_2^\epsilon(t)$, respectively, the first and second terms on the right side of (1). Then $\sum_{n=1}^{\infty} e^{-(\lambda_n + \epsilon)t} = \theta_1^\epsilon(t) + \theta_2^\epsilon(t) - e^{-\epsilon t}$.

Throughout what follows, when we say that a result holds uniformly in ϵ , we will mean that it holds uniformly for $\epsilon \in [0, 1]$, or what is the same thing, for $\alpha \in [1/2, \sqrt{5}/2]$.

The following two lemmas are obvious.

LEMMA 1. $\sum_{n=1}^{\infty} e^{-(\lambda_n + \epsilon)t} = O(e^{-\lambda_1 t})$ uniformly in ϵ , as $t \rightarrow \infty$.

LEMMA 2. $\theta_1^\epsilon(t) = O(e^{-t/4})$ uniformly in ϵ , as $t \rightarrow \infty$.

Combining Lemmas 1 and 2, we obtain the following lemma.

LEMMA 3. $\theta_2^\epsilon(t) - e^{-\epsilon t} = O(e^{-(\min(\lambda_1, 1/4))t})$ uniformly in ϵ , as $t \rightarrow \infty$.

LEMMA 4. $\theta_2^\epsilon(t)$ is of rapid decrease as $t \rightarrow 0$, uniformly in ϵ . I. e., for any negative integer k , $t^k \theta_2^\epsilon(t) \rightarrow 0$ as $t \rightarrow 0$, uniformly in ϵ .

PROOF. For any k , $t^k \theta_2^\epsilon(t)$ is equal to a convergent series of positive terms. Moreover, for t less than some η , which depends on k but can be taken independent of ϵ , the terms all tend monotonically to zero as $t \downarrow 0$. The result then follows from Beppo Levi's theorem.

LEMMA 5. Define $\theta_2^\epsilon(0) = 0$. Then for each $\epsilon \geq 0$, $\theta_2^\epsilon(t)$ is continuous, and the series for $\theta_2^\epsilon(t)$ converges uniformly to $\theta_2^\epsilon(t)$ on compact subsets of $[0, \infty)$.

PROOF. For any $\epsilon \geq 0$, $\theta_2^\epsilon(t)$ is continuous at $t = 0$ by Lemma 4. For $t > 0$, $\theta_2^\epsilon(t)$ is clearly continuous, being a linear combination of three continuous functions. Moreover, since all the terms of the series that defines $\theta_2^\epsilon(t)$ are positive, it follows from Dini's theorem that the series converges uniformly on compact subsets of $[0, \infty)$.

LEMMA 6. On any compact subset of $[0, \infty)$, $\theta_2^\epsilon(t) \rightarrow \theta_2^0(t)$ uniformly, as $\epsilon \rightarrow 0$.

PROOF. $\theta_2^\epsilon(t)$ increases monotonically to $\theta_2^0(t)$ as $\epsilon \downarrow 0$. The result thus follows from Dini's theorem.

Now suppose $\text{Re } s > 1$. Taking the Mellin transform of $\sum_{n=1}^\infty e^{-(\lambda_n + \epsilon)t}$, and adding $1/s$ to the result, we obtain

$$\Gamma(s)Z_\epsilon(s) + \frac{1}{s} = \frac{A\Gamma(s)}{4\pi} \int_{-\infty}^\infty (\alpha^2 + r^2)^{-s} r \tanh \pi r \, dr + \int_0^\infty (\theta_2^\epsilon(t) - e^{-\epsilon t}) t^s \frac{dt}{t} + \frac{1}{s},$$

or

$$(2) \quad \Gamma(s)Z_\epsilon(s) + \frac{1}{s} = \frac{A\Gamma(s)}{8(s-1)} \int_{-\infty}^\infty (\alpha^2 + r^2)^{1-s} \text{sech}^2 \pi r \, dr + \int_0^1 (\theta_2^\epsilon(t) + 1 - e^{-\epsilon t}) t^s \frac{dt}{t} + \int_1^\infty (\theta_2^\epsilon(t) - e^{-\epsilon t}) t^s \frac{dt}{t}.$$

In view of Lemmas 3 and 4, the right side of the last equation gives, for any $\epsilon \geq 0$, a meromorphic continuation of $\Gamma(s)Z_\epsilon(s) + 1/s$, and hence of $\Gamma(s)Z_\epsilon(s)$, into $\text{Re } s > -1$, and indeed, into the whole plane if $\epsilon = 0$ (since the first and third integrals are entire for any $\epsilon \geq 0$, and the second integral is holomorphic in $\text{Re } s > -1$, and entire if $\epsilon = 0$). If $\epsilon > 0$, and we observe, using the power series for $e^{-\epsilon t}$, that $\int_0^1 (1 - e^{-\epsilon t}) t^s dt/t$ can be continued to the left of $\text{Re } s > -1$, with simple poles at the negative integers, we obtain a meromorphic continuation of $\Gamma(s)Z_\epsilon(s)$ into the entire plane in this case as well. Since it is clear from this that the only possible poles of $\Gamma(s)Z_\epsilon(s)$ are simple poles at $1, 0, -1, -2, \dots$, with the pole at $s = 1$ always present, we

conclude that for any $\epsilon \geq 0$, $Z_\epsilon(s)$ can be meromorphically continued into the whole plane, with a single simple pole at $s = 1$, having residue $A/4\pi$ (since $\int_{-\infty}^{\infty} \operatorname{sech}^2 \pi r \, dr = 2/\pi$).

Suppose now $\operatorname{Re} s > -1$, and $s \neq 0, 1$. Then in view of Lemma 6, it is evident, by inspecting the right side of (2), bearing in mind Lemmas 3 and 4, that $\Gamma(s)Z_\epsilon(s) + 1/s \rightarrow \Gamma(s)Z(s) + 1/s$, and hence $\Gamma(s)Z_\epsilon(s) \rightarrow \Gamma(s)Z(s)$, as $\epsilon \rightarrow 0$.

On the other hand, if $\operatorname{Re} s > 0$,

$$\Gamma(s)Z_\epsilon(s) = \frac{A\Gamma(s)}{8(s-1)} \int_{-\infty}^{\infty} (\alpha^2 + r^2)^{1-s} \operatorname{sech}^2 \pi r \, dr + \int_0^\infty (\theta_2^\epsilon(t) - e^{-\epsilon t}) t^s \frac{dt}{t},$$

and if $\epsilon > 0$, it is permissible to split the last integral into two integrals, since it follows from Lemma 3 that for positive ϵ , $\theta_2^\epsilon(t)$ is of exponential decrease as $t \rightarrow \infty$. We thus obtain, at first for $\operatorname{Re} s > 0$, and then for the whole plane by analytic continuation,

$$\Gamma(s)Z_\epsilon(s) = \frac{A\Gamma(s)}{8(s-1)} \int_{-\infty}^{\infty} (\alpha^2 + r^2)^{1-s} \operatorname{sech}^2 \pi r \, dr + \int_0^\infty \theta_2^\epsilon(t) t^s \frac{dt}{t} - \Gamma(s) \epsilon^{-s}.$$

Now taking $-1 < \operatorname{Re} s < 0$, letting $\epsilon \rightarrow 0$, and bearing in mind that by Lemma 3, $\theta_2^\epsilon(t) = O(1)$ uniformly in ϵ , as $t \rightarrow \infty$, we find that

$$\Gamma(s)Z(s) = \frac{A\Gamma(s)}{8(s-1)} \int_{-\infty}^{\infty} (\frac{1}{4} + r^2)^{1-s} \operatorname{sech}^2 \pi r \, dr + \int_0^\infty \theta_2^0(t) t^s \frac{dt}{t}.$$

Since both integrals are holomorphic in $\operatorname{Re} s < 0$, we have obtained an expression for $\Gamma(s)Z(s)$ in the left half-plane.

Let us examine $\int_0^\infty \theta_2^0(t) t^s \, dt/t$, assuming $\operatorname{Re} s < 0$. Now

$$\int_0^\infty e^{-(t^2 + (nl_\gamma)^2)/4t} t^{s-1/2} \frac{dt}{t} = 2(nl_\gamma)^{s-1/2} K_{1/2-s}(\frac{1}{2}nl_\gamma) \quad [5, \text{p. 183}],$$

so we obtain the following result (the interchange of summation and integration being justified by Lemma 5):

THEOREM 1. *If $\operatorname{Re} s < 0$,*

$$\Gamma(s)Z(s) = \frac{A\Gamma(s)}{8(s-1)} \int_{-\infty}^{\infty} (\frac{1}{4} + r^2)^{1-s} \operatorname{sech}^2 \pi r \, dr + (4\pi)^{-1/2} \sum_{\{\gamma\}_p} \sum_{n=1}^{\infty} (l_\gamma/n)^{1/2} (\operatorname{csch} \frac{1}{2}nl_\gamma) (nl_\gamma)^s K_{1/2-s}(\frac{1}{2}nl_\gamma).$$

Now if $\operatorname{Re} s < \frac{1}{2}$,

$$(nl_\gamma)^s K_{1/2-s}(\frac{1}{2}nl_\gamma) = \pi^{-1/2} \Gamma(1-s)(nl_\gamma)^{1/2} \int_0^\infty ((\frac{1}{2}nl_\gamma)^2 + x^2)^{-(1-s)} \cos x \, dx$$

[5, p. 172],

so if we define $\phi_s(A)$, for $\text{Re } s > \frac{1}{2}$ and positive A , by setting $\phi_s(A) = (2\pi)^{-1} \int_0^\infty ((A/2)^2 + x^2)^{-s} \cos x \, dx$, we obtain the following reformulation of Theorem 1.

THEOREM 2. *If $\text{Re } s < 0$,*

$$\Gamma(s)Z(s) = \frac{A\Gamma(s)}{8(s-1)} \int_{-\infty}^\infty (\frac{1}{4} + r^2)^{1-s} \text{sech}^2 \pi r \, dr$$

(3)

$$+ \Gamma(1-s) \sum_{\{\gamma\}_p} \sum_{n=1}^\infty l_\gamma(\text{csch } \frac{1}{2}nl_\gamma) \phi_{1-s}(nl_\gamma).$$

Suppose now $\text{Re } s > 1$. Then

$$\frac{A\Gamma(s)}{8(s-1)} \int_{-\infty}^\infty (\frac{1}{4} + r^2)^{1-s} \text{sech}^2 \pi r \, dr = \frac{A\Gamma(s)}{2\pi} \int_0^\infty (\frac{1}{4} + r^2)^{-s} r \tanh \pi r \, dr.$$

Now as we have pointed out, it is well known that $\sum_{\lambda_n < T} 1 \sim AT/4\pi$, so it follows that $\sum_{0 < r_n < T} 1 \sim AT^2/4\pi$. But $(A/2\pi) \int_0^T r \tanh \pi r \, dr \sim AT^2/4\pi$, and in view of the trace formula, is the correct principal term in the asymptotic analysis of $\sum_{0 < r_n < T} 1$. This suggests defining a remainder term

$$R(T) = \sum_{0 < r_n < T} 1 - \frac{A}{2\pi} \int_0^T r \tanh \pi r \, dr.$$

Then if we denote the eigenvalues in $(0, \frac{1}{4}]$ by $\lambda_1, \dots, \lambda_N$, and define $\lambda(r) = \frac{1}{4} + r^2$, Theorem 2 and the previous arguments tell us that $\Gamma(s) \{ \sum_{n=1}^N \lambda_n^{-s} + \int_0^\infty (\lambda(r))^{-s} dR(r) \}$ can be meromorphically continued from $\text{Re } s > 1$ into the whole plane, and for $\text{Re } s < 0$, equals $\Gamma(1-s)\Phi(1-s)$, where $\Phi(s)$ is defined in the half-plane $\text{Re } s > 1$ by setting

$$\Phi(s) = \sum_{\{\gamma\}_p} \sum_{n=1}^\infty l_\gamma(\text{csch } \frac{1}{2}nl_\gamma) \phi_s(nl_\gamma).$$

If, now, we define $R^*(T) = R(\sqrt{T - \frac{1}{4}})$, integrate by parts, and make the change of variable $\lambda = \lambda(r)$, the previous statement becomes the statement that $\Gamma(s) \{ \sum_{n=1}^N \lambda_n^{-s} + s \int_{1/4}^\infty \lambda^{-s-1} R^*(\lambda) \, d\lambda \}$ can be meromorphically continued into the whole plane, and for $\text{Re } s < 0$, equals $\Gamma(1-s)\Phi(1-s)$.

Thus, setting

$$\Psi(s) = s \int_{1/4}^\infty \lambda^{-s-1} R^*(\lambda) \, d\lambda = s \int_{\log 1/4}^\infty e^{-\lambda s} R^*(e^\lambda) \, d\lambda = \int_{\log 1/4}^\infty e^{-\lambda s} dR^*(e^\lambda),$$

we find that $\Psi(s)$, the Laplace transform of the exponential form of the eigenvalue remainder measure, satisfies the following identity:

THEOREM 3. *If $\operatorname{Re} s < 0$, $\Gamma(s) \{ \sum_{n=1}^N \lambda_n^{-s} + \Psi(s) \} = \Gamma(1-s)\Phi(1-s)$.*

COROLLARY. *If there are no eigenvalues in $(0, \frac{1}{4}]$ and $\operatorname{Re} s < 0$, we have $\Gamma(s)\Psi(s) = \Gamma(1-s)\Phi(1-s)$.*

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