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ONE-DIMENSIONAL POLYHEDRAL IRREGULAR SETS OF HOMEOMORPHISMS OF 3-MANIFOLDS

BY

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ABSTRACT. Examples are given to show that there exist homeomorphisms of open 3-manifolds whose sets of irregular points are wildly embedded one-dimensional polyhedra. The main result of the paper is that a one-dimensional polyhedral set of irregular points can fail to be locally tame on, at most, a discrete subset of the set of points of order greater than one. Necessary and sufficient conditions are given so that the set of irregular points is locally tame at each point.

1. **Introduction.** Let h be a homeomorphism of a metric space (X, d) onto itself. h is *regular* (*positively regular*) *at* $x \in X$ if, for each $\epsilon > 0$, there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $d(h^i(x), h^i(y)) < \epsilon$ for each integer i $(i > 0)$. Let $\mathrm{Irr}(h)$ denote the set of points at which h fails to be regular. Suppose M is an open manifold which is an open mapping cylinder neighborhood of some continuum $C \subseteq M$. By using the product structure of $M - C$, one can define a homeomorphism h of M onto itself such that h is positively regular on all of M and $\mathrm{Irr}(h) = C$. The metric on M is the metric induced from the metric of the one point compactification of M. In [3], [6], Duvall and Husch investigated the converse of the situation; i.e. if M is an open manifold, h is a homeomorphism such that h is positively regular on M and $\mathrm{Irr}(h)$ is a nonempty compactum, need M be an open mapping cylinder neighborhood of $\mathrm{Irr}(h)$? If $\mathrm{Irr}(h)$ is compact zero dimensional, then $\mathrm{Irr}(h)$ is a singleton and the answer is yes by [7]. If $\mathrm{Irr}(h)$ is a compact polyhedron topologically embedded in M with codimension at least 3 and if the dimension of M is greater than 3, Duvall and Husch showed that $\mathrm{Irr}(h)$ could be wildly embedded in M. This provided strong evidence that the answer is no. However, they were able to give necessary and sufficient conditions so that $\mathrm{Irr}(h)$ would be tamely embedded and hence a positive answer for the dimensional range indicated. In [3], they claimed that $\mathrm{Irr}(h)$ was tamely embedded when the

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dimension of M is three, the dimension of $\operatorname{Irr}(h)$ is one and a certain cross-section hypothesis is added. In §5, we give a counterexample to their theorem. Also in §5, we give an example to show that M need not be the open mapping cylinder neighborhood of any embedding of $\operatorname{Irr}(h)$. We are able, however, to prove the following:

THEOREM A. *Let M be an open connected 3-manifold and let h be a homeomorphism of M onto itself such that h is positively regular on all of M and $C = \operatorname{Irr}(h)$ is a compact one-dimensional polyhedron topologically embedded in M. If C is the* 1*-sphere, suppose that there exists $x \in M$ such that $C \neq \limsup_{n\to+\infty} \{h^n(x)\}$. Then C fails to be locally tame at a discrete subset of the set of points of C of order $\neq$ one. In particular, if C has no points of order one, C fails to be locally tame on a finite subset.*

In our example in §5, C is an arc and C fails to be locally tame at a sequence of points which converges to an endpoint of C.

We show that there exists $r > 0$ such that $h^r|\operatorname{Irr}(h)$ is the identity and $f(x) = \lim_{i\to+\infty} h^{2ri}(x)$ exists for each $x \in M$ and f defines a retraction of M onto $\operatorname{Irr}(h)$. If M_0 is the orbit space of $h^{2r}|M - \operatorname{Irr}(h)$, then f induces a map $f_0\colon M_0 \longrightarrow \operatorname{Irr}(h)$.

THEOREM B. *Let M, h and C be as in Theorem* A *and let p be a point of order two in C. C is locally tame at p if and only if $f_0^{-1}(p)$ has property* AFG; *i.e. there exists a neighborhood W of $f_0^{-1}(p)$ in M_0 such that if U is a neighborhood of $f_0^{-1}(p)$ contained in W, then there exists a neighborhood V of $f_0^{-1}(p)$ contained in U such that each loop in V which is null-homologous in U is null-homotopic in U.*

Property AFG was developed in [3] and [6], and interested readers may consult [6] for some of the consequences. However, in this paper, we only use the definition.

2. **Preliminaries.** A *polyhedron* is a topological space which is homeomorphic to the underlying space of a locally finite simplicial complex. A subset P of a 3-manifold is a *topologically embedded polyhedron* if there exists a polyhedron Q and a homeomorphism of Q onto P. A topologically embedded polyhedron P contained in a 3-manifold M is *locally tame* at $p \in P$ if there exists a neighborhood U of p in M and a homeomorphism of the closure of U onto a subpolyhedron of M (with respect to the unique piecewise linear structure of M) such that the image of the closure of the intersection of U and P is a subpolyhedron of M. P is *locally tame* if P is locally tame at each of its points.

A 3-manifold M is *irreducible* if each locally tame 2-sphere in M is the boundary of a 3-cell in M. A locally tame connected 2-manifold S contained in M is *incompressible in* M provided (1) if D is a locally tame 2-cell in M such that $D \cap S = \text{bdry } D$, then bdry D bounds a 2-cell in S, and (2) if S is a 2-sphere, then S does not bound a 3-cell in M. Two disjoint 2-manifolds S_1, S_2 in M are (*topologically*) *parallel* if there exists a homeomorphism $h: S_1 \times [0, 1] \longrightarrow M$ such that $h(S_1 \times \{0\}) = S_1$ and $h(S_1 \times \{1\}) = S_2$.

We need the following two theorems of W. Haken [10], [11].

THEOREM 2.1. *If* M *is a compact connected* 3-*manifold, then there exists an integer* r (*called the Haken number of* M) *such that if* $N_1, \cdots, N_s$, $s > r$, *are pairwise disjoint locally tame closed connected incompressible surfaces in* M, *then at least two of the surfaces are topologically parallel in* M.

THEOREM 2.2. *Let* M *be a closed connected* 2-*manifold and let* $N \subseteq M \times (0, 1)$ *be a locally tame closed connected incompressible surface. Then* $M \times \{0\}$ *and* N *are topologically parallel in* $M \times [0, 1]$.

3. **Proof of Theorem A.** Let M, h and C be as in Theorem A. All homology groups in this section will be singular homology with Z_2 coefficients unless stated otherwise.

PROPOSITION 3.1. *If* C *is the* 1-*sphere, then* $h|C$ *is periodic.*

PROOF. By Corollary 31 of [4], the existence of an element x of M such that $\limsup_{n\to+\infty} \{h^n(x)\} \neq C$ implies that $\limsup_{n\to+\infty} \{h^n(y)\} \neq C$ for all $y \in M$. Since $h(C) = C$ and $h|C$ is positively regular, by Theorem 10 of [15], $h|C$ is regular on all of C. By [18], $h|C$ is topologically equivalent to either a rotation or the composition of rotation and a reflection and hence must be periodic.

PROPOSITION 3.2. *If* C *is not the* 1-*sphere, then* $h|C$ *is periodic.*

PROOF. Let C_0 be the set of points in C which do not have order two (in the usual graph theoretic sense). Since C_0 is finite, there exists $r > 0$ such that $h^r|C_0$ is the identity and if E is a component of $C - C_0$, $h^r(E) = E$. Let F be the fixed point set of h^r; note that F is closed. If $F \neq C$, let C_1 be a component of $C - F$. There exists a continuous map ϕ: $[0, 1] \longrightarrow C$ such that $\phi|(0, 1)$ is a homeomorphism whose image is C_1. Let $x \in C_1$ and suppose $\phi^{-1}h^r(x) < \phi^{-1}(x)$. Hence there exists $c \in [0, 1]$ such that

$$\lim_{n\to+\infty} \phi^{-1}h^m(x) = c.$$

Since $h^r\phi(c) = \phi(c)$, $c = 0$. Note, for each $y \in C_1$, $\lim_{n\to+\infty}\phi^{-1}h^{rn}(y) = 0$, and hence h^r is not positively regular at $\phi(1)$.

If $h^r|C$ is the identity, by replacing h^{2r} by h we may assume, without loss of generality, that $h|C$ is the identity and if U is an open orientable subset of M such that $h(U) = U$, then $h|U$ is orientation-preserving.

PROPOSITION 3.3. *For each* $x \in M$, $f(x) = \lim_{n\to+\infty} h^n(x)$ *exists and defines a retraction of* M *onto* C. *If* U *is an open subset of* C, *then the inclusion of* U *into* $f^{-1}U$ *is a homotopy equivalence. The natural projection* p *of* $M - C$ *onto the orbit space* M_0 *of* $h|M - C$ *is a covering map and* M_0 *is a closed 3-manifold.* f *induces a map* $f_0: M_0 \longrightarrow C$ *such that* $f_0 p = f|_{M-C}$.

PROOF. This is the content of Corollary 2.4 and Proposition 2.5 of [6] provided we know that h is not regular at ∞; i.e. the induced homeomorphism h_∞ of the one-point compactification of M, $M \cup \{\infty\}$, is not regular at ∞. However, if h_∞ is regular at ∞, then h_∞ is positively regular on $M \cup \{\infty\}$. By Theorem 10 of [15], h_∞ is regular on all of $M \cup \{\infty\}$, a contradiction.

PROPOSITION 3.4. *If* U *is a proper open subset of* C, *then* $f_0^{-1}U$ *is irreducible.*

PROOF. It suffices to show that $f^{-1}(U) - U$ is irreducible. Let $\Sigma \subseteq f^{-1}(U) - U$ be a locally tame 2-sphere; by Corollary 2.2 of [6], if V is a neighborhood of C, then there exists n such that $h^n(\Sigma) \subseteq V - C$. By [24], there exists a neighborhood V of C in M that is irreducible. It follows that Σ bounds a 3-cell in $f^{-1}(U) - U$.

If U is an open subset of C which is homeomorphic to an open interval, we shall call U an *open arc.*

PROPOSITION 3.5. *If* U *is an open arc in* C, *then* $f_0^{-1}U$ *has two ends. If* $p \in U$, *then* $f_0^{-1}(p)$ *is a continuum in* $f_0^{-1}U$ *which separates the two ends of* $f_0^{-1}U$; *i.e.,* $f_0^{-1}(U - p)$ *has two components, each of whose closures (in* $f_0^{-1}U$*) has one end.*

PROOF. If V is a connected neighborhood of p in U, it follows from Proposition 3.3 that $\mathrm{cl}(f_0^{-1}V)$ is a continuum. Since p can be expressed as the intersection of connected neighborhoods $\{V_i\}_{i=1}^{\infty}$, $V_i \supset V_{i+1}$ for all i, $f_0^{-1}(p) = \bigcap_{i=1}^{\infty} \mathrm{cl}(f_0^{-1}V_i)$ is a continuum. Note also, that f_0 is proper (i.e., if $A \subseteq C$ is compact, $f_0^{-1}A$ is compact).

Since U has two ends, it follows from [19] that $f_0^{-1}U$ has two ends. Since p separates U into two components it follows from Proposition 3.3 that $f_0^{-1}(p)$ separates $f_0^{-1}U$ into two components. We again apply [19] to prove that $f_0^{-1}(p)$ separates the ends of $f_0^{-1}U$.

If G is an Abelian group with a group of operators K generated by k, define

$$G_K = G/\{g - k(g) | g \in G\}, \qquad G^K = \{g \in G \,|\, k(g) = g\}.$$

PROPOSITION 3.6. *If $p \in U \subseteq V$ such that U and V are open arcs in C, then the homomorphism $H_1(f_0^{-1}U) \to H_1(f_0^{-1}V)$ induced by inclusion is an isomorphism and these groups are isomorphic to the direct sum of two copies of Z_2.*

PROOF. If G is one of the groups $H_0(f^{-1}(V) - V)$, $H_1(f^{-1}(V) - V)$, $H_0(f^{-1}(U) - U)$ or $H_1(f^{-1}(U) - U)$ and Z is the integers, define an operation of Z on G by $n(g) = h_*^n(g)$ where $g \in G$ and h_* is the homomorphism induced by h on G. From Serre [26], we get the following commutative diagram in which the rows are exact and the vertical maps are induced by inclusion.

$$\begin{array}{ccccccccc} 0 & \to & H_1(f^{-1}(U) - U)_Z & \to & H_1(f_0^{-1}U) & \to & H_0(f^{-1}(U) - U)^Z & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & H_1(f^{-1}(V) - V)_Z & \to & H_1(f_0^{-1}V) & \to & H_0(f^{-1}(V) - V)^Z & \to & 0 \end{array}$$

Since $f^{-1}U$ is contractible, $H_1(f^{-1}(U) - U) \cong H_2(f^{-1}U, f^{-1}(U) - U)$. By [27, pp. 340–342], $H_2(f^{-1}U, f^{-1}(U) - U) \cong H_c^1(U)$. Since $h|U =$ identity and the homomorphisms induced by h commute with the above isomorphisms, $H_1(f^{-1}(U) - U)_Z \cong H_1(f^{-1}(U) - U)$. Since $h|f^{-1}(U) - U$ is orientation-preserving, $H_0(f^{-1}(U) - U)^Z \cong H_0(f^{-1}(U) - U)$. It follows that $H_1(f_0^{-1}U) \cong Z_2 \oplus Z_2$.

Note that the two outside vertical maps of the above diagram are isomorphisms, hence $H_1(f_0^{-1}U) \to H_1(f_0^{-1}V)$ is an isomorphism.

REMARK. Proposition 3.6 is also true if homology with integral coefficients is used.

The following proposition is well known.

PROPOSITION 3.7. *If U is an open arc in C and p and q are distinct points of U, then there exists a locally tame closed connected 2-manifold $N \subseteq f_0^{-1}U$ such that N separates $f_0^{-1}(p)$ from $f_0^{-1}(q)$ in $f_0^{-1}U$.*

Let $p \in U$ where U is an open arc in C and let U_1 and U_2 be the components of $U - \{p\}$. Let $q_i \in U_i$ and let N_0^i be a surface in $f_0^{-1}U$ which separates $f_0^{-1}(q_i)$ from $f_0^{-1}(p)$ in $f_0^{-1}U$, $i = 1, 2$. Let V_0 be the compact 3-manifold in $f_0^{-1}U$ whose boundary is $N_0^1 \cup N_0^2$. Let H be the Haken number of V_0.

An ordered $(H + 1)$-tuple, $\Sigma = \{V_1, V_2, \cdots, V_{H+1}\}$ of compact 3-manifolds in $f_0^{-1}U$ is *admissible* if

(A1) $V_i \subset \operatorname{int} V_{i-1}$ for $i = 1, 2, \cdots, H + 1$;

(A2) $\mathrm{cl}(V_i - V_{i+1}) = R_i^1 \cup R_i^2$ where $R_i^1 \cap R_i^2 = \varnothing$, $R_i^j \cap R_{i+1}^k \neq \varnothing$ if and only if $j = k$, $i = 0, 1, \cdots, H$;

(A3) if $N_i^j = V_i \cap R_{i-1}^j$, then N_i^j separates the ends of $f_0^{-1}U$; $j = 1, 2$, $i = 1, 2, \cdots, H + 1$;

(A4) image $\{H_1(V_i) \longrightarrow H_1(V_{i-1})\}$ has rank $\leqslant 2$, $i = 1, 2, \cdots, H + 1$;

(A5) image $\{H_1(V_i \cup R_{i-1}^j \cup \cdots \cup R_k^j) \longrightarrow H_1(V_{i-1} \cup R_{i-2}^j \cup \cdots \cup R_{k-1}^j)\}$ has rank $\leqslant 2$, $j = 1, 2$, $i = 2, 3, \cdots, H + 1$; $k = 2, \cdots, i - 1$.

Define $\mathrm{bdry}\ \Sigma = \bigcup_{i=1}^{H+1} (N_i^1 \cup N_i^2)$.

PROPOSITION 3.8. *There exists an admissible* $(H + 1)$*-tuple* $\Sigma^0 = \{V_1^0, V_2^0, \cdots, V_{H+1}^0\}$.

PROOF. By means of a homeomorphism from $(0, 1)$ onto U, we can induce a linear ordering $<$ on U. Pick points $p_0^1 < p_1^1 < \cdots < p_{H+1}^1 < p < p_{H+1}^2 < p_H^2 < \cdots < p_0^2$ in U such that $f_0^{-1}(p_0^j)$, $j = 1, 2$, lies in $\mathrm{int}\ V_0$. By Proposition 3.7, there exist 2-manifolds N_i^j $(j = 1, 2; i = 1, 2, \cdots, H + 1)$ such that N_i^j separates $f_0^{-1}(p_i^j)$ from $f_0^{-1}(p_{i-1}^j)$ in $f_0^{-1}U$. Let V_i^0 be the compact submanifold of $f_0^{-1}U$ whose boundary is $N_i^1 \cup N_i^2$. Let

$$R_i^j = \mathrm{cl}(V_i - V_{i+1}) \cap f_0^{-1}U_j, \quad j = 1, 2, \quad i = 0, 1, \cdots, H.$$

It follows from Propositions 3.5 and 3.6 that Σ^0 is admissible.

Suppose that there exists a 2-cell $D \subset V_0$ such that, for some admissible $(H + 1)$-tuple $\Sigma = \{V_1, V_2, \cdots, V_{H+1}\}$, $D \cap \mathrm{bdry}\ \Sigma = \mathrm{bdry}\ D \subseteq N_i^j$ and such that $\mathrm{bdry}\ D$ bounds no 2-cell in N_i^j. Let N be a regular neighborhood of D such that $N \cap N_i^j = N \cap \mathrm{bdry}\ \Sigma$ is an annulus in $\mathrm{bdry}\ N$. If $N \subseteq V_i$, let $V_i' = \mathrm{cl}(V_i - N)$; if $N \subseteq R_{i-1}^j$, let $V_i' = V_i \cup N$. In the terminology of McMillan [22] we can apply a simple annexation of type 2 or a simple reduction to V_i in V_{i-1} without disturbing any V_j $(j \neq i)$. We say that $\Sigma' = \{V_1, \cdots, V_{i-1}, V_i', V_{i+1}, \cdots, V_{H+1}\}$ *is obtained by simplifying* Σ.

PROPOSITION 3.9. *If* Σ' *is obtained by simplifying an admissible* Σ, *then* Σ' *is admissible.*

PROOF. Suppose that we adopt the notation used above in the definition of "is obtained by simplifying Σ." If $N \subseteq V_i$, let $(R_i^j)' = \mathrm{cl}(R_i^j - N)$ and $(R_{i-1}^j)' = R_{i-1}^j \cup N$. If $N \subseteq R_{i-1}^j$, let $(R_i^j)' = R_i^j \cup N$ and $(R_{i-1}^j)' = \mathrm{cl}(R_{i-1}^j - N)$. It is clear that Σ' satisfies (A1) and (A2).

If $(N_i^j)'$ does not separate the ends of $f_0^{-1}U$, there is a piecewise linear embedding g of $(0, 1)$ into $f_0^{-1}U$ (as a closed subset) which "goes from one end to the other" [i.e., the closure of $g((0, 1))$ in the Freudenthal end point compactification [9] of $f_0^{-1}U$ is an arc] such that $g((0, 1))$ misses $(N_i^j)'$.

Since $\mathrm{cl}(N_i^j - (N_i^j)')$ is an annulus, we can get a similar embedding of (0, 1) into $f_0^{-1}U$ that misses N_i^j. Hence (A3) is verified for Σ'.

We will show that property (A5) is satisfied by considering four cases; the proof that (A4) is satisfied is similar. To simplify notation, let

$$R = V_{a+1} \cup R_a^c \cup \cdots \cup R_{b+1}^c \quad [\text{if } a = H + 1, \text{ omit } R \text{ in the following arguments}],$$

$$S = V_a \cup R_{a-1}^c \cup \cdots \cup R_b^c,$$

$$T = V_{a-1} \cup R_{a-2}^c \cup R_{a-3}^c \cup \cdots \cup R_{b-1}^c.$$

Case 1. $D \cap S = \text{bdry } D$. Consider the following diagram

$$\begin{tikzcd}
& H_1(R) \arrow[d, "r"] \arrow[dr, "t"] & & \\
H_1(N \cap S) \arrow[r] & H_1(S) \oplus H_1(N) \arrow[r, "s"] \arrow[d, "u"] & H_1(S \cup N) \arrow[r] \arrow[dl, "v"] & 0 \\
& H_1(T) & &
\end{tikzcd}$$

where the row is from the exact Mayer-Vietoris sequence (using reduced homology groups). Since $H_1(N) = 0$, the maps named in the diagram may be considered induced by inclusion. Since s is onto, rank image $t \leq$ rank image $r \leq 2$ and rank image v = rank image $u \leq 2$.

Case 2. $D \cap S = D$ and $D \cap \text{bdry } S = \text{bdry } D$. Consider the diagram

$$\begin{tikzcd}
& H_1(R) \arrow[dr, "r"] \arrow[drr, "t"] & & \\
H_1(N \cap \mathrm{cl}(S - N)) \arrow[r] & & H_1(N) \oplus H_1(\mathrm{cl}(S - N)) \arrow[r, "s"] \arrow[dl, "u"'] & H_1(S) \arrow[dll, "v"] \\
& H_1(T) & &
\end{tikzcd}$$

where again the row is from the exact Mayer-Vietoris sequence and the maps are induced by inclusion. Since $H_1(N \cap \mathrm{cl}(S - N)) = 0$, s is one-to-one. Therefore rank image r = rank image $t \leq 2$ and rank image $u \leq$ rank image $v \leq 2$.

Case 3. $D \cap \text{bdry } R \neq \varnothing$ or $D \cap \text{bdry } T \neq \varnothing$. This reduces to the cases above.

Case 4. $D \cap (\text{bdry } R \cup \text{bdry } S \cup \text{bdry } T) \neq \varnothing$. There are no changes in the required groups.

PROPOSITION 3.10. *There exists* $\Sigma^* = \{V_1^*, V_2^*, \cdots, V_{H+1}^*\}$ *which is admissible, which is obtained from* Σ^0 *by a finite number of simplifying operations and such that each component of* bdry Σ^* *is either a 2-sphere or is incompressible in* V_0.

PROOF. This follows from Lemma A of [22]; see the proof of Theorem 2 in [22].

If $\mathrm{cl}(V_i^* - V_{i+1}^*) = (R_i^1)^* \cup (R_i^2)^*$, we may assume that $N_0^j \subseteq (R_0^j)^*$, $j = 1, 2$.

PROPOSITION 3.11. $(N_i^j)^* = V_i^* \cap (R_{i-1}^j)^*$ *is not the union of 2-spheres.*

PROOF. It is easily seen that a collection of 2-spheres in an irreducible 3-manifold with two ends cannot separate the two ends.

PROPOSITION 3.12. *If* N *is a compact orientable 3-manifold with nonempty boundary and if* B *is the union of any subcollection of the components of* bdry N, *then* image $\{H_1(B) \longrightarrow H_1(N)\}$ *has rank at least as great as the total genus of* B.

PROOF. This is the corollary to Lemma 10.2 in [23].

PROPOSITION 3.13. $f_0^{-1}U$ *contains a collection of locally tame closed surfaces of genus one whose union separates the ends of* $f_0^{-1}U$.

PROOF. Suppose not. Then for each $i = 1, 2, \cdots, H + 1$, and $j = 1, 2$, there exists a component T_i^j of $(N_i^j)^*$ which has genus greater than one. By choice of H, $\{T_1^1, T_2^1, \cdots, T_{H+1}^1\}$ contains two surfaces which are parallel in V_0, say, T_i^1 and T_j^1, $i < j$. Since $(N_{i+1}^1)^*$ separates $(N_i^1)^*$ from $(N_j^1)^*$ if $j > i + 1$, some component, say S (if $j = i + 1$, S could be T_j^1), of $(N_{i+1}^1)^*$ lies in the region W bounded by T_i^1 and T_j^1 which is homeomorphic to $T_i^1 \times [0, 1]$. Note that we may assume that S is not a 2-sphere and hence S is incompressible in V_0. By Theorem 2.2, S and T_i^1 are parallel. We may assume that S is chosen such that no other component of $(N_{i+1}^1)^*$, except possibly 2-spheres, lies in the region bounded in W by S and T_i^1. If $S \neq T_{i+1}^1$, replace T_{i+1}^1 by S.

Similarly, there exists k such that T_k^2 and T_{k+1}^2 are parallel surfaces of genus greater than one. Suppose $i \leqslant k$; consider $A = V_{k+1}^* \cup (R_k^1)^* \cup \cdots \cup (R_{i+1}^1)^*$ and $B = V_k^* \cup (R_{k-1}^1)^* \cup (R_{k-2}^1)^* \cup \cdots \cup (R_i^1))^*$ if $i < k$ and consider $A = V_{k+1}^*$ and $B = V_k^*$ if $i = k$. By (A4) and (A5), image $\{H_1(A) \longrightarrow H_1(B)\}$ has rank $\leqslant 2$; hence,

$$\text{image}\,\{H_1(T_{i+1}^1 \cup T_{k+1}^2) \longrightarrow H_1(B)\}$$

has rank $\leqslant 2$. Since the parallelity components bounded by T_i^1, T_{i+1}^1 and T_k^2,

T_{k+1}^2 are contained in B except possibly for the interiors of a finite number of disjoint 3-cells, image $\{H_1(T_i^1 \cup T_k^2) \longrightarrow H_1(B)\}$ has rank $\leqslant 2$; this contradicts Proposition 3.12.

PROPOSITION 3.14. *If U is an open arc in C, then there exists a locally tame closed connected surface of genus one in $f_0^{-1}U$ which separates the ends of $f_0^{-1}U$.*

PROOF. Suppose $<$ is a linear ordering on U induced by means of a homeomorphism of U onto an open interval with the usual ordering.

Let $p_1 < p_2 < p_3 < p_4$ be elements of U and let $N_1 \subseteq f_0^{-1}(p_1, p_2)$ and $N_2 \subseteq f_0^{-1}(p_3, p_4)$ $[(x, y) = \{z \in U | x < z < y\}]$ be surfaces which are given in Proposition 3.7. Let V_0 be the compact submanifold of $f_0^{-1}U$ whose boundary is $N_1 \cup N_2$ and let H be the Haken number of V_0. Let $p_2 = q_0 < q_1 < \cdots < q_{2H+1} = p_3$ be elements of U, let $\mathcal{T}_i$, $i = 1, 2, \cdots, H + 1$, be a collection of locally tame closed surfaces of genus one in $f_0^{-1}(q_{2i-2}, q_{2i-1})$ whose union separates the ends of $f_0^{-1}(q_{2i-2}, q_{2i-1})$ and let W_i, $i = 1, 2, \cdots, H$, be surfaces in $f_0^{-1}(q_{2i-1}, q_{2i})$ given by Proposition 3.7. Note that if the union of elements in $\mathcal{T}_i$ separates the ends of $f_0^{-1}(q_{2i-2}, q_{2i-1})$, then the union also separates the ends of $f_0^{-1}U$. Hence there exists a component $T_i \in \mathcal{T}_i$ which is not the boundary of a compact submanifold of $f_0^{-1}U$. Since $f_0^{-1}U$ is irreducible, T_i is incompressible in $f_0^{-1}U$ and hence in V_0. By choice of H, $\{T_1, T_2, \cdots, T_{H+1}\}$ contains two surfaces which are parallel in V_0; say, T_i and T_j, $i < j$. Let R be the submanifold of V_0 whose boundary is $T_i \cup T_j$ and which is homeomorphic to $T_i \times [0, 1]$. Since W_i separates $\mathcal{T}_i$ and $\mathcal{T}_j$ in $f_0^{-1}U$, W_i separates T_i and $\mathcal{T}_j$ in R. It follows that T_i separates the ends of $f_0^{-1}U$.

From part of the proof above we get the following proposition.

PROPOSITION 3.15. *If U is an open arc in C, if V is an open connected subset of C which contains U and if each point of U separates V into two components whose closures in V are noncompact and if T is a locally tame closed connected surface of genus one in $f_0^{-1}U$ which separates the ends of $f_0^{-1}U$, then T is incompressible in $f_0^{-1}V$.*

PROPOSITION 3.16. *Let $Q = \{p \in C\,|$ either p has order > 2 or if p has order 2, then no neighborhood of $f_0^{-1}(p)$ is homeomorphic to $S^1 \times S^1 \times R\}$. Then Q is a discrete subset of the set of points of C of order $\geqslant 2$.*

PROOF. Suppose $p \in Q$ and the order of p is k. Let W be a neighborhood of p in C such that $W - \{p\}$ has k components, $\{\omega_i\}_{i=1}^k$ each of which is an open arc. Let q_{2i}, q_{2i+1} be distinct points of W_i and let N_i be a locally tame closed connected 2-manifold in $f_0^{-1}W_i$ such that N_i separates $f_0^{-1}(q_{2i})$ from $f_0^{-1}(q_{2i+1})$ in $f_0^{-1}W_i$. Let N be the compact submanifold of $f_0^{-1}W$ whose boundary is $\bigcup_{i=1}^{k} N_i$.

Let ϕ: $(0, 1] \longrightarrow C$ be a homeomorphism such that $\phi(1) = p$ and $\phi((0, 1)) \subseteq W_1 - f_0(N_1)$. Let $<$ be the induced ordering on $A =$ image ϕ. Choose a sequence of points $\{p_i\}_{i=1}^{\infty}$ in A such that $p_i < p_{i+1}$ for all i and $\lim_{i\to+\infty} p_i = p$. By Proposition 3.14, there exists a locally tame surface T_i of genus one contained in $f_0^{-1}(p_{2i-1}, p_{2i+1})$ which separates the ends of $f_0^{-1}(p_{2i-1}, p_{2i+1})$. By Proposition 3.15, T_i is incompressible in $f_0^{-1}W$ and hence in N.

We claim that there exists K such that if $i > K$, then T_i is parallel to T_K in N. Define $T_i \sim T_j$ if $T_i = T_j$ or T_i is parallel to T_j in N; "$\sim$" is an equivalence relation and, by Theorem 2.1, there exists only a finite number of equivalence classes. Suppose T_K is an element of an equivalence class which has infinitely many elements. If $i > K$, then there exists $j > i$ such that $T_K \sim T_j$. Since T_i separates T_K from T_j in N, it follows from Theorem 2.2 that $T_i \sim T_K$.

Hence, if $i, j > K$, T_i and T_j are parallel in N and, by Theorem 5 of [8], the component of $f_0^{-1}(p_1, p_2) - T_K$ which contains T_{K+1} is homeomorphic to $S^1 \times S^1 \times R$. Therefore, $(p_{2K+1}, p) \cap Q = \varnothing$ and it follows that p is an isolated point of Q.

PROPOSITION 3.17. *If $p \in C - Q$ is a point of order* 2, *then $f_0^{-1}(p)$ has arbitrarily small neighborhoods which are homeomorphic to $S^1 \times S^1 \times R$.*

PROOF. Let V and V_1 be neighborhoods of $f_0^{-1}(p)$ in M_0 such that V_1 is homeomorphic to $S^1 \times S^1 \times R$. By standard compactness arguments and Proposition 3.16, there exists an open arc $A \subseteq C - Q$ such that $f_0^{-1}(p) \subseteq f_0^{-1}A \subset V \cap V_1$. If A_1 and A_2 are the components of $A - \{p\}$, then there exist locally tame closed connected surfaces of genus one, T_i, contained in A_i, $i = 1, 2$, which separate the ends of $f_0^{-1}A_i$ and, hence, are incompressible in $f_0^{-1}A$ [Propositions 3.14 and 3.15]. Note that since $f_0^{-1}(p)$ separates V_1, T_i separates V_1 and the compact submanifold of V_1 bounded by $T_1 \cup T_2$ is homeomorphic to $S^1 \times S^1 \times [0, 1]$ [Theorem 2.2].

PROPOSITION 3.18. *If U is an open arc in $C - Q$, then $f_0^{-1}U$ is homeomorphic to $S^1 \times S^1 \times R$.*

PROOF. Let A be a closed interval contained in U; there exists a finite subset $\{p_1, \cdots, p_n\}$ of A, a finite number of open subsets $\{V_1, \cdots, V_n\}$ of $f_0^{-1}U$ and a finite number of open arcs $\{W_1, \cdots, W_n\}$ in U such that $f_0^{-1}(p_i) \subseteq f_0^{-1}W_i \subseteq V_i$, $A \subseteq \bigcup_{i=1}^{n} W_i$ and each V_i is homeomorphic to $S^1 \times$

$S^1 \times R$. Again, as in the proof of the previous proposition, $\bigcup_{i=1}^n V_i$ is homeomorphic to $S^1 \times S^1 \times R$. Hence there is a neighborhood Y of $f_0^{-1}A$ in U which is homeomorphic to $S^1 \times S^1 \times [0, 1]$. Since U can be expressed as the union of closed intervals $\{A_i\}_{i=1}^\infty$ such that $A_i \subseteq \operatorname{int} A_{i+1}$ for all i, again, by applying Propositions 3.14, 3.15, Theorem 2.2 and [8], $f_0^{-1}U$ is homeomorphic to $S^1 \times S^1 \times R$.

PROPOSITION 3.19. *If U is an open arc in $C - Q$ and $U_1 \subseteq U$ is an open arc, then the inclusion $f_0^{-1}U_1 \longrightarrow f_0^{-1}U$ is a homotopy equivalence.*

PROOF. Let T be a locally tame connected closed surface of genus one in $f_0^{-1}U_1$ which separates the ends of $f_0^{-1}U_1$. T is incompressible in both $f_0^{-1}U_1$ and $f_0^{-1}U$ and it follows from Theorem 2.2 and Proposition 3.18 that T is a deformation retract of both $f_0^{-1}U_1$ and $f_0^{-1}U$. The Proposition follows.

COROLLARY 3.20. *If U_1 and U satisfy the hypothesis of Proposition* 3.19. *then the inclusion $f^{-1}(U_1) - U_1 \longrightarrow f^{-1}(U) - U$ is a homotopy equivalence.*

PROPOSITION 3.21. *If $q \in C$ is a point of order > 1, then there exists a disk D in M which pierces C at q, $D \cap C = \{q\}$ and D is locally tame except possibly at q.*

PROOF. Let $\lambda: [0, 1] \longrightarrow C$ be an embedding such that $\lambda(0) = q$ and $\lambda((0, 1)) \subseteq C - Q$. By Proposition 3.18, there exists a homeomorphism $\phi: S^1 \times S^1 \times (0, 1) \longrightarrow f_0^{-1}\lambda((0, 1))$; we can assume that, for each positive integer i,

$$\phi(S^1 \times S^1 \times \{1/2i\}) \subseteq f_0^{-1}\lambda((1/(2i + 1), 1/(2i - 1))).$$

Let $\mu: S^1 \longrightarrow S^1 \times S^1$ be an embedding such that if $\mu_t: S^1 \longrightarrow S^1 \times S^1 \times \{t\}$ is the embedding defined by $\mu_t(x) = (\mu(x), t)$, $t \in (0, 1)$, then there exists a lifting $\widetilde{\mu}_t: S^1 \longrightarrow f^{-1}$ (image λ) of $\phi\mu_t$. The liftings $\widetilde{\mu}_t$ can be chosen so that the map $(x, t) \longrightarrow \widetilde{\mu}_t(x)$ is an embedding of $S^1 \times (0, 1)$ in f^{-1} (image λ).

There exists a sequence $\{K_i\}_{i=1}^\infty$ of positive integers, $K_i < K_{i+1}$, for all i, such that if A_i is the compact surface in $p^{-1}\phi(S^1 \times S^1 \times \{1/2i\})$ whose boundary is $h^{K_i}\widetilde{\mu}_{1/2i}(S^1) \cup h^{K_{i+1}}\widetilde{\mu}_{1/2i}(S^1)$ and

$$B_i = \bigcup_{1/(2i+2) \leq t \leq 1/2i} h^{K_{i+1}} (\text{image } \widetilde{\mu}_t),$$

then

$$\limsup_{i \to +\infty} A_i = \limsup_{i \to +\infty} B_i = \lambda(0).$$

$D = \bigcup_{i=1}^\infty A_i \cup \bigcup_{i=1}^\infty B_i \cup \{\lambda(0)\}$ is the desired disk.

PROPOSITION 3.22. *If $q \in C - Q$ is a point of order 2, then there exists a neighborhood basis $\{U_i\}_{i=1}^{\infty}$ at q such that $\pi_1(U_i - C)$ is isomorphic to the integers, $U_i \supseteq U_{i+1}$, and the inclusion map induces an isomorphism $\pi_1(U_i - C) \longrightarrow \pi_1(U_j - C)$ for all $i > j$.*

PROOF. Let $T \subseteq C - Q$ be an open arc which contains q. By Proposition 3.18, $f_0^{-1}T$ is homeomorphic to $S^1 \times S^1 \times R$ and hence $U_1 = f^{-1}(T) - T$ is homeomorphic to $S^1 \times R^2$. To prove the proposition, it suffices to show that if V is a neighborhood of q, $V \subseteq U_1$, then there exists a neighborhood U of q, $U \subseteq V$, such that the inclusion induces an isomorphism $\pi_1(U - C) \longrightarrow \pi_1(U_1 - C)$.

Let W be an open arc in C such that $q \in W \subseteq V \cap C$. By Proposition 3.18, there exists a homeomorphism $\phi\colon S^1 \times S^1 \times R \longrightarrow f_0^{-1}W$. Let $\pi\colon S^1 \times R \times R \longrightarrow S^1 \times S^1 \times R$ be a covering map which is the pullback of $p\colon f^{-1}(W) - W \longrightarrow f_0^{-1}W$; i.e. we have a commutative diagram

$$\begin{array}{ccc} S^1 \times R \times R & \xrightarrow{\tilde{\phi}} & f^{-1}(W) - W \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p} \\ S^1 \times S^1 \times R & \xrightarrow[\phi]{} & f_0^{-1}W \end{array}$$

There exists $\epsilon > 0$ such that $\phi(S^1 \times S^1 \times (-\epsilon, \epsilon)) \supseteq f_0^{-1}(q)$. Note that $\tilde{\phi}(S^1 \times R \times [-\epsilon, \epsilon])$ is a closed subset of $f^{-1}(W) - W$ such that $\mathrm{cl}\, \tilde{\phi}(S^1 \times R \times [-\epsilon, \epsilon])$ is a neighborhood of $f^{-1}(q)$. Also note that either $W_+ = \mathrm{cl}\, \tilde{\phi}(S^1 \times [0, +\infty) \times [-\epsilon, \epsilon])$ or $W_- = \mathrm{cl}\, \tilde{\phi}(S^1 \times (-\infty, 0] \times [-\epsilon, \epsilon])$ is a compact neighborhood of q; suppose that the former is true. For all i, $h^i(W_+)$ is a neighborhood of q. Since $f = \lim_{i \to +\infty} h^i$, there exists N such that $h^N(W_+) \subseteq V$. Let $U = h^N\tilde{\phi}(S^1 \times (0, +\infty) \times (-\epsilon, \epsilon)) \cup W$. The conclusion follows from Corollary 3.20.

PROPOSITION 3.23. *If $q \in T \subseteq C - Q$ where T is an open arc and if A is the closure of a component of $T - \{q\}$, then the neighborhood basis $\{U_i\}_{i=1}^{\infty}$ at q given by Proposition 3.22 can be chosen such that the inclusion induces the zero map $\pi_1(U_i - A) \longrightarrow \pi_1(U_j - A)$ for all $i > j$.*

PROOF. We shall use the notation introduced in the previous proof. It suffices to show that if $\lambda\colon S^1 \longrightarrow U - A$ is a loop, then λ is inessential in $V - A$. Note that if Z is an open arc in $C - Q$, then h induces the identity map on $\pi_1(f^{-1}(Z) - Z)$.

Choose a point $q' \in T - A$ such that if B is the open arc in T whose boundary is $\{q, q'\}$, then $f_0^{-1}B \subseteq \phi(S^1 \times S^1 \times (-\epsilon, \epsilon))$. Similar to the construction of U in Proposition 3.22, we can construct a set U' such that $U' \subseteq$

$U \cap f^{-1}B$, $U' \cap C = B$ and the inclusion of $U' - B$ into $f^{-1}(B) - B$ is a homotopy equivalence.

Note that λ is homotopic in $U - A$ to a loop $\lambda': S^1 \longrightarrow U - C$. Since the inclusion of $U' - B$ into $U - W$ is a homotopy equivalence, λ' is homotopic in $U - A$ to a loop $\lambda'': S^1 \longrightarrow U'$. Since U' is contractible, λ is homotopic in $U - A$ to the constant path.

PROPOSITION 3.24. *Suppose*

$$G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \xrightarrow{f_3} G_4 \xrightarrow{f_4} G_5$$

is a sequence of homomorphisms between groups such that $f_2 f_1$ *and* $f_4 f_3$ *are isomorphisms; then the image of* $f_3 f_2$ *is isomorphic to* G_1.

PROOF OF THEOREM A. We shall show that if $p \in C - Q$ has order 2, then C is locally tame at p. By Proposition 3.21 and [21], it suffices to show that there exists a closed neighborhood T of p in C such that $M - T$ has 1-ALG at each point $q \in T$; i.e., for each sufficiently small open set U containing q there is an open set V such that $q \in V \subseteq U$ and each loop in $V - T$ that bounds in $U - T$ is contractible in $U - T$.

Let T_0 be an open arc in $C - Q$ that contains p and such that $T = \mathrm{cl}\, T_0 \subseteq C - Q$ and contains no points of order 1 in C. Suppose $q \in T_0$ and let $\{U_i\}_{i=1}^{\infty}$ be a neighborhood basis at q given by Proposition 3.22. We may assume that $U_1 \cap C \subseteq T$. Suppose U is an open set containing q such that $U \subseteq U_1$. Hence there exists U_i such that $U_i \subseteq U$; let $V = \mathrm{int}\, U_i$. Then there exists U_j such that $U_j \subseteq V$. By Propositions 3.22 and 3.24, the image of $\pi_1(V - T) \longrightarrow \pi_1(U - T)$ is isomorphic to the integers and the result follows from [27, p. 389].

If $q \in T - T_0$, 1-ALG at q follows similarly from Proposition 3.23.

COROLLARY TO THEOREM A. *Let* M *be an open* 3*-manifold and let* h *be a homeomorphism of* M *onto itself such that* $\mathrm{Irr}(h)$ *is a one-dimensional connected polyhedron which is topologically embedded as a closed subset of* M, *and* h *is positively regular on* M. *If* $\mathrm{Irr}(h)$ *is the* 1*-sphere, suppose there exists* $x \in M$ *such that* $\mathrm{Irr}(h) \neq \limsup_{n \to +\infty} \{h^n(x)\}$. *If the one point compactification of* $\mathrm{Irr}(h)$ *is not a polyhedron, suppose that for some* r, $h^r | \mathrm{Irr}(h)$ *is the identity. Then* $\mathrm{Irr}(h)$ *fails to be locally tame at a discrete subset of the set of points of* $\mathrm{Irr}(h)$ *of order greater than one.*

OUTLINE OF PROOF. If $\mathrm{Irr}(h)$ is not homeomorphic to R, it follows from Proposition 3.2 that there exists r such that $h^r | \mathrm{Irr}(h)$ is the identity. If $\mathrm{Irr}(h)$ is homeomorphic to R and $h | \mathrm{Irr}(h)$ is not periodic, it follows from

Theorem 10 of [5], [12] and Proposition 3.2 that either $\mathrm{Irr}(h) = \varnothing$, an arc or a half-ray. Let r = period of $h \mid \mathrm{Irr}(h)$.

Let $M_L = \{x \in M \mid \lim_{i \to +\infty} h^{ri}(x) \text{ exists}\}$. Using the facts that $h^r \mid \mathrm{Irr}(h)$ is the identity and h is positively regular on all of M, one can show that M_L is a connected open subset of M which contains $\mathrm{Irr}(h)$. [See the comments following Corollary 4.5 about an example where $M_L \neq M$.] Note that $h(M_L) = M_L$ and $\mathrm{Irr}(h^r \mid M_L) = \mathrm{Irr}(h)$.

To apply the above proof of Theorem A to this case it suffices to show that if $f\colon M_L \longrightarrow \mathrm{Irr}(h)$ is defined by $f(x) = \lim_{i \to +\infty} h^{ri}(x)$ and if f_0 is the induced map from the orbit space of $h^r \mid M_L - \mathrm{Irr}(h)$ to $\mathrm{Irr}(h)$, then $f_0^{-1}(p)$ is compact for each $p \in \mathrm{Irr}(h)$. We let F be the Fruedenthal endpoint compactification of $f^{-1}(p)$. Note that p is not a limit point of $F - f^{-1}(p)$. Extend $h^r \mid f^{-1}(p)$ to a homeomorphism h_* of F onto itself. Note that h_* is regular except possibly on $(F - f^{-1}(p)) \cup \{p\}$. Since $h_*(F - f^{-1}(p)) = F - f^{-1}(p)$, it follows from [12] that $F - f^{-1}(p)$ is a singleton. The compactness of $f_0^{-1}(p)$ now follows from Proposition 7 of [5].

4. **Proof of Theorem B.** Let M, h and C be as in the Corollary to Theorem A. Define f, f_0 and Q as in the proof of the latter. Suppose that $h \mid C$ is the identity and assume that if U is an open orientable subset of M such that $h(U) = U$, then $h \mid U$ is orentation-preserving.

PROPOSITION 4.1. *If $p \in C$ is a point of order two and C is locally tame at p, then $f_0^{-1}(p)$ has property* AFG.

PROOF. Let U be an open arc in C such that $p \in U$ and C is locally tame at each point $q \in U$. There exists a neighborhood V of U in $f^{-1}U$ such that the pair (V, U) is homeomorphic to (R^3, R^1). Let $\phi\colon S^1 \longrightarrow f^{-1}(U) - U$ be a loop which is homologous to zero in $f^{-1}(U) - U$; i.e. there exists a compact connected 2-manifold T with connected boundary and a continuous map $\lambda\colon T \longrightarrow f^{-1}(U) - U$ such that $\lambda \mid \mathrm{bdry}\, T = \phi$. Since $f\lambda(T)$ is a compact subset of U, there exists n such that $h^n\lambda(T) \subseteq V - U$. But $h^n\phi$ is inessential in $V - U$ and hence ϕ is inessential in $f^{-1}(U) - U$. It follows that $\pi_1(f^{-1}(U) - U)$ is isomorphic to the integers Z. Since $h \mid f^{-1}(U) - U$ is orientation-preserving, $\pi_1(f_0^{-1}U)$ is isomorphic to $Z \oplus Z$. Hence $f_0^{-1}(p)$ has property AFG.

A similar proof gives the following.

PROPOSITION 4.2. *If $p \in C$ is a point of order one and C is locally tame at p, then $f_0^{-1}(p)$ has property* AFG.

PROPOSITION 4.3. *If $f_0^{-1}(p)$ has property* AFG *for some point $p \in C$ of order two, then $p \notin Q$.*

PROOF. By Proposition 3.16, there exists an arc U in C such that $U \cap Q = \{p\}$ or $\emptyset$. Since $f_0^{-1}(p)$ has AFG, there exists a neighborhood W of $f_0^{-1}(p)$ in $f_0^{-1}U$ such that if Y is any neighborhood of $f_0^{-1}(p)$ in W then there exists a neighborhood V of $f_0^{-1}(p)$ in Y such that each loop in V which is homologous to zero in Y is homotopic to zero in Y. Find an open arc U_1 in C such that $p \in U_1$ and $f_0^{-1}U_1 \subseteq W$. Let $Y = f_0^{-1}U_1$ and let V be given from the definition of AFG. Let U_2 be an open arc in C such that $f_0^{-1}(p) \subseteq f_0^{-1}U_2 \subseteq V$. By using Propositions 3.14, 3.15, 3.18 and Theorem 2.2, one can easily show that the inclusion induces an isomorphism $\pi_1(f_0^{-1}U_2) \longrightarrow \pi_1(f_0^{-1}U_1)$. Consider the following diagram

$$\begin{array}{ccccc} \pi_1(f_0^{-1}U_2) & \xrightarrow{i} & \pi_1(V) & \xrightarrow{j} & \pi_1(f_0^{-1}U_1) \\ & & \Big\downarrow{\scriptstyle h_1} & & \Big\downarrow{\scriptstyle h_2} \\ & & H_1(V) & \xrightarrow{k} & H_1(f_0^{-1}U_1) \end{array}$$

where i, j and k are induced by inclusions and h_1 and h_2 are the Hurewicz homomorphisms. Since ji is an isomorphism, image $j = \pi_1(f_0^{-1}U_1)$. By AFG, $h_2|$image j is one-to-one and, hence, h_2 is an isomorphism. By the remark to Proposition 3.6, $\pi_1(f_0^{-1}U_1)$ is then isomorphic to $Z \oplus Z$. $\pi_1(f_0^{-1}U_1)$ is homeomorphic to the interior of a compact irreducible 3-manifold F (Propositions 3.4 and 3.18) whose boundary is $S^1 \times S^1 \times \{0, 1\}$. By [28], F is homeomorphic to $S^1 \times S^1 \times [0, 1]$ and, hence, $p \notin Q$.

Theorem B follows immediately from Propositions 4.1 and 4.3.

THEOREM 4.4. *Let M be an open 3-manifold and let h be a homeomorphism of M onto itself which is positively regular on all of M. Suppose that $C = \operatorname{Irr}(h)$ is a one dimensional connected manifold which is a closed subset of M. If C is the 1-sphere, then suppose that there exists $x \in M$ such that $\limsup_{n\to+\infty} \{h^n(x)\} \neq C$. Then a necessary and sufficient condition that C be locally tame in M is that $f_0^{-1}(p)$ has property* AFG *for each $p \in C$ where f_0 is the map induced on the orbit space of $h^{2r}|M_L - C$ (r = period of $h|C$).* (See the proof of the Corollary of Theorem A for the definition of M_L.)

PROOF. The necessity of AFG follows from Propositions 4.1 and 4.2. It suffices to consider the case when p is of order one. Let U be an open connected subset of C which contains p and no other point of order one. By Propositions 4.3 and 3.18, $f_0^{-1}(U - \{p\})$ is homeomorphic to $S^1 \times S^1 \times R$. By an argument similar to that given in Proposition 4.3, $\pi_1(f_0^{-1}U)$ is isomorphic to Z. By [25], $f_0^{-1}U$ is homeomorphic to $S^1 \times R^2$ and, hence, $f^{-1}(U) - U$ is homeomorphic to R^3.

We now proceed as in the proof of Theorem A to show that C is 1-LC at p and the result follows from [21].

ADDENDUM TO THEOREM 4.4. *Suppose that* C *is locally tame. If* C *is the* 1-*sphere, then* M *is homeomorphic to* $S^1 \times R^2$ *or the twisted plane bundle over* S^1. *If* C *is homeomorphic to* $[0, 1]$, $[0, 1)$ *or* $(0, 1)$, M_L *is homeomorphic to* R^3. *In fact, if* C *is homeomorphic to* $[0, 1)$ *or* $(0, 1)$, *then the pair* (M_L, C) *is homeomorphic to* (R^3, R^1_+) *or* (R^3, R^1), *respectively.* $[R^1_+ = \{(x, 0, 0) | x \geqslant 0\}]$.

PROOF. If C is compact, then the result follows from the fact that for each compact set $D \subseteq M - C$, $\limsup_{n\to-\infty} \{h^n(D)\} = \infty$ (Proposition 2.1 of [6]) and from Theorem 5 of [1].

Suppose C is not compact; we shall first show that M_L is homeomorphic to R^3. Let D be a compact subset of M_L. Note that $f(D)$ is a compact subset of C; hence, there exists a closed arc $C_0 \subseteq C$ such that $f(D) \subseteq C_0$. Let N be a regular neighborhood of C_0 in M_L. By Proposition 2.1 of [6] or from the definition of f, there exists n such that $h^n(D)$ is contained in the interior of N. Hence D is contained in the interior of the 3-cell $h^{-n}(N)$ and M_L is homeomorphic to R^3 by [1].

Now consider the case when C is homeomorphic to $(0, 1)$; we leave the proof of the other case to the reader. Let $M^* = M_L \cup \{\infty\}$ be the one-point compactification of M_L and $C^* = C \cup \{\infty\}$. To show that (M_L, C) is homeomorphic to (R^3, R^1) it suffices to show that C^* is locally tame at ∞ since from the proof of Proposition 4.1, $\pi_1(M^* - C^*) = \pi_1(M_L - C) = Z$ [25]. We shall again use [21]. Let U be a neighborhood of ∞ in M^*; let Y be an open arc in C^* such that $\infty \in Y \subseteq U$. Let N be a regular neighborhood of $C - Y$ rel bdry$(C - Y)$ in M_L [13]. As before, there exists n such that $M^* - U \subseteq \operatorname{int} h^n(N)$. Let $V = M^* - h^n(N)$; note that $\pi_1(V - C^*)$ is isomorphic to Z and the inclusion of $V - C^*$ into $M^* - C^*$ induces an isomorphism of fundamental groups. It follows that C^* is 1-ALG at each point of C.

COROLLARY 4.5. *Let* h *be a homeomorphism of the* 3-*sphere* S^3 *onto itself such that* $\operatorname{Irr}(h)$ *is a* 1-*sphere. Suppose that there exists a point* $x \in \operatorname{Irr}(h)$ *such that* h *is positively regular on* $S^3 - \{x\}$ *and, for each* $y \in S^3 - \{x\}$, $\limsup_{n\to+\infty} \{h^n(y)\} \neq \{x\}$. *A necessary and sufficient condition that* $\operatorname{Irr}(h)$ *be locally tame in* S^3 *is that the map induced on the orbit space of* $h|S^3 - \operatorname{Irr}(h)$ *onto* $\operatorname{Irr}(h) - \{x\}$ *have property* AFG *for each point inverse.*

Duvall and Husch constructed an example in [6] of a homeomorphism h of S^3 such that $\operatorname{Irr}(h)$ is a 1-sphere, there exists $x \in \operatorname{Irr}(h)$ such that h is positively regular on $S^3 - \{x\}$ and $\operatorname{Irr}(h)$ is locally tame at each point except x. This example shows the necessity of adding the hypothesis that

$\limsup_{n\to+\infty}\{h^n(y)\} \neq \{x\}$ in Corollary 4.5.

5. **Wild irregular sets.** Let $\alpha: S^1 \longrightarrow S^1 \times I^2$ be a piecewise linear embedding of the 1-sphere into the interior of $S^1 \times I^2$ such that α is a homotopy equivalence and the pair $(S^1 \times I^2, \alpha(S^1))$ is not homeomorphic to $(S^1 \times I^2, S^1 \times (\frac{1}{2}, \frac{1}{2}))$. Let $\pi: R^1 \times I^2 \longrightarrow S^1 \times I^2$ be the universal covering map and let h_1 be a generator of the corresponding covering transformation group.

Let N be a regular neighborhood of $\alpha(S^1)$ in the interior of $S^1 \times I^2$, let $V' = \mathrm{cl}\,[(S^1 \times I^2) - N]$, $V'' = \pi^{-1}V'$, bdry $V'' = V_1 \cup V_2$ where $V_1 =$ bdry$(R^1 \times I^2)$, and let $V^* = V'' \cup \{0, \infty\}$ be the Freudenthal end point compactification of V''. $h_1 | V''$ induces a homeomorphism h^* of V^* such that, say $\lim_{i\to+\infty} h^{*i}(x) = 0$ and $\lim_{i\to-\infty} h^{*i}(x) = \infty$ for all $x \in V''$. It follows from [12] that $\mathrm{Irr}(h^*) = \{0, \infty\}$. Let $\phi_i: R^2 \longrightarrow V_i \cup \{0\}$ be a homeomorphism (onto) such that $\phi_i(\text{origin}) = 0$. Note that $\phi_i^{-1}(h^* | V_i \cup \{0\})\phi_i$ is a homeomorphism of R^2 which is regular except at the origin. By [18], there exists a homeomorphism μ_i of R^2 such that

$$\mu_i^{-1}\phi_i^{-1}(h^* | V_i \cup \{0\})\phi_i\mu_i(x) = x/2$$

for all $x \in R^2$.

Define $h_2: R^2 \times [0, +\infty) \longrightarrow R^2 \times [0, +\infty)$ by

$$\begin{aligned} h_2(x, y, z) &= (x/2, y/2, z) && \text{if } 0 \leqslant z \leqslant 1,\\ &= (x/2, y/2, z/2 + \tfrac{1}{2}) && \text{if } z \geqslant 1. \end{aligned}$$

Let M be the decomposition space obtained from the disjoint union of $V'' \cup \{0\}$ and $R^2 \times [0, +\infty) \times \{a_1, a_2\}$ where $V_1 \cup \{0\}$ is identified with $R^2 \times \{0\} \times \{a_1\}$ and $V_2 \cup \{0\}$ is identified with $R^2 \times \{0\} \times \{a_2\}$ by means of the homeomorphisms

$$x \longrightarrow (\mu_i^{-1}\phi_i^{-1}(x), 0, a_i), \qquad i = 1, 2.$$

Let $\rho: (V'' \cup \{0\}) \cup R^2 \times [0, +\infty) \times \{a_1, a_2\} \longrightarrow M$ be the natural projection. Define $h: M \longrightarrow M$ by

$$\begin{aligned} h(\rho(x)) &= \rho h^*(x) && \text{if } x \in V'' \cup \{0\},\\ &= \rho(h_2(x_1), a_i) && \text{if } x = (x_1, a_i) \in (R^2 \times [0, +\infty)) \times \{a_1, a_2\}. \end{aligned}$$

Note that

$$\mathrm{Irr}\,(h) = \rho(\{(0, 0, t) \in R^2 \times [0, +\infty) | 0 \leqslant t \leqslant 1\} \times \{a_1, a_2\}),$$

and h is positively regular on all of M. It is not difficult to see that M is homeomorphic to R^3 and $\mathrm{Irr}(h)$ is locally tame except at $\rho(0)$. Hence we have the following.

EXAMPLE 5.1. *There exists a homeomorphism h of R^3 onto itself such* *that h is positively regular on R^3 and* $\operatorname{Irr}(h)$ *is an arc which is locally tame except at one point.*

Suppose α is chosen such that if $x \in S^1$, then for any disk $D \subseteq S^1 \times I^2$ such that $\operatorname{bdry} D = \{x\} \times \operatorname{bdry} I^2$, $D \cap \alpha(S^1)$ contains at least two points. Let M be the disjoint union of $(V'' \cup \{0\}) \times \{b_1, b_2\}, R^2 \times [0, 1]$, $R^2 \times [0, +\infty) \times \{a_1, a_2\}$ where $R^2 \times \{0\} \times \{a_1\}$, $(V_2 \cup \{0\}) \times \{b_1\}, R^2 \times \{1\}$, $(V_1 \cup \{0\}) \times \{b_2\}$ are identified with $(V_1 \cup \{0\}) \times \{b_1\}$, $R^2 \times \{0\}$, $(V_2 \cup \{0\}) \times \{b_2\}$ and $R^2 \times \{0\} \times \{a_2\}$ respectively.

Define $h: M \longrightarrow M$ similar to the definition of h above [on $R^2 \times [0, 1]$, use the restriction of h_2]. Note that h is positively regular on M and $\operatorname{Irr}(h) = \rho((0, 0) \times [0, 1] \cup (0, 0) \times [0, 1] \times \{a_1, a_2\})$. However, in this construction, $\operatorname{Irr}(h)$ is not cellular in M and, hence, M is not homeomorphic to R^3.

EXAMPLE 5.2. *There exists a 3-manifold M not homeomorphic to R^3 and a homeomorphism h of M onto itself such that h is positively regular on M and* $\operatorname{Irr}(h)$ *is an arc.*

Our next example will have the property that $\operatorname{Irr}(h)$ is an arc which fails to be locally tame at an infinite number of points. To aid in the construction, we first consider a more general situation.

Let A and C be metric continua and let $\lambda: A \times [0, 1] \longrightarrow C$ be an onto continuous map such that $\lambda(a, 0) = \lambda(a, 1)$ for all $a \in A$. If d_A is a metric for A, define a metric d for $A \times R$ by $d((a, t), (a', t')) = d_A(a, a') + |t - t'|$. If $T: A \times R \longrightarrow A \times R$ is defined by $T(a, t) = (a, t + 1)$, then T is an isometry. If $V \subseteq A \times [0, 1]$, define $[V]_K = \bigcup_{i=K}^{\infty} T^i(V) - (A \times \{K\})$ for each integer K.

Suppose $A \times R$ and C are disjoint; let $M = C \cup (A \times R)$. Let $\mathfrak{U} = \{U\}$ be a basis for a topology on M defined by (i) U is an open subset of $A \times R$, or (ii) $U = V \cup [\lambda^{-1} V]_K$ where V is an open subset of C and K is an integer. M is a locally compact metrizable space; let $M_\infty = M \cup \{\infty\}$ be its one point compactification and let d' be a metric on M induced from a metric on M_∞.

PROPOSITION 5.3. *The inclusion map $i: (A \times R, d) \longrightarrow (M, d')$ is uniformly continuous.*

PROOF. Let $\epsilon > 0$ be given. Choose a finite number of points $c_1, c_2, \cdots, c_n \in C$ such that $V = \bigcup_{i=1}^n V_i$, $V_i = \epsilon/2$-open ball with center at c_i in M, covers C. There exists K_1 such that $A_1 = i(A \times [K_1, +\infty)) \subseteq V$. Let η be a Lebesgue number of the cover $\{(A_1 \cup C) \cap V_i\}$ of $A_1 \cup C$. There exists a finite number of open sets $\{U_i\}_{i=1}^m$ and an integer $K_2 > K_1$ such that $C \subseteq \bigcup_{i=1}^m U_i$ and the diameter of $U_j \cup i[\lambda^{-1} U_j]_{K_2}$ is less than η. Let $\eta' < 1$

be a Lebesgue number of the cover $\{\lambda^{-1}U_i \cup T\lambda^{-1}U_i\}_{i=1}^m$ of $A \times [0, 2]$. Note that η' is also a Lebesgue number of the cover $\{[\lambda^{-1}U_j]_{K_2}\}_{j=1}^m$ of $A \times [K_2 + 1, +\infty)$. Hence if $d(x, y) < \eta'$ and $x, y \in A \times [K_2 + 1, +\infty)$, then $d(i(x), i(y)) < \epsilon$.

There exists $K_3 < K_1$ such that $i(A \times (-\infty, K_3])$ is contained in $\epsilon/2$-neighborhood of ∞. Choose η'' from the uniform continuity of $i|A \times [K_3 - 1, K_2 + 2]$. Let $\delta = \text{minimum}\ \{\eta', \eta''\}$.

Define $h: M \longrightarrow M$ by $h(x) = T(x)$ if $x \in A \times R$ and $h(x) = x$ if $x \in C$. It is easily verified that h is a homeomorphism of M onto itself.

COROLLARY 5.4. $\mathrm{Irr}(h) = C$.

PROOF. First we show that h is regular at each $x \in A \times R$. Let $\epsilon > 0$ be given; choose δ_1 from the uniform continuity of i. Choose δ such that if $d'(x, y) < \delta$, then $d(x, y) < \delta_1$. Since $d(T^i(x), T^i(y)) = d(x, y)$, it follows that $d'(h^i(x), h^i(y)) < \epsilon$ for all i.

Suppose $x \in C$; choose $y \in A \times [0, 1]$ such that $\lambda(y) = x$. Note that $\lim_{i\to+\infty} h^i(y) = x$ and, hence, h is not regular at x.

PROPOSITION 5.5. *h is positively regular on all of M.*

PROOF. It suffices to show that h is positively regular on C; let $x \in C$ and let $\epsilon > 0$ be given. Let $V \cup [\lambda^{-1}V]_K$ be a basic open set containing x which lies in the ϵ-neighborhood of x. Choose $\delta > 0$ such that the δ-neighborhood of x lies in $V \cup [\lambda^{-1}V]_K$. Since $h^i(V \cup [\lambda^{-1}V]_K) \subseteq V \cup [\lambda^{-1}V]_K$, positive regularity of h at x is assured.

PROPOSITION 5.6. *If $f = \lim_{i\to+\infty} h^i$, then $f^{-1}(x) = \{x\} \cup \bigcup_{i=-\infty}^{+\infty} T^i(\lambda^{-1}(x))$ for each $x \in C$.*

PROOF. If $\lambda(z) = x$ and $\epsilon > 0$, then it follows from the definition of the topology of M that there exists an integer N such that $n > N$ implies that $h^n(z)$ is in the $\epsilon/2$-neighborhood of x; hence, $f(z)$ lies in the ϵ-neighborhood of x for any ϵ and $f(z) = x$.

Suppose $z \in M - C$ such that $f(z) = x$. There exists $z_1 \in A \times [0, 1]$ such that $T^i(z_1) = z$ for some i. Let $\epsilon > 0$ be given; let U be the ϵ-neighborhood of x in M. Since $f(z) = x$, for some j, K, $h^j(z) \in (U \cap C) \cup [\lambda^{-1}(U \cap C)]_K$. Hence $z_1 \in \lambda^{-1}(U \cap C)$; but $\bigcap_{\epsilon>0} \lambda^{-1}(U \cap C) = \lambda^{-1}(x)$ and thus, $\lambda(z_1) = x$.

Let $A(\epsilon) = \{(x, y) \in R^2 \mid x^2 + y^2 \leqslant \epsilon\}$ and let μ: $[0, 1] \longrightarrow A(1) \times [½, 1]$ be a piecewise linear embedding such that $\mu(0) = (0, 0, ½)$, $\mu(1) = (0, 0, 1)$ and $(A(1) \times [½, 1], \mu([0, 1]))$ is a knotted ball pair. Let N_i be a regular neighborhood of $\mu([0,1])$ in $A(1)\times[\tfrac12,1]$ such that $N_i\cap \operatorname{bdry}(A(1)\times[\tfrac12,1]) = A(2^{-i})\times\{\tfrac12,1\}$, $N_{i+1}\subseteq(\operatorname{int}N_i)\cup(A(1)\times\{\tfrac12,1\})$ and $\bigcap_{i=1}^{\infty}N_i = \mu([0,1])$, $i=1,2,\cdots$. Define $k\colon R^3\to R^3$ by $k(x,y,z)=(x/2,y/2,z/2)$. For $i=0,1,2,\cdots$, let

$$T_{2i} = A(2^{-2i-1})\times[0,2^{-i-1}]\cup\bigcup_{j=0}^{i}k^j(N_{2i-2j+1})$$

$$T_{2i+1} = A(2^{-2i-2})\times[0,2^{-i-1}]\cup\bigcup_{j=0}^{i}k^j(N_{2(i+1-j)}).$$

There exists a homeomorphism (onto) $\mu_i\colon \operatorname{cl}(A(2^{-2i-1})-A(2^{-2i-2}))\times[0,1]\to\operatorname{cl}(T_{2i}-T_{2i+1})$ such that $\mu_i|\operatorname{cl}(A(2^{-2i-1})-A(2^{-2i-2}))\times\{0,1\}$ is the identity. Define $\lambda_i\colon \operatorname{cl}(A(2^{-2i-1})-A(2^{-2i-2}))\times[0,1]\to[1/(i+2),1/(i+1)]$ by

$$\lambda_i(x,y,z)=\frac{2^{2i+2}(x^2+y^2)+i}{(i+1)(i+2)}$$

and define $\lambda\colon A(1)\times[0,1]\to[0,1]$ by

$$\begin{aligned}\lambda(z) &= \lambda_i\mu_i^{-1}(z) && \text{if } z\in\operatorname{cl}(T_{2i}-T_{2i+1}),\quad i=0,1,\cdots,\\ &= 1/(i+2) && \text{if } z\in\operatorname{cl}(T_{2i+1}-T_{2i+2}),\quad i=0,1,\cdots,\\ &= 1 && \text{if } z\notin T_0,\\ &= 0 && \text{if } z\in\operatorname{cl}\left(\bigcup_{j=0}^{\infty}k^j\mu([0,1])\right).\end{aligned}$$

It is easily verified that λ is a continuous map and $\lambda(a,0)=\lambda(a,1)$ for all $a\in A(1)$.

Let $M=(A(1)\times R)\cup[0,1]$ and $h\colon M\to M$ be constructed as above. Let $f=\lim_{i\to+\infty}h^i$ and consider $D_i=f^{-1}((1/(i+2),1/i))$, $i=1,2,\cdots$. Note that if we let $f=\lim_{i\to+\infty}h^i$ in Example 5.1, then

$$f^{-1}(\rho(\{(0,0,t)\in R^2\times[0,1)\}\times\{a_1,a_2\}))$$

is homeomorphic to D_i for a suitable choice of α and the two homeomorphisms called h are essentially the same.

Using this fact one can easily prove the following proposition.

PROPOSITION 5.7. *M is homeomorphic to $R^2\times[0,+\infty)$ and $[0,1]$ ($=M-(A(1)\times R)$) is locally tame except at $\{0\}\cup\{1/i\}_{i=1}^{\infty}$.*

EXAMPLE 5.8. *There exists a homeomorphism α of R^3 onto itself such that α is positively regular on R^3 and $\operatorname{Irr}(h)$ is an arc which fails to be locally tame at an infinite number of points.*

PROOF. Let $\beta: M \longrightarrow R^2 \times [0, +\infty)$ be a homeomorphism and let $\alpha_1 = \beta h \beta^{-1}$. Consider $\alpha_1 | R^2 \times \{0\}$; then $\alpha_1(x, 0) = (\alpha_1'(x), 0)$ for some homeomorphism α_1' of R^2 onto itself. Note that α_1' is regular except at 0 and hence by [18], there exists a homeomorphism t of R^2 onto itself such that $t^{-1}\alpha_1' t(x) = x/2$. Define $\alpha: R^3 \longrightarrow R^3$ by

$$\begin{aligned} \alpha(x, y) &= \alpha_1(x, y) && \text{if } (x, y) \in R^2 \times [0, +\infty), \\ &= (t(\tfrac{1}{2} \cdot t^{-1}(x)), y/2) && \text{if } (x, y) \in R^2 \times (-\infty, 0]. \end{aligned}$$

α is positively regular on all of R^3 and fails to be regular on $C = \beta([0, 1])$. (The metrics we use on R^3 and M are induced from the metrics of their one. point compactifications.)

Let h be the homeomorphisms of R^3 onto itself given by Example 5.1. Define $k: R^n \longrightarrow R^n$ by

$$\begin{aligned} k(y) &= (\tfrac{1}{2}\|y\| + \tfrac{1}{2})y && \text{if } \|y\| \geqslant 1, \\ &= y && \text{if } \|y\| < 1. \end{aligned}$$

Define $\lambda: R^3 \times R^n \longrightarrow R^3 \times R^n$ by $\lambda(x, y) = (h(x), k(y))$. We have the following.

EXAMPLE 5.9. *There exists a homeomorphism* λ *of* R^m $(m \geqslant 3)$ *onto itself such that* λ *is positively regular on all of* R^m. $\mathrm{Irr}(\lambda)$ *is an* $(m - 2)$*-cell which fails to be locally tame on an* $(m - 3)$*-cell.*

6. **Concluding remarks.** Using the construction techniques in Example 5.2, one can construct an infinite number of distinct 3-manifolds for which one can define a homeomorphism which is positively regular everywhere and whose set of irregular points is homeomorphic to $[0, 1]$. Since these manifolds are contractible and it is known that there are uncountably many distinct contractible 3-manifolds [20], the natural question arises whether there exist uncountably many such manifolds for which one can define a positively regular homeomorphism whose set of irregular points is an arc.

THEOREM 6.1. *If* C *is a compact one-dimensional polyhedron, then there exists at most a countably infinite collection of connected open 3-manifolds upon each of which it is possible to define a positively regular homeomorphism (onto) such that the set of irregular points is homeomorphic to* C.

PROPOSITION 6.2. *Let* M, C *and* h *be as in Theorem* A *and let* $p \in C$ *be a point of order one. If* p *is a limit point of the set of points at which* C *fails to be locally tame, then there exists a neighborhood* U *of* p *in* C *such that* $f^{-1}U$ *is homeomorphic to* R^3.

PROOF. Let $\lambda: (0, 1] \longrightarrow C$ be an imbedding such that $\lambda(1) = p$ and each point of the image of λ different from p has order two. Let $<$ be the induced ordering on $V =$ image of λ. Let $\{p_i\}_{i=1}^{\infty} \subseteq V$ be such that $p_i < p_{i+1}$ for all i and $\lim_{i\to+\infty} p_i = p$. Let $T_i \subseteq f_0^{-1}(p_i, p_{i+1})$ be the surface given by Proposition 3.14.

Case 1. There exist an infinite number of i's such that T_i is incompressible in the compact submanifold V_1 of $f_0^{-1}V$ which is bounded by T_1. By Theorems 2.1 and 2.2, it follows that there exists N such that if $i > N$, then T_i and T_N are topologically parallel in V_1. From the proof of Theorem A, (p_{N+1}, p) is locally tame at each point, a contradiction. Hence we must have the following.

Case 2. There exists N such that if $i \geqslant N$, T_i is not incompressible in V_1. There exists for each $i \geqslant N$, a locally tame disk D_i such that $D_i \cap T_i =$ bdry D_i and bdry D_i does not bound a 2-cell in T_i.

Since a 2-sphere cannot separate the ends of $f_0^{-1}\lambda(0, 1)$, $D_i \cap f_0^{-1}(p) \neq \varnothing$. By Proposition 3.4, $f_0^{-1}V$ is irreducible and, therefore, each T_i, $i \geqslant N$, is the boundary of a compact 3-manifold S_i in $f_0^{-1}V$ where S_i is homeomorphic to $S^1 \times I^2$. Suppose that the sequence $\{p_i\}$ is chosen such that $(p_i, p_{i+1}) \cap Q$ contains at most one point and $\{p_i\} \cap Q = \varnothing$. There is no loss of generality in assuming that $Q \cap (p_N, p_{N+1}) = \varnothing$; let $U = (p_N, p]$. $f_0^{-1}U$ is homeomorphic to $S^1 \times R^2$ and hence $f^{-1}(U) - U$ is homeomorphic to R^3.

$W = f^{-1}(p_N, p_{N+1})$ is homeomorphic to R^3 by the addendum to Theorem 4.4. Consider $f_0(T_N) \subseteq (p_N, p_{N+1})$. Since $Q \cap (p_N, p_{N+1}) = \varnothing$, $f_0(T_N)$ is a locally tame arc in W. Let W' be the decomposition space obtained from W by shrinking $f_0(T_N)$ to a point.

Let D'_N be a disk in $f^{-1}U$ such that $p(D'_N) = D_N$. Note that the image of $f^{-1}f_0(T_N)$ in W' is a locally tame plane whose intersection with the image of D'_N is the image of bdry D'_N. It follows that there is a locally tame 2-cell E in W such that $E \cap D'_N =$ bdry $E =$ bdry D'_N. Hence $E \cup D'_N$ is a locally tame 2-sphere in $f^{-1}U$ that bounds a 3-cell which contains $[p_{N+1}, p]$.

If K is a compact subset of $f^{-1}U$, $E \cup D'_N$ and an integer H can be chosen such that K lies in the interior of the 3-cell bounded by $h^H(E \cup D'_N)$. By [1], $f^{-1}U$ is homeomorphic to R^3.

Outline of the proof of 6.1. Let us first consider the case when C has no points of order one. As in the proof of Proposition 6.2, we can find a finite number of locally tame arcs in C such that when we shrink each of these arcs to points, we can find locally tame planes which are invariant under the homeomorphism induced by h and such that each component of the complement of the union of these planes contains a single element of Q [See Proposition 3.16]. Avoiding the introduction of new notations, we suppose that such planes exist for h. Let E be the closure of one of the above components; note that

$p(E - C)$ is a compact 3-dimensional manifold. By [2], there exists at most a countable number of compact 3-manifolds.

We have an exact sequence

$$1 \longrightarrow \pi_1(E - C) \longrightarrow \pi_1(p(E - C)) \longrightarrow Z \longrightarrow 1.$$

Since $\pi_1(p(E - C))$ is finitely generated, there exists a countable number of homomorphisms of $\pi_1(p(E - C))$ onto Z. Hence we have at most a countable number of choices for $\pi_1(E - C)$. By covering space theory, $\pi_1(E - C)$ determines $E - C$, hence there are at most a countable number of candidates for $E - C$. $E - C$ completely determines the decomposition space $E/E \cap C$ obtained from E by shrinking $E \cap C$ to a point. But E is obtained from $E/E \cap C$ by attaching a finite number of 3-cells along a disk of each [here we use the fact that $C \cap E$ is locally tame except possibly at one point]. Hence there exists at most a countable number of candidates for E and, therefore, for M.

We leave to the reader the completion of the proof of Theorem 6.1.

In [7], Duvall and Husch showed that if h is a positively regular homeomorphism of an open connected n-manifold M such that $\mathrm{Irr}(h)$ is a nonseparating compactum, then there exists a retraction $f: M \longrightarrow \mathrm{Irr}(h)$ so that $\mathrm{Irr}(h)$ is a strong deformation retract of M. If $h = 3$, $\mathrm{Irr}(h) = S^1$ and there exists $x \in M$ such that $\lim\sup_{n\to+\infty} \{h^n(x)\} = \mathrm{Irr}(h)$, there is no induced f_0 from the orbit space $h|M - \mathrm{Irr}(h)$ to S^1 and hence the techniques of this paper do not apply.

Conjecture. *If M, h and C are as in Theorem* A *and there exists $x \in M$ such that* $\lim\sup_{n\to+\infty}\{h^n(x)\} = C$, *then C is locally tame in M.*

Note that the constructions in Example 5.2 can be modified to give examples of homeomorphisms h such that $\mathrm{Irr}(h)$ can be any compact connected 2-dimensional polyhedron which is not locally tame at a finite number of points. In fact, by modifying Example 5.2, there is an open connected orientable 3-manifold M and a positively regular homeomorphism h of M into itself such that $\mathrm{Irr}(h)$ is a 2-sphere which fails to be locally tame at two points. M is not homeomorphic to $S^2 \times R$. If T is the product of r 1-spheres, define $h': M \times T \longrightarrow M \times T$ by $h'(x, t) = (h(x), t)$. h' is positively regular and $\mathrm{Irr}(h)$ is homeomorphic to $S^2 \times T$. We claim that $M \times T$ is not homeomorphic to an open mapping cylinder neighborhood of any locally tame embedding of $S^2 \times T$. Since $M \times T$ is orientable, it suffices to show that $M \times T$ is not homeomorphic to $S^2 \times T \times R$. If there existed such a homeomorphism, by Proposition 1.3 of [4], M is proper homotopy equivalent to $S^2 \times R$. By Corollary 2.2 of [6] and

[24], any inessential 2-sphere in M bounds a 3-cell (see also Proposition 3.4). By [16], M is homeomorphic to $S^2 \times R$.

PROBLEM. Extend the results of this paper to the case when $\operatorname{Irr}(h)$ is a 2-manifold.

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