ON THE DIMENSION OF VARIETIES OF SPECIAL DIVISORS

BY

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ABSTRACT. Let \( T_g \) denote the Teichmüller space and let \( V \) denote the universal family of Teichmüller surfaces of genus \( g \). Let \( V^{(n)}_g \) denote the \( n \)th symmetric product of \( V \) over \( T_g \) and let \( J \) denote the family of Jacobians over \( T_g \). Let \( f: V^{(n)}_g \rightarrow J \) be the natural relativization over \( T_g \) of the classical \( g \circ \) map defined by integrating holomorphic differentials. Let

\[ u: f^*\Omega^1/J \rightarrow \Omega^1/V^{(n)}_g \]

be the map induced by \( f \). We define \( G^r_n \) to be the analytic subspace of \( V^{(n)}_g \) defined by the vanishing of \( \Lambda^{n-r+1}u \).

Put \( \tau = (r + 1)(n - r) - rg \). We show that \( G^1_n - G^2_n \), if nonempty, is smooth of pure dimension \( 3g - 3 + \tau + 1 \). From this result, we may conclude that, for a generic curve \( X \), the fiber of \( G^1_n - G^2_n \) over the module point of \( X \), if nonempty, is smooth of pure dimension \( \tau + 1 \), a classical assertion.

Variational formulas due to Schiffer and Spencer and Rauch are employed in the study of \( G^r_n \).

0. Introduction. Let \( X \) be a complete, nonsingular curve of genus \( g \) over an algebraically closed field \( K \). Let \( X^{(n)} \) denote the \( n \)th symmetric product of \( X \). Let \( G^r_n(X) \) denote the subvariety of \( X^{(n)} \) of all divisors \( D \) of degree \( n \) such that \( \dim |D| \geq r \). (In the literature, e.g. [12], \( G^r_n(X) \) is often used to denote the subvariety of the Jacobian of \( X \) consisting of all linear systems of degree \( n \) and projective dimension at least \( r \).)

Put \( \tau \) equal to \( (r + 1)(n - r) - rg \). Brill and Noether [2] asserted that if \( \tau \) were nonnegative and \( X \) were a generic curve, then \( G^r_n(X) \) would have dimension \( \tau + r \). The recent work of Kleiman and Laksov ([10], [11]) and Kempf [8] shows that for \( X \) any curve, if \( \tau \geq 0 \), then \( G^r_n(X) \) has dimension \( \geq \tau + r \). We will show, in the case \( K = \mathbb{C} \), that if \( X \) is a generic curve, then \( G^1_n(X) - G^2_n(X) \), if nonempty, has dimension \( \tau + 1 \).
We work in the category of analytic spaces over \( \mathbb{C} \). We do this because we want to consider the Teichmüller space, an analytic, but not algebraic, variety [5]. We take the Séminaire Cartan, 1960–61, as our foundational reference. In particular, we allow the structure sheaf of an analytic space to contain nilpotents.

Let \( Y \) be an analytic space over \( \mathbb{C} \) and let \( E \) and \( F \) be locally free \( \mathcal{O}_Y \)-modules of ranks \( g \) and \( n \) respectively. Suppose we are given a map \( u: E \to F \). In §1, we define the analytic space \( Z^r(u) \) to be given by the vanishing of the map \( \bigwedge^{n-r+1} u \). We then study the infinitesimal structure of \( Z^r(u) \).

Let \( S \) be an analytic space over \( \mathbb{C} \) and let \( X \) be a family of nonsingular curves of genus \( g \) over \( S \). Let \( X_S^{(n)} \) denote the \( n \)th symmetric product of \( X \) over \( S \) and let \( J_S \) denote the family of Jacobians over \( S \) (cf. [7], [15]). Suppose we are given a map \( f: X_S^{(n)} \to J_S \). Let

\[
u: f^* \Omega^1_{J_S/S} \to \Omega^1_{X_S^{(n)}/S}
\]

be the map induced by \( f \). We study the analytic space \( Z^r(u) \subset X_S^{(n)} \) in the following situation: \( S = T_g \), the Teichmüller space, \( X \) is the universal family of Teichmüller surfaces of genus \( g \), and \( f \) is the natural relativization over \( T_g \) of the classical map from the \( n \)th symmetric product of a curve into its Jacobian defined by integrating a basis of homomorphic differentials (cf. §2). We let \( G^r_n \) denote \( Z^r(u) \) in this situation.

In order to understand explicitly the above map \( f \), we must use certain variational formulas which are similar to those derived by Schiffer and Spencer [19], but much closer in form to those appearing in Rauch [18]. We also need a theorem due to Patt [17] concerning local coordinates at a point of \( T_g \).

Our main result is:

**Theorem.** Suppose \( y \in G^1_n - G^2_n \). Then the dimension of the tangent space to \( G^1_n \) at \( y \) is \( 3g - 3 + \tau + 1 \).

From this result, we can conclude that if \( X \) is a generic compact Riemann surface, then \( G^1_n(X) - G^2_n(X) \), if nonempty, is smooth of pure dimension \( \tau + 1 \).

As an application, we show that the subvariety of \( T_g \), for \( g \geq 4 \), of curves with nonempty \( G^1_3 \) (so-called "trigonal" curves), is of dimension \( 2g + 1 \), a result which was known to Severi [22] and B. Segre [20].

In a sequel to this paper, we will show that if \( \tau \geq 0 \) then \( G^2_n \) (resp. \( G^3_n \)) has a component of dimension \( 3g - 3 + \tau + 2 \) (resp. \( 3g - 3 + \tau + 3 \)). The proof involves computations using the examples of Riemann surfaces given by Meis [16].

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1. $Z'(u)$ and its infinitesimal structure. Let $S$ be an analytic space over $C$. Denote by $((\text{An}/S))$ the category of analytic spaces over $S$. Let $Y$ be an analytic space over $S$ and let $E$ and $F$ be locally free $\mathcal{O}_Y$-modules of ranks $g$ and $n$ respectively. Suppose we are given a map $u: E \to F$. Define the functor $Z'(u): ((\text{An}/S))^0 \to ((\text{Sets}))$ by

$$Z'(u)(T) = \left\{ g \in \text{Hom}(T, Y) \mid \bigwedge^{n-r+1} g^*u = 0 \right\}.$$

We wish to show that this functor is represented by an analytic subspace of $Y$.

**Definition 1** [5]. Let $S$ be an analytic space and let $G: ((\text{An}/S))^0 \to ((\text{Sets}))$ be a functor. We say that $G$ is of a local nature if for every $T$ the presheaf $U \mapsto G(U)$, where $U$ runs through the open sets of $T$, is a sheaf.

**Remark.** This is the analog to the notion of a Zariski sheaf in the category of contravariant functors from $((\text{Schemes}))$ to $((\text{Sets}))$.

**Lemma 1.** Let $(S_i)$ be a covering of an analytic space $S$ by open sets. Let $G: ((\text{An}/S))^0 \to ((\text{Sets}))$ be a functor. Then $G$ is representable iff $G$ is of a local nature and for every $i$, the functor $G/S_i: ((\text{An}/S_i))^0 \to ((\text{Sets}))$ is representable.

**Proof.** [5, Corollary 5.7 of Exposé 7].

Our functor $Z'(u)$ is clearly of a local nature. Hence, by the lemma, its representability is a local question.

Let $y$ be a point of $Y$. Since $E$ and $F$ are locally free of ranks $g$ and $n$ respectively, the map $u$ is given locally at $y$ by an $n \times g$ matrix $\begin{bmatrix} f_{jk} \end{bmatrix}$ of functions regular at $y$. The functor $Z'(u)$ is then locally represented by the closed analytic subspace defined by the vanishing of the minors of order $n-r+1$ of the matrix $\begin{bmatrix} f_{jk} \end{bmatrix}$. Thus we have

**Proposition 1.** $Z'(u)$ is represented by a closed analytic subspace of $Y$.

We will also use $Z'(u)$ to denote this analytic subspace.

Put $\rho = \text{rank}(u \otimes \kappa(y))$. Locally at $y$, both $E$ and $F$ split off a direct summand of rank $\rho$, and $u$ maps one summand isomorphically onto the other. The map that $u$ induces on the other two summands is given by an $(n-\rho) \times (g-\rho)$ matrix $\begin{bmatrix} e_{jk} \end{bmatrix}$ of functions regular at $y$. The analytic space $Z'(u)$ is also defined locally at $y$ by the vanishing of the minors of order $(n-r+1-\rho)$ of the matrix $\begin{bmatrix} e_{jk} \end{bmatrix}$ (cf. [10]).

**Proposition 2.** Assume $r > 0$. Then the points of $Z^{r+1}(u)$ are singular points of $Z'(u)$. 
Proof. Suppose \( y \in Z^{r+1}(u) \). Then we have \( \rho < n - r \). By construction, the \( e_{jk} \) above vanish at \( y \), hence are in the maximal ideal \( m \) of \( O_{Y,y} \). The analytic space \( Z'(u) \) is defined locally at \( y \) by the vanishing of the minors of order \((n - r + 1 - \rho)\) of the matrix \([e_{jk}]\) and, since \( \rho < n - r \), all these minors are of order at least 2, hence are in \( m^2 \). Thus \( y \) cannot be a smooth point of \( Z'(u) \).

We want now to study the infinitesimal structure of \( Z'(u) \). Let \( \xi \) denote a tangent vector to \( Y \) at \( y \). We will also use \( \xi \) to denote the comorphism, which is a \( \mathbb{C} \)-homomorphism of local rings \( \xi : O_{Y,y} \to \mathbb{C}[\epsilon]/(\epsilon^2) \).

We are interested in seeing when \( \xi \) is a tangent vector to \( Z'(u) \) at \( y \). By definition, this will be true if \( \wedge^{n-r+1} \xi^* u = 0 \).

**Proposition 3.** \( \xi \) is a tangent vector to \( Z'(u) \) at \( y \) iff the minors of order \( n-r+1 \) of the matrix \([\xi(f_{jk})]\) are all zero.

**Proof.** It is easy to see that the map \( \xi^* u \) is given by the matrix \([\xi(f_{jk})]\). Thus we have \( \wedge^{n-r+1} \xi^* u = 0 \) iff the minors of order \( n-r+1 \) of \([\xi(f_{jk})]\) all vanish.

We now assume that \( Y \) is smooth of dimension \( m \) over \( \mathbb{C} \). Let \( y \in Y \) and let \( \sigma_1, \ldots, \sigma_m \) be local parameters on \( Y \) at \( y \). Let \( s_t \) in \( \mathbb{C} \) be given by

\[
\xi(\sigma_l) = s_l \epsilon, \quad l = 1, 2, \ldots, m.
\]

Then, by Taylor's Theorem, we have

\[
\xi(f_{jk}) = f_{jk}(y) + \epsilon \sum_{l=1}^{m} s_l \frac{\partial f_{jk}}{\partial \sigma_l}(y).
\]

The vanishing of the minors of order \( n-r+1 \) of the matrix \([\xi(f_{jk})]\) gives rise to linear equations in the \( s_l \). These equations must be satisfied for \( \xi \) to be a tangent vector to \( Z'(u) \) at \( y \). If we view \( s_1, \ldots, s_m \) as being unknowns, then the dimension of the solution space of this system of equations is the dimension of the tangent space to \( Z'(u) \) at \( y \).

If \( y \in Z'(u) - Z^{r+1}(u) \), we will want to use the following lemma.

**Lemma 2.** Let \( A \) be a commutative ring (with unit). Let \( M = [a_{jk}] \) be an \( m \times n \) matrix over \( A \). Suppose that a minor \( \mu \) of order \( r \) is a unit, and that every minor of order \( r+1 \) containing \( \mu \) vanishes. Then every minor of order \( r+1 \) vanishes.

**Proof.** Without loss of generality, we may assume that \( \mu \) is the leading (i.e., upper left) minor of order \( r \). Since \( \mu \) is a unit, we may perform column operations using the first \( r \) columns to change \( M \) to the matrix
where $N$ is an $(m - r) \times (n - r)$ matrix.

Then by row operations, using the first $r$ rows, we may change $M'$ to the matrix

$$M'' = \begin{bmatrix}
\mu & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & N
\end{bmatrix}$$

where $N$ is the same matrix as before.

Now, no minor containing $\mu$ is affected by performing these row and column operations. Hence, the minors of order $r + 1$ of $M''$ which contain $\mu$ are all zero. Thus $N$ is the zero matrix.

But this implies that every column of $M$ is a linear combination of the first $r$ columns of $M$. Hence, every minor of order $r + 1$ of $M$ is zero. □

Suppose now that $y \in Z'(u) - Z'^{r+1}(u)$. Then the matrix $[f_{jk}]$ has rank $n - r$. We may thus assume that the leading minor of order $n - r$ of $[f_{jk}]$, call it $\mu$, is nonzero. Let $\mu'$ denote the leading minor of order $n - r$ of $[\xi(f_{jk})]$. Then $\mu' = \mu + ce$, for some $c \in C$. Since $\mu$ is nonzero, $\mu'$ does not lie in the maximal ideal of $C[e]/(e^2)$, hence is a unit. We then have, by Proposition 3 and Lemma 2, that $\xi$ is a tangent vector to $Z'(u)$ at $y$ iff the minors of order $n - r + 1$ of $[\xi(f_{jk})]$ which contain $\mu'$ all vanish. Obviously, there are $r(g - n + r)$ such minors. If the equations in the $s_i$ given by the vanishing of these minors are linearly independent (over $C$), then the dimension of the tangent space to $Z'(u)$ at $y$ is $m - r(g - n + r)$. We could then conclude that $y$ is a smooth point of $Z'(u)$ by virtue of the following proposition.

**Proposition 4.** Either $Z'(u)$ is empty, or each component has codimension at most $r(g - n + r)$ in $Y$.

**Proof.** This is proved in [9] for $Y$ a scheme. With the obvious modifications, the proof is valid for $Y$ an analytic space.

2. The universal family of Teichmüller surfaces. In [5], Grothendieck proved the following

**Theorem 1.** There exist an analytic space $T_g$ and a family $V$ of Teichmüller surfaces of genus $g$ over $T_g$ which is universal in the following sense: for every family $X$ of Teichmüller surfaces of genus $g$ over an analytic space $S$, there exists a unique map $\Phi: S \rightarrow T_g$, such that $X$ is isomorphic (as a family of Teichmüller surfaces) to the pullback via $\Phi$ of $V/T_g$.  

$$M' = \begin{bmatrix}
\mu & a_{r+1,1} & \cdots & a_{r+1,r} & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
am_{m,1} & \cdots & a_{m,r} & N
\end{bmatrix}$$
$T_g$ is called the Teichmüller space (for Teichmüller surfaces of genus $g$). The Teichmüller space is a smooth, irreducible, and simply connected analytic space [5].

Let $h: V \to T_g$ denote the structural morphism. By well-known topological facts, since $T_g$ is simply connected, the fiber bundle $R^1 h_* \mathbb{Z}$ is trivial. Thus, there are sections of this bundle which give rise to cycles $\gamma_i(s), \delta_i(s), i = 1, \cdots, g$, which form a canonical homology basis for $H_1(V_s, \mathbb{Z}), s \in T_g$ [15].

Consider the sheaf $\Omega^1_{V/T_g}$. For all $s \in T_g$, we have

$$\dim H^0(V_s, \Omega^1_{V/T_g} \otimes \kappa(s)) = \dim H^0(V_s, \Omega^1_{V_s}) = g.$$ 

Hence, $h_*\Omega^1_{V/T_g}$ is a vector bundle of rank $g$ over $T_g$ and we have

$$h_*\Omega^1_{V/T_g} \otimes \kappa(s) \cong H^0(V_s, \Omega^1_{V_s})$$

by [4].

Choose holomorphic sections $d^*_i(s), i = 1, \cdots, g$, of $h_*\Omega^1_{V/T_g}$ such that

$$\{d^*_i(s)\}_{i=1}^g$$

is a basis for $H^0(V_s, \Omega^1_{V_s}), s \in T_g$ (cf. [15]). Put

$$a_{ij}(s) = \int \gamma_i(s) d^*_j(s), \quad b_{ij}(s) = \int \delta_i(s) d^*_j(s), \quad i, j = 1, \cdots, g.$$ 

For each $s \in T_g$, the matrix $[a_{ij}(s), b_{ij}(s)]$ is the period matrix of $V_s$.

Recall that the columns of this matrix generate a maximal lattice subgroup of $\mathbb{C}^g$. Let $J$ be the quotient of $T_g \times \mathbb{C}^g$ by this family of lattices. The induced projection $J \to T_g$ gives a complex analytic family of complex tori, the fiber $J_s$ being the Jacobian variety of the Teichmüller surface $V_s$ [15].

Since our concern will only be local, we assume that there exist sections of $V \to T_g$. Let $P_0(s)$ be such a section. As in [15], define a map $\psi: V \to J$ by

$$\psi(s, P) = \left( s, \int_{P_0(s)}^P d^*_1(s), \cdots, \int_{P_0(s)}^P d^*_g(s) \right) \mod \text{periods}$$

for $P \in V_s$.

Denote by $V_T^{(n)}$ the $n$th symmetric product of $V$ over $T_g$ (cf. [7]). Extend $\psi$ to a map $f: V_T^{(n)} \to J$ as follows. If $s \in T_g$ and $D \in (V_T^{(n)})_s$ is the divisor $\sum_{i=1}^n P_i$ on $V_s$, then

$$f(s, D) = \left( s, \sum_{i=1}^n \int_{P_0(s)}^{P_i} d^*_1(s), \cdots, \sum_{i=1}^n \int_{P_0(s)}^{P_i} d^*_g(s) \right) \mod \text{periods}.$$ 

Let
be the map induced by \( f \). Since \( J \) and \( V_{T_g}^{(n)} \) are smooth over \( T_g \) of relative dimensions \( g \) and \( n \) respectively, the sheaves

\[
f^*\Omega^1_{J/T_g} \text{ and } \Omega^1_{V_{T_g}^{(n)}}
\]

are locally free of ranks \( g \) and \( n \) respectively. Thus, we may consider the analytic subspace \( Z'(u) \subset V_{T_g}^{(n)} \) of \( \S 1 \). We will denote by \( G'_n \) the analytic space \( Z'(u) \) which arises in this situation. We will see in \( \S 4 \) that \( (G'_n)_{\ast} \) is what was denoted by \( G'_n(V_s) \) in \( \S 3 \).

We wish to study the infinitesimal structure of \( G'_n \). To do this, we need explicit knowledge of the above map \( f \). And to obtain this knowledge, we need certain variational formulas which are contained in the next section.

3. The variational formula. For a detailed treatment of the material in this section, the reader is referred to Rauch [18] or Patt [17].

Let \( X \) be a compact Riemann surface of genus \( g > 0 \). Let \( \Gamma = (\gamma_1, \cdots, \gamma_g) \) and \( \Delta = (\delta_1, \cdots, \delta_g) \) be a canonical homotopy basis and let \( \Pi \) be the simply connected surface obtained by the canonical dissection of \( X \) determined by \( \Gamma \) and \( \Delta \) (cf. [23]).

Let \( w \) be a point in the interior of \( \Pi \) and let \( \tau_{w,v}(z) \) denote the (normalized) elementary integral of the second kind with pole of order \( v + 1 \) at \( w \) and zero \( \Gamma \)-periods.

Let \( \xi \) be an Abelian integral of the first kind. Let \( a_i, i = 1, \cdots, g, \) denote the \( \Gamma \)-periods of \( d\xi \); that is,

\[
a_i = \int_{\gamma_i} d\xi, \quad i = 1, \cdots, g.
\]

The value of the derivatives of a determination of \( \xi \) at \( w \) and the periods of the differentials \( d\tau_{w,v} \) are related by

\[
\xi^{(\nu+1)}(w) = \frac{\nu!}{2\pi i} \sum_{j=1}^{g} a_j \int_{\delta_j} d\tau_{w,v}(z)
\]

which follows from the bilinear relation for differentials of the first and second kinds [23, p. 260].

Let \( Q_1, \cdots, Q_n \) be distinct points in the interior of \( \Pi \) such that all the \( Q_j \) are different from \( w \) and none of the \( Q_j \) is a zero of \( d\xi \). Let \( t_j, j = 1, \cdots, n, \) be a local parameter at \( Q_j \). Let \( D_1, \cdots, D_n \) be disjoint disks about \( Q_1, \cdots, Q_n \) respectively, such that \( D_j \) lies in the domain of \( t_j \), is completely
contained in the interior of $\Pi$ and such that no $D_j$ contains either $w$ or any zero of $d\zeta$.

Inside $D_j$, we can vary the local parameter $t_j$ to a new parameter $t_j^*$ given by

$$t_j^* = t_j + c_j/t_j, \quad j = 1, \ldots, n,$$

where $c_j$ is sufficiently small. This defines a new Riemann surface $X^*$, having the same canonical homotopy basis as $X$ has (since all variations take place in the interior of $\Pi$).

Let $\xi^*$ be the Abelian integral of first kind on $X^*$ with the same $\Gamma$-periods as $\xi$. We wish to compute

$$\Delta \xi^{(\nu+1)}(w) = \xi^*(\nu+1)(w) - \xi^{(\nu+1)}(w).$$

**Notation.** $dr_{w,\nu}d\zeta$ is a (not necessarily finite) quadratic differential on $X$. Locally at $Q_j$, we may write $dr_{w,\nu}d\zeta = h(t_j)dt_j^2$. We now introduce the notation $r'_{w,\nu}(Q_j)$ for $h(t_j)$.

Utilizing the techniques and formulas in [18] and [17], one can obtain the following proposition:

**Proposition 5.**

$$\Delta \xi^{(\nu+1)}(w) = \nu! \sum_{m=1}^n c_m r'_{w,\nu}(Q_m)\zeta'(Q_m) + O(c^2)$$

where $c = \max_{1\leq m\leq n}|c_m|$.

We will also want to use the following theorem, due to Patt [17]:

**Theorem 2.** One may choose $3g-3$ points $Q_1, \ldots, Q_{3g-3}$ on $X$ such that, if $c_m$ is the variation parameter at $Q_m$, then a neighborhood of the origin in the $c_1, \ldots, c_{3g-3}$ space describes a complex-analytic structure for a neighborhood of $X$ in the Teichmüller space. Moreover, the set of collections of $3g-3$ points with this property is open in $X^{3g-3}$.

**Proof.** The first assertion follows from Theorems 2 and 4 of [17]. Although Patt does not state the second assertion, his proofs demonstrate it, as was noted by Farkas [3, p. 885].

4. The equations which define the tangent space. Let $X$ be a compact Riemann surface of genus $g > 1$. Let $\{\gamma_j, \delta_j\}_{j=1}^g$ be a canonical homotopy basis and let $\{d\sigma_k\}_{k=1}^g$ be a basis of the holomorphic differentials. Put

$$A_{jk} = \int_{\gamma_j} d\sigma_k, \quad j, k = 1, \ldots, g.$$
Let $P$ be a point of $X$ and let $t$ be a local parameter on $X$ at $P$. Write

$$d\xi_k = \sum_{l=0}^{\infty} a_{k,l} t^l dt.$$ 

Fix a point $P_0$ different from $P$. Choose a point $(Q_1, \ldots, Q_{3g-3})$ from the open subset of $X^{3g-3}$ in Theorem 2 such that all the $Q_m$ are different from $P$ and $P_0$ and such that none of the $Q_m$ is a zero of any $d\xi_k$. Perform the variation described in §3, taking the disk about each $Q_m$ sufficiently small so that no two disks intersect and no disk contains $P, P_0$, or any zero of any $d\xi_k$. Let $c_m$ denote the variation parameter at $Q_m$, $m = 1, \ldots, 3g-3$, as in §3.

Let $s_0 \in T_g$ be the module point of $X$ (i.e., $V_{s_0} = X$). By definition of the variation, there exists a complex-analytic neighborhood $U$ of $s_0$ in $T_g$ such that, for all $s' \in U$, the curves $\{\gamma_j, \delta_j\}_{j=1}^g$ are a canonical homotopy basis on $V_{s'}$, the points $P_0$ and $P$ are on $V_{s'}$, and $t$ is a local parameter on $V_{s'}$ at $P$. Choose holomorphic sections $d\xi^*_k$, $k = 1, \ldots, g$, of $h_*\Omega^1_{V/T_g}$ such that

$$\int_{\gamma_j} d\xi^*_k(s') = A_{jk}, \quad s' \in U, \quad j, k = 1, \ldots, g$$

(cf. [15, §3]).

**Proposition 6.** With notation as in §3 and above, if we define $a^*_{k,l}$ by $d\xi^*_k = \sum_{l=0}^{\infty} a^*_{k,l} t^l dt$, then we have

$$a^*_{k,l} = a_{k,l} + \sum_{m=1}^{3g-3} c_m \tau_{P,l}(Q_m) \xi_k(Q_m) + O(c^2).$$

**Proof.** The variational formula (Proposition 5) shows that this equality holds in a complex-analytic neighborhood of $(s_0, P)$ on $V$. This is the main import of the variational formula.

In order to study the map

$$u: f^*\Omega^1_{J/T_g} \longrightarrow \Omega^1_{V^{(n)}/T_g}$$

of §2, we first consider the divisor $nP$ on $X$. Let $t_1, \ldots, t_n$ be $n$ copies of $t$, and let $\sigma_1, \ldots, \sigma_n$ denote the $n$ elementary symmetric functions in $t_1, \ldots, t_n$.

**Proposition 7.** Local parameters on $V^{(n)}_{T_g}$ at $(s_0, nP)$ are given by $c_1, \ldots, c_{3g-3}$, $\sigma_1, \ldots, \sigma_n$. 

Proof. By Theorem 2, local parameters on $T_g$ at $s_0$ are given by $c_1, \ldots, c_{3g-3}$. By [1], local parameters on $X^{(n)}$ at $nP$ are given by $\sigma_1, \ldots, \sigma_n$. By the definition of the variation in §3, local parameters on $(V^{(n)}_{T_g})_{s'}$ at $nP$, for $s' \in U$, are also given by $\sigma_1, \ldots, \sigma_n$. Thus, local parameters on $V^{(n)}_{T_g}$ at $(s_0, nP)$ are given by $c_1, \ldots, c_{3g-3}, \sigma_1, \ldots, \sigma_n$.

Put

$$\tau_j = t_1^j dt_1 + \cdots + t_n^j dt_n, \quad j = 0, 1, 2, \cdots.$$ 

We have

**Proposition 8.** The space of holomorphic 1-forms on $X$ is naturally isomorphic to the space of holomorphic 1-forms on $X^{(n)}$. Both these spaces are isomorphic to the space of symmetric holomorphic 1-forms on the Cartesian product $X^n$. If $d\xi = \Sigma_{i=0}^\infty a_i t^i dt$ is a holomorphic 1-form on $X$ and $d\tilde{\xi}$ is the corresponding symmetric holomorphic 1-form on $X^n$, then $d\tilde{\xi} = \Sigma_{i=0}^\infty a_i t^i$.

**Proof.** [14, pp. 226–227].

This result is easily seen to relativize to the following proposition.

**Proposition 9.** The space of relative holomorphic 1-forms on $V^{(n)}_{T_g}$ over $T_g$ and the space of relative holomorphic 1-forms on $V$ over $T_g$ are naturally isomorphic. Both spaces are isomorphic to the space of relative symmetric holomorphic 1-forms on $V^n$, the product over $T_g$ of $n$ copies of $V$, over $T_g$. If $d\tilde{\xi}_k^*$ is the relative symmetric holomorphic 1-form on $V^n_{T_g}$ over $T_g$ corresponding to $d\xi_k^*$ (cf. Proposition 6), then

$$d\tilde{\xi}_k^* = \Sigma_{i=0}^\infty a_{k,i}^* t^i.$$ 

We will identify relative symmetric holomorphic 1-forms on $V^n_{T_g}$ over $T_g$ and relative holomorphic 1-forms on $V^{(n)}_{T_g}$ over $T_g$.

Now, we can express $d\tilde{\xi}_k^*$ in terms of $d\sigma_1, \ldots, d\sigma_n$ by using the following identities [14]:

$$\sigma_k t_0 - \sigma_{k-1} t_1 + \cdots + (-1)^k t_k = d\sigma_{k+1}. \quad \sigma_k = 0 \quad \text{and} \quad d\sigma_k = 0 \quad \text{if} \quad k > n.$$ 

(By convention, $\sigma_k = 0$ and $d\sigma_k = 0$ if $k > n$.) Inverting these identities, and writing out only the linear terms, we obtain

$$\tau_k = (-1)^k (d\sigma_{k+1} - \sigma_1 d\sigma_k - \cdots - \sigma_k d\sigma_1) + \text{higher order terms}.$$ 

Thus we may write
\[ d\xi_k^* = \sum_{l=0}^{\infty} (-1)^l \left[ \left( a_{k,l} + \sum_{m=1}^{3g-3} c_m \tau_{P,l}(Q_m) \kappa_k'(Q_m) \right) \right] \]
\[(\#) \quad (d\sigma_{l+1} - \sigma_1 d\sigma_l - \cdots - \sigma_l d\sigma_1) + O(\sigma^2, c^2) \]

where \( O(\sigma^2, c^2) \) denotes higher order terms in the \( \sigma_j \) and the \( c_m \).

By definition of the map \( f: V_T^{(n)} \to J \) in §2, it is easy to see that \( f \) is given at \( (s_0, nP) \) by

\[ f(s_0, nP) = (s_0, \int_{P_0}^{P} d\xi_k^*(s_0), \ldots, \int_{P_0}^{P} d\xi_g^*(s_0)) \mod \text{periods} \]

where the integrals \( \int_{P_0}^{P} d\xi_k^*(s_0) \) are evaluated by recalling that \( t_1, \ldots, t_n \) are just copies of \( t \). Let \( \frac{\partial \xi_k^*}{\partial \sigma_j} \) be given by

\[ d\xi_k^* = \sum_{j=1}^{n} \frac{\partial \xi_k^*}{\partial \sigma_j} d\sigma_j. \]

Then we have

**PROPOSITION 10.** The map

\[ u: f^* \Omega^1_{X/\mathbb{P}^3} \to \Omega^1_{V_T^{(n)}/\mathbb{P}^3} \]

is given locally at \( (s_0, nP) \) by the matrix

\[ [\frac{\partial \xi_k^*}{\partial \sigma_j}], \quad j = 1, \ldots, n, k = 1, \ldots, g. \]

**PROOF.** This follows easily from the definitions of \( f \) and \( \frac{\partial \xi_k^*}{\partial \sigma_j} \). (Compare with [3] and [6].)

**REMARK.** Let \( J \) denote the Jacobian variety of \( X \) and let \( f_0: X(n) \to J \) be the classical map (i.e., the map \( f \otimes \kappa(s_0) \)). Then the matrix \( M = [(\frac{\partial \xi_k^*}{\partial \sigma_j})(s_0, nP)] \) is the matrix of the map \( u_0: f_0^* \Omega^1_{X(n)} \to \Omega^1_{X(n)} \) at \( nP \) (cf. [3], [6]). It is then easy to see that \( (G_n^r s_0) \) is what was denoted by \( G_n^r(X) \) in §0.

Now let \( \xi \) be a tangent vector to \( V_T^{(n)} \) at \( (s_0, nP) \). Let \( s_j \) and \( b_m \) in \( \mathbb{C} \) be given by

\[ \xi(s_j) = s_j e, \quad j = 1, \ldots, n, \]
\[ \xi(c_m) = b_m e, \quad m = 1, \ldots, 3g - 3. \]

Then, using Taylor's Theorem as in §1, we have
We will now use (4#) to compute the partial derivatives of \( \frac{\partial \xi_k^*}{\partial \sigma_j} \) with respect to \( \sigma_i \) and with respect to \( c_m \). (We remind the reader that the functions \( \sigma_i \) and \( c_m \) vanish at \( (s_0, nP) \).) We obtain

\[
\left( \frac{\partial \xi_k^*}{\partial \sigma_i} \right) (s_0, nP) = (-1)^{i+l}a_{k,i+l-1} \\
and \left( \frac{\partial \xi_k^*}{\partial c_m} \right) (s_0, nP) = \tau_p^{i-1}(Q_m)\xi_k^*(Q_m).
\]

Substituting these expressions for the partial derivatives evaluated at \( (s_0, nP) \) into (1) gives us

**Proposition 11.**

\[
\xi \left( \frac{\partial \xi_k^*}{\partial \sigma_j} \right) = \left( \frac{\partial \xi_k^*}{\partial \sigma_j} \right) (s_0, nP) + \epsilon \sum_{i=1}^{n} (-1)^i a_{k,i+l-1} \\
+ \epsilon \sum_{m=1}^{3g-3} b_m \tau_p^{i-1}(Q_m)\xi_k^*(Q_m).
\]

Now on to the general case. Consider a divisor \( D \) on \( X \) of the form \( D = m_1P_1 + \cdots + m_dP_d \). Assume \( D \) is in \( G^r_n(X) \) and choose a basis \( \{d_K\}_{k=1}^{r} \) of the holomorphic differentials on \( X \) such that the last \( i = \dim H^1(X, 0_X(D)) \) of them vanish on \( D \).

In performing the variation in §3, choose a point \( (Q_1, \cdots, Q_{3g-3}) \) from the open set in \( X^{3g-3} \) in Theorem 2 so that each \( Q_m \) is different from \( P_0, P_1, \cdots, P_d \) and any other zero of any \( d_K \). (The choice of this point will be further modified later.) Take the disk about each \( Q_m \) sufficiently small so that no two disks intersect and such that no disk contains \( P_0, P_1, \cdots, P_d \) or any other zero of any \( d_K \).

Let \( f_j: V_{T_g}^{(m_j)} \rightarrow J \) be the map defined in §2 and let

\[
u_j: f_j^*\Omega^1_{J/T_g} \rightarrow \Omega^1_{V_{T_g}/T_g}^{(m_j)}
\]
be the map induced by \( f_j \). The obvious map

\[
V_T^{(m_1)} \times_T V_T^{(m_2)} \times_T \cdots \times_T V_T^{(m_d)} \to V_T^{(n)}
\]

is a local analytic isomorphism by an argument analogous to that given in \([14]\) in the case of a curve over a field. Locally, the map \( f \) is the one induced by the \( f_j \) and the map \( u \) is the one induced by the \( u_j \). Thus, the matrix of \( u \) locally at \((s_0, D)\) is obtained by “stacking” the matrices of the \( u_j \) locally at \((s_0, m_j P_j)\).

Let \( \xi \) be a tangent vector to \( V_{Tg}^{(n)} \) at \((s_0, D)\) and let \( \xi_j \) be the tangent vector to \( V_{Tg}^{(m_j)} \) at \((s_0, m_j P_j)\) induced by \( \xi \), for \( j = 1, \cdots, d \). Then the matrix of \( \xi u \) is obtained by “stacking” the matrices of the \( \xi_j u_j \), for \( j = 1, \cdots, d \).

Let \( M' \) denote the matrix of \( \xi u \). Let \( \mu \) denote the leading minor of order \( n - r \) of \( M \), the matrix \( [(\partial^2\xi_k/\partial \sigma_j)(s_0, D)] \), and let \( \mu' \) denote the leading minor of order \( n - r \) of \( M' \). Then we have \( \mu' = \mu + ce \) for some \( c \) in \( \mathbb{C} \). Now, by our choice of a basis of the holomorphic differentials on \( X \), the last \( i \) columns of \( M \) are identically zero, hence the last \( i \) columns of \( M' \) contain “pure” \( e \) terms (i.e., members of the maximal ideal of \( \mathbb{C}[e]/(e^2) \)). Thus, in computing a minor of order \( n - r + 1 \) containing \( \mu' \), any \( e \)'s in the first \( n - r \) columns will be “killed” by the \( e \) in the last column of the minor of order \( n - r + 1 \). Hence, we have established

**Lemma 3.** For purposes of computing the minors of order \( n - r + 1 \) of \( M' \), we may replace the first \( n - r \) columns of \( M' \) by the first \( n - r \) columns of \( M \).

Let \( M \) denote the resulting matrix.

\( M \) has a particularly nice form in the case that \( D = P_1 + P_2 + \cdots + P_n \), with all points distinct. Let \( t_j \) be a local parameter at \( P_j \) and write \( d\xi_k = \varphi_{j,k} dt_j \). Then we have

\[
M = \begin{bmatrix}
\varphi_{j,k}(P_j) \\
j = 1, \cdots, n \\
k = 1, \cdots, g - i
\end{bmatrix} \begin{bmatrix}
e \left(s_j \varphi'_{j,k}(P_j) + \sum_{m=1}^{3g-3} b_m \tau'_{P_j,0}(Q_m) \xi_k'(Q_m) \right) \\
j = 1, \cdots, n \\
k = g - i + 1, \cdots, g
\end{bmatrix}
\]

Going back to the general case, recall that, by Proposition 1, \( \xi \) will be a tangent vector to \( G_n^r \) at \((s_0, D)\) iff the minors of order \( n - r + 1 \) of the matrix \( M \) all vanish. Assume \( D = m_1 P_1 + \cdots + m_d P_d \) is in \( G_n^r(X) - G_n^{r+1}(X) \). Then the matrix \( M \) has rank precisely \( n - r \). Hence, by permuting the rows of
$M$, if necessary, we end up with a matrix whose leading minor of order $n-r$, which we will denote by $\mu$, is nonzero. We will continue to denote this matrix by $M$, although its form may differ slightly from that specified earlier.

Perform the same row permutations as above on the matrix $M$ and denote the resulting matrix also by $M$. Then $\mu$ is also the leading minor of order $n-r$ of $M$, so we may apply Lemma 2. Thus, for all the minors of order $n-r+1$ of $M$ to vanish, it is sufficient that every minor of order $n-r+1$ which contains $\mu$ vanishes. The vanishing of each of these minors gives rise to a linear equation in the $s_j$ and the $b_m$.

Let $\mu_{j,k}$ denote the minor of order $n-r+1$ of $M$ obtained by adjoining to $\mu$ the first $n-r$ elements of the $(n-r+j)$th row of $M$ and the first $n-r$ elements and the $(n-r+k)$th element of the $(n-r+k)$th column of $M$ (thus $j$ runs from 1 through $r$ and $k$ runs from 1 through $i$). The equation $\mu_{j,k} = 0$ is of the form $\sum E_{j,k} = 0$ where $E_{j,k}$ is a linear equation in the $s_j$ and the $b_m$ with coefficients in $C$.

We will now view the $s_j$ and the $b_m$ as being unknowns (as in §1). Thus, $E_{j,k}$ is an equation in $3g-3+n$ unknowns. By the discussion after Proposition 1, the dimension of the tangent space to $G_n^r$ at $(s_0, D)$ is

$$3g-3+n-(\text{the number of } E_{j,k} \text{ which are linearly independent}).$$

Consider the coefficient of $b_m$ in $E_{j,k}$. This coefficient will be a linear combination of certain of the $\tau_{p_j,v}(Q_m)\xi_k(Q_m)$. That is, the coefficient of $b_m$ will be a certain quadratic differential (the above linear combination of certain of the $dt\tau_{p_j,v}d\xi_k$) evaluated at the point $Q_m$. It should be noted that, by the symmetry of the matrix $M$ in the $b_m$, this quadratic differential does not depend on $m$, but only on $j$ and $k$. The coefficient of $b_1$ in $E_{j,k}$ is the value of this quadratic differential at $Q_1$, the coefficient of $b_2$ in $E_{j,k}$ the value at $Q_2$, etc. Put $\alpha_{j,k}$ equal to the above linear combination of certain of the $dt\tau_{p_j,v}d\xi_k$. Then $\alpha_{j,k}$ is a (not necessarily finite) quadratic differential.

Notation. Choose a local parameter $u_m$ on $X$ at $Q_m$ and write $\alpha_{j,k} = g(u_m)du^2$. Then we will write $\alpha_{j,k}(Q_m)$ for $g(0)$. Hence, by the above discussion, $\alpha_{j,k}(Q_m)$ is the coefficient of $b_m$ in $E_{j,k}$.

Our aim now is to show that, in certain situations, by suitably choosing the point $(Q_1, \cdots, Q_{3g-3})$, we may conclude that the $E_{j,k}$ are linearly independent. Assume that $ri \leq 3g-3$. By elementary linear algebra, to conclude that the $E_{j,k}$ are linearly independent, it is sufficient to show that the matrix of coefficients

$$A = [\alpha_{j,k}(Q_m)], \quad j = 1, \cdots, r; k = 1, \cdots, i; m = 1, \cdots, ri,$$

is nonsingular.
Lemma 4. Assume that the $\alpha_{j,k}$ for $j = 1, \ldots, r$ and $k = 1, \ldots, i$, are linearly independent and that $r_i \leq 3g - 3$. Then we may choose a point $(Q_1, \ldots, Q_{3g-3})$ from the open set in $X^{3g-3}$ in Theorem 2 such that each $Q_m$ is different from $P_0$ and no $Q_m$ is a zero of $d\zeta_1, \ldots, d\zeta_g$ and such that the above matrix $A$ is nonsingular.

Proof. The lemma will follow readily from the following

Sublemma. Let $\beta_1, \ldots, \beta_n$ be $n$ linearly independent quadratic differentials on $X$. Let $U$ be an open set contained in $X^n$. Then we may choose a point $(P_1, \ldots, P_n) \in U$ such that each $P_m$ is different from a finite set of points of $X$ and such that the matrix $[\beta_j(P_k)]$ ($j = 1, \ldots, n, k = 1, \ldots, n$) is nonsingular.

Proof. By induction on $n$. If $n = 1$, then $\beta_1$ is a nontrivial quadratic differential. Hence, $\beta_1$ is nonzero and finite on a dense open set of $X$. So, given any open set in $X$, there exists a point in that set satisfying the requirements of the Sublemma.

Now suppose $U$ is an open set contained in $X^n$. Let $V$ be the projection of $U$ onto $X^{n-1}$. Then $V$ is open and, by induction, we may choose a point $(P_0, \ldots, P_{n-1}) \in V$ such that each $P_m$ is different from a finite set of points of $X$ and such that the leading subdeterminant of order $n - 1$ of the determinant

$$\begin{vmatrix}
\beta_1(P_1) & \cdots & \beta_1(P_{n-1}) & \beta_1 \\
\cdot & & \cdot & \\
\cdot & & \cdot & \\
\beta_n(P_1) & \cdots & \beta_n(P_{n-1}) & \beta_n
\end{vmatrix}$$

is nonzero. Expanding the full determinant by the last column, we obtain a nontrivial linear combination of $\beta_1, \ldots, \beta_n$. By the linear independence of these quadratic differentials, this linear combination is a nontrivial quadratic differential, hence is nonzero and finite on an open dense set $W$ contained in $X$. Since $U$ is open in $X^n$ and $W$ is dense in $X$, we may choose a point in the intersection of $U$ and $\{(P_0, \cdots, P_{n-1})\} \times W$ which satisfies the requirements of our Sublemma.

Now, since the set of points in $X^{3g-3}$ in Theorem 2 is open, it is easy to see that we may choose a point $(Q_1, \cdots, Q_{3g-3})$ in this set such that each $Q_m$ is different from $P_0$ and the zeros of $d\zeta_1, \cdots, d\zeta_g$ and so that $Q_1, \cdots, Q_{ri}$ make the matrix $A$ nonsingular. This completes the proof of the lemma.

We then have
Proposition 12. Suppose $D$ is in $G^r_n(X) - G^{r+1}_n(X)$ and that $r_i \leq 3g - 3$. Then if all the $\alpha_{j,k}$ are linearly independent, the dimension of the tangent space to $G^r_n$ at $(s_0, D)$ is $3g - 3 + r + r$.

Proof. By Lemma 4, we may choose a point $(Q_1, \cdots, Q_{3g-3})$ from the open set in Theorem 2 such that each of the $Q_m$ is different from $P_0$ and the zeros of $d\xi_1, \cdots, d\xi_{3g}$ (note that this latter set includes the points of $D$), and such that the equations $E_{j,k}$ are linearly independent. Thus the dimension of the tangent space to $G^r_n$ at $(s_0, D)$ is $3g - 3 + n - ir = 3g - 3 + r + r$.

In the next section, we show that if $D$ is in $G^1_n(X) - G^2_n(X)$, then the $\alpha_{j,k}$ are linearly independent. (Note that we have $i < 3g - 3$ if $g > 1$.)

5. The dimension of $G^1_n - G^2_n$. For simplicity, we will first treat a divisor consisting of $n$ distinct points. So assume $D = P_1 + P_2 + \cdots + P_n$, all points distinct, is in $G^1_n(X) - G^2_n(X)$. Recall that the matrix $M$ is

$$
M = \begin{bmatrix}
\varphi_{j,k}(P_j) & \vdots & \varepsilon(s_j\varphi_{j,k}(P_j) + \sum_{m=1}^{3g-3} b_m r'_{P_j,0}(Q_m) \xi'_k(Q_m)) \\
j = 1, \cdots, n & j = 1, \cdots, n \\
k = 1, \cdots, g - i & k = g - i + 1, \cdots, g
\end{bmatrix}
$$

Let $|\hat{n}|$ denote the minor of order $n - 1$ obtained by omitting the $j$th row from the matrix $[\varphi_{j,k}(P_j)]$ ($j = 1, \cdots, n$, $k = 1, \cdots, g - i$). Then we have

$$
\alpha_{1,k}(Q) = \sum_{j=1}^{n} (-1)^{j-i} |\hat{\varphi}_{j,k}(P_j)| \tau'_{P_j,0}(Q) \xi'_{n+k-1}(Q)
$$

for $k = 1, 2, \cdots, i$. Suppose we had a linear relation of the form $\Sigma_{k=1}^{I} a_k \alpha_{1,k} = 0$ with some $a_k$ nonzero. Then this would imply that

$$
(*) \quad \left( \sum_{j=1}^{n} (-1)^{j-i} |\hat{\varphi}_{j,k}(P_j)| \tau'_{P_j,0}(Q) \right) \left( \sum_{k=1}^{I} a_k \xi'_{n+k-1}(Q) \right) = 0.
$$

But the $d\tau_{P_j,0}$, $j = 1, \cdots, n$, are linearly independent, since they have poles at different points. This, together with the fact that $|\hat{n}| \neq 0$, implies that there is a dense open set of points of $X$ where the expression $\Sigma_{j=1}^{n} (-1)^{j-i} |\hat{\varphi}_{j,k}(P_j)| \tau'_{P_j,0}(Q)$ is nonzero.

And the linear independence of $d\xi_n, \cdots, d\xi_g$, together with the fact that some $a_k$ is nonzero, implies that the expression $\Sigma_{k=1}^{I} a_k \xi'_{n+k-1}(Q)$ is nonzero on a dense open set of points of $X$. Hence, we may choose a point $Q$ such that $(*$) is nonzero, contradicting the assumption that $\alpha_{1,1}, \cdots, \alpha_{1,I}$ are linearly dependent.
Now suppose $D = m_1P_1 + \cdots + m_dP_d$ is in $G_n^1(X) - G_n^2(X)$. Then we have

$$\alpha_{1,k}(Q) = \sum_{k=1}^{l} a_k \zeta_{n+k-1}(Q) \left( (-1)^n \tau_{P_1,0}^m(Q) + \cdots + (-1)^{n-1} \tau_{P_d,m_d-1}^m(Q) \right).$$

Hence, if there existed a linear relation $\sum_{k=1}^{l} a_k \alpha_{1,k} = 0$, we would have

$$\sum_{k=1}^{l} a_k \zeta_{n+k-1}(Q) \left( (-1)^n \tau_{P_1,0}^m(Q) + \cdots + (-1)^{n-1} \tau_{P_d,m_d-1}^m(Q) \right) = 0.$$

The same reasoning as in the case of simple points applies, since $d\tau_{P_1,0}^m, \ldots, d\tau_{P_d,m_d-1}^m$ are easily seen to be linearly independent (they either have poles at different points or have poles of differing orders at the same point).

**Remark.** The above reasoning shows that if $D \in G_n^r - G_n^{r+1}$, then the $\alpha_{j,k}$ for a fixed $j$ are linearly independent.

**Theorem 3.** $G_n^1 - G_n^2$, if nonempty, is smooth of pure dimension $3g - 3 + \tau + 1$.

**Proof.** Let $(s_0, D)$ be any point of $G_n^1 - G_n^2$. By Proposition 12 and the work of this section, we may conclude that the dimension of the tangent space to $G_n^1$ at $(s_0, D)$ is $3g - 3 + \tau + 1$. By Proposition 4, the dimension of $G_n^2$ at $(s_0, D)$ is at least $3g - 3 + \tau + 1$, hence $G_n^1$ is smooth at $(s_0, D)$ and has dimension precisely $3g - 3 + \tau + 1$.

**Remark.** Theorem 3 does not depend upon $\tau$ being nonnegative.

**Theorem 4.** Suppose that $G_n^1(X) - G_n^2(X)$ is nonempty for a generic curve $X$. Then $G_n^1(X) - G_n^2(X)$, for a generic $X$, is smooth of pure dimension $\tau + 1$.

**Proof.** Under our assumption, the image of $G_n^1 - G_n^2$ in $T_g$ would be a dense open subspace $U$. By Sard's Theorem, since $G_n^1 - G_n^2$ is smooth, the generic fiber of the map $G_n^1 - G_n^2 \to U$ is smooth. And since $U$ has dimension $3g - 3$ and $G_n^1 - G_n^2$ has dimension $3g - 3 + \tau + 1$, the generic fiber has dimension $\tau + 1$. Thus, for a generic curve, $G_n^1(X) - G_n^2(X)$ is smooth of dimension $\tau + 1$.

**Remark.** If $\tau > 0$, then by [10] we know that $G_n^r(X)$ is nonempty. If we knew that $G_n^r(X)$ were reduced for a generic $X$, then, since the points of $G_n^{r+1}$ are singular points of $G_n^r$, we could conclude that $G_n^r(X) - G_n^{r+1}(X)$ is nonempty for generic $X$ if $\tau > 0$.

6. Moduli of trigonal curves. A trigonal curve is a curve $X$ such that $G_3^1(X)$ is nonempty. We can use Theorem 3 to compute the moduli of trigonal curves. By Clifford's Theorem, $G_3^2$ is empty hence, by Theorem 3, $G_3^1$, if
nonempty, is smooth of pure dimension \(3g - \tau + 1\). Now \(\tau = 2(3 - 1) - g = 4 - g\), so \(G_1^1\), if nonempty, has dimension \(2g + 2\).

By Theorem 1 of [12], we have that, for \(g \geq 4\), if \(G_1^1(X)\) is nonempty, then every component has dimension at least \(5 - g\) and at most 2, with the upper bound occurring if and only if \(X\) is hyperelliptic. So, if there exists a nonhyperelliptic trigonal curve of genus \(g\), then we must have that the dimension of the generic fiber of the map \(G_1^1 \to T_g^g\) is 1. Examples of such curves (for every \(g \geq 3\)) are given in [1a, p. 196](1). Hence, the dimension of the subvariety of \(T_g^g\), for \(g \geq 4\), of trigonal curves is \(2g + 2 - 1 = 2g + 1\). This agrees with the number which appears in Segre [20] and Severi [22].

**BIBLIOGRAPHY**


(1) The author wishes to thank the referee for pointing out this reference.


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