COMPARISON THEOREMS FOR BOUNDED SOLUTIONS
OF $\Delta u = Pu$

BY

MOSES GLASNER

ABSTRACT. Let $P$ and $Q$ be $C^1$ densities on a hyperbolic Riemann
surface $R$. A characterization of isomorphisms between the spaces of bounded
solutions of $\Delta u = Pu$ and $\Delta u = Qu$ on $R$ in terms of the Wiener harmonic
boundary is given.

In 1959, H. Royden [6] proved the following comparison theorem: If $P$
and $Q$ are nonnegative $C^1$ densities on a Riemann surface $R$ such that there
is a constant $c$ with $c^{-1}P \leq Q \leq cP$ outside some compact subset of $R$, then
the spaces of bounded solutions of $\Delta u = Pu$ and $\Delta u = Qu$ on $R$ are isomor-
that the same conclusion follows under the assumption that $\int_R |P - Q| < +\infty$.
The conditions $\int_R |P - Q| < +\infty$ and $c^{-1}P \leq Q \leq cP$ outside a compact sub-
set are independent and neither is a necessary condition for the conclusion. Re-
cently A. Lahtinen [2] gave a necessary and sufficient condition for the bounded
solutions of $\Delta u = Pu$ on $R$ to be isomorphic to the harmonic bounded func-
tions. This result actually is implicit in the paper of Loeb and Walsh [3] in the
axiomatic setting.

In this paper a necessary and sufficient condition is given for the existence
of an isomorphism $S$ between the spaces of bounded solutions of $\Delta u = Pu$ and
$\Delta u = Qu$ on $R$ with $|u - Su|$ bounded by a potential. This result contains
the ones mentioned above as special cases.

1. Let $R$ be a Riemann surface and $P$ a nonnegative $C^1$ density on $R$.
To avoid trivial considerations we assume $R$ is hyperbolic. We denote by $P(U)$
the space of solutions of $\Delta u = Pu$ on an open subset $U$ of $R$. The subspace
of bounded solutions of $P(U)$ will be denoted by $PB(U)$. The superscript $c$
in the notation $P^c(U)$ and $PB^c(U)$ denotes the subspaces with continuous ex-
tensions to $\partial U$. It is conventional to use the symbol $H$ in case $P \equiv 0$. 

Received by the editors January 18, 1974.

AMS (MOS) subject classifications (1970). Primary 31A35; Secondary 30A50, 35J05.

Key words and phrases. Solution of $\Delta u = Pu$, Riemann surfaces, Green's function,
maximum principle, Wiener harmonic boundary.
The Wiener ideal boundary will play a fundamental role and therefore we briefly mention some of its properties here (for more details, cf. [7]). The Wiener algebra $N$ associated with $R$ is the set of bounded continuous harmonizable functions on $R$. The bounded continuous superharmonic functions on $R$, for example, are contained in $N$. It can be seen that $N$ is also a vector lattice. The potential subalgebra $N_\delta$ consists of the functions in $N$ with 0 harmonizations.

The Wiener compactification $R^*$ of $R$ is a compact Hausdorff space which contains $R$ as an open dense subset and such that the functions in $N$ extend continuously to $R^*$ and separate points there. The space $R^*$ is unique up to a homeomorphism fixing $R$ pointwise. In general we shall use the same symbol for a function in $N$ and its continuous extension to $R^*$. We use $\overline{A}$ to denote the closure of $A$ in $R^*$ and $\partial A$ for the boundary of $A$ with respect to $R$. If we use $A^*$ to denote a subset of $R^*$, then by $A$ we shall mean $A^* \cap R$.

The Wiener harmonic boundary $\delta$ is the set of common zeros of functions in $N_\delta$ and hence is a compact subset of $R^* \setminus R$. The main properties of $\delta$ that we shall need are the following: If $U$ is an open subset of $R$ with $\partial U$ piecewise analytic and $f \in N$, then there is a function $h \in N$ such that $h \in H^c(U)$ and $h - f\restriction(R \setminus U) \cup (\overline{U} \cap \delta) = 0$. The function $h$ is called the harmonic projection of $f$ with respect to $U$. If $s$ is bounded and subharmonic on $U$ with continuous extension to $\partial U$ then $s \leq \max_{\partial U \cup (\overline{U} \cap \delta)} s$. Therefore a superharmonic function on $U$ with continuous extension to $\partial U$ is a potential if and only if it vanishes on $\partial U \cup (\overline{U} \cap \delta)$.

We shall use the notation $\|\varphi\|_A$ for the supremum of the function $|\varphi|$ on $A$.

2. We consider solutions of $\Delta u = Pu$ on subregions $G$ with $\partial G$ piecewise analytic ($\partial G$ may be $\emptyset$). The fact that nonnegative solutions are subharmonic gives the following (cf. [3]).

**Lemma.** If $u \in P^c B(G)$, then $\|u\|_G \leq \|u\|_{\partial G \cup (\overline{G} \cap \delta)}$. Moreover, if $u|\partial G \cup (\overline{G} \cap \delta) \geq 0$, then $u \geq 0$.

We now describe the integral operator $T_G$ which is the basic tool here. Let $\Omega$ be a relatively compact region in $R$ with $\partial \Omega$ piecewise analytic. Define $\tau_{G \cap \Omega}$ on the bounded $C^1$ functions on $\Omega$ by setting $\tau_{G \cap \Omega} \varphi = \int_{\Omega} e_{G \cap \Omega}(\cdot, z)\varphi P$, where $e_{G \cap \Omega}(\cdot, z)$ denotes the Green's function of $G \cap \Omega$ with pole at $z$. Then $\tau_{G \cap \Omega} \varphi$ is a $C^2$ function on $G \cap \Omega$ vanishing continuously on $\partial(G \cap \Omega)$ and satisfies $\Delta \tau_{G \cap \Omega} \varphi = -\varphi P$.

Therefore setting $T_{G \cap \Omega} u = u + \tau_{G \cap \Omega} u$ gives an operator $T_{G \cap \Omega}$:
COMPARISON THEOREMS FOR BOUNDED SOLUTIONS OF $\Delta u = Pu$

$P^c(G \cap \Omega) \rightarrow H^c(G \cap \Omega)$ such that $u - T_G u \mid \partial(G \cap \Omega) = 0$. For $u \in P^cB(G)$ with $u \geq 0$, $T_G u = \lim_{n \to \infty} T_{G^n} u$ exists and is given by $T_G u = u + \tau_G u$, where $\tau_G u = \int_G \mathcal{g}_G(\cdot, z) u P$. If $u \in P^cB(G)$ is arbitrary, then $T_G u$ is defined to be $T_G u_1 - T_G u_2$ where $u = u_1 - u_2, u_i \in P^cB(G), u_i \geq 0$. We collect here some of the known properties of $T_G$ that will be needed in later arguments (cf. [4], [5]).

**Theorem.** The operator $T_G : P^cB(G) \rightarrow H^cB(G)$ gives an isometric isomorphism such that $u - T_G u \mid \partial G \cup (\overline{G} \cap \delta) = 0$. If $\int_G \mathcal{g}_G(\cdot, z) P < +\infty$, for every $z \in G$, then $T_G$ is onto.

3. We introduce the set $\delta^P = \{ p \in \delta \mid p \text{ has a nbd } U^* \text{ in } R^* \text{ with } \int_U \mathcal{g}_U(\cdot, z) P < +\infty \text{ for each } z \in U \}$. Here $\mathcal{g}_U(\cdot, z)$ for an arbitrary open set $U$ is defined as follows: Let the component of $U$ containing $z$ be denoted by $U_z$. Then $\mathcal{g}_U(\cdot, z)$ is the Green's function of $U_z$ on $U_z$ and zero on $U \setminus U_z$.

Clearly $\delta^P$ is an open subset of $\delta$. The significance of $\delta^P$ stems from the following

**Theorem.** The functions in $PB(R)$ restricted to $\delta$ vanish on $\delta \setminus \delta^P$.

It is sufficient to prove the assertion for $u \in PB(R)$ with $u \geq 0$. By Theorem 2, $\tau_R u(z) < +\infty$ for each $z \in R$. Suppose $p \in \delta$ and $u(p) \neq 0$. By the continuity of $u$ on $R^*$ there is a neighborhood $U^*$ of $p$ and an $\epsilon > 0$ such that $u \mid U^* > \epsilon$. Then

$$+\infty > \tau_R u(z) > \int_U \mathcal{g}_R(\cdot, z) u P \geq \epsilon \int_U \mathcal{g}_R(\cdot, z) P \geq \epsilon \int_U \mathcal{g}_U(\cdot, z) P,$$

for each $z \in U$. Thus $p \in \delta^P$.

Combining this with Lemma 2 gives the

**Corollary.** If $u \in PB(R)$, then $\|u\|_R \leq \|u\|_\delta P$. If in addition $u \mid \delta^P \geq 0$, then $u \geq 0$.

4. Denote by $C_C(A)$ the continuous functions with compact support in $A$ and by $C_0(A)$ the closure of $C_C(A)$ with respect to $\| \cdot \|_A$.

**Theorem.** The spaces $PB(R)$ and $C_0(\delta^P)$ are isomorphic vector lattices. The isomorphism is obtained by restriction to $\delta^P$, i.e. $PB(R) \mid \delta^P = C_0(\delta^P)$.

We begin the proof by observing that for any $f \in N$ with $K = \text{supp}(f \mid \delta)$ a compact subset of $\delta^P$, there exists a $u \in PB(R)$ with $u - f \mid \delta = 0$. In fact we may assume $f \geq 0$. Cover $K$ by a finite number of sets $U_i^*$, $i = 1, \cdots, m$ such that $\int_{U_i} \mathcal{g}_{U_i}(\cdot, z) P < +\infty$ for each $z \in U_i$. By taking $U_i^*$ slightly smaller if necessary we may assume that $\partial U_i$ is piecewise analytic.
By the Urysohn property for $N$ we can find $\varphi_i \in N$ such that $\text{supp} \varphi_i \subset U_i^*$, $\varphi_i \geq 0$ and $\sum_i \varphi_i | K = 1$. Let $f_i = \varphi_i f \in N$ and note that $f = \sum_i f_i$ on $K$. Denote by $h_i$ the harmonic projection of $f_i$ with respect to $U_i$, i.e. $h_i \in H^P(B(U_i))$ and $0 = f_i | \partial U_i = h_i | \partial U_i f_i | U_i^* \cap \delta = h_i | U_i^* \cap \delta$. Let $G$ be any component of $U_i$. Then $\int_G g_G(\cdot, z)P < +\infty$ for every $z \in G$. Thus by Theorem 2 there is a function $u \in P^cB(G)$ such that $u | \partial G = 0$, $u | G \cap \delta = h_i | G \cap \delta$. Repeating this in each component of $U_i$ we obtain $v_i \in P^cB(U_i)$ with $v_i > 0$, $v_i | \partial U_i = 0$ and $v_i - f_i | U_i^* \cap \delta = 0$. Setting $v_i = 0$ on $R \setminus U_i$ gives a subsolution on $R$.

Let $k_i$ be the least harmonic majorant of $v_i$. Take an exhaustion $\{\Omega_n\}$ of $R$ by regular regions. Denote by $u_{in}$ the function in $P^c(\Omega_n)$ such that $u_{in} - v_i | \partial \Omega_n = 0$. Then we have $0 \leq v_i \leq u_{in} \leq u_{in+1} \leq k_i$. Thus $u_i = \lim_n u_{in}$ is in $PB(R)$ and $0 \leq v_i \leq u_i \leq k_i$. Since $k_i - v_i$ is a potential on $R$ it vanishes on $\delta$ and consequently $k_i - u_i$ also vanishes there. That is, $u_i - v_i | \delta = 0$. The function $u = \sum_i u_i \in PB(R)$ has the property that $u | \delta = f$.

For an arbitrary $f \in C_0(\delta^P)$ there is a sequence $\{f_k\}$ of functions of the sort considered above such that $\|f - f_k\|_P \rightarrow 0$. Then the corresponding sequence of solutions $\{u_k\}$ is a Cauchy sequence with respect to $\|\cdot\|_R$ by Corollary 3. Thus there is a $u \in PB(R)$ such that $\|u - u_k\|_R \rightarrow 0$. Let $u | \delta^P = g$ and for a given $\varepsilon > 0$ take $k$ such that $\|u - u_k\|_R < \varepsilon$ and $\|f - f_k\|_P < \varepsilon$. Then the denseness of $R$ in $R^*$ gives $\|g - f\|_P < \varepsilon$. This means $g = f$ on $\delta^P$.

Thus far we have shown $C_0(\delta^P) \subset PB(R)|\delta^P$. For the proof of the reverse inclusion take $u \in PB(R)$ which without loss of generality can be assumed to be nonnegative. By Theorem 3, $u | \delta \setminus \delta^P = 0$. Set $K_n = \{p \in \delta \mid u(p) > 1/n\}$. $K_n$ is a closed and hence compact subset of $\delta$. Since $K_n \subset \delta^P$ it is compact in $\delta^P$. Now choose $\varphi_n \in N$ such that $\varphi_n K_n = 1$, $\varphi_n | \delta \setminus K_{n+1} = 0$ and $0 \leq \varphi_n \leq 1$. Then $\varphi_n u \in C_0(\delta^P)$ and $\|u - \varphi_n u\|_P \leq 1/n$.

Corollary 3 implies that the mapping of $PB(R)$ onto $C_0(\delta^P)$ obtained by restriction to $\delta^P$ preserves order and sup norm.

A slightly more tractable description of $\delta^P$ can be derived from this theorem.

**Corollary.** $\delta^P = \{p \in \delta \mid p \text{ has a nbd } U^* \text{ in } R^* \text{ such that } \int Ug_R(\cdot, z)P < +\infty \text{ for each } z \in U\}$.

Since $g_R(\cdot, z) \geq g_U(\cdot, z)$ for $z \in U$ we need only show that for each $p \in \delta^P$ there is a neighborhood $U^*$ of $p$ such that $\int Ug_R(\cdot, z)P < +\infty$. But by the theorem there is a function $u \in PB(R)$ such that $u(p) = 2$. Set $U^* = \{q \in R^* \mid u(q) > 1\}$. By Theorem 2, $\tau_R u$ exists and, in particular, $+\infty > \tau_R u(z) \geq \int Ug_R(\cdot, z)P$, for $z \in U$. 
5. The main result is as follows:

**Theorem.** Suppose $P$ and $Q$ are nonnegative $C^1$ densities on a hyperbolic Riemann surface $R$. There is an isomorphism $S$ between $PB(R)$ and $QB(R)$ such that $|u - Su|$ is bounded by a potential on $R$ if and only if $\delta^P = \delta^Q$.

If $\delta^P = \delta^Q$, then both $PB(R)$ and $QB(R)$ are isomorphic to $C_0(\delta^P)$ by restriction to $\delta^P$. Thus define $S: PB(R) \rightarrow QB(R)$ by $u - Su \ |\delta^P = 0$. In order to show that $|u - Su|$ is bounded by a potential, express $u$ as $u = u_1 - u_2$ with $u_i \in PB(R)$ and $u_i > 0$. Note that $u |\delta^P = u_1 |\delta^P - u_2 |\delta^P$ which implies that $Su = Su_1 - Su_2, Su_i > 0$. Take $h_i \in HB(R)$ with $h_i |\delta = u_i |\delta, \ i = 1, 2$. Since $u_i - h_i$ and $Su_i - h_i$ are bounded subharmonic functions on $R$ which vanish on $\delta$ we have $u_i - h_i \leq 0$, $Su_i - h_i \leq 0$. Thus $|u - Su|$ is bounded by the potential $(h_1 - u_1) + (h_1 - Su_1) + (h_2 - u_2) + (h_2 - Su_2)$.

Conversely, suppose an isomorphism $S$ as described in the theorem exists. Then $|u - Su| \ |\delta = 0$ for each $u \in PB(R)$. If $p \in \delta^P$, then by Theorem 4 we can find $u \in PB(R)$ with $u(p) = 1$. This implies that $Su(p) = 1$ and in view of Theorem 3 we conclude that $p \in \delta$, i.e. $\delta^P \subseteq \delta^Q$. By symmetry we obtain $\delta^P = \delta^Q$.

The assumption on $|u - Su|$ implies that $S$ preserves the behavior of functions on $\delta^P$. In view of Theorem 4 this means that $S$ commutes the lattice operations. If the assumption on $|u - Su|$ is replaced by the assumption that $S$ is a vector lattice isomorphism, then by the Kakutani theorem we see that $\delta^P$ and $\delta^Q$ are homeomorphic.


**Corollary.** If $P$ and $Q$ are $C^1$ densities on a hyperbolic Riemann surface $R$ such that $c^{-1}P \leq Q \leq cP$ outside some compact subset and for some constant $c$, then $PB(R)$ and $QB(R)$ are isomorphic.

**Corollary.** If $P$ and $Q$ are $C^1$ densities on a hyperbolic Riemann surface $R$ and $\int_R |P - Q| < +\infty$, then $PB(R)$ and $QB(R)$ are isomorphic.

In the first case it is clear that the hypothesis implies that $\delta^P = \delta^Q$. In the second case note that $g_R(\cdot, z) |\delta = 0$ and hence there is a neighborhood $V^*$ of $\delta$ in $R^*$ with $g_R(\cdot, z) |V^* \leq 1$. Thus $\int_V g_R(\cdot, z) |P - Q| < +\infty$. By the Harnack inequality this is also valid if $z$ is allowed to vary and the conclusion $\delta^P = \delta^Q$ now follows.

Actually the corollaries followed from the slightly weaker hypotheses $c^{-1}P \leq Q \leq cP$ in $V, \int_V |P - Q| < +\infty$, where $V^*$ is a neighborhood of $\delta$ in $R^*$.

Denote by $w$ the greatest solution of $\Delta u = Pu$ on $R$ which is less than 1 on $R$. The following result is due to Lahtinen [2] and Loeb and Walsh [3].
Corollary. HB(R) and PB(R) are isomorphic vector lattices if and only if 1 is the least harmonic majorant of \( w \) on \( R \).

Let \( h \) be the least harmonic majorant of \( w \). Then \( h - w \) is a potential and hence vanishes on \( \delta \). Therefore, \( h \) is the constant 1 if and only if \( w|\delta = 1 \). This in turn is equivalent to \( \delta^P = \delta \) which is equivalent to \( PB(R) \) being isomorphic to \( HB(R) \).

6. Denote by \( PBE(R) \) (resp. \( PBD(R) \)) the subspace of \( PB(R) \) such that \( E(u) = \int_R du \wedge \ast du + u^2 P < + \infty \) (resp. \( D(u) = \int_R du \wedge \ast du < + \infty \)). Denote by \( \Delta \) the Royden harmonic boundary of \( R, R^* \) the corresponding compactification and define

\[
\Delta^P = \left\{ p \in \Delta | p \text{ has a nbd } U^* \text{ in } R^* \text{ with } \int_U P < + \infty \right\},
\]

\[
\Delta_p = \left\{ p \in \Delta | p \text{ has a nbd } U^* \text{ in } R^* \text{ with } \int \int_{U \times U} g_R(x, y) P(x) P(y) < + \infty \right\}.
\]

These definitions lead to criteria for isomorphisms between the closures with respect to the sup norm of the bounded energy finite or bounded Dirichlet finite solutions.

Theorem. Suppose \( P \) and \( Q \) are nonnegative \( C^1 \) densities on \( R \). There is an isomorphism \( S \) between \( PBE(R) \) and \( QBE(R) \) (resp. \( PBD(R) \) and \( QBD(R) \)) such that \( |u - Su| \) is bounded by a potential on \( R \) if and only if \( \Delta^P = \Delta^Q \) (resp. \( \Delta_p = \Delta_p \)).

The proof is analogous to that of Theorem 5 and therefore we only mention some differences. The operator \( T_G \) defined in §2 also maps the spaces \( P^c BE(G) \) and \( P^c BD(G) \) into \( H^c BD(G) \). If \( \int_G P < + \infty \), then

\[
T_G(P^c BE(G)) = H^c BD(G)
\]

(cf. [1]) and if \( \int_G \int_G g_G(x, y) P(x) P(y) < + \infty \), then

\[
T_G(P^c BD(G)) = H^c BD(G)
\]

(cf. [5]). This is the motivation for the choice of \( \Delta^P \) and \( \Delta_p \). The proper analogue of Theorem 4 is that the closure of \( PBE(R) \) (resp. \( PBD(R) \)) with respect to the sup norm restricted to \( \Delta^P \) (resp. \( \Delta_p \)) is the space \( C_0(\Delta^P) \) (resp. \( C_0(\Delta_p) \)) but this causes no complications. In the proof some complications do occur because of the need to establish the convergence of sequences of functions in the \( D \) or \( E \) norm.

Added in proof. M. Nakai (Banach spaces of bounded solutions of \( \Delta u = Pu \) on hyperbolic Riemann surfaces, Nagoya Math. J. 53 (1974), 141–155) has
simultaneously discovered Theorem 5 and also has given a more detailed analysis of its consequences.

REFERENCES


DEPARTMENT OF MATHEMATICS, PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PENNSYLVANIA 16802