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THE THEORY OF COUNTABLE ANALYTICAL SETS[1]

BY

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ABSTRACT. The purpose of this paper is the study of the structure of countable sets in the various levels of the analytical hierarchy of sets of reals. It is first shown that, assuming projective determinacy, there is for each odd n a largest countable Π^1_n set of reals, $\mathcal{C}_n$ (this is also true for n even, replacing Π^1_n by Σ^1_n and has been established earlier by Solovay for $n = 2$ and by Moschovakis and the author for all even $n > 2$). The internal structure of the sets $\mathcal{C}_n$ is then investigated in detail, the point of departure being the fact that each $\mathcal{C}_n$ is a set of Δ^1_n-degrees, wellordered under their usual partial ordering. Finally, a number of applications of the preceding theory is presented, covering a variety of topics such as specification of bases, ω-models of analysis, higher-level analogs of the constructible universe, inductive definability, etc.

It is a classical theorem of effective descriptive set theory that a Σ^1_1 thin (i.e. containing no perfect set) subset of the continuum is countable and in fact contains only Δ^1_1 reals. As a consequence, among the countable Σ^1_1 sets of reals there is no largest one. Solovay [41] showed that every thin Σ^1_2 set contains only constructible reals and therefore (in sharp contrast with the previous case), assuming a measurable cardinal exists, there is a largest countable Σ^1_2 set of reals, namely the set of constructible ones. In [19] Moschovakis and the author extended Solovay's theorem to all even levels of the analytical hierarchy. It was proved there that, assuming projective determinacy (PD), there is a largest countable Σ^1_{2n} set for all $n \geqslant 1$, which we denote by C_{2n}. The first main result we prove in this paper (see §1) is the existence of a largest countable Π^1_{2n+1} set (which we denote by C_{2n+1}) for all $n \geqslant 0$, assuming PD again. For $n = 0$ our proof shows, in ZF + DC only, the existence of a largest *thin* Π^1_1 set of reals C_1, a fact which was also independently discovered by D. Guaspari [12] and G. E. Sacks [38]. (To complete the picture, we remark here that it was shown in [17], using PD, that no largest countable Σ^1_{2n+1} or Π^1_{2n} sets exist.)

Once the existence of the largest countable sets C_n is established the rest of §1 is devoted to the study of their internal structure. We show here that C_n is actually a set of Δ^1_n-degrees which is well ordered under the usual ordering of Δ^1_n-degrees. Let us denote by $\{d^n_\xi\}_{\xi<\rho_n}$ the increasing hierarchy of the Δ^1_n-

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degrees of (elements of) C_n. As we (hope to) illustrate in this paper the hierarchy $\{d_\xi^n\}_{\xi<\rho_n}$ plays a significant role in the general structure theory of the analytical sets of the nth level. It provides a common frame of reference and a starting point for investigations in a variety of diverse topics such as specification of bases, ω-models of analysis, higher-level analogs of L, inductive definability, etc.

After the general theory is discussed, we examine in more detail in §2 the structure of C_1, the largest thin Π_1^1 set (part of the results here have been independently discovered by D. Guaspari [12] and G. E. Sacks [38]). We show, among other things, that $\{d_\xi^1\}_{\xi<\rho_1=\aleph_1^L}$ coincides with the hierarchy of the hyperdegrees of the Boolos-Putnam complete sets of integers. A natural quasi-hierarchy of hyperdegrees is then defined, which extends (through *all* the constructible reals) the natural hierarchy of hyperdegrees and it is shown to be exactly $\{d_\xi^1\}_{\xi<\aleph_1^L}$.

§ 3 accounts for the structural differences between C_{2n+1} with $n > 0$ and C_1. The basic reason for them is the fact that the notion of "well ordering on ω" is Δ_{2n+1}^1, if $n > 0$, but not Δ_1^1. Martin and Solovay [29] made a remarkable use of this fact to produce a counterexample to the well-known conjecture that the Kleene basis theorem for Σ_1^1 generalizes, under PD, in a straightforward fashion to all Σ_{2n+1}^1, with $n > 0$. The Martin-Solovay discovery revealed for the first time an important structural difference between the first and the higher odd levels of the analytical hierarchy. In the first part of § 3 we deal with reflecting pointclasses and we unearth more such structural differences. This eventually leads us to the subject of Q-theory (see §3B), which was developed by Martin and Solovay and independently by the author. And we end § 3 with a brief summary of the connection between countable analytical sets and higher level analogs of L.

Finally, in § 4, questions concerning countable analytical sets in inner models like L are discussed and various consistency and independence results are obtained.

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0. Preliminaries.

0A. Let $\omega = \{0, 1, 2, \ldots\}$ be the set of natural numbers and $R = {}^\omega\omega$ the set of all functions from ω to ω or (for simplicity) *reals*. Letters $i, j, k, l, m, \ldots$ denote elements of ω and $\alpha, \beta, \gamma, \delta, \ldots$ reals. Subsets of ω are many times below identified with their characteristic functions. We study subsets of the *product spaces*

$$X = X_1 \times X_2 \times \ldots \times X_k,$$

where X_i is ω or $\mathcal{R}$. We call such subsets *pointsets.* Sometimes we think of them as relations and we write interchangeably

$$x \in A \;\Leftrightarrow\; A(x).$$

A *pointclass* is a class of pointsets, usually in all product spaces. We shall be concerned primarily in this paper with the *analytical* pointclasses Σ^1_n, Π^1_n, Δ^1_n and their corresponding *projective* pointclasses $\boldsymbol{\Sigma}^1_n$, $\boldsymbol{\Pi}^1_n$, $\boldsymbol{\Delta}^1_n$. For information about them we refer to [37], [39] and the introduction of [30] from which we also draw some of our nonstandard notation.

If Γ is a pointclass we put

$$\check{\Gamma} = \{\mathcal{X} - A : A \in \Gamma, A \subseteq \mathcal{X}\} = \text{the } \textit{dual} \text{ of } \Gamma,$$

and $\Delta = \Gamma \cap \check{\Gamma}$, while for each $a \in \mathcal{R}$,

$$\Gamma(a) = \{A\colon \text{ For some } B \in \Gamma,\ A(x) \Leftrightarrow B(x, a)\}$$

and $\Delta(a) = \Gamma(a) \cap \check{\Gamma}(a)$. We also let

$$\boldsymbol{\Gamma} = \bigcup_{a \in \mathcal{R}} \Gamma(a) \quad \text{and} \quad \boldsymbol{\Delta} = \boldsymbol{\Gamma} \cap \check{\boldsymbol{\Gamma}}.$$

If $A \subseteq \mathcal{X} \times \omega$, put

$$\exists^\omega A = \{x\colon \exists n A(x, a)\}, \qquad \forall^\omega A = \{x\colon \forall n A(x, n)\}$$

and if $A \subseteq \mathcal{X} \times \mathcal{R}$, put

$$\exists^{\mathcal{R}} A = \{x\colon \exists a A(x, a)\}, \qquad \forall^{\mathcal{R}} A = \{x\colon \forall a A(x, a)\}.$$

If Φ is any operation on pointsets (like e.g. $\exists^\omega$ above), let $\Phi\Gamma = \{\Phi A\colon A \in \Gamma\}$ and say that Γ is *closed under* Φ if $\Phi\Gamma \subseteq \Gamma$.

Let Γ be a pointclass, $\mathcal{X}$ a product space ($\mathcal{X} = \omega$ or $\mathcal{R}$ is enough for this definition). We say that Γ is *$\mathcal{X}$-parametrized* if for any product space $\mathcal{Y}$ there is a $G \in \Gamma$, $G \subseteq \mathcal{Y} \times \mathcal{X}$ so that if $G_x = \{y\colon (y, x) \in G\}$ then

$$\{G_x\colon x \in \mathcal{X}\} = \{A \subseteq \mathcal{Y}\colon A \in \Gamma\}.$$

In this case we say that G is *$\mathcal{X}$-universal for* Γ *subsets* of $\mathcal{Y}$. If $A = G_x$, x is called a *code* of A.

A pointclass Γ is *adequate* if it contains all the recursive sets (in all product spaces) and is closed under conjunction, disjunction, number quantification of both kinds and substitution by total recursive functions. It is clear that all analytical pointclasses are adequate and all Σ^1_n, Π^1_n are ω-parametrized while all projective pointclasses are adequate and all $\boldsymbol{\Sigma}^1_n$, $\boldsymbol{\Pi}^1_n$ are $\mathcal{R}$-parametrized. In general if Γ is adequate and ω-parametrized, $\boldsymbol{\Gamma}$ is adequate and $\mathcal{R}$-parametrized Moreover the $\mathcal{R}$-universal sets for $\boldsymbol{\Gamma}$ can be taken to be in Γ.

Whenever we work with a parametrized pointclass which has also some closure properties we shall always assume that universal sets for this class are chosen so that the pointclass is uniformly closed (e.g. if Γ is assumed closed under $A \cap B$, a fixed recursive function will compute a code for $A \cap B$ from codes of A, B).

0B. Let A be a pointset. A *norm* on A is a map $\phi\colon A \twoheadrightarrow \lambda$ from A

onto an ordinal λ, the *length* of ϕ (notation: length $(\phi) = |\phi| = \lambda$). With each such norm we associate the *prewellordering* (i.e. the reflexive, transitive, connected, wellfounded relation) $\leqslant^{\phi}$ on A, defined by

$$x \leqslant^{\phi} y \Leftrightarrow \phi(x) \leqslant \phi(y).$$

Put also

$$x <^{\phi} y \Leftrightarrow \phi(x) < \phi(y).$$

Conversely each prewellordering $\precsim$ with field A gives rise to a unique norm $\phi: A \twoheadrightarrow \lambda$ such that $\precsim \, = \, \leqslant^{\phi}$; we then call λ the length of $\precsim$.

If Γ is a pointclass and ϕ is a norm on A, we say that ϕ is a Γ-*norm* if there exist relations $\leqslant^{\phi}_{\Gamma}, \leqslant^{\phi}_{\check{\Gamma}}$ in $\Gamma, \check{\Gamma}$ respectively so that

$$y \in A \Rightarrow \forall x \{[x \in A \; \& \; \phi(x) \leqslant \phi(y)] \Leftrightarrow x \leqslant^{\phi}_{\Gamma} y \Leftrightarrow x \leqslant^{\phi}_{\check{\Gamma}} y\}.$$

Notice that this is equivalent to saying that the following two relations are in Γ:

$$x \leqslant^{*}_{\phi} y \Leftrightarrow x \in A \; \& \; [y \notin A \vee \phi(x) \leqslant \phi(y)],$$
$$x <^{*}_{\phi} y \Leftrightarrow x \in A \; \& \; [y \notin A \vee \phi(x) < \phi(y)].$$

(The meaning of $\leqslant^{*}_{\phi}, <^{*}_{\phi}$ becomes clear if one defines ϕ to be ∞ outside A.) If ϕ is a Γ-norm on A, there exist relations $<^{\phi}_{\Gamma}, <^{\phi}_{\check{\Gamma}}$ in $\Gamma, \check{\Gamma}$ respectively so that

$$y \in A \Rightarrow \forall x \{[x \in A \; \& \; \phi(x) < \phi(y)] \Leftrightarrow x <^{\phi}_{\Gamma} \Leftrightarrow x <^{\phi}_{\check{\Gamma}} y\}.$$

Let now

Prewellordering $(\Gamma) \Leftrightarrow$ *Every pointset in* Γ *admits a* Γ-*norm.*

The prewellordering property was formulated (in an equivalent form) by Moschovakis; see [30] for further details. Prewellordering (Π^1_1) can be proved by classical arguments and Prewellordering (Σ^1_2) was proved by Moschovakis; see [37]. Finally Martin [27] and independently Moschovakis [4] proved Prewellordering (Π^1_{2n+1}), Prewellordering (Σ^1_{2n+2}) for any n, assuming projective determinacy (PD) if $n > 0$. Here PD is the hypothesis that every projective set is determined. In general Determinacy (Γ) abbreviates: Every $A \in \Gamma$ is determined. For information about games, determinacy, etc., see [35], [30] or the recent survey article [10].

If a pointclass has the prewellordering property and satisfies some mild closure conditions (which is always the case with the pointclasses we are interested in) then Γ has several other nice properties and a rich structure theory. We refer the reader to [4], [33] or [18] for more on the prewellordering results and techniques. We shall use most of them without explicit mentioning in the sequel.

0C. Let A be a pointset. A *scale* on A is a sequence $\{\phi_n\}_{n \in \omega}$ of norms on A with the following *limit property*:

If $x_i \in A$, for all i, if $\lim_{i \to \infty} x_i = x$ and if for each n and all large enough i, $\phi_n(x_i) = \lambda_n$, then $x \in A$ and for each n, $\phi_n(x) \leqslant \lambda_n$.

If Γ is a pointclass and $\{\phi_n\}_{n \in \omega}$ is a scale on A we say that $\{\phi_n\}_{n \in \omega}$

is a Γ-*scale* if there exist relations $S_\Gamma, S_{\check{\Gamma}}$ in $\Gamma, \check{\Gamma}$ respectively so that

$$y \in A \Rightarrow \forall x\{[x \in A \ \& \ \phi_n(x) \leqslant \phi_n(y)] \Leftrightarrow S_\Gamma(n, x, y) \leftrightarrow S_{\check{\Gamma}}(n, x, y)\}.$$

Put

$$\textit{Scale}\,(\Gamma) \Leftrightarrow \textit{Every pointset in } \Gamma \textit{ admits a } \Gamma\textit{-scale}.$$

The notions of scale and scale property were formulated by Moschovakis in [31]. He proved there that Scale (Π^1_{2n+1}), Scale (Σ^1_{2n+2}) hold for any n, assuming PD when $n > 0$. His original application was in proving uniformization (see [31]). Since then many other applications of scales have been found and various "scale techniques" have been developed, see e.g. [20] or [18].

0D. We shall often talk about trees on $\omega \times \lambda$, where λ is an ordinal. A *tree* on a set X is a set T of finite sequences from X, closed under subsequences i.e.

$$(x_0 \ldots x_n) \in T \ \& \ k \leqslant n \Rightarrow (x_0 \ldots x_k) \in T.$$

The empty sequence is thus always a member of a nonempty tree. A *branch* of T is a sequence $f \in {}^\omega X$ such that for every n, $(f(0), \ldots, f(n)) \in T$. The set of all branches of T is denoted by

$$[T] = \{f \in {}^\omega X: \ \forall n(f(0), \ldots, f(n)) \in T\}.$$

A tree T is *wellfounded* if $[T] = \varnothing$ i.e. T has no branches.(2)

If T is a tree on $\omega \times \lambda$, T contains elements of the form $((k_0, \xi_0), \ldots, (k_n, \xi_n))$, where $k_i \in \omega$, $\xi_i < \lambda$. A branch of such a tree is a sequence $f \in {}^\omega(\omega \times \lambda)$ which for convenience will be represented by the unique pair (a, g) such that $f(n) = (a(n), g(n))$. The *first projection* of $[T]$, in symbols $p[T]$ is

$$p[T] = \{a: \exists g(a, g) \in [T]\}.$$

Thus $p[T] \subseteq R$. If $a \in R$, let

$$T(a) = \{(\xi_0, \ldots, \xi_n): ((a(0), \xi_0), \ldots, (a(n), \xi_n)) \in T\}.$$

Then for each $a \in R$, $T(a)$ is a tree on λ and $a \in p[T] \Leftrightarrow T(a)$ is not wellfounded.

If $\{\phi_n\}_{n\in\omega}$ is a scale on a set of reals A we define its *associated tree* T by

$$T = \{((a(0), \phi_0(a)), \ldots, (a(n), \phi_n(a))): a \in A\}.$$

Then it is easy to see that $A = p[T]$.

0E. Let ω have the discrete topology. We give $R = {}^\omega\omega$ the natural product topology. This is generated by the basic neighborhoods

$$N_s = \{a: \bar{a}(lhs) = s\},$$

where s is a finite sequence from ω. We give the product spaces X the product topology. It is then well known that every product space X which contains at least one copy of R is naturally homeomorphic to R via a recursive homeomorphism.

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(2) The *length* of a wellfounded tree T is defined as usual and is denoted by $|T|$.

A *perfect* subset of R is a nonempty closed set with no isolated points. Every perfect set has cardinality $2^{\aleph_0}$. A set of reals is called *thin* if it contains no perfect subset. We shall be mainly studying here thin and countable analytical sets.

It is a classical result that every uncountable Σ_1^1 set contains a perfect subset. Thus for Σ_1^1 sets the notions of thin and countable coincide. Assuming PD, M. Davis [9] proved that every uncountable projective set contains a perfect subset. Thus under PD "countable" and "thin" coincide for projective sets. (Nevertheless in L there are uncountable thin Π_1^1 sets.)

There are much finer effective versions of the above mentioned results. For example, it is well known that every countable Σ_1^1 set contains only Δ_1^1 reals (see e.g. [25], [14]). This result has been recently generalized (unpublished) to all odd levels of the analytical hierarchy by D. A. Martin who proved

THEOREM (D. A. MARTIN). *Assume* PD *if* $n > 1$. *If* $n \geqslant 1$ *is odd then every countable* Σ_n^1 *set of reals contains only* Δ_n^1 *reals.*

Earlier Moschovakis [32] has proved the weaker version of this theorem resulting from replacing Σ_n^1 by Δ_n^1.

Occasionally we shall talk about Lebesgue measure on ${}^\omega 2$. We mean then, as usual, the product measure on ${}^\omega 2$, where $2 = \{0, 1\}$ has the measure $\nu(\{0\}) = \nu(\{1\}) = ½$. For more see e.g. [17].

0F. Our set theoretic notation and terminology will be standard, when possible. In particular $\xi, \eta, \theta, \lambda, \ldots$ denote ordinals, $P(X)$ denotes the power set of X and when M is a model of set theory and τ a term, τ^M denotes the interpretation of τ in M. Finally card X denotes the cardinality of X.

The notions of admissible set and ordinal appear frequently in § 2. For information about them see [5], [21], [2] or [33]. If A is a set of ordinals, an ordinal $\lambda > \omega$ is called *A-admissible* if $\langle L_\lambda[A], \epsilon, A \cap L_\lambda[A]\rangle$ is an admissible structure. Sacks [38] proved that if $\lambda > \omega$ is a countable admissible ordinal then for some $A \subseteq \omega$, λ is the first A-admissible ordinal $> \omega$. The following (unpublished) theorem of Jensen extends substantially Sacks' theorem and will be needed in § 2.(3)

THEOREM (R. JENSEN). *Assume* $B \subseteq \omega$, ν *is a countable ordinal and* $\{a_\theta\}_{\theta<\nu}$ *a sequence of countable ordinals such that for each* $\theta < \nu$, a_θ *is* $(\{a_\xi\}_{\xi<\theta}, B)$*-admissible. Then there is a set* $A \subseteq \omega$ *such that, for all* $\theta < \nu$,

(i) a_θ *is the* θ*th* A*-admissible ordinal* $> \omega$,

(ii) a_θ *is countable in* $L_{a_{\theta+1}}[A]$ (*if* $\theta + 1 < \nu$),

(iii) B *is hyperarithmetic in* A.

(3) We wish to thank L. Harrington for explaining to us Jensen's theorem and its ramifications.

Moreover such an A can be found in any admissible set containing B, $\{a_\theta\}_{\theta<\nu}$, in which ν and all a_θ are countable.

0G. The whole discussion in this paper takes place in ZF + DC, Zermelo-Fraenkel set theory with dependent choices:

(DC) $\quad \forall u \in x\, \exists v (u, v) \in r \Rightarrow \exists f \forall n (f(n), f(n+1)) \in r.$

Every additional hypothesis is stated explicitly. For simplicity we have used full PD as a general hypothesis for many of our theorems. One can usually trace easily how much PD was needed in each particular proof.

1. **General theory of countable analytical sets.** We prove in this section a number of results about countable analytical sets which establish their main structural properties. Among these are the existence of largest countable Π^1_n, if n is odd, and Σ^1_n, if n is even, sets and the fact that the Δ^1_n-degrees of elements of these sets are wellordered.

1A. *Existence of largest countable sets in certain pointclasses.* We shall state and prove the main result here in an abstract form, partly in order to illustrate better the basic ideas, but also because we want it to be applicable in a wider range of situations.

DEFINITION. Let Γ be a pointclass. Let J be a collection of pointsets such that

$$A \in J \,\&\, B \subseteq A \Rightarrow B \in J.$$

We call J Γ-*additive* iff for any sequence $\{A_\xi\}_{\xi<\theta}$ for which the *associated prewellordering*

$$x \leqslant y \Leftrightarrow x, y \in \bigcup_{\xi<\theta} A_\xi \,\&\, \text{least } \xi[x \in A_\xi] \leqslant \text{least } \xi[y \in A_\xi]$$

is in Γ, we have

$$\forall \xi < \theta [A_\xi \in J] \Rightarrow \bigcup_{\xi<\theta} A_\xi \in J.$$

We think of the elements of J as "small" sets. Typical examples of interesting J's are the σ-ideals of null or meager sets of reals and closer to our subject the σ-ideal of countable sets and the class of thin sets. It was noticed in [17] that, assuming PD if $n > 1$, the σ-ideals of null and meager sets are Γ-additive, where $\Gamma = \Sigma^1_n$ or Π^1_n. We prove a similar result here for thin sets. (The reader should probably recall here that determinacy implies: thin = countable.)

THEOREM (1A-1). *Assume* PD *if* $n > 1$. *For all* $n \geqslant 1$, *the class of thin sets is* Σ^1_n *and* Π^1_n*-additive.*

PROOF. Assume not, towards a contradiction. Let Γ be Σ^1_n or Π^1_n below. Pick a sequence $\{A_\xi\}_{\xi<\theta}$ of thin sets with least possible θ so that the associated prewellordering $\leqslant$ is in Γ, but $\bigcup_{\xi<\theta} A_\xi$ is not thin. Let then $P \subseteq \bigcup_{\xi<\theta} A_\xi$ be a perfect set. Find g: ${}^\omega 2 \to P$ continuous and 1-1. Put

$$a \leqslant' \beta \Leftrightarrow a, \beta \in {}^{\omega}2 \;\&\; g(a) \leqslant g(\beta).$$

Then $\leqslant'$ is a prewellordering in Γ and any initial segment of it is thin by the minimality of θ. Then $\leqslant' \subseteq {}^{\omega}2 \times {}^{\omega}2$ has measure 0 by Fubini's theorem and thus for almost all $a \in {}^{\omega}2$ $\{\beta: a \leqslant' \beta\}$ has measure 0. But then ${}^{\omega}2$ has measure 0. Contradiction. □

The next theorem is a basic fact concerning the existence of largest "small" sets in certain pointclasses. One of the key ideas used in its proof traces to [19].

THEOREM (1A-2). *Let* Γ *be an adequate, ω-parametrized pointclass having the prewellordering property. Let* $\mathcal{J}$ *be a collection of pointsets closed under subsets and assume* $\mathcal{J}$ *is* Γ*-additive. Finally, for some* $G \subseteq \mathcal{R} \times \mathcal{R}$, $G \in \Gamma$ *which is* $\mathcal{R}$*-universal for* Γ *subsets of* $\mathcal{R}$, *assume that the relation*

$$M_{\mathcal{J}}(a) \Leftrightarrow \mathcal{R} - G_a \in \mathcal{J}$$

is in Γ. *Then there exists a largest set of reals in* $\Gamma \cap \mathcal{J}$.

PROOF. Let $\mathcal{P} \subseteq \mathcal{R} \times \omega$, $\mathcal{P} \in \Gamma$ be ω-universal for Γ subsets of $\mathcal{R}$. Put

$$\mathcal{T}(a, n) \Leftrightarrow \mathcal{P}(a, n) \;\&\; \{\beta: (\beta, n) \in \mathcal{P} \;\&\; \phi(\beta, n) \leqslant \phi(a, n)\} \in \mathcal{J},$$

where ϕ is a Γ-norm on $\mathcal{P}$. Our hypotheses clearly imply that $\mathcal{T} \in \Gamma$. Let $\mathcal{C} = \{a: \exists n \mathcal{T}(a, n)\}$. Again $\mathcal{C} \in \Gamma$ and if $A \in \Gamma \cap \mathcal{J}$ then $A \subseteq \mathcal{C}$. To complete the proof we show that $\mathcal{C} \in \mathcal{J}$.

Let ψ be a Γ-norm on $\mathcal{T}$. Let $\sup\{\psi(a, n): (a, n) \in \mathcal{T}\} = \theta$. For each $\xi < \theta$, let $A_\xi = \{a: \exists n(\mathcal{T}(a, n) \& \psi(a, n) = \xi)\}$. Clearly $\mathcal{C} = \bigcup_{\xi<\theta} A_\xi$. The prewellordering associated with $\{A_\xi\}_{\xi<\theta}$ is given by

$$\begin{aligned} a \leqslant \beta &\Leftrightarrow a, \beta \in \mathcal{C} \;\&\; \text{least } \xi[a \in A_\xi] \leqslant \text{least } \xi[\beta \in A_\xi] \\ &\Leftrightarrow a, \beta \in \mathcal{C} \;\&\; \min\{\psi(a, n): (a, n) \in \mathcal{J}\} \leqslant \min\{\psi(\beta, m): (\beta, m) \in \mathcal{T}\} \\ &\Leftrightarrow a, \beta \in \mathcal{C} \;\&\; \exists n \forall m((a, n) \leqslant^*_\psi (\beta, m)). \end{aligned}$$

Thus $\leqslant \in \Gamma$. To prove that $\mathcal{C} \in \Gamma$ it is therefore enough to show that for each $\xi < \theta, A_\xi \in \mathcal{J}$. Fix ξ. Then $A_\xi = \bigcup_{n<\omega} \{a: (a, n) \in \mathcal{T} \;\&\; \psi(a, n) = \xi\}$. For any $n < \omega$, $\{a: \mathcal{T}(a, n)\} \in \mathcal{J}$ as we can easily see using the Γ-additivity of $\mathcal{J}$. Thus for each $n < \omega$, $\{a: \mathcal{T}(a, n) \;\&\; \psi(a, n) = \xi\} = A_{\xi,n} \in \mathcal{J}$. To finish the proof we now check that the prewellordering $\leqslant'$ associated with $\{A_{\xi,n}\}_{n<\omega}$ is in Γ. In fact

$$\begin{aligned} a \leqslant' \beta \Leftrightarrow a, \beta \in A_\xi \;\&\; \forall m[(\beta, m) \in \mathcal{T} \;\&\; \psi(\beta, m) = \xi \Rightarrow \\ (\exists n \leqslant m)((a, n) \in \mathcal{T} \;\&\; \psi(a, n) = \xi)] \end{aligned}$$

so that actually $\leqslant' \in \Gamma \cap \breve{\Gamma} = \Delta$. □

For the next corollary assume that $A \in \mathcal{J} \;\&\; n \in \omega \Rightarrow A \times \{n\} \in \mathcal{J}$.

COROLLARY (1A-3). *Under the hypotheses of* (1A-2) *the pointclass* $\Gamma \cap \mathcal{J}$ *is ω-parametrized.*

PROOF. It is clear that the relation $\mathcal{T} \subseteq \mathcal{R} \times \omega$ defined in the proof of

(1A-2) provides a parametrization for the sets of reals in $\Gamma \cap \mathcal{J}$. To show that $\mathcal{T} \in \mathcal{J}$ let ψ, θ be as in (1A-2). If for $\xi < \theta$, $\mathcal{T}_\xi = \{(a, n): (a, n) \in \mathcal{T} \,\&\, \psi(a, n) = \xi\}$ then $\mathcal{T}_\xi \in \mathcal{J}$ and $\mathcal{T} = \bigcup_{\xi<\theta} \mathcal{T}_\xi$. But the prewellordering associated with $\{\mathcal{T}_\xi\}_{\xi<\theta}$ is clearly in Γ, so $\mathcal{T} \in \mathcal{J}$ □

REMARKS. (1) It is clear that in (1A-2) we could replace the hypothesis "$M_{\mathcal{J}} \in \Gamma$" by the following alternative one: There is a relation $M'_{\mathcal{J}} \in \Gamma$ such that when a "codes" a Δ subset of $R\Delta_a$ then $M'_{\mathcal{J}}(a) \Leftrightarrow \Delta_a \in \mathcal{J}$.

(2) The following simple fact justifies the indirect method of proof of (1A-2).

PROPOSITION. *Let* $\Gamma, \mathcal{J}, G$ *be as in the statement of* (1A-2) *and assume no basic neighbourhood* $N_s = \{a: \bar{a}(lhs) = s\}$ *is in* $\mathcal{J}$. *Then* $S_{\mathcal{J}}(a) \Leftrightarrow G_a \in \mathcal{J}$ *is not in* Γ.

PROOF. Assume $S_{\mathcal{J}} \in \Gamma$ towards a contradiction. Let $s_0 = (0), s_1 = (1, 0), s_2 = (1, 1, 0), s_3 = (1, 1, 1, 0), \ldots$ and let $A \subseteq \omega$ be a set in $\Gamma - \breve{\Gamma}$. Put $B = \{a: \exists k(k \in A \,\&\, a \in N_{s_k})\}$. Then

$$k \in A \Leftrightarrow B \cap N_{s_k} \notin \mathcal{J},$$

so $A \in \breve{\Gamma}$ a contradiction. □

We have seen in (1A-1) that the collection of thin sets is Γ-additive when $\Gamma = \Sigma^1_n$ or Π^1_n, assuming PD if $n > 1$. In order to apply now (1A-2), in the specific cases we are interested in, we have only to estimate the complexity of the predicate $M_{\mathcal{J}}$.

THEOREM (1A-4) [17]. *Assume* PD *if* $n > 1$. *For each* $n \geqslant 1$ *let* $G \subseteq R \times R, G \in \Pi^1_n$ *be* R*-universal for* Π^1_n *subsets of* R. *Then the relation*

$$M^n(a) \Leftrightarrow R - G_a \text{ is thin}$$

is Π^1_n.

The proof of this fact in [17] used measure theoretic ideas. An alternative proof for odd n (which is the most interesting case) can be given as an application of Martin's result (see 0E). We have: $M^n(a) \Leftrightarrow R - G_a$ is thin ($\Leftrightarrow R - G_a$ is countable) $\Leftrightarrow \forall\beta(\beta \notin G_a \Rightarrow \beta \in \Delta^1_n(a))$.

Combining now (1A-2) and (1A-4) we have

THEOREM (1A-5). (1) (*Proved also independently by D. Guaspari* [12], *G. E. Sacks* [38].) *There exists a largest thin* Π^1_1 *set.*

(2) *Assume* PD. *For any* $n > 0$, *there exists a largest countable* Π^1_{2n+1} *set.*

We cannot apply (1A-2) directly to prove the existence of largest countable Σ^1_{2n} sets ($n > 0$). This is because to say that a Δ^1_{2n} set is countable is not a Σ^1_{2n} statement (otherwise by the uniformization theorem to say that a Σ^1_{2n} set is countable would also be a Σ^1_{2n} statement). Nevertheless it is still true that largest countable Σ^1_{2n} sets exist, under PD.

THEOREM (1A-6). (1) (Solovay [41]). *Assume there are only countable many* *constructible reals. Then there exists a largest countable* Σ^1_2 *set, namely the set of constructible reals* $\mathcal{R} \cap L$.

(2) (Kechris-Moschovakis [19]). *Assume* PD. *For each* $n > 0$ *there exists a largest countable* Σ^1_{2n} *set of reals.*

PROOF. (2) Using the uniformization theorem (Moschovakis [31]) notice that the closure of the largest countable Π^1_{2n+1} set under "recursive in" must be the largest countable Σ^1_{2n+2} set. □

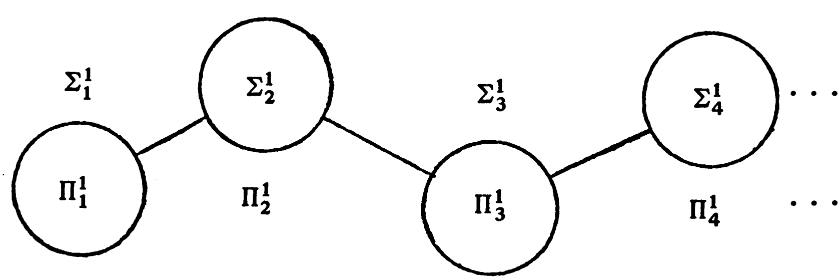
NOTATION. Let C_1 be the largest thin Π^1_1 set of reals, let $C_2 = \mathcal{R} \cap L$ and assuming PD if $n > 2$ let C_n = *largest countable* Σ^1_n *or* Π^1_n *set according as* n *is even or odd.* Thus $C_{2n+2} = \{\alpha: \exists\beta(\beta \in C_{2n+1}\ \&\ \alpha$ recursive in $\beta)\}$.

The reader might now wonder what happens with countable Σ^1_{2n+1} and Π^1_{2n} sets. Here is an answer.

THEOREM (1A-7) [19]. *Assume* PD *if* $n > 1$. *For each* $n \geqslant 0$ *there is no largest countable* Σ^1_{2n+1} *or* Π^1_{2n} *set.*

The proof for Σ^1_{2n+1} in [17] used again measure theoretic ideas (while the proof for Π^1_{2n} is a trivial consequence of the basis theorem for Σ^1_{2n}). An obvious alternative proof can again be based on Martin's theorem (see 0E). Martin's result seems to answer any reasonable question about the structure of countable Σ^1_{2n+1} sets. There is still an interesting open problem about countable Π^1_{2n} sets, namely: Does every countable Π^1_{2n} set contain a Π^1_{2n} singleton (assuming any reasonable hypotheses)? This problem has its origins in Tanaka's result [43] that every countable arithmetic set contains an arithmetic singleton.

And we conclude this subsection by drawing the following picture which summarizes its contents. A circled pointclass has a largest countable set while an uncircled one does not (PD is assumed again).



1B. *The wellordering of the* Δ^1_n*-degrees in* C_n. We now study the internal structure of the C_n's. We need first a few definitions.

DEFINITION. If $\alpha, \beta \in \mathcal{R}$ let

$$\alpha \leqslant_n \beta \Leftrightarrow \alpha \in \Delta^1_n(\beta), \qquad \alpha <_n \beta \Leftrightarrow \alpha \leqslant_n \beta\ \&\ \beta \nleqslant_n \alpha,$$
$$\alpha =_n \beta \Leftrightarrow \alpha \leqslant_n \beta\ \&\ \beta \leqslant_n \alpha.$$

The Δ^1_n-*degree* of a real α, in symbols $[\alpha]_n$, is $[\alpha]_n = \{\beta: \beta =_n \alpha\}$. Put also

$$[\alpha]_n \leqslant [\beta]_n \Leftrightarrow \alpha \leqslant_n \beta, \quad [\alpha]_n < [\beta]_n \Leftrightarrow \alpha <_n \beta.$$

Clearly $\leqslant$ is a partial ordering on the set of all Δ^1_n-degrees with least element $[\lambda t o]_n$.

We first notice the following

THEOREM (1B-1). *Assume* PD *if* $n > 1$. *For each* $n \geqslant 1$, $\alpha \in C_n$ & $\beta =_n \alpha \Rightarrow \beta \in C_n$.

PROOF. Consider the set $C^*_n = \{\beta: \exists \alpha \in \Delta^1_n(\beta)[\beta \in \Delta^1_n(\alpha)\ \&\ \alpha \in C_n]\}$. Clearly $C_n \subseteq C^*_n$ and $C_n \in \Sigma^1_n$ or Π^1_n according as n is even or odd. To complete the proof we show that C^*_n is thin. For $n > 1$ this is clear, since C^*_n is then countable. Consider thus C^*_1. If C^*_1 is not thin let $P \subseteq C^*_1$ be perfect. Then

$$\forall \alpha \in P \exists n[n \text{ codes a } \Delta^1_1(\alpha) \text{ real } \Delta^\alpha_n\ \&\ \Delta^\alpha_n =_1 \alpha\ \&\ \Delta^\alpha_n \in C_1].$$

Since the matrix is Π^1_1 we can find (by Kreisel's selection theorem) $f: P \to \omega$, $f \in \Delta^1_1$ such that $\forall \alpha \in P[f(\alpha)$ codes a $\Delta^1_1(\alpha)$ real $\Delta^\alpha_{f(\alpha)}$ & $\Delta^\alpha_{f(\alpha)} =_1 \alpha$ & $\Delta^\alpha_{f(\alpha)} \in C_1]$. But P is uncountable so we can find $n_0 \in \omega$ such that $f^{-1}[\{n_0\}] = S$ is uncountable. For all $\alpha \in S$, n_0 codes a $\Delta^1_1(\alpha)$ real $\Delta^\alpha_{n_0}$ and $\{\Delta^\alpha_{n_0}: \alpha \in S\} \subseteq C_1$ is Σ^1_1 so it is countable. But clearly the map $\alpha \mapsto \Delta^\alpha_{n_0}$ is countable-to-one on S (since $\Delta^\alpha_{n_0} =_1 \alpha$ if $\alpha \in S$). Thus card $S \leqslant \aleph_0$. Contradiction. □

THEOREM (1B-2). *Assume* PD *if* $n > 2$ *below. Let* $n \geqslant 2$ *be even. Then* C_n *is closed under* $\leqslant_n$ (*i.e.* $\alpha \in C_n\ \&\ \beta \leqslant_n \alpha \Rightarrow \beta \in C_n$).

PROOF. Let $C^*_n = \{\beta: \exists \alpha(\beta \in \Delta^1_n(\alpha)\ \&\ \alpha \in C_n)\}$. Then C^*_n is countable and $C^*_n \in \Sigma^1_n$. Thus $C^*_n \subseteq C_n$. □

REMARK. It is easy to see that (1B-2) fails for odd n.

We have thus established that each C_n is actually a collection of Δ^1_n-degrees. We show now that the Δ^1_n-degrees in C_n are in fact wellordered under their natural partial ordering. This is the main structural result about C_n and it will be the starting point of our later investigations.

THEOREM (1B-3). *Assume* PD *if* $n > 2$. *For each* $n \geqslant 1$ C_n *is prewellordered by the relation* $\alpha \leqslant_n \beta$ *i.e. the* Δ^1_n-*degrees of elements of* C_n *are wellordered under* $\leqslant$.

PROOF. Let first n be odd. It is enough to show that $\leqslant_n \restriction C_n \times C_n$ is connected and wellfounded. Let ϕ be a Π^1_n-norm on C_n. Let $\alpha, \beta \in C_n$. Assume $\phi(\alpha) \leqslant \phi(\beta)$ without loss of generality. Then $\alpha \in \{\alpha': \phi(\alpha') \leqslant \phi(\beta)\} = S$ and $S \in \Delta^1_n(\beta)$ is thin, so it contains only $\Delta^1_n(\beta)$ reals. In particular $\alpha \in \Delta^1_n(\beta)$. This proves connectedness. To prove wellfoundedness assume

$\alpha_0, \alpha_1, \alpha_2, \ldots$ is an infinite descending $<_n$-chain i.e. $\alpha_0 >_n \alpha_1 >_n \alpha_2 >_n \ldots$. If $\alpha, \beta \in C_n$ then $\alpha <_n \beta \Rightarrow \phi(\alpha) < \phi(\beta)$ so $\phi(\alpha_0) > \phi(\alpha_1) > \phi(\alpha_2) > \ldots$, a contradiction.

Consider now the case n is even. We have

$$C_n = \{\alpha \colon \exists \beta (\beta \in C_{n-1} \mathbin{\&} \alpha \leqslant_T \beta)\},$$

where $\alpha \leqslant_T \beta \Leftrightarrow \alpha$ is recursive in β. Let $P(\alpha, \beta) \Leftrightarrow \alpha \leqslant_T \beta \mathbin{\&} \beta \in C_{n-1}$. Uniformize $P \in \Pi^1_{n-1}$ by $P^* \in \Pi^1_{n-1}$ (i.e. $P^* \subseteq P \mathbin{\&} \exists \beta P(\alpha, \beta) \Leftrightarrow \exists ! \beta P^*(\alpha, \beta)$). For any $\alpha \in C_n$ let α^* be the unique real in C_{n-1} such that $P^*(\alpha, \alpha^*)$. Then $\alpha =_n \alpha^*$. Given now $\alpha, \beta \in C_n$ assume without loss of generality that $\alpha^* \leqslant_{n-1} \beta^*$. Then clearly $\alpha \leqslant_n \beta$. So $\leqslant_n \restriction C_n \times C_n$ is connected. To prove it is wellfounded notice that $\alpha <_n \beta \Rightarrow \alpha^* <_{n-1} \beta^*$ and use the previous case. □

Since $\leqslant_n \restriction C_n \times C_n$ is a prewellordering let ψ_n be the associated norm. Put

$$d^n_\xi = \psi_n^{-1}[\{\xi\}] = \{\alpha \in C_n \colon \psi_n(\alpha) = \xi\}.$$

Thus d^n_ξ is the ξth Δ^1_n-degree in C_n. Let length $(\psi_n) = \rho_n$. Then $C_n = \bigcup_{\xi < \rho_n} d^n_\xi$. By the proof of (1B-3) it is clear that $\rho_n \leqslant \rho_{n-1}$ for n even. We shall actually see later that $\rho_n = \rho_{n-1}$ when n is even (see 3C). It is not hard to see that $\rho_n < \rho_{n+1}$ if n is even so that $\rho_1 = \rho_2 \; (= \aleph^L_1) < \rho_3 = \rho_4 < \rho_5 = \ldots$.

Our next aim is to study the hierarchy $\{d^n_\xi\}_{\xi < \rho_n}$. Our main result for the moment concerns the passage from d^n_ξ to $d^n_{\xi+1}$. The passage from $\{d^n_\xi\}_{\xi < \lambda}$ to d^n_λ when λ is limit is much more complicated and we shall give a full description of it only for $n = 1$ in the next section.

First let us notice the following trivial fact.

PROPOSITION (1B-4). *Assume* PD *if* $n > 2$. *Then for all* $n \geqslant 1$, $d^n_0 = [\lambda t o]_n$.

To study the behavior of $\{d^n_\xi\}_{\xi < \rho_n}$ at successor stages we need now a definition.

DEFINITION. If $d = [\alpha]_n$ is a Δ^1_n-degree the Δ^1_n*-jump* of d, in symbols d', is the Δ^1_n-degree of a complete $\Pi^1_n(\alpha)$ subset of ω. [Recall that $A \subseteq \omega$ is Γ*-complete* iff for all $B \subseteq \omega$, $B \in \Gamma$ there is a recursive $f \colon \omega \to \omega$ such that $n \in B \Leftrightarrow f(n) \in A$.]

Concerning Δ^1_n-jumps we have the following lemma.

LEMMA (1B-5). *Assume* PD *if* $n > 2$. *For any* $n \geqslant 1$, C_n *is closed under* Δ^1_n*-jumps i.e.* $d \subseteq C_n \Rightarrow d' \subseteq C_n$.

PROOF. We consider the case n is odd, the case of n even being entirely similar. Let $\mathcal{P} \subseteq \mathcal{R} \times \omega$ be Π^1_n and ω-universal for Π^1_n subsets of $\mathcal{R}$ such that moreover $\mathcal{P}^\alpha = \{n \colon \mathcal{P}(\alpha, n)\}$ is a complete $\Pi^1_n(\alpha)$ set. We show that

$$\alpha \in C_n \Rightarrow \mathcal{P}^\alpha \in C_n.$$

Let ϕ be a Π^1_n-norm on P. Let

$$\begin{aligned} I(\beta, a) &\Leftrightarrow \text{“}\beta \text{ codes a } \phi\text{-initial segment of } P^a\text{”} \\ &\Leftrightarrow \beta \in {}^{\omega}2 \ \& \ \forall k(\beta(k) = 0 \Rightarrow P(a, k)) \\ &\qquad \& \ \forall k \forall m(\beta(k) = 0 \ \& \ \phi(m, a) \leqslant \phi(k, a) \Rightarrow \beta(m) = 0). \end{aligned}$$

Clearly for all a, $I_a = \{\beta: I(\beta, a)\}$ is countable, $I \in \Pi^1_n$ and $I(P^a, a)$ holds. Let now

$$C'_n = \{\beta: \exists a \in \Delta^1_n(\beta)\ (I(\beta, a) \ \& \ a \in C_n)\}.$$

Clearly $C'_n \in \Pi^1_n$ and C'_n is thin (by a proof similar to that of (1B-1)). Thus $C'_n \subseteq C_n$ and we are done. □

We are now ready to describe $d^n_{\xi+1}$ in terms of d^n_ξ. We proved the next result first for $n = 1$ with a proof that did not seem immediately extendible to all *odd* n. (Our proof of the even case is clearly general.) Moschovakis then invented the argument used below which works for *all odd* n. Alternative proofs for all $n > 1$ can be given using the ideas of § 3.

THEOREM (1B-6). *Assume* PD *if* $n > 1$. *For any* $n \geqslant 1$, $d^n_{\xi+1} = (d^n_\xi)'$.

PROOF. Assume first n is odd. Let P be as in the proof of (1B-5). It is enough to show that if $a, \beta \in C_n$ then $a <_n \beta \Rightarrow P^a \leqslant_n \beta$. Let $a \in C_n \Leftrightarrow (a, n_0) \in P$. Let ϕ be a Γ-norm on P. If a, β are in C_n and $a <_n \beta$, then we have $m \in P^a \Leftrightarrow \phi(a, m) \leqslant \phi(\beta, n_0)$. Because otherwise, for some $m_0 \in P^a$ $\phi(a, m_0) > \phi(\beta, n_0)$, thus the set $A = \{\beta': \phi(\beta', n_0) < \phi(a, m_0)\}$ is a countable $\Delta^1_n(a)$ set and $\beta \in A$, therefore $\beta \leqslant_n a$, a contradiction. Thus $P^a \leqslant_n \langle a, \beta\rangle \leqslant_n \beta$.

Let now n be even. Let ϕ be a Σ^1_n-norm on C_n and $a \in d^n_\xi$. Find $\beta \in d^n_{\xi+1}$ with least $\phi(\beta)$. Then

$$\gamma \leqslant_n a \Leftrightarrow \exists \eta \leqslant \xi(\gamma \in d^n_\eta) \Leftrightarrow \phi(\gamma) < \phi(\beta).$$

Thus $\{\gamma: \gamma \leqslant_n a\}$ is $\Delta^1_n(\beta)$. Let $A \subseteq \omega$ be $\Sigma^1_n(a)$-complete and ψ a $\Sigma^1_n(a)$-norm on A. Let $I(\delta) \Leftrightarrow$ "δ is the characteristic function of a ψ-initial segment of A". Then $I \in \Sigma^1_n(a)$, $I(A)$ holds and $I - \{\gamma: \gamma \leqslant_n a\} = \{A\}$. So $\{A\} \in \Sigma^1_n(\langle a, \beta\rangle) = \Sigma^1_n(\beta)$, thus $A \leqslant_n \beta$ and $A \in d^n_{\xi+1}$. □

REMARK. For each n odd and $a \in R$ let $\lambda^1_{n,a} = \sup\{\xi: \xi$ is the length of a $\Delta^1_n(a)$ prewellordering of $R\}$. Then (see [30])

$$\lambda^1_{n,a} = \sup\{\phi(a, n): (a, n) \in P\},$$

where $P \subseteq R \times \omega$, $P \in \Pi^1_n$ is ω-universal for Π^1_n subsets of R and ϕ is a Π^1_n-norm on P (PD is of course assumed if $n > 1$). The ordinals $\lambda^1_{n,a}$ are the analogs of the Church-Kleene ordinals ω^a_1 $(= \lambda^1_{1,a})$ and the analog of the spector criterion is true i.e.

$$a \leqslant_n \beta \Rightarrow [\lambda^1_{n,a} < \lambda^1_{n,\beta} \Leftrightarrow P^a \leqslant_n \beta].$$

From this it follows that when $\alpha, \beta \in C_n$,

$$\alpha \leqslant_n \beta \Leftrightarrow \lambda^1_{n,\alpha} \leqslant \lambda^1_{n,\beta}, \quad \alpha <_n \beta \Leftrightarrow \lambda^1_{n,\alpha} < \lambda^1_{n,\beta}.$$

1C. *The* Δ^1_n*-good wellordering on* C_n. Another important structural property of the C_n's is that they admit a nice wellordering which properly refines $\leqslant_n \upharpoonright C_n \times C_n$. For $n = 1$ or $n = 2$ since $C_1 \subseteq C_2 \subseteq L$ this wellordering can be taken to be, as one should probably expect, the restriction of the usual wellordering of L. Nevertheless for $n \geqslant 1$ there is no a priori given model like L in which C_{2n+1}, C_{2n+2} are embedded, so another way of attack has to be developed. And it is rather interesting to note here that this "model independent" definition of a wellordering on C_{2n+1}, C_{2n+2} was actually the first step in the creation of higher-level analogs of L in which C_{2n+1}, C_{2n+2} are embedded, thereby reversing the connection between C_1, C_2 and L. We shall say more on this in 3C.

The key idea behind the construction of a nice wellordering on C_n can be isolated in the use of a theorem of Mansfield [25] and one of its proofs. Mansfield originally proved his theorem via forcing. Solovay obtained later a new forcing-free proof, which is essentially the one given below with one simple alteration: we replaced Solovay's "inductive analysis" by the dual notion of "derivation"; as a result the proof becomes very similar to Cantor's proof of the Cantor-Bendixson theorem (see [22]).

THEOREM (1C-1) (MANSFIELD [25]). *Assume* T *is a tree on* $\omega \times \lambda$ *and* $A = p[T]$. *Then if* A *contains an element not in* $L[T]$, A *contains a perfect subset.*

PROOF. For any tree J on $\omega \times \lambda$, define the *derivative* J' of J as follows:

$((k_0, \xi_0), \ldots, (k_n, \xi_n)) \in J'$
$\Leftrightarrow$ There are two incompatible in the first coordinate extensions of $((k_0, \xi_0), \ldots, (k_n, \xi_n))$ both in J [i.e. we can find $((k'_0, \xi'_0), \ldots, (k'_m, \xi'_m)) \in J$, $((k''_0, \xi''_0), \ldots, (k''_m, \xi''_m)) \in J$ both extending $((k_0, \xi_0), \ldots, (k_n, \xi_n))$ such that $(k'_0, \ldots, k'_m)$, $(k''_0, \ldots, k''_m)$ are incompatible].

Notice that $J' \subseteq J$ and J' is also a tree. We define now à la Cantor the ξth derivative of T by

$$T^0 = T, \quad T^{\xi+1} = (T^\xi)', \quad T^\lambda = \bigcap_{\xi<\lambda} T^\xi, \text{ if } \lambda \text{ is limit.}$$

Clearly $\xi \to T^\xi$ is a function absolute for any model of set theory containing T, in particular $L[T]$. Moreover $T^0 \supseteq T^1 \supseteq \ldots \supseteq T^\xi \supseteq T^{\xi+1} \supseteq \ldots$. Let $\theta_T =$ least θ such that $T^\theta = T^{\theta+1}$.

Case 1. $T^{\theta_T} = \varnothing$. Let then $\alpha \in A$. Since $A = p[T]$, we can find $f \in {}^\omega\lambda$ such that $(\alpha, f) \in [T]$. Since $(\alpha, f) \notin [T^{\theta_T}] = \varnothing$ let $\xi < \theta_T$ be such that

$(a, f) \in [T^\xi] - [T^{\xi+1}]$. Let n be the least integer such that $((a(0), f(0)), \ldots, (a(n), f(n))) \notin T^{\xi+1} = (T^\xi)'$. Then all branches of T^ξ extending $((a(0), f(0)), \ldots, (a(n), f(n)))$ have the same real part i.e. a. So

$$p[T_{((a(0),f(0)),\ldots,(a(n),f(n)))}] = \{a\} \text{ [4]}$$

thus $a \in L[T]$. So in this case $A \subseteq L[T]$.

Case 2. $T^{\theta^T} \neq \varnothing$. Then $T^{\theta^T} = (T^{\theta^T})' \neq \varnothing$ i.e. every sequence in T^{θ^T} has two extensions in T^{θ^T} incompatible in the 1st coordinate. It is easy then to see that $p[T^{\theta^T}]$, and therefore A, contains a perfect subset. □

The following more precise version of (1C-1) is a corollary of its proof and was probably first noticed by Solovay. It will be needed in § 2.

COROLLARY (1C-2). *Assume T is a tree on $\omega \times \lambda$ and $A = p[T]$. If for any set X, X^+ denotes the smallest admissible set containing X as a member, we have:*

$$A - T^+ \neq \varnothing \Rightarrow A \text{ contains a perfect subset.}$$

We are now ready to prove the main result of this section.

DEFINITION. A prewellordering $\leqslant$ on a pointset A is Δ-*good*, where Γ is a pointclass and $\Delta = \Gamma \cap \check{\Gamma}$, if for each $a \in A$ $\{\beta: \beta \leqslant a\}$ is countable and the relation

$$\mathrm{InSeg}_{\leqslant}(\gamma, a) \Leftrightarrow \{\gamma_n: n \in \omega\} = \{\beta: \beta \leqslant a\} \text{ [5]}$$

is in Δ for $a \in A$ i.e. for P, Q in $\Gamma, \check{\Gamma}$ respectively

$$a \in A \Rightarrow [\mathrm{InSeg}_{\leqslant}(\gamma, a) \Leftrightarrow P(\gamma, a) \Leftrightarrow Q(\gamma, a)].$$

THEOREM (1C-3). *Assume* PD *if $n > 2$. For all $n \geqslant 1$, C_n admits a Δ^1_n-good wellordering.*[6]

PROOF. Assume we have shown that $C_n = C$, where $n \geqslant 1$ is *odd*, admits a Δ^1_n-good wellordering $\leqslant$. We show then that $C_{n+1} = C^*$ admits a Δ^1_{n+1}-good wellordering. Recall that

$$C^* = \{a: \exists \beta \in C (a \leqslant_T \beta)\}.$$

Let $P(a, \beta) \Leftrightarrow a \leqslant_T \beta \;\&\; \beta \in C$. Then $P \in \Pi^1_n$. Uniformize P by $P^* \in \Pi^1_n$ (so that $P^* \subseteq P \;\&\; \exists \beta(a, \beta) \in P \Leftrightarrow \exists! \beta(a, \beta) \in P^*$). If P^* contains a perfect set F, then $A = \{\beta: \exists a(a, \beta) \in F\}$ is Σ^1_1 and $A \subseteq C$. So A is countable. Thus $B = \{a: \exists \beta(a, \beta) \in F\}$ is also countable (since $(a, \beta) \in F \Rightarrow a \leqslant_T \beta$). But B is in a 1-1 correspondence with F, a contradiction. So P^* is thin. Thus C^* is the 1-1 recursive image of a thin Π^1_n set $E^* \subseteq C$ i.e. for some $f: R \to R$ recursive, f is 1-1 on E^* and $f[E^*] = C^*$. Let then for $a, a' \in C^*$

(4) When T is a tree and $u \in T$ we put $T_u = \{v \in T: v \text{ extends } u\}$.

(5) For any real γ, $\gamma_n(m) = \gamma(\langle n, m \rangle)$ where $\langle\, , \rangle$ is a 1-1 recursive correspondence between $\omega \times \omega$ and ω.

(6) Our original computation (see [15]) of the wellordering defined in the proof below, *when $n > 1$ is odd*, gave it as Δ^1_{n+1}-good. Later Martin and Solovay and independently the author noticed that this wellordering was actually Δ^1_n-good.

$$a \leqslant^* a' \Leftrightarrow \exists\beta\exists\beta'(\beta, \beta' \in E^* \;\&\; \beta \leqslant \beta' \;\&\; f(\beta) = a \;\&\; f(\beta') = a').$$

If $a \in C^*$ we have

$$\begin{aligned}\text{InSeg}_{\leqslant^*}(\gamma, a) \Leftrightarrow{}& \forall n(\gamma_n \leqslant^* a) \;\&\; \exists\delta\exists\beta[\beta \in E^* \;\&\; f(\beta) = a \;\&\; \text{InSeg}_{\leqslant}(\delta, \beta) \\ &\qquad \&\forall m(\delta_m \in E^* \Rightarrow \exists n(\gamma_n = f(\delta_m)))] \\ \Leftrightarrow{}& \forall\delta\forall\beta\{\beta \in E^* \;\&\; f(\beta) = a \;\&\; \text{InSeg}_{\leqslant}(\delta, \beta) \Rightarrow \\ &\qquad [\forall m(\delta_m \in E^* \Rightarrow \exists n(\gamma_n = f(\delta_m))) \\ &\qquad \& \;\forall n\exists m(\delta_m \in E^* \;\&\; \gamma_n = f(\delta_m))]\}.\end{aligned}$$

Thus $\text{InSeg}_{\leqslant^*}(\gamma, a)$ is Δ^1_{n+1} for $a \in C^* = C_{n+1}$ and we are done.

We are now faced with the more difficult task of showing that $C = C_n$ admits a Δ^1_n-good wellordering when $n \geqslant 1$ is *odd*. The key idea here is the use of (1C-1).

Let $\{\phi_m\}_{m\in\omega}$ be a Π^1_n-scale on C. Let T be the associated tree (as in 0D). For $\xi <$ length (ϕ_0) put

$$T_\xi = \{((k_0, \xi_0), \ldots, (k_n, \xi_n)) \in T: \xi_0 \leqslant \xi\}.$$

Using the limit property of scales it is not hard to see that if $\phi_0(a) = \xi$, then

$$p[T_\xi] = \{\beta \in C: \phi_0(\beta) \leqslant \phi_0(a)\} \stackrel{\text{def}}{=} I_a.$$

Since I_a is countable so is T_ξ. In particular (using the notation of the proof of (1C-1)), $\theta_{T_\xi} \stackrel{\text{def}}{=} \theta_\xi$ is countable and $T_\xi^{\theta_\xi} = \varnothing$. We then define the following wellordering on $p[T_\xi]$, denoted by $<_\xi$:

$a <_\xi \beta \Leftrightarrow$ (1) For some $\theta < \theta_\xi$, $(a, \vec{\phi}(a)) \in [T_\xi^\theta] - [T_\xi^{\theta+1}]$ while $(\beta, \vec{\phi}(\beta)) \in T_\xi^{\theta+1}$, where $\vec{\phi}(a) = (\phi_0(a), \phi_1(a), \ldots)$

or (2) For some $\theta < \theta_\xi$, $(a, \vec{\phi}(a)), (\beta, \vec{\phi}(\beta)) \in [T_\xi^\theta] - [T_\xi^{\theta+1}]$ and if $((a(0), \phi_0(a)), \ldots, (a(n), \phi_n(a)))$, $((\beta(0), \phi_0(\beta)), \ldots, (\beta(m), \phi_m(\beta)))$ are the shortest sequences which do not belong to $T_\xi^{\theta+1}$, then $n < m$ or ($n = m$ & the sequence "with the a" precedes the sequence "with the β" lexicographically).

The promised wellordering $<$ on C is now defined as follows: If $a, \beta \in C$ put

$$a < \beta \Leftrightarrow \phi_0(a) < \phi_0(\beta) \;\; or \;\; [\phi_0(a) = \phi_0(\beta) \;\&\; a <_{\phi_0(a)} \beta].$$

The proof that $<$ is a Δ^1_n-good wellordering on C consists of a simple key observation followed by a messy but rather straightforward computation (of which we shall indicate only the main steps below).

Notice first that $\leqslant^{\varphi_0}$ is a Δ_n^1-good prewellordering on C. This is because for $a \in C$,

$$\begin{aligned}\text{InSeg}_{\leqslant}\, \phi_0(\delta, a) &\Leftrightarrow \forall m(\phi_0(\delta_m) \leqslant \phi_0(a)) \,\&\, \forall\beta(\phi_0(\beta) \leqslant \phi_0(a) \Rightarrow \exists m(\beta = \delta_m))\\ &\Leftrightarrow \forall m(\phi_0(\delta_m) \leqslant \phi_0(a))\\ &\qquad \& \,\forall\beta \in \Delta_n^1(a)(\phi_0(\beta) \leqslant \phi_0(a) \Rightarrow \exists m(\beta = \delta_m)).\end{aligned}$$

The last equivalence follows from the fact that when $a \in C$, $\{\beta\colon \phi_0(\beta) \leqslant \phi_0(a)\} \in \Delta_n^1(a)$ and is countable, so it contains only $\Delta_n^1(\alpha)$ reals.

Now given $a \in C$ and δ such that $\text{InSeg}_{\leqslant}\, \phi_0(\delta, a)$ we can get easily (i.e. in a Δ_n^1 fashion) a real coding $T_{\phi_0(a)}$ and thus we can find Π_n^1, Σ_n^1 relations R_Π, R_Σ respectively, so that whenever $a \in C$ and $\text{InSeg}_{\leqslant}\, \phi_0(\delta, a)$ we have

$$\beta \leqslant a \Leftrightarrow R_\Pi(\delta, \beta, a) \Leftrightarrow R_\Sigma(\delta, \beta, a).$$

Then if $a \in C$ we have

$$\begin{aligned}\text{InSeg}_{\leqslant}\, (\gamma, a) &\Leftrightarrow (\exists\delta)[\text{InSeg}_{\leqslant}\, \phi_0(\delta, a) \;\&\; \forall m R_\Sigma(\delta, \gamma_m, a) \;\&\\ &\qquad\qquad \forall m(R_\Pi(\delta, \delta_m, \beta) \Rightarrow \exists k(\delta_m = \gamma_k))]\\ &\Leftrightarrow \forall\delta[\text{InSeg}_{\leqslant}\, \phi_0(\delta, a) \Rightarrow\\ &\qquad [\forall m R_\Pi(\delta, \gamma_m, a) \;\&\; \forall m(R_\Sigma(\delta, \delta_m, \beta) \Rightarrow\\ &\qquad\qquad \exists k(\delta_m = \gamma_k))]].\end{aligned}$$

These computations show that $\leqslant$ is Δ_n^1-good and complete the proof. □

1D. *Remarks on further generalizations.* Our main goal in this paper is to study the countable analytical sets. Nevertheless our methods are quite general and we can extract from them some more abstract results which seem interesting and probably useful (this depends very much of course on finding more and more interesting pointclasses satisfying the hypotheses in the statements of the theorems given below). We state these results now.

THEOREM (1D-1). *Let* Γ *be adequate, ω-parametrized and closed under* $\forall^R$. *If* Scale (Γ) *is true and the collection of thin sets is* Γ*-additive, then* Γ *has a largest thin set and the class of thin* Γ *sets is ω-parametrized.*

The only new thing (beyond the ideas in (1A-2)) which is needed here is the estimate that the following relation is in Γ:

$M'(a) \Leftrightarrow$ "a codes a set of reals (say Δ_a) in Δ & Δ_a is thin". This can be done using Scale (Γ).

The next generalization we have in mind is straightforward but rather cumbersome to write down in detail. We can nevertheless summarize it easily by saying that the results in the previous sections concerning Π_n^1, with n odd, would go through for any pointclass Γ which is adequate, ω-parametrized, closed under $\forall^R$ provided that Scale (Γ) is true, the class of thin sets is Γ-additive and every thin set in $\Delta(a) = \Gamma(a) \cap \breve{\Gamma}(a)$ contains only $\Delta(a)$ reals. Moreover the results about Σ^1_n with n even are true for any class of the form $\exists^R\Gamma$ where Γ is as above.

2. **The largest thin Π^1_1 set C_1.** We restrict now our attention to the study of the structure of C_1, the largest thin Π^1_1 set. By Solovay's theorem (see [41]) this set is contained in L. Since it is clearly a proper subset of $R \cap L = C_2$ our first task is to find out which constructible reals get in C_1. We shall give several characterizations of C_1 among which one that says that C_1 is the hyperdegree closure of the Boolos-Putnam complete sets (see [8]). This leads eventually to a complete description of the behaviour of the sequence $\{d^1_\xi\}_{\xi<\rho_1=\aleph_1^L}$ of the Δ^1_1-degrees of C_1 at limit stages. Finally we connect C_1 with other known concepts and results in recursion theory, hoping to illustrate the fact that C_1 is a natural "universe" in which many recursion theoretic problems can be discussed. Some of the results below have been independently discovered by D. Guaspari and G. E. Sacks.

2A. *Various characterizations of* C_1. Since C_1 is a thin Π^1_1 set clearly $C_1 \subseteq L$. Also

$$C_1 \subseteq C_2(= L \cap R) = \{a\colon \exists\beta(\beta \in C_1 \;\&\; a \leqslant_T \beta)\},$$

so that C_1 is a "trunk" for the hierarchy of constructible reals. Which of them get in C_1? We shall see the answer below in various forms all of which suggest the naturalness of C_1.

Sacks [38] has shown that if ξ is a countable admissible ordinal then $\xi = \omega_1^a$ for some a, where

$$\omega_1^a = \sup\,\{\eta\colon \eta \text{ is the length of a } \Delta^1_1(a) \text{ wellordering of } \omega\}.$$

In general no such a belongs to $L_\xi = L_{\omega_1^a}$. This leads to the set of "good" a's namely $\{a\colon a \in L_{\omega_1^a}\}$. It turns out that this set is exactly C_1. This has been proved by G. E. Sacks and independently by D. Guaspari. Their proofs are forcing or omitting type arguments. We give below another proof based on (1C-2). It is due jointly to Moschovakis and the author and is much in the spirit of elementary recursion theory and classical descriptive set theory.

THEOREM (2A-1) (GUASPARI [12], SACKS [38]). *The largest thin* Π^1_1 *set* C_1 *is equal to* $\{a\colon a \in L_{\omega_1^a}\}$.

PROOF. We show first that if $A \in \Pi^1_1$ and A is thin then, for all $a \in A$, $a \in L_{\omega_1^a}$.

For some recursive tree T on $\omega \times \omega$ we have $a \in A \Leftrightarrow T(a)$ is well-founded. Fix $a_0 \in A$. Then $T(a_0)$ is a tree recursive in a_0, thus

$|T(a_0)| = \xi < \omega_1^{a_0}$. Let

$$a \in A' \Leftrightarrow a \in A \ \& \ |T(a)| \leqslant \xi.$$

Then $a_0 \in A'$ and by a Shoenfield type argument

$$a \in A' \Leftrightarrow S(a) \text{ is not wellfounded,}$$

where S is a tree on $\omega \times \xi$ and $S \in L_{\omega_1^{a_0}}$. But $A' \subseteq A$ so A' is thin, thus by (1C-2) $A' \subseteq S^+ \subseteq L_{\omega_1^{a_0}}$, therefore $a_0 \in L_{\omega_1^{a_0}}$.

To prove now that $B = \{a: a \in L_{\omega_1^a}\} \subseteq C_1$ it is enough to show that $B \in \Pi_1^1$ and B is thin. The first assertion is clear since

$$a \in L_{\omega_1^a} \Leftrightarrow \exists \beta \in \Delta_1^1(a)(\beta \in WO \ \& \ a \in L_{|\beta|}).(^7)$$

To see that B is thin notice that if $P \subseteq B$ is perfect and $g: {}^\omega 2 \to P$ is 1-1 and continuous, the relation

$$a \leqslant \beta \Leftrightarrow \omega_1^{g(a)} \leqslant \omega_1^{g(\beta)}$$

is a Σ_1^1 prewellordering on ${}^\omega 2$, thus it is Lebesgue measurable. But its initial segments are countable contradicting as usual Fubini's theorem. □

Using (2A-1) and a simple forcing argument one can show that

$$a \in C_1 \Leftrightarrow \forall \beta(\omega_1^a \leqslant \omega_1^\beta \Leftrightarrow a \leqslant_1 \beta).$$

This has been also observed by Guaspari and Sacks and gives another characterization of C_1 which has the advantage that it does not mention L. It is not therefore inconceivable that it will have a "soft" recursion theoretic proof. We give such a proof below:

THEOREM (2A-2). (Due also independently to Guaspari [12], Sacks [38].) *For all* a, $a \in C_1 \Leftrightarrow \forall \beta(\omega_1^a \leqslant \omega_1^\beta \Leftrightarrow a \leqslant_1 \beta)$.

PROOF. Let $a \in C_1$ and $\omega_1^a \leqslant \omega_1^\beta$. Let $a \in C_1 \Leftrightarrow f(a) \in WO$, where $f: R \to R$ is recursive. Then $|f(a)| < \omega_1^a \leqslant \omega_1^\beta$. Find $\gamma \in \Delta_1^1(\beta)$ so that $\gamma \in WO \ \& \ |f(a)| \leqslant |\gamma|$. Then $A' = \{a: f(a) \in WO \ \& \ |f(a)| \leqslant |\gamma|\}$ is a $\Delta_1^1(\gamma)$ subset of C_1, so contains only $\Delta_1^1(\gamma)$ reals. Thus $a \in \Delta_1^1(\gamma)$ and therefore $a \in \Delta_1^1(\beta)$.

Let now $B = \{a: \forall \beta(\omega_1^a \leqslant \omega_1^\beta \Rightarrow a \leqslant_1 \beta)\}$. It is easy to see that $B \in \Pi_1^1$. To see that B is thin note that if $P \subseteq B$ is perfect and $g: {}^\omega 2 \to P$ is 1-1 and continuous, the prewellordering

$$a \leqslant \beta \Leftrightarrow \omega_1^{g(a)} \leqslant \omega_1^{g(\beta)}$$

is Δ_1^1 hence it has countable length, a contradiction. □

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(7) Here $\beta \in WO \Leftrightarrow$ "$\{(m,n): \beta(\langle m,n\rangle) = 0\} = <_\beta$ is a wellordering on ω" and for $\beta \in WO$, $|\beta| = \text{length}(<_\beta)$.

We finally come to the characterization of C_1 in terms of the Boolos-Putnam sets.

DEFINITION. An ordinal $\xi < \aleph_1^L$ is called an *index* iff $\exists a(a \in L_{\xi+1} - L_\xi)$. Let $\mathrm{IND} = \{\xi : \xi \text{ is an index}\}$. If $a \in L_{\xi+1} - L_\xi$ we call ξ *the index of* a.

The following is a basic result of Boolos and Putnam:

THEOREM (2A-3) (BOOLOS-PUTNAM [8]). *Assume* $\xi \in \mathrm{IND}$. *Then*:

(a) *There exists a real* E_ξ *of index* ξ *such that all the reals in* $L_{\xi+1}$ *are arithmetic in* E_ξ *(roughly speaking* E_ξ *codes* $\epsilon \restriction L_\xi$). *Call* E_ξ *a complete set of order* ξ.

(b) *There is an* $a \in WO$ *of index* $\leqslant \xi$ *such that* $|a| = \xi$.

(c) *If* $\xi \in \mathrm{IND}$, *then every ordinal in the interval* $[\xi, \omega_1^{E_\xi}]$ *is an index and* $\mathcal{O}^{E_\xi}$ *is complete of index* $\omega_1^{E_\xi}$, *where* $\mathcal{O}^{E_\xi}$ *is the relativized Kleene's* $\mathcal{O}$.

Using the above information we can now easily see that

LEMMA (2A-4). *For any* $\xi \in \mathrm{IND}$, $E_\xi \in C_1$.

PROOF. Let $\xi \in \mathrm{IND}$. Let $a \in L_{\xi+1}$ be such that $a \in WO$ & $|a| = \xi$. Then $a \leqslant_1 E_\xi$, so $\xi = |a| < \omega_1^{E_\xi}$ and therefore $E_\xi \in L_{\omega_1^{E_\xi}}$. □

Thus all the complete Boolos-Putnam sets are in C_1. In fact, as far as Δ_1^1-degrees are concerned, these are all the elements of C_1.

In order to state the next and the following results we adopt the following terminology and notation:

We write d_ξ instead of d_ξ^1 for all $\xi < \aleph_1^L = \rho_1$ and $[a]_h$ instead of $[a]_1$. We call $[a]_h$ the *hyperdegree* of a instead of Δ_1^1-degree. We write $a \leqslant_h \beta$ instead of $a \leqslant_1 \beta$ and we let $\omega_1^{[a]_h} = \omega_1^a$.

THEOREM (2A-5). (a) $C_1 = \bigcup_{\xi \in \mathrm{IND}} [E_\xi]_h$. (b) *If* $\eta < \aleph_1^L$,

$$d_{\eta+1} = [E_{\omega_1^{d_\eta}}]_h \quad \textit{and} \quad d_\eta = [E_\theta]_h,$$

where $\theta =$ *least index* $\geqslant \sup\{\omega_1^{d_\xi} : \xi < \eta\}$, *if* η *is limit.*

PROOF. Clearly (a) follows from (b). To prove (b) we proceed by induction on $\eta < \aleph_1^L$. Assume $d_\eta = [E_\xi]_h$ where $\xi \in \mathrm{IND}$. By (2A-4)

$$[E_\xi]_h' = [E_{\omega_1^{E_\xi}}]_h = [E_{\omega_1^{d_\eta}}]_h.$$

But also by (1B-6) $(d_\eta)' = d_{\eta+1}$. So $d_{\eta+1} = [E_{\omega_1^{d_\eta}}]_h$. Consider now the limit case. Clearly $d_\eta \leqslant [E_\theta]_h$, because $E_\theta \in C_1$ and if $a \in d_\xi$, where $\xi < \eta$, then $a \in L_{\omega_1^{d_\xi}} \subseteq L_\theta$. To prove $[E_\theta]_h \leqslant d_\eta$, let $a \in d_\eta$. Then $a \notin L_{\omega_1^\beta}$ for any $\beta \in d_\xi$ where $\xi < \eta$. Thus the index of a is $\geqslant \theta$. But $a \in L_{\omega_1^a}$, so $\theta < \omega_1^a$ and therefore $E_\theta \in L_{\theta+1} \subseteq L_{\omega_1^a}$ so that $E_\theta \leqslant_h a$. □

REMARK. It is easy to see from the above that

$$\omega_1^{d_{\xi+1}} = (\omega_1^{d_\xi})^+ = \text{least admissible} > \omega_1^{d_\xi}.$$

The fact that every real in C_1 has the same hyperdegree as a Boolos-Putnam real provides a very convenient way of thinking about C_1 and can be used to study further properties of C_1. Finally, let us notice that the restriction of the natural wellordering of $L, \leqslant_L$, to C_1 is Δ_1^1-good wellordering (this was also observed by D. Guaspari). Because, if $a \in C_1$ we have, using (2A-1) (where ϕ is a Π_1^1-norm on C_1):

$$\begin{aligned}\text{InSeg}_{\leqslant_L}(\gamma, a) \Leftrightarrow\ & \forall\beta[\{(\phi(\beta) < \phi(a)\ \&\ a \nleqslant_h \beta) \vee (\omega_1^\beta = \omega_1^a\ \&\forall\delta \leqslant_h a \\ & \quad (\text{If } \delta \text{ codes some } L_\theta \text{ with } a \in L_\theta, \text{ then } L_\theta \models \beta <_L a))\} \\ & \quad \Rightarrow \exists n(\beta = \gamma_n)]\ \&\ \forall n[(\phi(\gamma_n) < \phi(a)\ \&\ \omega_1^{\gamma_n} < \omega_1^a) \vee \\ & \quad \exists\delta \leqslant_h a\ (\delta \text{ codes some } L_\theta \text{ with } a \in L_\theta \\ & \qquad \&\ L_\theta \models \gamma_n <_L a\ \&\ a =_h \gamma_n)] \\ \Leftrightarrow\ & \forall\beta \leqslant_h a[\{(\phi(\beta) < \phi(a)\ \&\ \omega_1^\beta < \omega_1^a) \vee \\ & \quad (a =_h \beta\ \&\ \exists\delta \leqslant_h a\ (\delta \text{ codes some } L_\theta\ \&\ a \in L_\theta \\ & \qquad \&\ L_\theta \models \beta <_L a))\} \Rightarrow \exists n(\beta = \gamma_n)] \\ & \quad \&\ \forall n[(\phi(\gamma_n) < \phi(a)\ \&\ a \nleqslant_h \gamma_n) \vee \forall\delta \leqslant_h a\ (\text{If } \delta \text{ codes} \\ & \quad \text{some } L_\theta\ \&\ a \in L_\theta, \text{ then } L_\theta \models \gamma_n <_L a\ \&\ \omega_1^a = \omega_1^{\gamma_n})].\end{aligned}$$

Using now *any* Δ_1^1-good wellordering $\leqslant$ on C_1 we can also prove that C_1 has a Π_1^1 *cross section* B i.e., a $B \subseteq C_1$ such that $\forall a \in C_1 \exists! \beta \in B(a =_h \beta)$. This is also proved independently by D. Guaspari [12]. In fact, let

$$\beta \in B \Leftrightarrow \beta \in C_1\ \&\ \forall a(a < \beta \Rightarrow \omega_1^a < \omega_1^\beta).$$

REMARK. This last result is peculiar to C_1, since it fails for all other C_n with $n > 1$ odd. (See the end of 3A.)

2B. *A natural quasi-hierarchy of hyperdegrees.* The *natural hierarchy of hyperdegrees* (Richter [36]) is defined by

$$h_0 = [\lambda to]_h, \qquad h_{\xi+1} = (h_\xi)', \qquad h_\lambda = \text{l.u.b. } \{h_\xi\}_{\xi<\lambda}, \quad \text{if } \lambda \text{ is limit.}$$

Here l.u.b. refers to the partial ordering of hyperdegrees. Let π_0 be the least (obviously limit) ordinal π for which this hierarchy stops i.e. l.u.b. $\{h_\xi\}_{\xi<\pi_0}$ does not exist. Richter [36] proved

$$\pi_0 \geqslant \text{least recursively inaccessible} = \omega_1^{E_1},$$

where E_1 is the Tugué type-2 object. Then Sacks [38] and subsequently Kripke and Richter showed that actually $\pi_0 = \omega_1^{E_1}$. It is clear from (2A-2) that C_1 is closed under l.u.b.'s so that $h_\xi = d_\xi$ for all $\xi < \omega_1^{E_1}$ i.e. the natural hierarchy of hyperdegrees is an initial segment of the hierarchy of hyperdegrees in C_1.

We study here an extension of the natural hierarchy of hyperdegrees into a natural quasi-hierarchy by taking η-l.u.b.'s when l.u.b.'s do not exist (the precise definition is given below). We show that this natural quasi-hierarchy coincides with the hierarchy of hyperdegrees in C_1. This gives, among other things, a (pseudo) construction of C_1 from below and resolves the problem of the behaviour of $\{d_\xi\}_{\xi<\aleph_1^L}$ at limit stages.

DEFINITION. Let X be a set of hyperdegrees. A hyperdegree e is an *η-upper bound* of X if there is a sequence $\{e_\xi\}_{\xi<\eta}$ of hyperdegrees such that e_0 is an upper bound of $X, e_{\xi+1} = (e_\xi)', e_\lambda$ is an upper bound of $\{e_\xi\}_{\xi<\lambda}$ if λ is limit and $e_\eta = e$. An *η-least upper bound* of X (in symbols, η-l.u.b. (X)) is an η-upper bound of X which is $\leq$ to any other η-upper bound of X.

DEFINITION. The *natural quasi-hierarchy* of hyperdegrees is defined as follows:

$$h_0 = [\lambda t o]_h, \qquad h_{\xi+1} = (h_\xi)',$$
$$h_\lambda = \eta_0\text{-l.u.b. } \{h_\xi\}_{\xi<\lambda}, \text{ where } \eta_0 \text{ is the least } \eta \text{ such that } \eta\text{-l.u.b. } \{h_\xi\}_{\xi<\lambda} \text{ exists, where } \lambda \text{ is limit.}$$

Let λ_0 be the least ordinal λ for which this hierarchy stops i.e. for all η, η-l.u.b. $\{h_\xi\}_{\xi<\lambda}$ does not exist.

DEFINITION. If $\lambda < \aleph_1^L$, λ limit, let

$$\text{Obstr}(\lambda) = \text{order type of the set of admissibles in the interval } [\eta_\lambda, \omega_1^{d_\lambda}), \text{ where } \eta_\lambda = \lim_{\xi<\lambda} \omega_1^{d_\xi}.$$

THEOREM (2B-1). (a) $\lambda_0 = \aleph_1^L$.

(b) *For all* $\xi < \aleph_1^L, h_\xi = d_\xi$.

(c) *If* $\lambda < \aleph_1^L$ *is limit, then*

$$d_\lambda = \eta\text{-l.u.b. } \{d_\xi\}_{\xi<\lambda} \Leftrightarrow \text{Obstr}(\lambda) = \eta.$$

REMARK. A particular instance of (c) is;

$$d_\lambda = \text{l.u.b. } \{d_\xi\}_{\xi<\lambda} \Leftrightarrow \text{Obstr}(\lambda) = 0$$
$$\Leftrightarrow \eta_\lambda \text{ is not recursively inaccessible.}$$

This instance has been also proved independently by D. Guaspari and G. E. Sacks.

PROOF OF THE THEOREM. We show the following lemmas from which the full proof is obvious.

LEMMA A. *Let* λ *be limit,* $\lambda < \aleph_1^L$. *Then* $\text{Obstr}(\lambda) = \eta \Rightarrow d_\lambda = \eta$-l.u.b. $\{d_\xi\}_{\xi<\lambda}$.

PROOF. Let e be any η-upper bound of $\{d_\xi\}_{\xi<\lambda}$. Then $e = e_\eta$, where $\{e_\theta\}_{\theta\leq\eta}$ demonstrates that e is an η-upper bound of $\{d_\xi\}_{\xi<\lambda}$. Let

$a_0, a_1, \ldots, a_\theta, \ldots$ $(\theta < \eta)$ be the increasing enumeration of the admissibles in $[\eta_\lambda, \omega_1^{d_\lambda}]$. Then clearly $\omega_1^{e_\theta} \geqslant a_\theta$ for all $\theta < \eta$, so

$$\omega_1^{e_\eta} = \omega_1^e \geqslant \lim_{\theta<\eta}(\omega_1^{e_\theta} + 1) \geqslant \lim_{\theta<\eta}(a_\theta + 1).$$

But $\omega_1^{e_\eta}$ is admissible, thus $\omega_1^e \geqslant \omega_1^{d_\lambda}$, so by (2A-2) $e \geqslant d_\lambda$.

We show now that d_λ is an η-upper bound of $\{d_\xi\}_{\xi<\lambda}$. The key to this proof is Jensen's theorem (see 0F). Assume first η is limit. Let $a_0 \in L_{\omega_1^{d_\lambda}}$ be such that for all $\theta < \eta$:

(a) a_θ is the θth admissible in a_0 ordinal $> \omega$,

(b) a_θ is countable in $L_{a_{\theta+1}}[a_0]$.

Let f_θ be the hyperdegree of a Boolos-Putnam (relativized to a_0) complete real appearing in $L_{a_{\theta+1}}[a_0] - L_{a_\theta}[a_0]$. Then f_0 is an upper bound for $\{d_\xi\}_{\xi<\lambda}$, $\forall\theta, \theta'(\theta \leqslant \theta' \Rightarrow f_\theta \leqslant f_{\theta'})$ and $f_{\theta+1} = (f_\theta)'$. Finally $f_\theta \leqslant d_\lambda$ for all $\theta < \eta$. Let $f_\eta = d_\lambda$. Clearly $\{f_\theta\}_{\theta\leqslant\eta}$ demonstrates that d_λ is an η-upper bound of $\{d_\xi\}_{\xi<\lambda}$.

Assume now η is a successor, say $\eta = \eta_0 + n$, where η_0 is limit. Find $a_0 \in L_{\omega_1^{d_\lambda}}$, $\{f_\theta\}_{\theta<\eta_0}$ exactly as before (so that a_θ is the θth admissible in a_0, for *all* $\theta < \eta$). Applying again Jensen's theorem find $\beta_0 \in L_{\omega_1^{d_\lambda}}$ such that $a_0 \leqslant_h \beta_0$ and $\omega_1^{\beta_0} = a_{\eta_0}$, while $\omega_k^{\beta_0} \overset{\text{def}}{=} k$th admissible in $\beta_0 = a_{\eta_0+(k-1)}$. Let, for $k < n, f_{\eta_0+k}$ be the $(k+1)$th hyperjump of $[\beta_0]_h$ and $f_\eta = f_{\eta_0+n} = d_\lambda$. Again $\{f_\theta\}_{\theta\leqslant\eta}$ shows that d_λ is an η-upper bound. □

LEMMA B. *If* $\lambda < \aleph_1^L$, λ *limit then*

$$d_\lambda = \eta\text{-l.u.b.}\{d_\xi\}_{\xi<\lambda} \Rightarrow \text{Obstr}(\lambda) = \eta.$$

PROOF. Let $d_\lambda = \eta$-l.u.b.$\{d_\xi\}_{\xi<\lambda}$. Let Obstr$(\lambda) = \eta'$. Then by Lemma A, $d_\lambda = \eta'$-l.u.b.$\{d_\xi\}_{\xi<\lambda}$. So $\eta = \eta'$. □

LEMMA. C. Let $\lambda \leqslant \aleph_1^L$, λ *limit. If* $d = \eta$-l.u.b.$\{d_\xi\}_{\xi<\lambda}$ *exists, then* $d \subseteq C_1$ *and* Obstr$(\lambda) \leqslant \eta$.

PROOF. Let $a_0, a_1, \ldots, a_\theta, \ldots, \theta < \eta'$, be the increasing enumeration of the admissibles in $[\eta_\lambda, \omega_1^d)$. Clearly $\eta' \geqslant \eta$. If $\eta' > \eta$ then as in Lemma A we can construct a sequence of hyperdegrees $\{f_\theta\}_{\theta\leqslant\eta+1}$ such that $f_\theta \leqslant d$ for all $\theta \leqslant \eta + 1$ which demonstrates that $f_{\eta+1}$ is an $(\eta + 1)$-upper bound of $\{d_\xi\}_{\xi<\lambda}$. Then $f_\eta \geqslant d$, a contradiction. So $\eta' = \eta$.

To prove $d \subseteq C_1$ it is enough to show (by (2A-2)) that $\forall e(\omega_1^d \leqslant \omega_1^e \Rightarrow d \leqslant e)$. Let $\omega_1^d \leqslant \omega_1^e$. Exactly as before an η-upper bound of $\{d_\xi\}_{\xi<\lambda}$ belongs in $L_{\omega_1^e}[e]$. Thus $d \leqslant e$. Finally let $d = d_{\lambda'}$ for some $\lambda' \geqslant \lambda$. Since

$[\eta_\lambda, \omega_1^{d_\lambda}) \subseteq [\eta_\lambda, \omega_1^{d_{\lambda'}})$, we have Obstr $(\lambda) \leqslant \eta$. □

REMARK. L. Harrington noticed that as λ varies over limit ordinals $< \aleph_1^L$, Obstr (λ) takes on every value $< \aleph_1^L$. This is because if $\eta < \aleph_1^L$ is given and λ is the least limit ordinal such that $\eta_\lambda > \eta$ and Obstr $(\lambda) \geqslant \eta$, we have Obstr $(\lambda) = \eta$.

Leeds and Putnam [23] and recently M. Lucian in her Ph. D. thesis [24] have proved intrinsic characterizations of the hierarchy of *arithmetic degrees* of the complete sets E_ξ which have the same general flavor as the one we give here about their *hyperdegrees*. The main essential difference is in the definition of upper bounds. In their case an upper bound of a collection of arithmetic degrees X must (roughly speaking) code in a uniform way all of X. This uniformity clearly disappears in our case and makes our hierarchy conceptually simpler but at the same time less fine than theirs. Thus looking successively at the hierarchies of Turing degrees (see [8]), arithmetic degrees, hyperdegrees and Δ_2^1-degrees of the E_ξ's we obtain smoother and smoother hierarchies at the expense, of course, of fineness.

2C. Δ_1^1 *reals and* Π_1^1 *singletons*. We start with a characterization of the Δ_1^1 reals which is immediate from the previous results.

THEOREM (2C-1). *The set of* Δ_1^1 *reals is the unique countable* Π_1^1 *set of reals which is closed under* $\leqslant_h$.

PROOF. Clearly the set of all Δ_1^1 reals is countable, Π_1^1 and closed under $\leqslant_h$. Let A be any other such set. Then $A = \bigcup_{\xi<\eta} d_\xi$ for some η. We show $\eta = 1$. Because if $\eta > 1$ then the complete Π_1^1 set of integers P is in A and thus every real $\alpha \leqslant_h P$ is in A. But by Kleene's basis theorem one such real is in $\mathcal{R} - A$. □

REMARK. L. Harrington [13] proved that every Δ_1^1 set of reals with at least one non-Δ_1^1 real contains an element with the same hyperdegree as the complete Π_1^1 set of numbers. From this it follows easily that the hypothesis of countability can be dropped in (2C-1) i.e. *the class of all* Δ_1^1 *reals is the unique nontrivial* Π_1^1 *set closed under* $\leqslant_h$.

COROLLARY (2C-2) (GANDY-KREISEL-TAIT [11]). *The set of all* Δ_1^1 *reals is the intersection of all* ω*-models of analysis.*

PROOF. It is easy to show that every ω-model of analysis is closed under $\leqslant_h$.

Put now

$$\alpha \in A \Leftrightarrow \forall\beta[\{\beta_n : n \in \omega\} \text{ is an } \omega\text{-model of analysis} \Rightarrow \exists n(\alpha = \beta_n)].$$

Then $A \in \Pi_1^1, A$ is countable and since A = intersection of the ω-models of analysis, A is closed under $\leqslant_h$. Thus $A = \{a: a \in \Delta_1^1\}$. □

COROLLARY (2C-3) (KREISEL). *If A is a nonempty Σ_1^1 set and $\beta \notin \Delta_1^1$, then A contains a real a such that $\beta \nleqslant_h a$.*

PROOF. Let $H(A) = \{\beta: \forall a(a \in A \Rightarrow \beta \leqslant_h a)\}$. Then $H(A)$ contains only Δ_1^1 reals. □

It is well known that the above result has also the following corollary.

COROLLARY (2C-4) (GANDY). *If A is a nonempty Σ_1^1 set, A contains a real a with $\omega_1^a = \omega_1$.*

PROOF. If for all $a \in A$, $\omega_1^a > \omega_1$ then for all $a \in A$ the complete Π_1^1 set of integers is $\leqslant_h a$, contradicting (2C-3). □

Our final corollary has to do with bases for Σ_1^1. Call a set of reals A a *basis* for Σ_1^1 if for all $B \subseteq R$, $B \in \Sigma_1^1$, $B \neq \varnothing \Rightarrow B \cap A \neq \varnothing$.

COROLLARY (2C-5). *There is a least Σ_1^1 basis for Σ_1^1, namely $\{a: \omega_1^a = \omega_1\}$.*

PROOF. Let $A = \{a: \omega_1^a = \omega_1\}$. Then $A \in \Sigma_1^1$ and is a basis by (2C-4). Let $A' \in \Sigma_1^1$ be some other basis. We show that $A \subseteq A'$. If $A \nsubseteq A'$ let $a_0 \notin A'$, $\omega_1^{a_0} = \omega_1$. Now $B = R - A'$ is a Π_1^1 set, say $a \in B \Leftrightarrow f(a) \in WO$, where $f: R \to R$ is recursive. Let $|f(a_0)| = \xi < \omega_1^{a_0} = \omega_1$. If $B' = \{a: f(a) \in WO \;\&\; |f(a)| \leqslant \xi\}$ clearly $B' \neq \varnothing \;\&\; B' \in \Delta_1^1$. So $B' \cap A' \neq \varnothing$ which contradicts $B' \subseteq R - A'$. □

Our final result has to do with the problem of locating the Π_1^1 singletons inside C_1. It is obvious that they are all contained in C_1. It is also easy to see that if $a \in d_\xi$ and a is a Π_1^1 singleton so is every other member of d_ξ. Thus we can talk about the hyperdegrees of Π_1^1 singletons. It is easy to see that $d_0, d_1, \ldots, d_\omega, \ldots$ are all hyperdegrees of Π_1^1 singletons. Moreover there is a $\sigma < \delta_2^1$ = supremum of the Δ_2^1 wellorderings of ω, such that d_σ is not the hyperdegree of a Π_1^1 singleton, while on the other hand $\{\sigma: d_\sigma$ is the hyperdegree of a Π_1^1 singleton$\}$ is a cofinal subset of δ_2^1.

DEFINITION. Let σ_0 be the least ordinal σ such that d_σ is not the hyperdegree of a Π_1^1 singleton.

It follows from results of Suzuki (see [42]) that σ_0 is limit. We identify in our next theorem σ_0 with a very familiar ordinal in recursion theory.

DEFINITION. An ordinal ξ is Π_1^1 (Σ_1^1) *reflecting* if for every Π_1^1 (Σ_1^1) formula ϕ in $\langle L_\xi, \epsilon\rangle$ with parameters in L_ξ we have

$$L_\xi \models \phi \Rightarrow \exists \eta < \xi (L_\eta \models \phi).$$

Let π_1^1 (σ_1^1) be the least Π_1^1 (Σ_1^1) reflecting ordinal. Aczel and Richter [3] proved that

$$\pi_1^1 = |\Pi_1^1| \overset{\text{def}}{=} \text{closure ordinal of } \Pi_1^1\text{-nonmonotone inductive definitions on } \omega$$

and similarly $\sigma_1^1 = |\Sigma_1^1|$.

THEOREM (2C-6). $\sigma_0 = \pi_1^1$.

REMARK. D. Guaspari [12] has independently shown that σ_0 is Π_1^1-reflecting.

PROOF. We first show that σ_0 is a Π_1^1 reflecting ordinal. Let θ_0 be the least ordinal such that $d_{\sigma_0} = [E_{\theta_0}]_h$. We show that θ_0 is Π_1^1 reflecting which implies that $\sigma_0 = \theta_0$ because σ_0 = order type of admissibles $< \theta_0$.

Let ϕ be a Π_1^1-norm on C_1 and put

$$\alpha \in P \Leftrightarrow \alpha \in C_1 \ \& \ \forall \beta(\phi(\beta) < \phi(\alpha) \ \& \ \alpha \not\leq_h \beta \Rightarrow \beta \text{ is a } \Pi_1^1 \text{ singleton}).\text{(8)}$$

Then $P = \bigcup_{\xi \leq \sigma_0} d_\xi$ and $P \in \Pi_1^1$, because "β is a Π_1^1 singleton" is Π_1^1 as a simple application of the Novikoff-Kondo-Addison uniformization theorem shows. Assume $L_{\theta_0} \models \forall X \phi(X, \xi)$, where X varies over subsets of L_{θ_0}, ϕ is first order over $\langle L_{\theta_0}, \epsilon\rangle$ and $\xi < \theta_0$. Since $\theta_0 < \beta_0$ = least nonindex = least β such that L_β is a β-model of analysis (see [8]), it is clear that $\theta_0 = \lim_{\sigma < \sigma_0} \omega_1^{d_\sigma} = \eta_{\sigma_0}$. So for some $\sigma < \sigma_0$, some $\alpha_\sigma \in d_\sigma$ and some $\beta = \{m\}^{\alpha_\sigma}$, with $\beta \in WO$, we have $|\beta| = \xi$. Let $\{\alpha_\sigma\} = \{\alpha: R(\alpha)\}$, where $R \in \Pi_1^1$. Then $S(E_{\theta_0})$ holds where

$$S(\alpha) \Leftrightarrow \alpha \text{ codes an } L_\theta \ \& \ L_\theta \models \forall X \phi(X, \xi), \text{ where } \xi < \theta \ \& \ \exists \gamma \in \Delta_1^1(\alpha)[R(\gamma) \ \& \ \{m\}^\gamma \in WO \ \& \ |\{m\}^\gamma| = \xi].$$

Since $S \in \Pi_1^1$ we cannot have $P \cap S \subseteq d_{\sigma_0}$ (otherwise d_{σ_0} is the hyperdegree of a Π_1^1 singleton). Thus $S \cap \bigcup_{\sigma < \sigma_0} d_\sigma \neq \varnothing$. Say $\alpha \in S \cap d_{\sigma'}$, where $\sigma' < \sigma_0$. If α codes L_θ, we have $\theta < \theta_0$ and $L_\theta \models \forall X \phi(X, \xi)$.

We prove now that $\sigma_0 = \theta_0$ is the least reflecting ordinal. For that it is enough to show that $E_{\pi_1^1}$ is not a Π_1^1 singleton. If it was, then for some $A \in \Pi_1^1$, $\{E_{\pi_1^1}\} = \{\alpha: \alpha \in A\}$. Say $\alpha \in A \Leftrightarrow \forall Y \phi(\alpha, Y)$, where ϕ is an arithmetical formula and Y varies over subsets of ω. Since $E_{\pi_1^1} \in L_{\pi_1^1+1}$, let $\psi(n, \xi)$ be a first order formula over $\langle L_{\pi_1^1}, \epsilon\rangle$, such that

$$n \in E_{\pi_1^1} \Leftrightarrow L_{\pi_1^1} \models \psi(n, \xi).$$

Then

$$L_{\pi_1^1} \models \forall X[X \subseteq \omega \ \& \ \forall n(n \in X \Leftrightarrow \psi(n, \xi)) \Rightarrow \forall Y(Y \subseteq \omega \Rightarrow \phi(X, Y))].$$

(8) Note that if $\leq$ is a Δ_1^1-good wellordering on C_1 and $\alpha \in P' \Leftrightarrow \alpha \in C_1 \ \& \ \forall \beta(\beta < \alpha \Rightarrow \beta$ is a Π_1^1 singleton) then $P' \in \Pi_1^1$ and all its elements except *one* are Π_1^1 singletons.

Since π_1^1 is Π_1^1 reflecting find $\theta < \pi_1^1$ such that

$$L_\theta \models \forall X[X \subseteq \omega \;\&\; \forall n(n \in X \Leftrightarrow \psi(n, \xi)) \Rightarrow \forall Y(Y \subseteq \omega \Rightarrow \phi(X, Y))].$$

Let $E = \{n \in \omega : L_\theta \models \psi(n, \xi)\}$. Then $E \in A$, so $E = E_{\pi_1^1}$. But $E \in L_{\theta+1}$, a contradiction. □

COROLLARY (2C-7) (RICHTER [36]). *The hyperdegrees in the natural hierarchy of hyperdegrees are all hyperdegrees of* Π_1^1 *singletons.*

PROOF. $\omega_1^{E_1} < \pi_1^1$. □

2D. *Remarks on generalizations.* Some of the results of §2 (and §1) generalize on an arbitrary countable acceptable structure $\mathfrak{A}$ instead of ω (for the terminology see Moschovakis [33]). In particular the existence of largest thin Π_1^1 set, the prewellordering of the Δ_1^1-degrees with the successor stages as in (1B-5), (1B-6) are true for any such $\mathfrak{A}$. The same is true for (2A-2). It is not known if (2C-1) holds in this abstract setting (since basis theorems are not available in general). Nevertheless there is a weaker version of it (due to Moschovakis) which holds and is enough to give the analog of (2C-2)—see [33] for details—as well as (2C-3) and (2C-4).

3. **Higher level countable analytical sets.** Our main purpose in this section is to study those aspects of the structure of C_n for $n > 1$ odd, which are different from C_1. The basic reason for these differences is the fact that the notion of wellordering is Δ_n^1 if $n > 1$ but not Δ_1^1. Our study stems from the Martin-Solovay [29] discovery of a counterexample to the well-known conjecture that the reals recursive in a Σ_n^1 set of integers are a basis for Σ_n^1, when $n > 1$ is odd. (For $n = 1$ this is the classical Kleene basis theorem.) In a sense the results below are the outcome of the author's faith to *a* generalization of the Kleene basis theorem and the Martin-Solovay discovery. Many of the theorems in this section are independently due to Martin and Solovay [29].

3A. *Reflecting pointclasses.*

DEFINITION. Let H be a collection of sets of integers and Θ a collection of sets of reals. We say that H *reflects with respect to* Θ, in symbols $\mathrm{Refl}(H, \Theta)$, if for every $A \subseteq \omega, A \in H$ and every $P \subseteq \mathcal{R}$, $P \in \Theta$ we have

$$P(A) \Rightarrow \exists X \subseteq A(X \in H \cap \check{H} \;\&\; P(X)).$$

We put $\mathrm{Refl}(\Gamma, \Gamma')$ for $\mathrm{Refl}(\Gamma \cap \mathcal{P}(\omega), \Gamma' \cap \mathcal{P}(\mathcal{R}))$, when Γ, Γ' are pointclasses and we call a pointclass Γ *reflecting*, in symbols $\mathrm{Refl}(\Gamma)$, iff $\mathrm{Refl}(\Gamma, \Gamma)$.[9]

Reflection phenomena are very common in recursion theory and we have tried

(9) A notion similar to our Refl (H, Θ) was independently developed by Y. N. Moschovakis [44] in his study of nonmonotone inductive definability. Classes Γ which are reflecting in our sense arise also naturally in recursion theory in higher types as Harrington [45] first discovered.

to capture in our definition at least those which are relevant to our specific subject. Before we come to them however let us give a few other examples:

(a) Let Γ' be an adequate, ω-parametrized pointclass. Let Γ be the pointclass consisting of the sets inductively definable from (nonmonotone) operators in Γ' (see e.g. [3]). Then it is easy to see that $\mathrm{Refl}(\Gamma, \Gamma')$ holds.

(b) Let Γ be an adequate pointclass having the prewellordering property. Let C_Γ be the collection of Γ-*closed* sets of reals (see [34]), where $P \subseteq \mathcal{R}$ is Γ-closed iff for all $R(m,n) \in \Gamma, R' = \{n: P(\{m: R(m,n)\})\}$ is in Γ. Then $\mathrm{Refl}(\Gamma, C_\Gamma)$ holds. [Proof. Let $A \subseteq \omega$ be in Γ and assume $P(A)$ holds, where $P \subseteq \mathcal{R}$ is in C_Γ, while for no $X \subseteq A, X \in \Delta, P(X)$ holds. Then $n \notin A \Leftrightarrow P(\{m: m <^*_\phi n\})$, where ϕ is a Γ-norm on A. Thus $A \in \Delta$, a contradiction.] This fact has some interesting applications. For more, we refer to Moschovakis [34].

The next result gives many examples of reflecting pointclasses. We have originally proved it assuming also that Γ is closed under $\exists^{\mathcal{R}}$ *or* $\forall^{\mathcal{R}}$. The present stronger version is due to L. Harrington.

THEOREM (3A-1). *Let Γ be an adequate pointclass. If $WO \in \Delta$, every set of integers in Γ admits a Γ-norm and, for every $a \in \mathcal{R}$, $\{a\} \in \Delta \Rightarrow a \in \Delta$, then Γ is reflecting.*

PROOF. Let $A \subseteq \omega, A \in \Gamma, P \subseteq \mathcal{R}, P \in \Gamma$ and assume $P(A)$ holds. Suppose towards a contradiction, that for no $X \subseteq A, X \in \Delta, P(X)$ is true. Let ϕ be a Γ-norm on A.

Put $I(\lesssim) \Leftrightarrow \lesssim$ is a (not necessarily proper) initial segment of $\leqslant^\phi$. Then $I \in \Gamma$ and if $P' = \{\lesssim: \lesssim$ is a prewellordering & $\{m: m \lesssim m\} \in P\}, I \cap P' = \{\leqslant^\phi\}$, so (since $P' \in \Gamma$) $\{\leqslant^\phi\} \in \Gamma$.

Similarly (following Martin and Solovay [28]) let

$E(\lesssim) \Leftrightarrow \lesssim$ is a prewellordering which is a (not necessarily proper) end extension of $\leqslant^\phi$.(10)

Then $E \in \check{\Gamma}$ and if

$$G(\lesssim) \Leftrightarrow E(\lesssim) \;\&\; \forall m(m \lesssim m \Rightarrow \neg P(\{n: n \lesssim m \;\&\; m \not\lesssim n\}))$$

we have $G \in \check{\Gamma}$ and $\{\leqslant^\phi\} = G$, so $\{\leqslant^\phi\} \in \check{\Gamma}$ and thus $\{\leqslant^\phi\} \in \Delta$. This shows that $A \in \Delta$, a contradiction. □

COROLLARY (3A-2). $\mathrm{Refl}(\Sigma^1_2)$; $\mathrm{PD} \Rightarrow \mathrm{Refl}(\Pi^1_n)$, $\mathrm{Refl}(\Sigma^1_n)$,(11) *according as $n > 1$ is odd or even.*

(10) A linear preordering is a reflexive, transitive and connected relation. A linear preordering $\lesssim'$ is an *end extension* of linear preordering $\lesssim$ iff (i) $\lesssim \subseteq \lesssim'$, (ii) $\forall x \forall y$ ($x \in \mathrm{Field}(\lesssim)$ & $y \lesssim' x \Rightarrow y \lesssim x$).

(11) $\mathrm{Refl}(\Sigma^1_n)$ for n even is of course also a trivial consequence of the basis theorem.

Thus under PD *the complete* $\Pi^1_n, n > 1$ *odd, set of integers is not a* Π^1_n *singleton,* while of course the complete Π^1_1 set of integers is a Π^1_1 singleton (in particular $\neg\,\mathrm{Refl}\,(\Pi^1_1)$).

The above reflection phenomenon true about all Π^1_n, with $n > 1$ odd, is the source of many interesting properties of these pointclasses which make them look different in some respects from Π^1_1. We mention a few now, omitting most of the details when they take us too far afield.

(A) Translating reflexivity from sets of integers to ordinals via norms we can easily see that if π^1_n (σ^1_n) = least Π^1_n (Σ^1_n) reflecting ordinal (as in Aczel-Richter [3]) and $\delta^1_n = \sup\{\xi: \xi$ is a Δ^1_n ordinal$\}$, then (using PD)

$$\delta^1_{2n+1} = \pi^1_{2n+1} < \sigma^1_{2n+1} \quad \text{for} \quad n > 0$$

(and also $\delta^1_{2n} = \sigma^1_{2n} < \pi^1_{2n}$). For $n = 1$ it is known that $\delta^1_1 < \pi^1_1 < \sigma^1_1$; see Aanderaa [1].

(B) An ω-model of the language of analysis is as usual identified with a set of reals. If M is such a model and σ is a sentence in the language of analysis, $M \models \sigma$ means that σ is true in M. An ω-model of the language of analysis M is called Σ^1_n*-correct* if Σ^1_n formulas of analysis are absolute for M. The next result answers a question of Moschovakis (we abbreviate here $\mathcal{D}^1_n = \{a: a \in \Delta^1_n\}$).

THEOREM. *Assume* PD. *If* $n > 1$ *is odd,* $\mathcal{D}^1_n \not\models$ Determinacy (Δ^1_{n-1}).

PROOF. Assume the conclusion fails, towards a contradiction. Consider

$$\begin{aligned} M(a) \Leftrightarrow \exists\beta \in \Delta^1_n(a)[[\beta] = \{\beta_n: n \in \omega\} \text{ is an } \omega\text{-model of} \\ \Delta^1_n\text{-comprehension } \& \ \Delta^1_{n-1}\text{-determinacy} \\ \text{which is } \Sigma^1_{n-1}\text{-correct}]. \end{aligned}$$

Then $M \in \Pi^1_n$ and if $A \subseteq \omega$ is Π^1_n complete, $M(A)$ holds. Then by $\mathrm{Refl}\,(\Pi^1_n)$, $M(X)$ holds for some $X \in \Delta^1_n$. This contradicts the result of Moschovakis that every ω-model of Δ^1_n-comprehension & Δ^1_{n-1}-determinacy which is Σ^1_{n-1}-correct contains all the Δ^1_n reals. □

REMARK. It is not hard to see that $\mathcal{D}^1_1 \models$ Determinacy (Δ^1_0), where $\Delta^1_0 = \Delta^0_1$ = clopen.

(C) One can easily show, using the Martin-Solovay trick mentioned in the proof of (3A-1), that if

$$\delta^1_{n,a} = \sup\{\xi: \xi \text{ is the length of a } \Delta^1_n(a) \text{ wellordering of } \omega\}$$

then for every n (and assuming PD if $n \geqslant 2$)

$$\delta^1_n < \delta^1_{n,a} \Rightarrow T \leqslant_n a,$$

where T is a complete Π_n^1 or Σ_n^1 set of integers, according as n is odd or even. The same is true for $\lambda_{n,\alpha}^1$ if n is odd instead of $\delta_{n,\alpha}^1$ and was mentioned after (1B-6). Let now n be odd > 1 and P be a complete Π_n^1 set of integers. Then P is not a Π_n^1 singleton, so $\{\alpha: \delta_n^1 < \delta_{n,\alpha}^1\}$ is not Π_n^1 and similarly with λ_n^1. In particular $\{(\alpha, \beta): \delta_{n,\alpha}^1 \leqslant \delta_{n,\beta}^1\}$ is not Σ_n^1 (and similarly for λ_n^1). This of course contrasts the fact that $\omega_1^\alpha \leqslant \omega_1^\beta$ is a Σ_1^1 relation. This latter fact was used repeatedly in the study of C_1 and one can start suspecting that some facts about C_1 will not generalize in a straightforward fashion to $C_n, n > 1$ odd. For example, it is no longer true that $C_3 = \{\alpha: \forall\beta(\delta_{3,\alpha}^1 \leqslant \delta_{3,\beta}^1 \Rightarrow \alpha \leqslant_3 \beta)\}$ (in fact the set on the right is equal to the set Q_3 of the next subsection). It is unknown if replacing $\delta_{3,\alpha}^1$ by $\lambda_{3,\alpha}^1$ will make this equation true. It is known that $C_n \subseteq \{\alpha: \forall\beta(\lambda_{n,\alpha}^1 \leqslant \lambda_{n,\beta}^1 \Rightarrow \alpha \leqslant_n \beta)\}$ and can be proved as in (2A-2). Another result which, although true for C_1, fails for all C_n ($n > 1$ odd) is the uniformization type theorem mentioned at the end of 2A. If it was true then the complete Π_n^1 set of integers would be a Π_n^1 singleton.

3B. *Q-theory.* The reflection phenomena on $\Pi_n^1, n > 1$ odd, discussed in 3A, viewed from the perspective of the theory of countable sets produce a number of interesting results with unexpected consequences in various directions. The most important consequence is related to the basis problem for $\Sigma_n^1, n > 1$ odd, which was settled by Martin and Solovay [29]. Others include questions relevant to higher level analogs of L, inductive definability, etc. Many of the results below have been also proved independently by Martin and Solovay [29].

We fix in this subsection an odd integer $n > 1$.

It is well known that there are quite a few hyperdegrees below the hyperdegree of a complete Π_1^1 set of integers. The following is in sharp contrast with this fact.

THEOREM (3B-1). *Assume* PD. *If* $n > 1$ *is odd, the complete* Π_n^1 *set of integers has minimal* Δ_n^1*-degree.*

PROOF. Let $S = \{\alpha \in WO: |\alpha| \geqslant \delta_n^1\}$. Then if $P \subseteq \omega$ is Π_n^1 complete one can see, using the Martin-Solovay trick mentioned in (3A-1), that $\forall\alpha \in S(P \leqslant_n \alpha)$. Let $H(S) = \{\beta: \forall\alpha \in S(\beta \leqslant_n \alpha)\}$. Then, since $S \in \Sigma_n^1, H(S) \in \Pi_n^1$ and in fact $H(S)$ is a countable initial segment of the Δ_n^1-degrees, so in particular it is an initial segment of C_n, say $H(S) = \bigcup_{\xi<\xi_0} d_\xi^n$, where, since $P \in H(S)$, we have $\xi_0 \geqslant 2$. If now d is a Δ_n^1-degree and $d < d_1^n = [P]_n$ we have $d \subseteq H(S)$, therefore $d = d_0^n = [\lambda t o]_n$.

REMARK. It is easy to see using (1B-2), (1B-6) that for $n \geqslant 2$ even, the complete Σ_n^1 set of integers has minimal Δ_n^1-degree too.

Theorem (3B-1) shows that there are nontrivial initial segments of C_n which are Δ_n^1-closed i.e. they are initial segments of the Δ_n^1-degrees. Let τ_n be the greatest τ such that $\bigcup_{\xi<\tau} d_\xi^n$ is closed under $\leqslant_n$. Then a simple extension of (3B-1) shows that $\tau_n \geqslant \omega$. It will turn out that τ_n is actually very large. To prove this and other things we need first to make explicit an idea involved in the proof of (3B-1).

DEFINITION. Let $\emptyset \neq S \subseteq R$ be Σ_n^1. The *hull* of S, in symbols $H(S)$, is given by

$$H(S) = \{a: \forall\beta(\beta \in S \Rightarrow a \leqslant_n \beta)\}.$$

Clearly $H(S)$ is a countable Π_n^1 set and an initial segment of the Δ_n^1-degrees. We call any set $H(S)$ with $S \neq \emptyset$, $S \in \Sigma_n^1$ a Σ_n^1*-hull.*

THEOREM. (3B-2). *Assume* PD *if* $n > 1$. *For any odd* n, *if* $A \in \Pi_n^1$ *is a* Σ_n^1*-hull, then* A *contains no* Π_n^1 *singleton which is not* Δ_n^1.

PROOF. Assume $\{a\} \in \Pi_n^1$ and $a \in A$. Then if $A = H(S)$ we have

$$a(n) = m \Leftrightarrow \forall\beta[\beta \in S \Rightarrow \exists a' \in \Delta_n^1(\beta)(a' \in \{a\} \,\&\, a'(n) = m)].$$

Thus $a \in \Delta_n^1$. □

Another interesting property of hulls is the following: Let $H(S)$ be a hull and $P(a, \beta) \in \Pi_n^1$. Then $\exists a \in H(S) P(a, \beta)$ is also Π_n^1 (being equivalent to $\forall a[a \in S \Rightarrow \exists a' \leqslant_n a(a' \in H(S) \,\&\, P(a', \beta))]$.

DEFINITION. A set $A \subseteq R$ is called Π_n^1*-bounded* if for all $P(a, \beta)$ in Π_n^1, the set $\exists a \in A P(a, \beta)$ is also Π_n^1.

REMARK. One can see using the results in 2C that $A \subseteq R$ is Π_1^1-bounded iff $A \subseteq \{a: a \in \Delta_1^1\}$ and $A \in \Pi_1^1$ i.e. A is a Σ_1 subset of L_{ω_1}.

It is obvious that if A is Π_n^1-bounded then $A \in \Pi_n^1$, while $\bar{A} = \{\beta: \exists a \in A(\beta \leqslant_n a)\}$ is also Π_n^1-bounded and closed under $\leqslant_n$. Every Σ_n^1-hull $H(S)$ is a countable Π_n^1-bounded set closed under $\leqslant_n$. It is also easy to see that every countable Π_n^1-bounded set A is contained in a Σ_n^1-hull. [Take $S = \{\beta: \forall a \in A(a \leqslant_T \beta)\}$. Then $A \subseteq H(S)$.] Thus

$$\bigcup_{S\in\Sigma_n^1, S\neq 0} H(S) = \bigcup\{A: A \text{ is a } \Pi_n^1\text{-bounded set}\} \overset{\text{def}}{=} \bigcup_{\xi<q_n} d_\xi^n.$$

Clearly $q_n \leqslant \tau_n$. In fact, as Martin and Solovay [29] proved, $q_n = \tau_n$.

At this stage we introduce a very natural hull, called Q_n, which eventually turns out to be the largest one. Its most intuitive description is probably the following: Q_n is the set of all reals which belong to every standard model of ZFC + PD which is Σ_{n-1}^1-correct, i.e. makes Σ_{n-1}^1 formulas absolute. We shall nevertheless choose a slightly more technical definition of Q_n, which makes its study more straightforward (and has some advantages when generalizations are considered). *The definition of* Q_n *and all the results in the rest of this subsection are due to Martin and Solovay* [29] *and independently to the author, except when otherwise explicitly stated.*

DEFINITION. Let

$$a \in S_n \Leftrightarrow [a] = \{a_m : m \in \omega\} \text{ is closed under pairing and (i) } [a] \text{ is closed under } \leqslant_n \text{ and the } \Delta^1_n\text{-jump, \& (ii) } C_n \cap [a] \text{ is countable in } [a].$$

It is easy to see that $S \in \Sigma^1_n$ [(i) translates to $\forall m \forall \beta \leqslant_n a_m \exists k(\beta = a_k)$ & $\forall m \exists k(a_k \in WO \;\&\; \delta^1_{n,a_m} \leqslant |a_k|)$, while (ii) translates to $\exists k \forall m[a_m \in C_n \Rightarrow \exists l(a_{k,l} = a_m)]]$.

Let now $Q_n = \{\beta : \forall a \in S(\beta \in [a])\}$. Clearly Q_n is closed under $\leqslant_n$ and the Δ^1_n-jump and is a Π^1_n-bounded countable set.

LEMMA. *Assume* PD & $n > 2$ *odd. If* ϕ *is a* Π^1_n*-norm on the countable* Π^1_n*-bounded set* A, *then* $\forall a \in A \forall \beta(\beta \in WO \;\&\; \phi(a) \leqslant |\beta| \Rightarrow a \leqslant_n \beta)$.

PROOF. Let

$$T(\gamma, \lesssim) \Leftrightarrow \lesssim \text{ is a prewellordering on } \omega \;\&\; \forall m \forall n(m \neq n \Rightarrow \gamma_m \neq \gamma_n) \;\&\; \forall a \in A \exists m(a = \gamma_m) \;\&\; \forall m \forall n(\gamma_m, \gamma_n \in A \Rightarrow [\phi(\gamma_m) \leqslant \phi(\gamma_n) \Leftrightarrow m \lesssim n])$$

Clearly $T \in \Sigma^1_n$. Now notice that if $\beta \in WO$ is such that $|\beta| < |\phi|$ we have

$$a \in A \;\&\; \phi(a) < |\beta| \Leftrightarrow \exists \gamma \exists \lesssim [T(\gamma, \lesssim) \;\&\; (\exists n)(n \lesssim n \;\&\; \{m : m \lesssim n\} \text{ has length } < |\beta| \;\&\; \phi(a) = \phi(\gamma_n))]$$

so $\{a \in A : \phi(a) < |\beta|\}$ is a countable $\Sigma^1_n(\beta)$ set, so it contains only $\Delta^1_n(\beta)$ reals. □

THEOREM (3B-3) (MARTIN-SOLOVAY [29]). *Assume* PD & $n > 1$ *odd. Then* Q_n = *largest* Σ^1_n*-hull* (= *largest* Π^1_n*-bounded set*).

PROOF. Let A be a Σ^1_n-hull. Then $A \subseteq C_n$. Let $a \in S_n$. We have to prove $A \subseteq [a]$. Let ξ be the least ordinal such that $d^n_\xi \subseteq A - [a]$, towards a contradiction. Let ϕ be a Π^1_n-norm on C_n. Pick $\beta \in d^n_\xi$ with least $\phi(\beta)$. Then $\phi(\beta) = \sup \{\delta^1_{n,\gamma} : \gamma \in \bigcup_{\eta<\xi} d^n_\eta\}$. But $\bigcup_{\eta<\xi} d^n_\eta \subseteq C_n \cap [a]$ which is countable in $[a]$. Thus we can find k such that $a_k \in WO$ and $|a_k| \geqslant \sup \{\delta^1_{n,\gamma} : \gamma \in \bigcup_{\eta<\xi} d^n_\eta\} = \phi(\beta)$. Then by the lemma $\beta \leqslant_n a_k$ so $\beta \in [a]$, a contradiction. □

Thus $Q_n = \bigcup_{\xi<q_n} d^n_\xi$ is the largest Σ^1_n-hull. In particular no Π^1_n

singleton which is not Δ^1_n belongs to Q_n. Martin and Solovay [29] proved that $d^n_{q_n}$ is the first place where a Π^1_n nontrivial singleton appears and moreover that this first nontrivial Π^1_n singleton is a basis for Σ^1_n sets of reals i.e. every non-empty Σ^1_n set of reals contains a real Δ^1_n in it. This is the appropriate generalization of the Kleene basis theorem, since of course the complete Π^1_1 set of integers is the first (in the $\leqslant_1$ sense) nontrivial Π^1_1 singleton. From the Martin-Solovay result it is clear that Q_n is the largest initial segment of C_n closed under $\leqslant_n$ and thus also the largest countable Π^1_n set closed under $\leqslant_n$ (in particular $q_n = \tau_n$). It has also many other characterizations like for example $Q_n = \{a\colon a$ is Δ^1_n in some ordinal$\}$, [where "a is Δ^1_n in ξ" iff "$a \leqslant_n \beta$ for all $\beta \in WO$ with $|\beta| = \xi$"], for which we refer the reader to [29].

Looking in the periodicity picutre of the analytical hierarchy, we see that Q_n plays in many (but not all) respects the role of $\{a\colon a \in \Delta^1_1\}$, for the odd levels $n > 1$. In particular many characterizations (but not all) of $\{a\colon a \in \Delta^1_1\}$ hold (appropriately generalized) for Q_n e.g. Q_n = the intersection of all ω-models of analysis + PD, which are Σ^1_{n-1}-correct. Another interesting aspect of the Q-theory is the notion of degree it creates. Relativize Q_n to any real β and denote it by Q^β_n. Write $a \leqslant^Q_n \beta \Leftrightarrow a \in Q^\beta_n$. The Q_n-degree of a, in symbols, $[a]^Q_n$ is $\{\beta\colon a \leqslant^Q_n \beta \mathbin{\&} \beta \leqslant^Q_n a\}$. It turns out that there is a reasonable Q_n-jump (for example the Q_n-jump of the trivial degree is the Q_n-degree of the first nontrivial Π^1_n singleton). Moreover there is a natural (uncountable) ordinal assignment associated with the Q_n-degrees that plays the same role that the assignment $a \mapsto \omega^a_1$ plays in the theory of Δ^1_1-degrees. In particular it seems very probable (and has been already done in some cases) that many results concerning Δ^1_1-degrees (and their connection with countable Π^1_1 sets) which are known to fail when naively generalized to Δ^1_n-degrees, will find appropriate generalizations to Q_n-degrees. We shall not pursue this matter here.

And we conclude this section by proving (as we promised earlier) that q_n is a fairly large ordinal i.e. Q_n goes considerably beyond $\{a\colon a \in \Delta^1_n\}$.

Definition. For each limit $\lambda < \rho_n$ = length of the hierarchy of Δ^1_n-degrees in C_n, let

$$\eta_\lambda = \sup\left\{\delta^1_{n,a}\colon a \in \bigcup_{\xi<\lambda} d^n_\xi\right\}.$$

Theorem (3B-4). *Assume* PD & $n > 1$ *odd. Let* $Q_n = \bigcup_{\xi<q_n} d^n_\xi$. *Then* $q_n = \eta_{q_n}$.

Proof. Assume not, i.e. $q_n < \eta_{q_n}$. Pick $a^* \in Q_n$ such that $a^* \in WO \mathbin{\&} |a^*| = q_n$. Let $<$ be the Δ^1_n-good wellordering on C_n. Define a real β as follows:

For each $n \in \omega$,

$$\beta_m = \text{the } <\text{-least element of } d^n_{|m|_{a^*}},$$

where thinking of a^* as coding a wellordering $\leqslant_{a^*}$ on ω, $|m|_{a^*}$ = length of the initial segment determined by m. Clearly $\beta \notin Q_n$. If now $a \in S_n$ we have $a^* \in [a]$, thus if a_k enumerates Q_n, β is Δ^1_n in $\langle a^*, a_k\rangle$, so $\beta \in [a]$ i.e. $\beta \in Q_n$, a contradiction. □

REMARK. One can actually show that q_n is a limit of λ's such that $\eta_\lambda = \lambda$ etc. See also Theorem (3C-2).

3C. *Connections with higher-level analogs of* L. We have seen already that C_2 is the set of reals in L and C_1 is a nice "trunk" for the hierarchy of hyperdegrees of constructible reals. A similar situation is true in higher levels, only that this time the models were introduced after the C_n's have been defined. We summarize most of the known facts about even higher-level analogs of L and their connections to countable analytical sets in the next theorem, whose proof will appear elsewhere (see also [15], [16]). Parts (1), (2) are due to Moschovakis and (3), (4) to the author.

THEOREM (3C-1) (KECHRIS-MOSCHOVAKIS). *Assume* PD. *For* $n \geqslant 2$ *even, let* C_n *be the largest countable* Σ^1_n *set and put*

$$L^n = L(C_n) = \textit{smallest model of ZF containing all the ordinals and } C_n \textit{ as an element.}$$

Then:

(1) $C_n = L^n \cap R$.
(2) Σ^1_n *formulas are absolute for* L^n.
(3) $L^n \models$ GCH + "*There is a* Δ^1_n*-good wellordering of* R".
(4) $L^n \models$ Determinacy $(\Delta^1_{n-1}) + \neg$ Determinacy (Σ^1_{n-1}).

REMARK. From this fact it is now clear that for each even n,

$$\rho_n = \rho_{n-1} = \aleph_1^{L^n}.$$

Thus the existence of reasonable higher-level analogs for L has been established for all even levels. What about odd levels? If a model M of set theory does not contain all reals and is Σ^1_n-correct, with $n > 1$ odd, then $M \cap R \notin \Sigma^1_n$ (see [17]) and $M \cap R \notin \Pi^1_n$ (obvious). Instead of M being Σ^1_n-closed we can require only that M is closed under Δ^1_n (after all if n is even this is equivalent). Again by [17] $M \cap R$ cannot be Σ^1_n. Could it be Π^1_n? The rather surprising answer is yes as the next theorem shows. Its proof is another application of the Q-theory and will be omitted here.

THEOREM (3C-2) (KECHRIS, MARTIN-SOLOVAY). *Assume* PD & $n > 1$

odd. Let $L^n = L(Q_n)$. *Then*

(i) $Q_n = L^n \cap \mathcal{R}$,

(ii) L^n *is* Δ^1_n*-closed (thus in particular it is* Σ^1_{n-1}*-correct),*

(iii) $L^n \models$ GCH + *"There is a* Δ^1_n*-good wellordering of* $\mathcal{R}$*".*

The zig-zag picture for higher-level analogs of L looks now like

$$L = L^2 \qquad\qquad L(C_4) = L^4$$

$$L(Q_3) = L^3 \qquad\qquad L(Q_5) = L^5$$

and suggests a certain similarity between Q_{2n+1} and C_{2n}, when $n > 0$ which also extends to other directions.

There is a very nice "duality" property of the models $L(Q_n)$, which was noted also independently by Martin and Solovay. If $P(\alpha)$ is a Π^1_n relation, there is a Σ^1_n relation $P^*(\alpha)$ (gotten explicitly from P) so that

$$\alpha \in L^n \Rightarrow [P(\alpha) \Leftrightarrow L^n \models P^*(\alpha)],$$

and similarly interchanging Σ^1_n with Π^1_n. For example if $A \subseteq \omega$ then

$$A \in \Sigma^1_n \Leftrightarrow L^n \models \text{"}A \text{ is } \Pi^1_n\text{"}.$$

In a sense, every Π^1_n (Σ^1_n) property passing through the threshold of the $L(Q_n)$ world is immediately transformed to a Σ^1_n (Π^1_n) property. This illustrates nicely the duality between the theories "ZFC + PD" and "ZFC + There exists a Δ^1_n-good wellordering of $\mathcal{R}$" and has some interesting and amusing consequences.

4. **Countable analytical sets in models of set theory.** We have studied in the previous sections the structure of countable analytical sets in the real world. We look now at the situation in various models of set theory, especially inner models. This also leads to independence and consistency results.

4A. *On countable* Σ^1_3 *sets.* We have seen that (under PD) the only analytical pointclasses that have largest countable sets are exactly those which have the prewellordering property, namely Π^1_n for n odd and Σ^1_n for n even. In particular there is no largest countable Σ^1_3 set. It is thus of some interest to notice here that there is a model of set theory having a measurable cardinal in which there is a largest countable Σ^1_3 set. As we shall see later, in $L[\mu]$, where μ is a normal measure on a measurable cardinal, there is no largest countable Σ^1_3 set. In particular the existence of a largest countable Σ^1_3 set cannot be settled in "ZFC + There exists a measurable cardinal".

PROPOSITION (4A-1). *Assume there is a countable standard model of* "ZFC + *There exists a measurable cardinal*". *Then there is a standard model of* *the same theory in which a largest countable* Σ^1_3 *set exists.*

PROOF. Let M be a countable standard model of ZFC + $V = L[\mu]$, where $M \models$ "μ is a normal measure on a measurable cardinal κ". Let G be a generic map collapsing $\aleph_1^M$ to ω. Let $N = M[G]$. *In* N, let T be an ordinal definable tree on $\omega \times \text{ORD}$, so that, if $S \subseteq R$ is a complete Σ^1_3 set, $S = p[T]$. (For the construction of such a T, see Mansfield [26].) Work now in N. If $A \in \Sigma^1_3$ and $A - L[T] \neq \varnothing$, then A contains a perfect set. So if $A \in \Sigma^1_3$ is countable, $A \subseteq L[T]$. But $L[T]$ consists of ordinal definable sets, thus $L[T] \subseteq M$. Now $M = L[\mu]$, so that $R \cap M = R \cap L[\mu] \in \Sigma^1_3$ by Silver [40]. Clearly $R \cap M$ is countable (in N), since $M \models$ CH, so $R \cap M \subseteq L[T]$. Thus $R \cap M = R \cap L[T] = R \cap L[\mu]$ is the largest countable Σ^1_3 set.

4B. *Countable analytical sets in inner models.* As one should probably expect, the picture of countable analytical sets in inner models is radically different from the one we have seen in the previous sections, assuming PD. We have here

THEOREM (4B-1) (KECHRIS-MOSCHOVAKIS). *Assume there is a* Δ^1_n*-good wellordering of* R. *Then there is no largest countable* Σ^1_n *or* Π^1_{n-1} *set of reals.*

PROOF. Assume $\leqslant$ is a Δ^1_n-good wellordering of R. Let, towards a contradiction, C be the largest countable Σ^1_n set of reals. Notice that C is then a $\leqslant$-initial segment of R. Let

$$\begin{aligned} P(\alpha) &\Leftrightarrow \alpha \text{ codes an } \leqslant\text{-initial segment of } C \\ &\Leftrightarrow \forall n(\alpha_n \in C) \;\&\; \forall n \forall \beta < \alpha_n \exists m(\beta = \alpha_m). \end{aligned}$$

Clearly $P \in \Sigma^1_n$. Let

$$A(\alpha) \Leftrightarrow \alpha \in P \;\&\; \forall \beta < \alpha(\{\alpha_n : n \in \omega\} \neq \{\beta_n : n \in \omega\}).$$

Then $A \in \Sigma^1_n$ and is countable. So $A \subseteq C$. But clearly some $\beta \in A$ codes all of C, a contradiction.

To prove now the result about Π^1_{n-1} we need only establish the following lemma (of independent interest in the study of inner models). Its proof is an adaptation of a trick of Solovay (see [40, p. 440]).

LEMMA. *Assume there is a* Δ^1_n*-good wellordering of* R. *Then for any* $A \subseteq R$,

$$A \in \Sigma^1_n \Leftrightarrow A \textit{ is the 1-1 recursive image of a } \Pi^1_{n-1} \textit{ set.}$$

PROOF. Let $A \subseteq R$. Then $\Leftarrow$ is obvious. To prove $\Rightarrow$ let $A \in \Sigma^1_n$, say $\alpha \in A \Leftrightarrow \exists \beta(\alpha, \beta) \in B$, with $B \in \Pi^1_{n-1}$. Call a real γ *nice* if $[\gamma] = \{\gamma_n : n \in \omega\}$ makes Π^1_{n-1} formulas absolute and for all l we can find m, k such that $P(\alpha_m, \alpha_k, \alpha_l)$ and $[\alpha_k] \subseteq [\alpha]$, where $\text{InSeg}_{\leqslant}(\delta, \alpha) \Leftrightarrow \exists \epsilon P(\epsilon, \delta, \alpha)$, with $P \in \Pi^1_{n-1}$. Clearly $N(\gamma) \Leftrightarrow$ "γ is nice" is Π^1_{n-1}. Moreover by a simple Skolem-Löwenheim argument we have $\forall \alpha \exists \gamma(\alpha \in [\gamma] \;\&\; N(\gamma))$. Define now

$(\alpha, \beta) \in B^* \Leftrightarrow$ (i) β_0 is the $<$-least element of $\{\beta: (\alpha, \beta) \in B\}$,
(ii) β_{i+1} is the $<$-least element of N such that $\alpha, \beta_i \in [\beta_{i+1}]$.

Clearly $\alpha \in A \Leftrightarrow \exists\beta(\alpha, \beta) \in B^* \Leftrightarrow \exists!\beta(\alpha, \beta) \in B^*$, while it is easy to see that $B^* \in \Pi^1_{n-1}$. □

REMARKS. (A) Since the existence of a Δ^1_n-good wellordering of $\mathcal{R}$ implies the existence of an uncountable $A \subseteq \mathcal{R}$ with $A \in \Delta^1_n$ such that both $A, \mathcal{R} - A$ are thin it is clear that under this assumption no largest thin Σ^1_n set can exist.

(B) D. Guaspari has discovered a very nice extension of (4B-1) namely:

If there is a Δ^1_n-good wellordering of $\mathcal{R}$ then the union of all countable Σ^1_n sets is exactly the set of all Δ^1_{n+1} reals.

His theorem can be also proved by the methods used here, noticing that if $\leqslant$ is a Δ^1_n-good wellordering of $\mathcal{R}$ then every Π^1_n singleton $\{\alpha\}$ belongs to the countable Σ^1_n set $\{\beta: \beta \leqslant \alpha\}$, while every Δ^1_{n+1} real is recursive in a Π^1_n singleton.

As a corollary to (4B-1) we obtain the converse to Solovay's theorem that

$$\text{card}\,(\mathcal{R} \cap L) = \aleph_0 \Rightarrow \text{There exists a largest countable } \Sigma^1_2 \text{ set of reals.}$$

COROLLARY (4B-2). *If there exists a largest countable Σ^1_2 set of reals then there are only countably many constructible reals.*

PROOF. Let $\mathcal{C}$ be the largest countable Σ^1_2 set of reals. Then $\mathcal{C} \subseteq L$ and $\mathcal{C}$ is an initial segment of the Δ^1_2-good wellordering of $L \cap \mathcal{R}$. If it is a proper one, then $L \models$ "$\mathcal{C}$ is the largest countable Σ^1_2 set of reals", contradicting (4B-1). So $L \cap \mathcal{R} = \mathcal{C}$. □

REMARKS. (A) There is an appropriate generalization of this corollary to higher levels (see e.g. [15]).

(B) D. Guaspari has proved that if there is a largest thin Σ^1_2 set then $\mathcal{R} \cap L$ is thin.

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