

DECOMPOSABLE BRAIDS AS SUBGROUPS OF BRAID GROUPS⁽¹⁾

BY

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ABSTRACT. The group of all decomposable 3-braids is the commutator subgroup of the group I_3 of all 3-braids which leave strand positions invariant. The group of all 2-decomposable 4-braids is the commutator subgroup of I_4 , and the group of all decomposable 4-braids is explicitly characterized as a subgroup of the second commutator subgroup of I_4 .

Introduction. A braid on n strands is called *k-decomposable* iff whenever k arbitrary strands are removed, the remaining braid on $n - k$ strands is deformable into the identity braid. The set of all *k-decomposable n-braids* is denoted D_{kn} , and it shall be the task of this paper to determine D_{kn} as a subgroup of the braid group B_n in the cases where $n = 3, k = 1$, and $n = 4, k = 1, 2$. Based upon these cases, a reasonable conjecture is drawn as to the remainder of the D_{kn} .

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Decomposable 3-braids. We shall confine our attention to the subgroup of the braid group consisting of those braids which leave strand positions invariant. Denote this group by I_n .

Notation. For any elements u, v of a group:

$$(u, v) = u^{-1}v^{-1}uv; \quad u^v = v^{-1}uv.$$

For any n elements u_1, u_2, \dots, u_n ,

$$(u_1, u_2, \dots, u_n) = (\dots ((u_1, u_2), u_3), \dots, u_n).$$

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For all normal subgroups N_1, N_2, \dots, N_n of a group: (N_1, N_2, \dots, N_n) shall denote the normal subgroup generated by the set of all (u_1, u_2, \dots, u_n) , where $u_i \in N_i$. Let P_n denote the subgroup of B_n consisting of all “ n -pure” braids (i.e. those braids in which strands numbered 1 through $n - 1$ are uninvolved with each other and only strand n weaves its way among its lesser indexed straight companions).

LEMMA 1 (ARTIN). P_n is normal in I_n .

P_n is free on $n - 1$ free generators, t_1, t_2, \dots, t_{n-1} defined by

$$t_i = \sigma_i \sigma_{i+1} \cdots \sigma_{n-2} \sigma_{n-1}^2 \sigma_{n-2}^{-1} \cdots \sigma_{i+1}^{-1} \sigma_i^{-1},$$

where the σ are the generators for B_n given in [1]. This is obvious since P_n is the fundamental group of the space between two planes from which $n - 1$ straight line segments, parallel to each other, joining the two planes have been removed.

If x_1, x_2, \dots, x_n are the free generators of the free group on which B_n acts, then

$$t_i(x_j) = (x_i, x_n) x_j (x_n, x_i) \text{ for } i < j < n,$$

$$t_i(x_i) = x_n^{-1} x_i x_n = x_i (x_i, x_n),$$

$$t_i(x_n) = x_n^{-1} x_i^{-1} x_n x_i x_n = (x_n, x_i) x_n.$$

LEMMA 2. $D_{k,n}$ is normal in B_n .

PROOF. The conjugates of any k -decomposable n -braid are k -decomposable since if any k strands were removed the remnants of a conjugating braid and its inverse would be separated from each other by only a trivial braid, and would therefore be in a position to annihilate each other. Q.E.D.

LEMMA 3. $D_{1,n}$ is a normal subgroup of P_n .

PROOF. According to Artin [1] every braid may be written as a product, $\rho\pi$, where $\pi \in P_n$ and ρ is a braid in the subgroup I_{n-1} of I_n , generated by the first $n - 2$ of the $n - 1$ σ_i 's generating I_n . ρ leaves the n th strand straight and uninvolved with any other of the first $n - 1$ strands. The removal of the n th strand reduces π to a trivial braid on the first $n - 1$ strands, and leaves ρ unchanged. Thus if $\pi\rho \in D_{1,n}$, then $\rho = 1$.

Since $D_{1,n}$ is normal in B_n , it is normal in every subgroup of B_n in which it is contained, in particular in P_n . Q.E.D.

LEMMA 4. Let $\theta_i, i = 1, 2, \dots, n - 1$, be the normal closure of t_i in P_n . Then $D_{1,n}$ is the intersection of the groups θ_i .

PROOF. $D_{1,n}$ is obviously a subset of the intersection of the θ_i . The removal of the i th strand of an element of P_n has the effect of mapping it into its coset with respect to the normal closure of t_i . We now note that

$$D_{1,n} \supseteq \prod_{\substack{\text{all permutations} \\ (i_1, \dots, i_{n-1}) \text{ of } (1, \dots, n-1)}} (\theta_{i_1}, \theta_{i_2}, \dots, \theta_{i_{n-1}}). \text{ Q.E.D.}$$

We now consider the case where $n = 3$.

THEOREM 1. $D_{1,3} = I'_3$ (the commutator subgroup of I_3). $D_{1,3}$ is generated by the conjugates of $(t_1, t_2) = (\sigma_1\sigma_2^2\sigma_1^{-1}, \sigma_2^2)$; i.e. by conjugates of the braid $(\sigma_1\sigma_2^{-1})^3$.

PROOF. From the preceding lemmas, it follows that $D_{1,3}$ is generated by $\theta_1 \cap \theta_2$. Obviously this is (θ_1, θ_2) , i.e. the commutator subgroup of P_3 . Since I_3 is generated [1] by σ_1^2, σ_2^2 , and $\sigma_1\sigma_2^2\sigma_1^{-1}$, and since

$$\begin{aligned} \sigma_1\sigma_2^2\sigma_1^{-1}\sigma_1^2\sigma_2^2 &= \sigma_1\sigma_2^2\sigma_1\sigma_2^2 = \sigma_1\sigma_2(\sigma_2\sigma_1\sigma_2)\sigma_2 \\ &= \sigma_1\sigma_2(\sigma_1\sigma_2\sigma_1)\sigma_2 = (\sigma_1\sigma_2)^3 \end{aligned}$$

is in the center of B_3 , it follows that I'_3 is the normal closure of the commutator of any two of the elements $\sigma_1^2, \sigma_2^2, \sigma_1\sigma_2^2\sigma_1^{-1}$. We choose the last two which we had denoted t_1 and t_2 respectively. It must now be shown that (t_1, t_2) is a conjugate of $(\sigma_1\sigma_2^2\sigma_1^{-1}, \sigma_2^2)$.

Using the relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$, from B_3 ,

$$\begin{aligned} (t_1, t_2) &= \sigma_1\sigma_2^{-2}\sigma_1^{-1}\sigma_2^{-2}\sigma_1\sigma_2^2\sigma_1^{-1}\sigma_2^2 \\ &= \sigma_1\sigma_2^{-1}(\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1})\sigma_2^{-1}(\sigma_1\sigma_2)\sigma_2\sigma_1^{-1}\sigma_2^2 \\ &= \sigma_1\sigma_2^{-1}(\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1})\sigma_2^{-1}(\sigma_2\sigma_1\sigma_2\sigma_1^{-1})\sigma_2\sigma_1^{-1}\sigma_2^2 \\ &= \sigma_1\sigma_2^{-1}\sigma_1^{-2}\sigma_2\sigma_1^{-1}\sigma_2^2 \\ &= \sigma_2^{-1}((\sigma_2\sigma_1\sigma_2^{-1})\sigma_1^{-2}\sigma_2\sigma_1^{-1}\sigma_2)\sigma_2 \\ &= \sigma_2^{-1}((\sigma_1^{-1}\sigma_2\sigma_1)\sigma_1^{-2}\sigma_2\sigma_1^{-1}\sigma_2)\sigma_2 \\ &= \sigma_2^{-1}(\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2)\sigma_2 \\ &= \sigma_2^{-1}\sigma_1^{-1}(\sigma_1\sigma_2^{-1})^{-3}\sigma_1\sigma_2. \text{ Q.E.D.} \end{aligned}$$

Decomposable 4-braids.

THEOREM 2. $D_{1,4}$ is the product $((\theta_1, \theta_2), \theta_3)((\theta_1, \theta_3), \theta_2) = \theta^*$.

PROOF. We shall prove $D_{1\ 4} = ((\theta_1, \theta_2), \theta_3)((\theta_1, \theta_3), \theta_2)((\theta_2, \theta_3), \theta_1)$. According to P. Hall [3] each of the three factors is contained in the product of the other two. In order to simplify notation, denote t_1, t_2 , and t_3 by a, b , and c respectively. We wish to find the intersection of A, B , and C , the respective normal closures of a, b , and c .

The normal closure of a is freely generated by the elements a^α , where α runs through all freely reduced words in b and c . (This follows from elementary combinatorial arguments.) We now characterize which elements of A are also in B . If we map $b \rightarrow 1$, this has the result that $\alpha \rightarrow c^s$, where s is the exponent sum of the c 's in α . We denote c^s by $\bar{\alpha}$, and observe that $a^\alpha a^{-\bar{\alpha}}$ is indeed in both A and B . We wish to show that $A \cap B = (A, B)$. For this purpose we write $\alpha^{-1} = b^{e_1} c^{f_1} b^{e_2} c^{f_2} \dots b^{e_k} c^{f_k}$, where the e_i and f_i are nonzero integers with the possible exceptions of e_1 and f_k . Set $e = \sum_{i=1}^k e_i$ and $f = \sum_{i=1}^k f_i$, and rewrite α^{-1} as

$$(1) \quad \alpha^{-1} = b^{e_1} (c^{f_1} b c^{-f_1})^{e_2} (c^{f_1+f_2} b c^{-f_1-f_2})^{e_3} \dots (c^{f_1+\dots+f_{k-1}} b c^{-f_1-\dots-f_{k-1}})^{e_k} c^{f_1}$$

Now we use the identity

$$(2) \quad \begin{aligned} \Omega_k &= v_k^{-1} v_{k-1}^{-1} \dots v_1^{-1} u v_1 v_2 \dots v_{k-1} v_k u^{-1} \\ &= v_k^{-1} v_{k-1}^{-1} \dots v_1^{-1} u v_1 v_2 \dots v_{k-1} u^{-1} v_k (v_k, u^{-1}). \end{aligned}$$

Putting

$$(3) \quad u = c^f a c^{-f}, \quad \text{and} \quad v_k = (c^{f_1+\dots+f_{k-1}} b c^{-f_1-\dots-f_{k-1}})^{e_k},$$

we observe that $(v_k, u^{-1}) \in (A, B) = (B, A)$. Therefore $\Omega_k \in (A, B)$ if we can show that

$$(4) \quad \Omega_{k-1} = v_{k-1}^{-1} \dots v_1^{-1} u v_1 \dots v_{k-1} u^{-1} \in (A, B),$$

since

$$(5) \quad \Omega_k = v_k^{-1} \Omega_{k-1} v_k (v_k, u^{-1}).$$

However, this follows by induction since

$$(6) \quad \Omega_1 = v_1^{-1} u v_1 u^{-1} \text{ is obviously in } (A, B) \text{ if } v_1 = b^e.$$

We now characterize which products of elements $a^\alpha a^{-\bar{\alpha}}$ are also in C . Mapping $c \rightarrow 1$ has the effect of mapping $a^\alpha a^{-\bar{\alpha}} \rightarrow b^e a b^{-e} a^{-1}$. We must show that $a^\alpha a^{-\bar{\alpha}} a b^e a^{-1} b^{-e} \in \theta^*$.

We first show that for $v = c^\phi b^e c^{-\phi}$, and $u = c^\gamma a c^{-\gamma}$, that $(v, u^{-1}) \equiv (b, a^{-1}) \pmod{\theta^*}$.

$$\begin{aligned} (v, u^{-1})(a^{-1}, b^\epsilon) &= c^\phi b^{-\epsilon} c^{-\phi} c^\gamma a c^{-\gamma} c^\phi b^\epsilon c^{-\phi} c^\gamma a^{-1} c^{-\gamma} (ab^{-\epsilon} a^{-1} b^\epsilon) \\ &= c^\phi (b^{-\epsilon} c^\delta a c^{-\delta} b^\epsilon c^\delta a^{-1} c^{-\delta}) c^{-\phi} (ab^{-\epsilon} a^{-1} b^\epsilon), \end{aligned}$$

for $\delta = \gamma - \phi$. Thus

$$(v, u^{-1})(a^{-1}, b^\epsilon) = c^\phi (b^\epsilon, c^\delta a^{-1} c^{-\delta}) c^{-\phi} (b^\epsilon, a^{-1})^{-1}.$$

Let $c^\phi = r$, $b^{-\epsilon} = s$, and $c^\delta a c^{-\delta} = t$. Then

$$(v, u^{-1})(a^{-1}, b^\epsilon) = r s^{-1} t^{-1} s t s^{-1} a s a^{-1} r^{-1} (r, a s^{-1} a^{-1} s^{-1}).$$

Note that $(r, a s^{-1} a^{-1} s^{-1})$ is an element of θ^* .

To show that $(s, t)(s, a^{-1}) \in \theta^*$, note that $t = c^\delta a c^{-\delta}$. By the Witt-Hall identities [3]

$$(s, t)(s, a^{-1}) = (s, t a^{-1})((s, t), a^{-1}).$$

$t a^{-1} \in (A, C)$, $s \in B$ together imply that $(s, t a^{-1}) \in (B, (A, C)) \subseteq \theta^*$.

$((s, t), a^{-1}) \in \theta^*$ also. Substitution into the argument in lines (1) through (6) yields the desired result. Q.E.D.

From this, we make the reasonable conjecture that D_{kn} is an $(n - k - 1)$ -fold commutator subgroup of elements which are commutators from $n - k - 1$ distinct subgroups in $D_{k+1 n}$, each isomorphic to $D_{k n-1}$. The proof of this is a problem in the commutator calculus which may, perhaps, be solved by mimicking algebraically the geometric constructions in [2].

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