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WEAKLY STARLIKE MEROMORPHIC UNIVALENT FUNCTIONS

BY

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ABSTRACT. A weakly starlike meromorphic univalent function is one of the form $f(z) = -\rho z g(z)[(z-\rho)(1-\rho z)]^{-1}$ for $0 < \rho < 1$ and $g(z)$ a meromorphic starlike function. The behavior of coefficients and growth of this class of functions and of a subset are studied.

1. **Introduction.** Let $f(z)$ be meromorphic in the open unit disk defined by $|z| < 1$ and hereafter called Δ with a simple pole at ρ, $0 < |\rho| < 1$, and otherwise regular in Δ. $f(z)$ is in $\Lambda(\rho)$ if and only if there is a number ρ_1, $|\rho| < \rho_1 < 1$, such that

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < 0 \tag{1.1}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} d\theta = -1, \tag{1.2}$$

for $\rho_1 < |z| < 1$ with $z = re^{i\theta}$. On the other hand, $f(z)$ is in $\Lambda_1(\rho)$ if and only if $f(z)$ is regualr in $\overline{\Delta}$, the closure of Δ, except again for a simple pole at ρ and (1.1) and (1.2) are satisfied on $\partial\Delta$, the latter being the boundary of Δ. Clearly $\Lambda_1(\rho)$ is a subset of $\Lambda(\rho)$.

It is no restriction on the geometric conditions given in (1.1) and (1.2) to assume that $f(z)$ in $\Lambda(\rho)$ be normalized so that $f(0) = 1$ and ρ be real; hence we shall hereafter make these assumptions. Also, it is clear that functions in the class $\Lambda(\rho)$ are univalent.

Conditions (1.1) and (1.2) taken together require that the origin be omitted by every function in $\Lambda(\rho)$. Meromorphic functions with a normalization similar to the above have been studied by Ladegast in an interesting paper which apparently has been overlooked [5]. Furthermore, the normalization taken for $\Lambda(\rho)$ can be viewed as an analog of the Montel normalization for regular univalent functions [6].

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If $f(z)$ is in $\Lambda_1(\rho)$, then the function

$$g(z) = (z - \rho)(1 - \rho z)f(z)/ - \rho z \tag{1.3}$$

is meromorphic in $\bar{\Delta}$ with a pole of residue 1 at the origin. Furthermore on $\partial\Delta$,

$$\operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} = \operatorname{Re}\left\{\left(\frac{\rho}{z-\rho}\right) - \left(\overline{\frac{\rho}{z-\rho}}\right) + \frac{zf'(z)}{f(z)}\right\}$$

$$= \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < 0.$$

Consequently, $g(z)$ is in Σ^* the class of meromorphic, normalized, starlike univalent functions ([1], [7]).

However, if we choose $f(z)$ in $\Lambda(\rho)$ we can likewise show that $f(z)$ has a representation like (1.3). For if we let $f_r(z) = f(rz)$, then $f_r(z)$ is in $\Lambda_1(\rho/r)$ for $\rho_1 < r \leqslant 1$ and by the argument given above we infer the existence of a function in Σ^* such that

$$f_r(z) = \frac{-\rho z/r}{(z - \rho/r)(1 - \rho z/r)}\, g_r(z).$$

Σ^* is normal and compact [2], therefore we can find a sequence of increasing real numbers $\{r_n\}$ converging to 1 such that $\{g_{r_n}(z)\}$ converges to a function $g(z)$ in Σ^* *in compacta.* Letting $n \to \infty$, we get the representation

$$f(z) = -\rho z g(z)/(z - \rho)(1 - \rho z) \tag{1.4}$$

for every function $f(z)$ in $\Lambda(\rho)$.

(1.4) gives a particularly useful representation for studying the properties of $\Lambda(\rho)$; for this reason we define the following classes of functions. $f(z)$ is in $\Lambda^*(\rho)$ if and only if $f(z)$ satisfies (1.4) for some function $g(z)$ in Σ^*; and $f(z)$ is in $\Lambda_1^*(\rho)$ if and only if $f(z)$ is defined by (1.4) with $g(z)$ in Σ^* but with the further restriction that $g(z)$ be regular and satisfy (1.1) in $\bar{\Delta}$. Clearly $\Lambda_1^*(\rho) \subset \Lambda^*(\rho)$.

It is evident that $\Lambda(\rho) \subset \Lambda^*(\rho)$ for each value of ρ. We shall show however that $\Lambda(\rho)$ is a proper subset of $\Lambda^*(\rho)$ for some values of ρ.

Functions in $\Lambda^*(\rho)$ are the reciprocals of weakly starlike, regular, univalent functions introduced by Hummel ([3], [4]). It is for this reason that we refer to members of $\Lambda^*(\rho)$ as weakly starlike, meromorphic univalent functions. $\Lambda(\rho)$ and $\Lambda^*(\rho)$ are particular cases of classes of functions studied by Styer [8].

It is evident from the above that $\Lambda_1^*(\rho) = \Lambda_1(\rho)$, for all ρ. Furthermore, for a given $f(z)$ in $\Lambda^*(\rho)$ there exists an increasing sequence of numbers $\{r_n\}$ converging to 1 and a corresponding sequence $\{f_n(z)\}$ such that $f_n(z)$ is in

$\Lambda(\rho/r_n)$, for each n, and $\lim_{n\to\infty} f_n(z) = f(z)$ in Δ.

2. **Relationship between $\Lambda(\rho)$ and $\Lambda^*(\rho)$.** The work of Styer [8] shows that $\Lambda(\rho) = \Lambda^*(\rho)$ if $\rho < 2 - \sqrt{3}$. We begin by improving this range slightly.

THEOREM 1. *If $\rho < (3 - 2\sqrt{2})^{1/2}$, then $\Lambda(\rho) = \Lambda^*(\rho)$.*

PROOF. Let $f(z)$ be in $\Lambda^*(\rho)$ and let

$$\psi(z) = \psi(z; \rho) = -\rho z/(z - \rho)(1 - \rho z), \tag{2.1}$$

then (1.4) assumes the form $f(z) = \psi(z)g(z)$ for $g(z)$ in Σ^*.

For $\rho \geqslant 2 - \sqrt{3}$, Hummel [3, p. 548] has essentially shown that

$$\operatorname{Re}\left\{\frac{z\psi'(z)}{\psi(z)}\right\} \leqslant \frac{[(1-\rho^2r^2)^{1/2} - (r^2-\rho^2)^{1/2}]^2}{2(1-r^2)(1-\rho^2)} \tag{2.2}$$

for $|z| = r$ and $\rho < r < 1$. If $g(z)$ is in Σ^*, it is well known [2] that

$$\operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} \leqslant -\left(\frac{1-r}{1+r}\right), \tag{2.3}$$

for $|z| = r$. Using the representation for $f(z)$ and the above we get

$$\begin{aligned}\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} &= \operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} + \operatorname{Re}\left\{\frac{z\psi'(z)}{\psi(z)}\right\} \\ &\leqslant \frac{(1-\rho^2)(-1+4r-r^2) - 2(1-\rho^2r^2)^{1/2}(r^2-\rho^2)^{1/2}}{2(1-r^2)(1-\rho^2)}.\end{aligned} \tag{2.4}$$

Calculation shows that for $r > \rho > 2 - \sqrt{3}$, the numerator in (2.4) is negative provided that $(r-1)^2 Q(r) < 0$, where

$$Q(r) = (1+\rho^2)^2 r^2 - [6\rho^4 - 20\rho^2 + 6]r + (1+\rho^2)^2.$$

To insure that $Q(r)$ be negative in some annulus $\rho_1 < |z| < 1$ it is sufficient that $Q(1) < 0$; this is the case if $\rho < (3 - 2\sqrt{2})^{1/2}$.

THEOREM 2. *If $\rho > 1/2$, then $\Lambda(\rho)$ is a proper subset of $\Lambda^*(\rho)$.*

The proof consists of giving an example and of appealing to the discussion in the introduction. (N. B. $(3 - 2\sqrt{2})^{1/2} \sim 0.4$; the authors were not able to show the exact relationship between $\Lambda(\rho)$ and $\Lambda^*(\rho)$ when $(3 - 2\sqrt{2})^{1/2} \leqslant \rho \leqslant 1/2$.)

For any natural number k and any real number θ we define

$$f(z) = -\rho(1 - e^{i\theta}z^k)^{2/k}/(z-\rho)(1-\rho z), \tag{2.5}$$

a member of $\Lambda^*(\rho)$. Letting $z = e^{i\phi}$, $0 \leqslant \phi < 2\pi$, we find that

$$zf'(z)/f(z) = 2iV(\phi); \tag{2.6}$$

where

$$V(\phi) = \frac{-\sin(\theta + k\phi)}{2 - 2\cos(\theta + k\phi)} + \frac{\rho \sin\phi}{1 - 2\rho\cos\phi + \rho^2}. \tag{2.7}$$

By the conformal properties of (2.6) we conclude that if $V(\phi)$ is nondecreasing, then $f(z)$ is in $\Lambda(\rho)$; on the other hand, if there is an interval over which $V(\phi)$ is decreasing, then $f(z)$ cannot be in $\Lambda(\rho)$. In the discussion which follows we examine the behavior of $V(\phi)$ for arbitrary k, θ and ρ and then choose appropriate values of these parameters to give a proof of Theorem 2.

Differentiating $V(\phi)$ we find that $V'(\phi) = Q(\phi)/P(\phi)$, where $P(\phi) \geqslant 0$ and

$$\begin{aligned} Q(\phi) &= (2k - 2k\cos(\theta + k\phi))(1 - 2\rho\cos\phi + \rho^2)^2 \\ &\quad + (\rho\cos\phi - 2\rho^2 + \rho^3\cos\phi)(2 - 2\cos(\theta + k\phi))^2 \\ &= 2(1 - \cos(\theta + k\phi))H(\phi), \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} H(\phi) &= (1 - 2\rho\cos\phi + \rho^2)^2 k \\ &\quad + (\rho\cos\phi - 2\rho^2 + \rho^3\cos\phi)(2 - 2\cos(\theta + k\phi)). \end{aligned} \tag{2.9}$$

At this point we can show that $f(z)$ is not in $\Lambda(\rho)$ if $H(\phi) < 0$ for some value of ϕ.

Choose $k = 1$ and $\theta = \pi$, then

$$H(\phi) = (1 + \rho)^2[2\rho\cos^2\phi - 2\rho\cos\phi + (1 - \rho)^2]. \tag{2.10}$$

Replacing $\cos\phi$ by x, $-1 \leqslant x \leqslant 1$, the bracketed expression in (2.10) reduces to the quadratic $B(x) = 2\rho x^2 - 2\rho x + (1 - \rho)^2$. $B(x)$ is strictly decreasing for $-1 \leqslant x < 1/2$ and strictly increasing for $1/2 < x \leqslant 1$. $B(1/2) \geqslant 0$, if $0 < \rho \leqslant 1/2$ and $B(1/2) < 0$, if $1/2 < \rho < 1$; therefore we conclude that $f(z)$ is in $\Lambda(\rho)$ when $0 < \rho \leqslant 1/2$ and not otherwise.

The following functions

$$F(z) = -\rho(1 + z)^2/(z - \rho)(1 - \rho z) \tag{2.11}$$

and

$$f(z) = -\rho(1 - z)^2/(z - \rho)(1 - \rho z), \tag{2.12}$$

obtained by appropriate choices for k and θ in (2.5), are useful examples as extremals. As we have already shown $F(z)$ is in $\Lambda(\rho)$ for $\rho \leqslant 1/2$ but is in $\Lambda^*(\rho)\backslash\Lambda(\rho)$ for $\rho > 1/2$. As we will now show $f(z)$ is in $\Lambda(\rho)$ for all ρ. To this end we choose $k = 1$ and $\theta = 0$ in (2.9); then

$$H(\phi) = (1-\rho)^2[-2\rho\cos^2\phi - 2\rho\cos\phi + (1+\rho)^2].$$

Replacing $\cos\phi$ by x, we obtain

$$D(x) = -2\rho x^2 - 2\rho x + (1+\rho)^2,$$

having removed the positive factor $(1-\rho)^2$. An examination of the quadratic $D(x)$ reveals that $D(x)$ is positive for all x in the interval $[-1, 1]$ and all admissible ρ. This is sufficient to ensure that $f(z)$, (2.12), is in $\Lambda(\rho)$ for all ρ.

Similar calculations show that when $k = 2$ and $\theta = 0$, $f(z)$, defined in (2.5), is in $\Lambda(\rho)$ for all ρ; and if $k = 2$ and $\theta = \pi$, then $f(z)$ is in $\Lambda(\rho)$ for $0 < \rho < [4\sqrt{2} - \sqrt{5}]/3\sqrt{3}$, otherwise $f(z)$ is in $\Lambda^*(\rho)$. Also, (2.9) shows that if k is sufficiently large, $f(z)$ is in $\Lambda(\rho)$ for all ρ and any value of θ.

3. **Bounds and coefficients.** We begin this section by giving bounds on the growth of weakly starlike meromorphic functions and then apply these to determine bounds on their coefficients.

LEMMA 1. *If $f(z)$ is in $\Lambda^*(\rho)$ and $|z| = r$, then*

$$(3.1)\qquad |f(z)| \leqslant \begin{cases} \rho(1+r)^2/(\rho - r)(1-\rho r), & r < \rho, \\ \rho(1+r)^2/(r-\rho)(1-\rho r), & r > \rho, \end{cases}$$

and

$$(3.2)\qquad |f(z)| \geqslant \rho(1-r)^2/(r+\rho)(1+\rho r), \qquad 0 < r < 1, \qquad r \neq \rho.$$

These bounds follow from the representation of $f(z)$ in terms of meromorphic starlike functions whose growth bounds are known [2]. Choosing $g(z) = (1+z)^2/z$ in (1.4) shows the bounds are sharp.

LEMMA 2. *If $f(z)$ is in $\Lambda^*(\rho)$ and $f(z) = 1 + \Sigma_{n=1}^{\infty} a_n z^n$ for $|z| < \rho$, then*

$$(3.3)\qquad |\rho a_n - (1+\rho^2)a_{n-1} + \rho a_{n-2}| \leqslant 2\rho/n, \qquad n = 1, 2, 3, \cdots,$$

and in particular,

$$(3.4)\qquad (1-\rho)^2/\rho \leqslant |a_1| \leqslant (1+\rho)^2/\rho.$$

PROOF. Rewriting (1.4) we get

$$f(z)(z-\rho)(1-\rho z)/-\rho z = g(z), \tag{3.5}$$

where $g(z)$ is in Σ^*. If we let $g(z) = 1/z + \Sigma_{k=0}^{\infty} b_k z^k$ for z in Δ, then it is known [1] that $|b_k| \leq 2/(k+1)$ for all k; and this along with some calculation in (3.5) gives (3.3). (3.4) is obtained from (3.3) by choosing n to be 1.

The function defined by

$$f_0(z) = -\rho(1-z)^2/(z-\rho)(1-\rho z)$$

which is in $\Lambda(\rho)$ shows that (3.3) is sharp for 1 and 2 and that the lower bound of (3.4) is sharp. The functions

$$f_n(z) = -\rho(1-e^{i\theta}z^n)^{2/n}/(z-\rho)(1-\rho z)$$

show that (3.3) is sharp for all n.

THEOREM 3. *If* $f(z)$ *is in* $\Lambda^*(\rho)$ *and*

$$f(z) = \sum_{n=-\infty}^{\infty} A_n z^n, \qquad \rho < |z| < 1; \tag{3.6}$$

then

$$|A_{-n}| \leq \rho^n((1+\rho)/(1-\rho)), \qquad n = 1, 2, \cdots; \tag{3.7}$$

and

$$|A_n| = O(1/\sqrt{n}). \tag{3.8}$$

PROOF. For any function $f(z)$ with a simple pole at ρ and otherwise regular in Δ, we may write $f(z) = \rho/(z-\rho) \cdot h(z)$ is regular in Δ. If $h(z) = \Sigma_{k=0}^{\infty} d_k z^k$, then for $\rho < |z| < 1$,

$$f(z) = \left(\sum_{k=1}^{\infty} \left(\frac{\rho}{z}\right)^k\right)\left(\sum_{k=0}^{\infty} d_k z^k\right). \tag{3.9}$$

Comparing coefficients in (3.9) and (3.6) we then have

$$\begin{aligned} A_{-n} &= \rho^n d_0 + \rho^{n+1} d_1 + \rho^{n+2} d_2 + \cdots \\ &= \rho^n [d_0 + d_1\rho + d_2\rho^2 + \cdots] \\ &= \rho^n h(\rho); \end{aligned}$$

consequently

$$|A_{-n}| = \rho^n |h(\rho)|, \qquad n = 1, 2, 3, \cdots. \tag{3.10}$$

If $f(z)$ is now chosen to be in $\Lambda^*(\rho)$, then $h(z) = -zg(z)/(1-\rho z)$, with $g(z)$ in Σ^*, the class of normalized meromorphic starlike functions. Hence we may write

$$|h(\rho)| = \frac{\rho}{(1-\rho^2)}\,|g(\rho)| \leqslant \frac{\rho}{(1-\rho^2)}\,\frac{(1+\rho)^2}{\rho} = \frac{1+\rho}{1-\rho}\,.$$

This with (3.10) gives (3.7).

Next we prove (3.8) by first showing it is true when $f(z)$ is in $\Lambda_1(\rho)$ and later remove this restriction. Consequently, let $f(z)$ be in $\Lambda_1(\rho)$, then

$$\frac{-zf'(z)}{f(z)} \cdot \frac{(z-\rho)(1-\rho z)}{z} = P(z)$$

is regular throughout $\bar{\Delta}$ and since the factor $(z-\rho)(1-\rho z)/z$ is positive on $\partial\Delta$ it follows that $\mathrm{Re}\{P(z)\} > 0$ for $|z| = 1$, therefore $\mathrm{Re}\{P(z)\} > 0$ for z in $\bar{\Delta}$. Furthermore if

$$P(z) = \sum_{n=0}^{\infty} c_n z^n, \quad z \in \Delta, \qquad f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n \quad \text{for } |z| < \rho,$$

then $c_0 = \rho a_1$ and $\mathrm{Re}\{\rho a_1\} > 0$.

Using Parseval's identity and the known bounds $|c_n| \leqslant 2|c_0| = 2\rho|a_1|$, $n = 1, 2, 3, \cdots$ (see for example [2]) we have, with $z = re^{i\theta}$, that

$$\begin{aligned}(3.11)\qquad \frac{1}{2\pi}\int_0^{2\pi} |P(z)|^2 d\theta &= |c_0|^2 + \sum_{k=1}^{\infty} |c_k|^2 r^{2k} \\ &\leqslant \rho^2|a_1|^2 + 4\rho^2|a_1|^2 r^2/(1-r^2) \\ &= \rho^2|a_1|^2[1 + 4r^2/(1-r^2)] \\ &= \rho^2|a_1|^2[(1+3r^2)/(1-r^2)].\end{aligned}$$

Now restricting z so that $\rho < |z| < 1$ and making use of (3.1), (3.11) and the Schwarz inequality we have

$$\begin{aligned}(3.12)\qquad \frac{1}{2\pi}\int_0^{2\pi} |zf'(z)|\,d\theta &\leqslant \frac{\rho r(1+r)^2}{(r-\rho)^2(1-\rho r)^2} \cdot \frac{\rho|a_1|(1+3r^2)^{1/2}}{(1-r^2)^{1/2}} \\ &\leqslant \frac{8\rho^2|a_1|}{(1-\rho)^2(r-\rho)^2(1-r^2)^{1/2}} \leqslant \frac{B(\rho)}{(r-\rho)^2(1-r^2)^{1/2}},\end{aligned}$$

where $B(\rho) = 8\rho^2(1-\rho)^{-2}(\sup|a_1|) \leqslant 8\rho((1+\rho)/(1-\rho))^2$, as was shown above, (3.4).

From this we obtain

(3.13) $$|nA_n| \leqslant \frac{1}{2\pi r^n} \int_0^{2\pi} |zf'(z)|\,d\theta \leqslant \frac{1}{r^n} \frac{B(\rho)}{(r-\rho)^2(1-r^2)^{1/2}}.$$

Letting

(3.14) $$h_n(r) = r^n(r-\rho)^2(1-r^2)^{1/2},$$

we now maximize $h_n(r)$ for fixed n and $\rho < r < 1$. Differentiation of (3.14) gives $h_n'(r) = r^{n-1}(r-\rho)(1-r^2)^{-1/2}q_n(r)$ with

(3.15) $$q_n(r) = -(n+3)r^3 + (n\rho + \rho)r^2 + (n+2)r - n\rho.$$

A calculation shows that $q_n(r)$ has a unique root r_n, $\rho < r_n < 1$, such that $h_n(r_n)$ maximizes $h_n(r)$ over the interval $(\rho, 1)$.

Summarizing these results we have

(3.16) $$n|A_n| \leqslant B(\rho)/r_n^n(r_n-\rho)^2(1-r_n^2)^{1/2}$$

for each n; we proceed to study the right side of (3.16). Solving the equation $q_n(r_n) = 0$ for n we have

(3.17) $$n = r_n^2/(1-r_n^2) - 2r_n/(r_n-\rho).$$

Comparing (3.17) for the cases n and $n+1$ shows that $\{r_n\}$ is an increasing sequence which necessarily converges to 1.

Using (3.17) we write r_n^n solely in terms of r_n and ρ, then we conclude that

(3.18) $$\lim_{n\to\infty} r_n^n = e^{-1/2}.$$

Rewriting (3.17) yields the relation

(3.19) $$1 - r_n^2 = r_n^2(r_n-\rho)/(n(r_n-\rho) + 2r_n);$$

and using (3.19) we reassemble (3.16) to appear as

(3.20) $$\sqrt{n}\,|A_n| \leqslant \frac{B(\rho)[(r_n-\rho) + 2r_n^n/n]^{1/2}}{r_n^n r_n (r_n-\rho)^{5/2}}.$$

Using (3.18) and the fact that $\lim_{n\to\infty} r_n = 1$, we see that the right side of (3.20) converges to $B(\rho)e^{1/2}(1-\rho)^{-2}$.

It therefore follows that there is a constant $C(\rho)$, independent of $f(z)$, such that

(3.21) $$\sqrt{n}\,|A_n| \leqslant C(\rho)$$

for all n and all $f(z)$ in $\Lambda_1(\rho)$.

Suppose $f(z)$ is in $\Lambda(\rho)$, then $f(tz)$ is in $\Lambda_1(\rho/t)$ for $\rho < t < 1$. Using this together with the fact that $(z-\rho/t)(1-\rho z/t)/(z-\rho)(1-\rho z)$ is real and positive when $|z| = 1$, we conclude that

$$F_t(z) = \frac{t(z-\rho/t)(1-\rho z/t)}{(z-\rho)(1-\rho z)} f(tz) \tag{3.22}$$

is in $\Lambda_1(\rho)$. Letting $F_t(z) = \Sigma_{n=-\infty}^{\infty} B_n(t) z^n$ for $\rho < |z| < 1$, we may rewrite (3.21) to read

$$\sqrt{n}\,|B_n(t)| \leqslant C(\rho), \tag{3.23}$$

for all admissible t. Now as t approaches 1, $F_t(z)$ approaches $f(z)$ and $B_n(t)$ approaches A_n; this gives the bound (3.21) for all $f(z)$ in $\Lambda(\rho)$. This concludes the proof of (3.8) for $\Lambda(\rho)$.

The proof is easily extended to the class $\Lambda^*(\rho)$. If $f(z)$ is in $\Lambda^*(\rho)$ it has the form (1.4) and

$$f_t(z) = -\rho z(tg(tz))/(z-\rho)(1-\rho z)$$

is in $\Lambda_1^*(\rho)$ which is $\Lambda_1(\rho)$. Letting $f_t(z) = \Sigma_{n=-\infty}^{\infty} B_n(t) z^n$ we see that (3.23) holds, and letting t approach 1 enables us to conclude that (3.8) holds for $\Lambda^*(\rho)$ as well as $\Lambda(\rho)$.

The bounds in (3.7) are rendered sharp by the function

$$f(z) = -\rho(1+z)^2/(z-\rho)(1-\rho z).$$

THEOREM 4. *If $f(z)$ is in $\Lambda^*(\rho)$, then for z in Δ and $z \neq \rho$*

$$\frac{(1-|a|)^2}{|a|(1-|z|^2)} \leqslant \left|\frac{f'(z)}{f(z)}\right| \leqslant \frac{(1+|a|)^2}{|a|(1-|z|^2)}, \tag{3.24}$$

and

$$|\tfrac{1}{2}(1-|z|^2)^2(f''(z)/f'(z)) - (\bar{z}(1-|z|^2) + a^{-1}(1+|a|^2)(1-|z|^2))| \leqslant 2|f(z)/f'(z)| \leqslant 2|a|(1-|z|^2)/(1-|a|^2), \tag{3.25}$$

where

$$a = (\rho - z)/(1-\rho\bar{z}). \tag{3.26}$$

PROOF. If $f(z)$ is in $\Lambda_1(\rho)$ and $z_0 \neq \rho$, let

$$g(z) = \frac{(z-a)(1-\bar{a}z)}{-zaf(z_0)} f\left(\frac{z+z_0}{1+\bar{z}_0 z}\right), \tag{3.27}$$

with $a = (\rho - z_0)/(1 - \rho\bar{z}_0)$. Since $(z + z_0)/(1 + \bar{z}_0 z)$ maps a into ρ, $f((z + z_0)/(1 + \bar{z}_0 z))$ has a simple pole at a, therefore $g(z)$ has a simple pole at the origin with its residue there equal to 1. Differentiating (3.27) logarithmically and restricting z so that $|z| = 1$ we have

$$\frac{zg'(z)}{g(z)} = 2i\,\mathrm{Im}\left(\frac{a}{z-a}\right) + \frac{1-|z_0|^2}{|1+\bar{z}_0 z|^2}\cdot\frac{wf'(w)}{f(w)}, \qquad w = \frac{z+z_0}{1+\bar{z}_0 z}. \tag{3.28}$$

The real part of the last term in (3.28) is negative on $\partial\Delta$, therefore we conclude that $g(z)$ is in Σ^*.

The Laurent expansion of $g(z)$ in Δ is

$$g(z) = \frac{1}{z} + \left[\frac{f'(z_0)(1-|z_0|^2)}{f(z_0)} - \left(\frac{1+|a|^2}{a}\right)\right]$$
$$+\left[\frac{f''(z_0)(1-|z_0|^2)^2 - 2\bar{z}_0(1-|z_0|^2)f'(z_0)}{2f(z_0)} - \frac{(1+|a|^2)(1-|z_0|^2)f'(z_0)}{af(z_0)} + a^{-1}\bar{a}\right]z + \cdots. \tag{3.29}$$

Using well-known bounds on the coefficients of functions in Σ^* ([1], [7]), we write (3.24) and (3.25) for $\Lambda_1(\rho)$. (3.24) and (3.25) can be extended to all of $\Lambda(\rho)$ by observing that if $f(z)$ is in $\Lambda(\rho)$, then $f(tz)$ is in $\Lambda_1(\rho/t)$ for t sufficiently close to 1. If $f(z)$ is in $\Lambda^*(\rho)$, then, as was noted in the introduction, we can choose a sequence of functions $\{f_n(z)\}$ and a sequence of increasing real numbers $\{r_n\}$ converging to 1 such that $f_n(z)$ is in $\Lambda_1(\rho/r_n)$ and $f(z)$ is the uniform limit of the sequence $\{f_n(z)\}$ on appropriate subsets of Δ.

The function $F(z) = -\rho(1 + z)^2/(z - \rho)(1 - \rho z)$ gives equality on the right-hand side of (3.24) for $z = -r$, $r \neq \rho$; and $f(z) = \rho(1-z)^2/(z-\rho)(1-\rho z)$ gives the left side of (3.24), when $z = -r$, $r \neq \rho$.

Ladegast gives a bound for the quotient $f''(z)/f'(z)$ in the case where $f(z)$ is assumed to be univalent only [5, (17), p. 133]. His relation is in form similar to (3.25), however his bound is unbounded in the vicinity of ρ whereas the right side of (3.25) is not.

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