

## WEAKLY STARLIKE MEROMORPHIC UNIVALENT FUNCTIONS

BY

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**ABSTRACT.** A weakly starlike meromorphic univalent function is one of the form  $f(z) = -\rho z g(z)[(z - \rho)(1 - \rho z)]^{-1}$  for  $0 < \rho < 1$  and  $g(z)$  a meromorphic starlike function. The behavior of coefficients and growth of this class of functions and of a subset are studied.

**1. Introduction.** Let  $f(z)$  be meromorphic in the open unit disk defined by  $|z| < 1$  and hereafter called  $\Delta$  with a simple pole at  $\rho$ ,  $0 < |\rho| < 1$ , and otherwise regular in  $\Delta$ .  $f(z)$  is in  $\Lambda(\rho)$  if and only if there is a number  $\rho_1$ ,  $|\rho| < \rho_1 < 1$ , such that

$$(1.1) \quad \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} < 0$$

and

$$(1.2) \quad \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} d\theta = -1,$$

for  $\rho_1 < |z| < 1$  with  $z = re^{i\theta}$ . On the other hand,  $f(z)$  is in  $\Lambda_1(\rho)$  if and only if  $f(z)$  is regular in  $\bar{\Delta}$ , the closure of  $\Delta$ , except again for a simple pole at  $\rho$  and (1.1) and (1.2) are satisfied on  $\partial\Delta$ , the latter being the boundary of  $\Delta$ . Clearly  $\Lambda_1(\rho)$  is a subset of  $\Lambda(\rho)$ .

It is no restriction on the geometric conditions given in (1.1) and (1.2) to assume that  $f(z)$  in  $\Lambda(\rho)$  be normalized so that  $f(0) = 1$  and  $\rho$  be real; hence we shall hereafter make these assumptions. Also, it is clear that functions in the class  $\Lambda(\rho)$  are univalent.

Conditions (1.1) and (1.2) taken together require that the origin be omitted by every function in  $\Lambda(\rho)$ . Meromorphic functions with a normalization similar to the above have been studied by Ladegast in an interesting paper which apparently has been overlooked [5]. Furthermore, the normalization taken for  $\Lambda(\rho)$  can be viewed as an analog of the Montel normalization for regular univalent functions [6].

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If  $f(z)$  is in  $\Lambda_1(\rho)$ , then the function

$$(1.3) \quad g(z) = (z - \rho)(1 - \rho z)f(z) / -\rho z$$

is meromorphic in  $\bar{\Delta}$  with a pole of residue 1 at the origin. Furthermore on  $\partial\Delta$ ,

$$\begin{aligned} \operatorname{Re} \left\{ \frac{zg'(z)}{g(z)} \right\} &= \operatorname{Re} \left\{ \left( \frac{\rho}{z - \rho} \right) - \left( \frac{\rho}{z - \rho} \right) + \frac{zf'(z)}{f(z)} \right\} \\ &= \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} < 0. \end{aligned}$$

Consequently,  $g(z)$  is in  $\Sigma^*$  the class of meromorphic, normalized, starlike univalent functions ([1], [7]).

However, if we choose  $f(z)$  in  $\Lambda(\rho)$  we can likewise show that  $f(z)$  has a representation like (1.3). For if we let  $f_r(z) = f(rz)$ , then  $f_r(z)$  is in  $\Lambda_1(\rho/r)$  for  $\rho_1 < r \leq 1$  and by the argument given above we infer the existence of a function in  $\Sigma^*$  such that

$$f_r(z) = \frac{-\rho z/r}{(z - \rho/r)(1 - \rho z/r)} g_r(z).$$

$\Sigma^*$  is normal and compact [2], therefore we can find a sequence of increasing real numbers  $\{r_n\}$  converging to 1 such that  $\{g_{r_n}(z)\}$  converges to a function  $g(z)$  in  $\Sigma^*$  in compacta. Letting  $n \rightarrow \infty$ , we get the representation

$$(1.4) \quad f(z) = -\rho z g(z) / (z - \rho)(1 - \rho z)$$

for every function  $f(z)$  in  $\Lambda(\rho)$ .

(1.4) gives a particularly useful representation for studying the properties of  $\Lambda(\rho)$ ; for this reason we define the following classes of functions.  $f(z)$  is in  $\Lambda^*(\rho)$  if and only if  $f(z)$  satisfies (1.4) for some function  $g(z)$  in  $\Sigma^*$ ; and  $f(z)$  is in  $\Lambda_1^*(\rho)$  if and only if  $f(z)$  is defined by (1.4) with  $g(z)$  in  $\Sigma^*$  but with the further restriction that  $g(z)$  be regular and satisfy (1.1) in  $\bar{\Delta}$ . Clearly  $\Lambda_1^*(\rho) \subset \Lambda^*(\rho)$ .

It is evident that  $\Lambda(\rho) \subset \Lambda^*(\rho)$  for each value of  $\rho$ . We shall show however that  $\Lambda(\rho)$  is a proper subset of  $\Lambda^*(\rho)$  for some values of  $\rho$ .

Functions in  $\Lambda^*(\rho)$  are the reciprocals of weakly starlike, regular, univalent functions introduced by Hummel ([3], [4]). It is for this reason that we refer to members of  $\Lambda^*(\rho)$  as weakly starlike, meromorphic univalent functions.  $\Lambda(\rho)$  and  $\Lambda^*(\rho)$  are particular cases of classes of functions studied by Styer [8].

It is evident from the above that  $\Lambda_1^*(\rho) = \Lambda_1(\rho)$ , for all  $\rho$ . Furthermore, for a given  $f(z)$  in  $\Lambda^*(\rho)$  there exists an increasing sequence of numbers  $\{r_n\}$  converging to 1 and a corresponding sequence  $\{f_{r_n}(z)\}$  such that  $f_{r_n}(z)$  is in

$\Lambda(\rho/r_n)$ , for each  $n$ , and  $\lim_{n \rightarrow \infty} f_n(z) = f(z)$  in  $\Delta$ .

2. Relationship between  $\Lambda(\rho)$  and  $\Lambda^*(\rho)$ . The work of Styer [8] shows that  $\Lambda(\rho) = \Lambda^*(\rho)$  if  $\rho < 2 - \sqrt{3}$ . We begin by improving this range slightly.

**THEOREM 1.** *If  $\rho < (3 - 2\sqrt{2})^{1/2}$ , then  $\Lambda(\rho) = \Lambda^*(\rho)$ .*

**PROOF.** Let  $f(z)$  be in  $\Lambda^*(\rho)$  and let

$$(2.1) \quad \psi(z) = \psi(z; \rho) = -\rho z / (z - \rho)(1 - \rho z),$$

then (1.4) assumes the form  $f(z) = \psi(z)g(z)$  for  $g(z)$  in  $\Sigma^*$ .

For  $\rho \geq 2 - \sqrt{3}$ , Hummel [3, p. 548] has essentially shown that

$$(2.2) \quad \operatorname{Re} \left\{ \frac{z\psi'(z)}{\psi(z)} \right\} \leq \frac{[(1 - \rho^2 r^2)^{1/2} - (r^2 - \rho^2)^{1/2}]^2}{2(1 - r^2)(1 - \rho^2)}$$

for  $|z| = r$  and  $\rho < r < 1$ . If  $g(z)$  is in  $\Sigma^*$ , it is well known [2] that

$$(2.3) \quad \operatorname{Re} \left\{ \frac{zg'(z)}{g(z)} \right\} \leq -\left( \frac{1 - r}{1 + r} \right),$$

for  $|z| = r$ . Using the representation for  $f(z)$  and the above we get

$$(2.4) \quad \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} = \operatorname{Re} \left\{ \frac{zg'(z)}{g(z)} \right\} + \operatorname{Re} \left\{ \frac{z\psi'(z)}{\psi(z)} \right\} \\ \leq \frac{(1 - \rho^2)(-1 + 4r - r^2) - 2(1 - \rho^2 r^2)^{1/2}(r^2 - \rho^2)^{1/2}}{2(1 - r^2)(1 - \rho^2)}.$$

Calculation shows that for  $r > \rho > 2 - \sqrt{3}$ , the numerator in (2.4) is negative provided that  $(r - 1)^2 Q(r) < 0$ , where

$$Q(r) = (1 + \rho^2)^2 r^2 - [6\rho^4 - 20\rho^2 + 6]r + (1 + \rho^2)^2.$$

To insure that  $Q(r)$  be negative in some annulus  $\rho_1 < |z| < 1$  it is sufficient that  $Q(1) < 0$ ; this is the case if  $\rho < (3 - 2\sqrt{2})^{1/2}$ .

**THEOREM 2.** *If  $\rho > 1/2$ , then  $\Lambda(\rho)$  is a proper subset of  $\Lambda^*(\rho)$ .*

The proof consists of giving an example and of appealing to the discussion in the introduction. (N. B.  $(3 - 2\sqrt{2})^{1/2} \sim 0.4$ ; the authors were not able to show the exact relationship between  $\Lambda(\rho)$  and  $\Lambda^*(\rho)$  when  $(3 - 2\sqrt{2})^{1/2} \leq \rho \leq 1/2$ .)

For any natural number  $k$  and any real number  $\theta$  we define

$$(2.5) \quad f(z) = -\rho(1 - e^{i\theta} z^k)^{2/k} / (z - \rho)(1 - \rho z),$$

a member of  $\Lambda^*(\rho)$ . Letting  $z = e^{i\phi}$ ,  $0 \leq \phi < 2\pi$ , we find that

$$(2.6) \quad zf'(z)/f(z) = 2iV(\phi),$$

where

$$(2.7) \quad V(\phi) = \frac{-\sin(\theta + k\phi)}{2 - 2\cos(\theta + k\phi)} + \frac{\rho \sin \phi}{1 - 2\rho \cos \phi + \rho^2}.$$

By the conformal properties of (2.6) we conclude that if  $V(\phi)$  is nondecreasing, then  $f(z)$  is in  $\Lambda(\rho)$ ; on the other hand, if there is an interval over which  $V(\phi)$  is decreasing, then  $f(z)$  cannot be in  $\Lambda(\rho)$ . In the discussion which follows we examine the behavior of  $V(\phi)$  for arbitrary  $k$ ,  $\theta$  and  $\rho$  and then choose appropriate values of these parameters to give a proof of Theorem 2.

Differentiating  $V(\phi)$  we find that  $V'(\phi) = Q(\phi)/P(\phi)$ , where  $P(\phi) \geq 0$  and

$$(2.8) \quad \begin{aligned} Q(\phi) &= (2k - 2k \cos(\theta + k\phi))(1 - 2\rho \cos \phi + \rho^2)^2 \\ &\quad + (\rho \cos \phi - 2\rho^2 + \rho^3 \cos \phi)(2 - 2\cos(\theta + k\phi))^2 \\ &= 2(1 - \cos(\theta + k\phi))H(\phi), \end{aligned}$$

where

$$(2.9) \quad \begin{aligned} H(\phi) &= (1 - 2\rho \cos \phi + \rho^2)^2 k \\ &\quad + (\rho \cos \phi - 2\rho^2 + \rho^3 \cos \phi)(2 - 2\cos(\theta + k\phi)). \end{aligned}$$

At this point we can show that  $f(z)$  is not in  $\Lambda(\rho)$  if  $H(\phi) < 0$  for some value of  $\phi$ .

Choose  $k = 1$  and  $\theta = \pi$ , then

$$(2.10) \quad H(\phi) = (1 + \rho)^2 [2\rho \cos^2 \phi - 2\rho \cos \phi + (1 - \rho)^2].$$

Replacing  $\cos \phi$  by  $x$ ,  $-1 \leq x \leq 1$ , the bracketed expression in (2.10) reduces to the quadratic  $B(x) = 2\rho x^2 - 2\rho x + (1 - \rho)^2$ .  $B(x)$  is strictly decreasing for  $-1 \leq x < 1/2$  and strictly increasing for  $1/2 < x \leq 1$ .  $B(1/2) \geq 0$ , if  $0 < \rho \leq 1/2$  and  $B(1/2) < 0$ , if  $1/2 < \rho < 1$ ; therefore we conclude that  $f(z)$  is in  $\Lambda(\rho)$  when  $0 < \rho \leq 1/2$  and not otherwise.

The following functions

$$(2.11) \quad F(z) = -\rho(1+z)^2/(z-\rho)(1-\rho z)$$

and

$$(2.12) \quad f(z) = -\rho(1-z)^2/(z-\rho)(1-\rho z),$$

obtained by appropriate choices for  $k$  and  $\theta$  in (2.5), are useful examples as extremals. As we have already shown  $F(z)$  is in  $\Lambda(\rho)$  for  $\rho \leq 1/2$  but is in  $\Lambda^*(\rho) \setminus \Lambda(\rho)$  for  $\rho > 1/2$ . As we will now show  $f(z)$  is in  $\Lambda(\rho)$  for all  $\rho$ . To this end we choose  $k = 1$  and  $\theta = 0$  in (2.9); then

$$H(\phi) = (1 - \rho)^2 [-2\rho \cos^2 \phi - 2\rho \cos \phi + (1 + \rho)^2].$$

Replacing  $\cos \phi$  by  $x$ , we obtain

$$D(x) = -2\rho x^2 - 2\rho x + (1 + \rho)^2,$$

having removed the positive factor  $(1 - \rho)^2$ . An examination of the quadratic  $D(x)$  reveals that  $D(x)$  is positive for all  $x$  in the interval  $[-1, 1]$  and all admissible  $\rho$ . This is sufficient to ensure that  $f(z)$ , (2.12), is in  $\Lambda(\rho)$  for all  $\rho$ .

Similar calculations show that when  $k = 2$  and  $\theta = 0$ ,  $f(z)$ , defined in (2.5), is in  $\Lambda(\rho)$  for all  $\rho$ ; and if  $k = 2$  and  $\theta = \pi$ , then  $f(z)$  is in  $\Lambda(\rho)$  for  $0 < \rho < [4\sqrt{2} - \sqrt{5}]/3\sqrt{3}$ , otherwise  $f(z)$  is in  $\Lambda^*(\rho)$ . Also, (2.9) shows that if  $k$  is sufficiently large,  $f(z)$  is in  $\Lambda(\rho)$  for all  $\rho$  and any value of  $\theta$ .

**3. Bounds and coefficients.** We begin this section by giving bounds on the growth of weakly starlike meromorphic functions and then apply these to determine bounds on their coefficients.

LEMMA 1. *If  $f(z)$  is in  $\Lambda^*(\rho)$  and  $|z| = r$ , then*

$$(3.1) \quad |f(z)| \leq \begin{cases} \rho(1+r)^2/(\rho-r)(1-\rho r), & r < \rho, \\ \rho(1+r)^2/(r-\rho)(1-\rho r), & r > \rho, \end{cases}$$

and

$$(3.2) \quad |f(z)| \geq \rho(1-r)^2/(r+\rho)(1+\rho r), \quad 0 < r < 1, \quad r \neq \rho.$$

These bounds follow from the representation of  $f(z)$  in terms of meromorphic starlike functions whose growth bounds are known [2]. Choosing  $g(z) = (1+z)^2/z$  in (1.4) shows the bounds are sharp.

LEMMA 2. *If  $f(z)$  is in  $\Lambda^*(\rho)$  and  $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$  for  $|z| < \rho$ , then*

$$(3.3) \quad |\rho a_n - (1 + \rho^2)a_{n-1} + \rho a_{n-2}| \leq 2\rho/n, \quad n = 1, 2, 3, \dots,$$

and in particular,

$$(3.4) \quad (1 - \rho)^2/\rho \leq |a_1| \leq (1 + \rho)^2/\rho.$$

PROOF. Rewriting (1.4) we get

$$(3.5) \quad f(z)(z - \rho)(1 - \rho z)/- \rho z = g(z),$$

where  $g(z)$  is in  $\Sigma^*$ . If we let  $g(z) = 1/z + \sum_{k=0}^{\infty} b_k z^k$  for  $z$  in  $\Delta$ , then it is known [1] that  $|b_k| \leq 2/(k+1)$  for all  $k$ ; and this along with some calculation in (3.5) gives (3.3). (3.4) is obtained from (3.3) by choosing  $n$  to be 1.

The function defined by

$$f_0(z) = -\rho(1-z)^2/(z-\rho)(1-\rho z)$$

which is in  $\Lambda(\rho)$  shows that (3.3) is sharp for 1 and 2 and that the lower bound of (3.4) is sharp. The functions

$$f_n(z) = -\rho(1 - e^{i\theta} z^n)^{2/n}/(z - \rho)(1 - \rho z)$$

show that (3.3) is sharp for all  $n$ .

**THEOREM 3.** *If  $f(z)$  is in  $\Lambda^*(\rho)$  and*

$$(3.6) \quad f(z) = \sum_{n=-\infty}^{\infty} A_n z^n, \quad \rho < |z| < 1;$$

then

$$(3.7) \quad |A_{-n}| \leq \rho^n((1+\rho)/(1-\rho)), \quad n = 1, 2, \dots;$$

and

$$(3.8) \quad |A_n| = O(1/\sqrt{n}).$$

PROOF. For any function  $f(z)$  with a simple pole at  $\rho$  and otherwise regular in  $\Delta$ , we may write  $f(z) = \rho/(z - \rho) \cdot h(z)$  is regular in  $\Delta$ . If  $h(z) = \sum_{k=0}^{\infty} d_k z^k$ , then for  $\rho < |z| < 1$ ,

$$(3.9) \quad f(z) = \left( \sum_{k=1}^{\infty} \left(\frac{\rho}{z}\right)^k \right) \left( \sum_{k=0}^{\infty} d_k z^k \right).$$

Comparing coefficients in (3.9) and (3.6) we then have

$$\begin{aligned} A_{-n} &= \rho^n d_0 + \rho^{n+1} d_1 + \rho^{n+2} d_2 + \dots \\ &= \rho^n [d_0 + d_1 \rho + d_2 \rho^2 + \dots] \\ &= \rho^n h(\rho); \end{aligned}$$

consequently

$$(3.10) \quad |A_{-n}| = \rho^n |h(\rho)|, \quad n = 1, 2, 3, \dots$$

If  $f(z)$  is now chosen to be in  $\Lambda^*(\rho)$ , then  $h(z) = -zg(z)/(1 - \rho z)$ , with  $g(z)$  in  $\Sigma^*$ , the class of normalized meromorphic starlike functions. Hence we may write

$$|h(\rho)| = \frac{\rho}{(1 - \rho^2)} |g(\rho)| \leq \frac{\rho}{(1 - \rho^2)} \frac{(1 + \rho)^2}{\rho} = \frac{1 + \rho}{1 - \rho}.$$

This with (3.10) gives (3.7).

Next we prove (3.8) by first showing it is true when  $f(z)$  is in  $\Lambda_1(\rho)$  and later remove this restriction. Consequently, let  $f(z)$  be in  $\Lambda_1(\rho)$ , then

$$\frac{-zf'(z)}{f(z)} \cdot \frac{(z - \rho)(1 - \rho z)}{z} = P(z)$$

is regular throughout  $\bar{\Delta}$  and since the factor  $(z - \rho)(1 - \rho z)/z$  is positive on  $\partial\Delta$  it follows that  $\text{Re}\{P(z)\} > 0$  for  $|z| = 1$ , therefore  $\text{Re}\{P(z)\} > 0$  for  $z$  in  $\bar{\Delta}$ . Furthermore if

$$P(z) = \sum_{n=0}^{\infty} c_n z^n, \quad z \in \Delta, \quad f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n \quad \text{for } |z| < \rho,$$

then  $c_0 = \rho a_1$  and  $\text{Re}\{\rho a_1\} > 0$ .

Using Parseval's identity and the known bounds  $|c_n| \leq 2|c_0| = 2\rho|a_1|$ ,  $n = 1, 2, 3, \dots$  (see for example [2]) we have, with  $z = re^{i\theta}$ , that

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} |P(z)|^2 d\theta &= |c_0|^2 + \sum_{k=1}^{\infty} |c_k|^2 r^{2k} \\ &\leq \rho^2 |a_1|^2 + 4\rho^2 |a_1|^2 r^2 / (1 - r^2) \\ (3.11) \quad &= \rho^2 |a_1|^2 [1 + 4r^2 / (1 - r^2)] \\ &= \rho^2 |a_1|^2 [(1 + 3r^2) / (1 - r^2)]. \end{aligned}$$

Now restricting  $z$  so that  $\rho < |z| < 1$  and making use of (3.1), (3.11) and the Schwarz inequality we have

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} |zf'(z)| d\theta &\leq \frac{\rho r(1 + r)^2}{(r - \rho)^2(1 - \rho r)^2} \cdot \frac{\rho |a_1| (1 + 3r^2)^{1/2}}{(1 - r^2)^{1/2}} \\ (3.12) \quad &\leq \frac{8\rho^2 |a_1|}{(1 - \rho)^2 (r - \rho)^2 (1 - r^2)^{1/2}} \leq \frac{B(\rho)}{(r - \rho)^2 (1 - r^2)^{1/2}}, \end{aligned}$$

where  $B(\rho) = 8\rho^2(1 - \rho)^{-2}(\sup|a_1|) \leq 8\rho((1 + \rho)/(1 - \rho))^2$ , as was shown above, (3.4).

From this we obtain

$$(3.13) \quad |nA_n| \leq \frac{1}{2\pi r^n} \int_0^{2\pi} |zf'(z)| d\theta \leq \frac{1}{r^n} \frac{B(\rho)}{(r-\rho)^2(1-r^2)^{1/2}}.$$

Letting

$$(3.14) \quad h_n(r) = r^n(r-\rho)^2(1-r^2)^{1/2},$$

we now maximize  $h_n(r)$  for fixed  $n$  and  $\rho < r < 1$ . Differentiation of (3.14) gives  $h'_n(r) = r^{n-1}(r-\rho)(1-r^2)^{-1/2}q_n(r)$  with

$$(3.15) \quad q_n(r) = -(n+3)r^3 + (n\rho + \rho)r^2 + (n+2)r - n\rho.$$

A calculation shows that  $q_n(r)$  has a unique root  $r_n$ ,  $\rho < r_n < 1$ , such that  $h_n(r_n)$  maximizes  $h_n(r)$  over the interval  $(\rho, 1)$ .

Summarizing these results we have

$$(3.16) \quad n|A_n| \leq B(\rho)/r_n^n(r_n - \rho)^2(1 - r_n^2)^{1/2}$$

for each  $n$ ; we proceed to study the right side of (3.16). Solving the equation  $q_n(r_n) = 0$  for  $n$  we have

$$(3.17) \quad n = r_n^2/(1 - r_n^2) - 2r_n/(r_n - \rho).$$

Comparing (3.17) for the cases  $n$  and  $n + 1$  shows that  $\{r_n\}$  is an increasing sequence which necessarily converges to 1.

Using (3.17) we write  $r_n^n$  solely in terms of  $r_n$  and  $\rho$ , then we conclude that

$$(3.18) \quad \lim_{n \rightarrow \infty} r_n^n = e^{-1/2}.$$

Rewriting (3.17) yields the relation

$$(3.19) \quad 1 - r_n^2 = r_n^2(r_n - \rho)/(n(r_n - \rho) + 2r_n);$$

and using (3.19) we reassemble (3.16) to appear as

$$(3.20) \quad \sqrt{n} |A_n| \leq \frac{B(\rho)[(r_n - \rho) + 2r_n^n/n]^{1/2}}{r_n^n r_n (r_n - \rho)^{5/2}}.$$

Using (3.18) and the fact that  $\lim_{n \rightarrow \infty} r_n = 1$ , we see that the right side of (3.20) converges to  $B(\rho)e^{1/2}(1-\rho)^{-2}$ .

It therefore follows that there is a constant  $C(\rho)$ , independent of  $f(z)$ , such that

$$(3.21) \quad \sqrt{n} |A_n| \leq C(\rho)$$



for all  $n$  and all  $f(z)$  in  $\Lambda_1(\rho)$ .

Suppose  $f(z)$  is in  $\Lambda(\rho)$ , then  $f(tz)$  is in  $\Lambda_1(\rho/t)$  for  $\rho < t < 1$ . Using this together with the fact that  $(z - \rho/t)(1 - \rho z/t)/(z - \rho)(1 - \rho z)$  is real and positive when  $|z| = 1$ , we conclude that

$$(3.22) \quad F_t(z) = \frac{t(z - \rho/t)(1 - \rho z/t)}{(z - \rho)(1 - \rho z)} f(tz)$$

is in  $\Lambda_1(\rho)$ . Letting  $F_t(z) = \sum_{n=-\infty}^{\infty} B_n(t)z^n$  for  $\rho < |z| < 1$ , we may re-write (3.21) to read

$$(3.23) \quad \sqrt{n} |B_n(t)| \leq C(\rho),$$

for all admissible  $t$ . Now as  $t$  approaches 1,  $F_t(z)$  approaches  $f(z)$  and  $B_n(t)$  approaches  $A_n$ ; this gives the bound (3.21) for all  $f(z)$  in  $\Lambda(\rho)$ . This concludes the proof of (3.8) for  $\Lambda(\rho)$ .

The proof is easily extended to the class  $\Lambda^*(\rho)$ . If  $f(z)$  is in  $\Lambda^*(\rho)$  it has the form (1.4) and

$$f_t(z) = -\rho z(tg(tz))/(z - \rho)(1 - \rho z)$$

is in  $\Lambda_1^*(\rho)$  which is  $\Lambda_1(\rho)$ . Letting  $f_t(z) = \sum_{n=-\infty}^{\infty} B_n(t)z^n$  we see that (3.23) holds, and letting  $t$  approach 1 enables us to conclude that (3.8) holds for  $\Lambda^*(\rho)$  as well as  $\Lambda(\rho)$ .

The bounds in (3.7) are rendered sharp by the function

$$f(z) = -\rho(1 + z)^2/(z - \rho)(1 - \rho z).$$

**THEOREM 4.** *If  $f(z)$  is in  $\Lambda^*(\rho)$ , then for  $z$  in  $\Delta$  and  $z \neq \rho$*

$$(3.24) \quad \frac{(1 - |a|)^2}{|a|(1 - |z|^2)} \leq \left| \frac{f'(z)}{f(z)} \right| \leq \frac{(1 + |a|)^2}{|a|(1 - |z|^2)},$$

and

$$(3.25) \quad \begin{aligned} &|\frac{1}{2}(1 - |z|^2)^2(f''(z)/f'(z)) - (\bar{z}(1 - |z|^2) + a^{-1}(1 + |a|^2)(1 - |z|^2))| \\ &\leq 2|f(z)/f'(z)| \leq 2|a|(1 - |z|^2)/(1 - |a|^2), \end{aligned}$$

where

$$(3.26) \quad a = (\rho - z)/(1 - \rho\bar{z}).$$

**PROOF.** If  $f(z)$  is in  $\Lambda_1(\rho)$  and  $z_0 \neq \rho$ , let

$$(3.27) \quad g(z) = \frac{(z - a)(1 - \bar{a}z)}{-zaf(z_0)} f\left(\frac{z + z_0}{1 + \bar{z}_0z}\right),$$

with  $a = (\rho - z_0)/(1 - \rho\bar{z}_0)$ . Since  $(z + z_0)/(1 + \bar{z}_0z)$  maps  $a$  into  $\rho$ ,  $f((z + z_0)/(1 + \bar{z}_0z))$  has a simple pole at  $a$ , therefore  $g(z)$  has a simple pole at the origin with its residue there equal to 1. Differentiating (3.27) logarithmically and restricting  $z$  so that  $|z| = 1$  we have

$$(3.28) \quad \frac{zg'(z)}{g(z)} = 2i \operatorname{Im} \left( \frac{a}{z-a} \right) + \frac{1 - |z_0|^2}{|1 + \bar{z}_0z|^2} \cdot \frac{wf'(w)}{f(w)}, \quad w = \frac{z + z_0}{1 + \bar{z}_0z}.$$

The real part of the last term in (3.28) is negative on  $\partial\Delta$ , therefore we conclude that  $g(z)$  is in  $\Sigma^*$ .

The Laurent expansion of  $g(z)$  in  $\Delta$  is

$$(3.29) \quad g(z) = \frac{1}{z} + \left[ \frac{f'(z_0)(1 - |z_0|^2)}{f(z_0)} - \left( \frac{1 + |a|^2}{a} \right) \right] \\ + \left[ \frac{f''(z_0)(1 - |z_0|^2)^2 - 2\bar{z}_0(1 - |z_0|^2)f'(z_0)}{2f(z_0)} \right. \\ \left. - \frac{(1 + |a|^2)(1 - |z_0|^2)f'(z_0)}{af(z_0)} + a^{-1}\bar{a} \right] z + \dots$$

Using well-known bounds on the coefficients of functions in  $\Sigma^*$  ([1], [7]), we write (3.24) and (3.25) for  $\Lambda_1(\rho)$ . (3.24) and (3.25) can be extended to all of  $\Lambda(\rho)$  by observing that if  $f(z)$  is in  $\Lambda(\rho)$ , then  $f(tz)$  is in  $\Lambda_1(\rho/t)$  for  $t$  sufficiently close to 1. If  $f(z)$  is in  $\Lambda^*(\rho)$ , then, as was noted in the introduction, we can choose a sequence of functions  $\{f_n(z)\}$  and a sequence of increasing real numbers  $\{r_n\}$  converging to 1 such that  $f_n(z)$  is in  $\Lambda_1(\rho/r_n)$  and  $f(z)$  is the uniform limit of the sequence  $\{f_n(z)\}$  on appropriate subsets of  $\Delta$ .

The function  $F(z) = -\rho(1+z)^2/(z-\rho)(1-\rho z)$  gives equality on the right-hand side of (3.24) for  $z = -r$ ,  $r \neq \rho$ ; and  $f(z) = \rho(1-z)^2/(z-\rho)(1-\rho z)$  gives the left side of (3.24), when  $z = -r$ ,  $r \neq \rho$ .

Ladegast gives a bound for the quotient  $f''(z)/f'(z)$  in the case where  $f(z)$  is assumed to be univalent only [5, (17), p. 133]. His relation is in form similar to (3.25), however his bound is unbounded in the vicinity of  $\rho$  whereas the right side of (3.25) is not.

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