TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 202, 1975

DECOMPOSABLE BRAIDS AS SUBGROUPS OF BRAID GROUPS[1]

BY

H. LEVINSON

ABSTRACT. The group of all decomposable 3-braids is the commutator subgroup of the group I_3 of all 3-braids which leave strand positions invariant. The group of all 2-decomposable 4-braids is the commutator subgroup of I_4, and the group of all decomposable 4-braids is explicitly characterized as a subgroup of the second commutator subgroup of I_4.

Introduction. A braid on n strands is called *k-decomposable* iff whenever k arbitrary strands are removed, the remaining braid on $n - k$ strands is deformable into the identity braid. The set of all k-decomposable n-braids is denoted D_{kn}, and it shall be the task of this paper to determine D_{kn} as a subgroup of the braid group B_n in the cases where $n = 3, k = 1$, and $n = 4, k = 1, 2$. Based upon these cases, a reasonable conjecture is drawn as to the remainder of the D_{kn}.

I wish to thank Professor W. Magnus who posed problems treated in this paper and proved most of Theorem 2.

Decomposable 3-braids. We shall confine our attention to the subgroup of the braid group consisting of those braids which leave strand positions invariant. Denote this group by I_n.

Notation. For any elements u, v of a group:

$$(u, v) = u^{-1}v^{-1}uv; \qquad u^v = v^{-1}uv.$$

For any n elements $u_1, u_2, \cdots, u_n$,

$$(u_1, u_2, \cdots, u_n) = (\cdots((u_1, u_2), u_3), \cdots, u_n).$$

Received by the editors June 28, 1972.

AMS (MOS) subject classifications (1970). Primary 55A25.

(1) The results in this paper constituted part of the author's Ph. D. thesis written at the Courant Institute of Mathematical Sciences, New York University, under the direction of Professor Wilhelm Magnus. The author further wishes to acknowledge and thank the Rutgers University Research Council for having provided funds for the typing of the manuscript of this paper.

For all normal subgroups $N_1, N_2, \cdots, N_n$ of a group: $(N_1, N_2, \cdots, N_n)$ shall denote the normal subgroup generated by the set of all $(u_1, u_2, \cdots, u_n)$, where $u_i \in N_1$. Let P_n denote the subgroup of B_n consisting of all "n-pure" braids (i.e. those braids in which strands numbered 1 through $n-1$ are uninvolved with each other and only strand n weaves its way among its lesser indexed straight companions).

LEMMA 1 (ARTIN). *P_n is normal in I_n.*

P_n is free on $n-1$ free generators, $t_1, t_2, \cdots, t_{n-1}$ defined by

$$t_i = \sigma_i\sigma_{i+1} \cdots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}^{-1} \cdots \sigma_{i+1}^{-1}\sigma_i^{-1},$$

where the σ are the generators for B_n given in [1]. This is obvious since P_n is the fundamental group of the space between two planes from which $n-1$ straight line segments, parallel to each other, joining the two planes have been removed.

If $x_1, x_2, \cdots, x_n$ are the free generators of the free group on which B_n acts, then

$$t_i(x_j) = (x_i, x_n)x_j(x_n, x_i) \quad \text{for} \quad i < j < n,$$

$$t_i(x_i) = x_n^{-1}x_ix_n = x_i(x_i, x_n),$$

$$t_i(x_n) = x_n^{-1}x_i^{-1}x_nx_ix_n = (x_n, x_i)x_n.$$

LEMMA 2. *$D_{k,n}$ is normal in B_n.*

PROOF. The conjugates of any k-decomposable n-braid are k-decomposable since if any k strands were removed the remnants of a conjugating braid and its inverse would be separated from each other by only a trivial braid, and would therefore be in a position to annihilate each other. Q.E.D.

LEMMA 3. *$D_{1,n}$ is a normal subgroup of P_n.*

PROOF. According to Artin [1] every braid may be written as a product, $\rho\pi$, where $\pi \in P_n$ and ρ is a braid in the subgroup I_{n-1} of I_n, generated by the first $n-2$ of the $n-1$ σ_i's generating I_n. ρ leaves the nth strand straight and uninvolved with any other of the first $n-1$ strands. The removal of the nth strand reduces π to a trivial braid on the first $n-1$ strands, and leaves ρ unchanged. Thus if $\pi\rho \in D_{1,n}$, then $\rho = 1$.

Since $D_{1,n}$ is normal in B_n, it is normal in every subgroup of B_n in which it is contained, in particular in P_n. Q.E.D.

LEMMA 4. *Let $\theta_i, i = 1, 2, \cdots, n-1$, be the normal closure of t_i in P_n. Then $D_{1,n}$ is the intersection of the groups θ_i.*

PROOF. $D_{1,n}$ is obviously a subset of the intersection of the θ_i. The removal of the ith strand of an element of P_n has the effect of mapping it into its coset with respect to the normal closure of t_i. We now note that

$$D_{1,n} \supseteq \prod_{\substack{\text{all permutations} \\ (i_1, \cdots, i_{n-1}) \text{ of } (1, \cdots, n-1)}} (\theta_{i_1}, \theta_{i_2}, \cdots, \theta_{i_{n-1}}). \quad \text{Q.E.D.}$$

We now consider the case where $n = 3$.

THEOREM 1. *$D_{1\ 3} = I_3'$ (the commutator subgroup of I_3). $D_{1\ 3}$ is generated by the conjugates of $(t_1, t_2) = (\sigma_1\sigma_2^2\sigma_1^{-1}, \sigma_2^2)$; i.e. by conjugates of the braid $(\sigma_1\sigma_2^{-1})^3$.*

PROOF. From the preceding lemmas, it follows that $D_{1\ 3}$ is generated by $\theta_1 \cap \theta_2$. Obviously this is (θ_1, θ_2), i.e. the commutator subgroup of P_3. Since I_3 is generated [1] by σ_1^2, σ_2^2, and $\sigma_1\sigma_2^2\sigma_1^{-1}$, and since

$$\begin{aligned}\sigma_1\sigma_2^2\sigma_1^{-1}\sigma_1^2\sigma_2^2 = \sigma_1\sigma_2^2\sigma_1\sigma_2^2 &= \sigma_1\sigma_2(\sigma_2\sigma_1\sigma_2)\sigma_2 \\ &= \sigma_1\sigma_2(\sigma_1\sigma_2\sigma_1)\sigma_2 = (\sigma_1\sigma_2)^3\end{aligned}$$

is in the center of B_3, it follows that I_3' is the normal closure of the commutator of any two of the elements $\sigma_1^2, \sigma_2^2, \sigma_1\sigma_2^2\sigma_1^{-1}$. We choose the last two which we had denoted t_1 and t_2 respectively. It must now be shown that (t_1, t_2) is a conjugate of $(\sigma_1\sigma_2^2\sigma_1^{-1}, \sigma_2^2)$.

Using the relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$, from B_3,

$$\begin{aligned}(t_1, t_2) &= \sigma_1\sigma_2^{-2}\sigma_1^{-1}\sigma_2^{-2}\sigma_1\sigma_2^2\sigma_1^{-1}\sigma_2^2 \\ &= \sigma_1\sigma_2^{-1}(\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1})\sigma_2^{-1}(\sigma_1\sigma_2)\sigma_2\sigma_1^{-1}\sigma_2^2 \\ &= \sigma_1\sigma_2^{-1}(\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1})\sigma_2^{-1}(\sigma_2\sigma_1\sigma_2\sigma_1^{-1})\sigma_2\sigma_1^{-1}\sigma_2^2 \\ &= \sigma_1\sigma_2^{-1}\sigma_1^{-2}\sigma_2\sigma_1^{-1}\sigma_2^2 \\ &= \sigma_2^{-1}((\sigma_2\sigma_1\sigma_2^{-1})\sigma_1^{-2}\sigma_2\sigma_1^{-1}\sigma_2)\sigma_2 \\ &= \sigma_2^{-1}((\sigma_1^{-1}\sigma_2\sigma_1)\sigma_1^{-2}\sigma_2\sigma_1^{-1}\sigma_2)\sigma_2 \\ &= \sigma_2^{-1}(\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2)\sigma_2 \\ &= \sigma_2^{-1}\sigma_1^{-1}(\sigma_1\sigma_2^{-1})^{-3}\sigma_1\sigma_2. \quad \text{Q.E.D.}\end{aligned}$$

Decomposable 4-braids.

THEOREM 2. *$D_{1\ 4}$ is the product $((\theta_1, \theta_2), \theta_3)((\theta_1, \theta_3), \theta_2) \hat{=} \theta^*$.*

PROOF. We shall prove $D_{1\,4} = ((\theta_1, \theta_2), \theta_3)((\theta_1, \theta_3), \theta_2)((\theta_2, \theta_3), \theta_1)$. According to P. Hall [3] each of the three factors is contained in the product of the other two. In order to simplify notation, denote t_1, t_2, and t_3 by a, b, and c respectively. We wish to find the intersection of A, B, and C, the respective normal closures of a, b, and c.

The normal closure of a is freely generated by the elements a^α, where α runs through all freely reduced words in b and c. (This follows from elementary combinatorial arguments.) We now characterize which elements of A are also in B. If we map $b \longrightarrow 1$, this has the result that $\alpha \longrightarrow c^s$, where s is the exponent sum of the c's in α. We denote c^s by $\bar{\alpha}$, and observe that $a^\alpha a^{-\bar{\alpha}}$ is indeed in both A and B. We wish to show that $A \cap B = (A, B)$. For this purpose we write $\alpha^{-1} = b^{e_1} c^{f_1} b^{e_2} c^{f_2} \cdots b^{e_k} c^{f_k}$, where the e_i and f_i are nonzero integers with the possible exceptions of e_1 and f_k. Set $e = \Sigma_{i=1}^k e_i$ and $f = \Sigma_{i=1}^k f_i$, and rewrite α^{-1} as

$$
\begin{aligned}
\alpha^{-1} &= b^{e_1}(c^{f_1} b c^{-f_1})^{e_2}(c^{f_1+f_2} b c^{-f_1-f_2})^{e_3} \\
&\qquad \cdots (c^{f_1+\cdots+f_{k-1}} b c^{-f_1 \cdots -f_{k-1}})^{e_k} c^f.
\end{aligned} \tag{1}
$$

Now we use the identity

$$
\begin{aligned}
\Omega_k &= v_k^{-1} v_{k-1}^{-1} \cdots v_1^{-1} u v_1 v_2 \cdots v_{k-1} v_k u^{-1} \\
&= v_k^{-1} v_{k-1}^{-1} \cdots v_1^{-1} u v_1 v_2 \cdots v_{k-1} u^{-1} v_k (v_k, u^{-1}).
\end{aligned} \tag{2}
$$

Putting

$$
u = c^f a c^{-f}, \quad \text{and} \quad v_k = (c^{f_1+\cdots+f_{k-1}} b c^{-f_1-\cdots-f_{k-1}})^{e_k}, \tag{3}
$$

we observe that $(v_k, u^{-1}) \in (A, B) = (B, A)$. Therefore $\Omega_k \in (A, B)$ if we can show that

$$
\Omega_{k-1} = v_{k-1}^{-1} \cdots v_1^{-1} u v_1 \cdots v_{k-1} u^{-1} \in (A, B), \tag{4}
$$

since

$$
\Omega_k = v_k^{-1} \Omega_{k-1} v_k (v_k, u^{-1}). \tag{5}
$$

However, this follows by induction since

$$
\Omega_1 = v_1^{-1} u v_1 u^{-1} \text{ is obviously in } (A, B) \text{ if } v_1 = b^e. \tag{6}
$$

We now characterize which products of elements $a^\alpha a^{-\bar{\alpha}}$ are also in C. Mapping $c \longrightarrow 1$ has the effect of mapping $a^\alpha a^{-\bar{\alpha}} \longrightarrow b^e a b^{-e} a^{-1}$. We must show that $a^\alpha a^{-\bar{\alpha}} a b^e a^{-1} b^{-e} \in \theta^*$.

We first show that for $v = c^\phi b^\epsilon c^{-\phi}$, and $u = c^\gamma a c^{-\gamma}$, that $(v, u^{-1}) \equiv (b, a^{-1}) \bmod \theta^*$.

$$(v, u^{-1})(a^{-1}, b^{\epsilon}) = c^{\phi}b^{-\epsilon}c^{-\phi}c^{\gamma}ac^{-\gamma}c^{\phi}b^{\epsilon}c^{-\phi}c^{\gamma}a^{-1}c^{-\gamma}(ab^{-\epsilon}a^{-1}b^{\epsilon})$$
$$= c^{\phi}(b^{-\epsilon}c^{\delta}ac^{-\delta}b^{\epsilon}c^{\delta}a^{-1}c^{\delta})c^{-\phi}(ab^{-\epsilon}a^{-1}b^{\epsilon}),$$

for $\delta = \gamma - \phi$. Thus

$$(v, u^{-1})(a^{-1}, b^{\epsilon}) = c^{\phi}(b^{\epsilon}, c^{\delta}a^{-1}c^{-\delta})c^{-\phi}(b^{\epsilon}, a^{-1})^{-1}.$$

Let $c^{\phi} = r$, $b^{-\epsilon} = s$, and $c^{\delta}ac^{-\delta} = t$. Then

$$(v, u^{-1})(a^{-1}, b^{\epsilon}) = rs^{-1}t^{-1}sts^{-1}asa^{-1}r^{-1}(r, as^{-1}a^{-1}s^{-1}).$$

Note that $(r, as^{-1}a^{-1}s^{-1})$ is an element of θ^*.

To show that $(s, t)(s, a^{-1}) \in \theta^*$, note that $t = c^{\delta}ac^{-\delta}$. By the Witt-Hall identities [3]

$$(s, t)(s, a^{-1}) = (s, ta^{-1})((s, t), a^{-1}).$$

$ta^{-1} \in (A, C), s \in B$ together imply that $(s, ta^{-1}) \in (B, (A, C)) \subseteq \theta^*$. $((s, t), a^{-1}) \in \theta^*$ also. Substitution into the argument in lines (1) through (6) yields the desired result. Q.E.D.

From this, we make the reasonable conjecture that D_{kn} is an $(n-k-1)$-fold commutator subgroup of elements which are commutators from $n-k-1$ distinct subgroups in $D_{k+1\ n}$, each isomorphic to $D_{k\ n-1}$. The proof of this is a problem in the commutator calculus which may, perhaps, be solved by mimicking algebraically the geometric constructions in [2].

BIBLIOGRAPHY

1. E. Artin, *Theory of braids,* Ann. of Math. (2) 48 (1947), 101–126. MR 8, 367.

2. H. Levinson, *Decomposable braids and linkages,* Trans. Amer. Math. Soc. 178 (1973), 111–126.

3. W. Magnus, A. Karrass and D. Solitar, *Combinatorial group theory: Presentations of groups in terms of generators and relations,* Pure and Appl. Math., vol. 13, Interscience, New York, 1966. Theorem 5.2, p. 290 and Equation 9, p. 290. MR 34 #7617.

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NEW JERSEY 08903