

DUALITY THEORIES FOR METABELIAN LIE ALGEBRAS. II

BY

MICHAEL A. GAUGER

ABSTRACT. In this paper I have replaced one of the axioms given in my *Duality theory for metabelian Lie algebras* (Trans. Amer. Math. Soc. **187** (1974), 89–102) concerning duality theories by a considerably more natural assumption which yields identical results—a uniqueness theorem.

1. Introduction. In [3] I defined and proved a uniqueness theorem for what I referred to as an algebraic duality theory for metabelian Lie algebras. In this paper I will investigate the duality theories which arise using axioms (I), (II), (III) and (V) of [3]. Axiom (V) was an easy consequence of the first four, and is really a more natural thing than (IV) to require of a duality theory. It will be shown that these weaker yet more natural axioms give rise to nearly identical results. To classify all metabelian Lie algebras N such that $\text{cod } N^2 = g$ one must consider the cases where $0 \leq \dim N^2 \leq (\frac{g}{2})$. A duality theory nearly cuts this problem in half in the sense that classifying those N such that $\dim N^2 = p$, is the same as classifying their duals N_D which satisfy $\dim N_D^2 = (\frac{g}{2}) - p$. A duality theory shuffles among themselves those N for which $\dim N^2 = (\frac{g}{2})/2$, and apparently contributes nothing to their classification. Using axioms (I), (II), (III), and (V) for the definition of an algebraic duality theory, we prove

MAIN THEOREM. *There is only one algebraic duality theory.*

The assumption that the ground field k is algebraically closed and of characteristic zero, together with the notation of §§1 and 2 of [3], remain in force throughout this article. [2], [3] and [5] provide a suitable background.

Suppose D_1, D_2 are duality theories satisfying axioms (I), (II), (III) and (V) of [3] and let V be a finite-dimensional k -vector space. Let $G_p(\wedge^2 V)$ represent the projective Grassmann variety of all p -dimensional subspaces of $\wedge^2 V$. Consider $D_1, D_2 : G_p(\wedge^2 V) \rightarrow G_{m-p}(\wedge^2(V^*))$ where $m = \dim \wedge^2 V$. By (II), $D_2^{-1} \circ D_1 = \rho$ is an automorphism of $G_p(\wedge^2 V)$. We can assume $p \leq \dim \wedge^2 V/2$ for otherwise we could consider $D_1 \circ D_2^{-1}$. Suppose $S_1, S_2 \in G_p(\wedge^2 V)$ and $V \oplus \wedge^2 V/S_1 \cong V \oplus \wedge^2 V/S_2$ (see §1 and Theorem 1 of [3]). Then by

Received by the editors July 24, 1973.

AMS (MOS) subject classifications (1970). Primary 17B30.

Copyright © 1975, American Mathematical Society

(V) for D_1 we have

$$V^* \oplus \wedge^2(V^*)/D_1(S_1) \cong V^* \oplus \wedge^2(V^*)/D_1(S_2).$$

By (V) and (III) for D_2 , we see that

$$V \oplus \wedge^2 V/D_2^{-1}(D_1(S_1)) \cong V \oplus \wedge^2 V/D_2^{-1}(D_1(S_2)).$$

Thus, due to Theorem 1 of [3], the automorphism ρ of $G_p(\wedge^2 V)$ has the property that if $S_1, S_2 \in G_p(\wedge^2 V)$ and $S_1 \equiv S_2(GL(V))$, then $\rho(S_1) \equiv \rho(S_2)(GL(V))$. That is, ρ induces a bijection of the quotient set $G_p(\wedge^2 V)/GL(V)$. (Axiom (IV) of [3] had the effect of making ρ centralize the image of $GL(V)$ in $\text{Aut } G_p(\wedge^2 V)$. See §1 of [3].)

To establish a framework for the solution of this problem I wish to digress and consider the following situation. Let G be an algebraic group acting on a variety V by a morphism $G \times V \rightarrow V$. Then there is an induced map $G \rightarrow \text{Aut } V$ —the group of automorphisms of V . I wish to consider those automorphisms of V compatible with the G -orbit structure. The set V/G cannot always be given the structure of a variety in such a way that $V \rightarrow V/G$ is a morphism (in particular this would require orbits in V to be closed; see [1]). Therefore, the best one seems to be able to ask in general is: what is the subgroup S of $\text{Aut } V$ consisting of those automorphisms which induce bijections on V/G . S is called the orbit shuffling group. Certainly $S \supseteq G$ (identifying G with its image in $\text{Aut } V$), and sitting in between these two groups is the group F which consists of those automorphisms stabilizing each orbit. F is called the orbit fixing group. Thus $G \subseteq F \subseteq S$, and F is normal in S since it is the kernel of the homomorphism $S \rightarrow \text{Bij}(V/G)$. In fact, S is the normalizer of F in $\text{Aut } V$. For this, suppose $\tau \in \text{Aut } V$ normalizes F , and let $v_1, v_2 \in V$ satisfy $\theta(v_1) = v_2$ for some $\theta \in G$. Then $\tau(v_2) = \tau\theta(v_1) = \theta'\tau(v_1)$ for some $\theta' \in F$. Since θ' fixes G -orbits, there is a $\theta'' \in G$ such that $\theta'(\tau(v_1)) = \theta''(\tau(v_1))$ and this completes the argument that $\tau \in S$.

Consider this situation for the action $GL(V) \times G_p(\wedge^2 V) \rightarrow G_p(\wedge^2 V)$ where $p \leq \dim \wedge^2 V/2$, and let S, F be the orbit shuffling and fixing groups respectively. From the preceding discussion, the determination of the possible duality theories satisfying (I), (II), (III) and (V) of [3] for this particular V and p is precisely the problem of determining F and S , each coset of F in S giving rise to a distinct theory in the following way. Let $\rho \in S$, let

$$D: G_p(\wedge^2 V) \rightarrow G_{m-p}(\wedge^2(V^*)), \quad D: G_{m-p}(\wedge^2(V^*)) \rightarrow G_p(\wedge^2 V)$$

($m = \dim \wedge^2 V$) be the Scheuneman-Gauger duality (see §1 of [3]). Then as ρ runs over a set of distinct coset representatives of F in S , the pairs

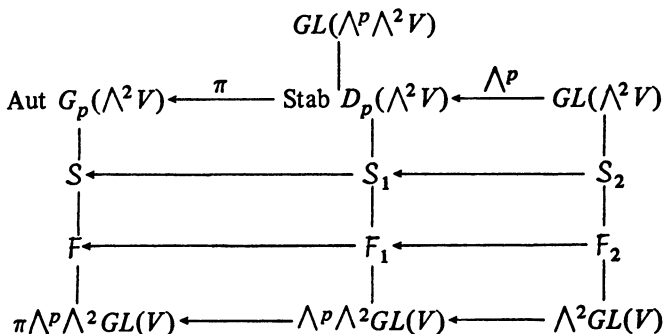
$$D \circ \rho: G_p(\wedge^2 V) \rightarrow G_{m-p}(\wedge^2(V^*)), \quad \rho^{-1} \circ D: G_{m-p}(\wedge^2(V^*)) \rightarrow G_p(\wedge^2 V)$$

exhaust the possible theories for this V and p . If two such theories for a ρ_1 and ρ_2 were to agree, that is,

$$V^* \oplus \wedge^2(V^*)/D(\rho_1(S)) \cong V^* \oplus \wedge^2(V^*)/D(\rho_2(S)) \quad \text{for all } S \in G_p(\wedge^2 V),$$

then by a routine calculation (Theorem 1 of §1 of [3]) $\rho_1^{-1} \circ \rho_2 \in F$, $\rho_1 F = \rho_2 F$ which is impossible.

So let V be a g -dimensional k -vector space and let $1 \leq p \leq (\frac{g}{2})$. Consider the diagram



where π is the natural map, $\text{Stab } D_p(\wedge^2 V)$ is the stabilizer of the set $D_p(\wedge^2 V)$ of decomposable p -vectors of $\wedge^p \wedge^2 V$, $S_1 = \pi^{-1}(S)$, $F_1 = \pi^{-1}(F)$, $S_2 = (\wedge^p)^{-1}(S_1)$, $F_2 = (\wedge^p)^{-1}(F_1)$ and all other symbols have the same significance as in §§1 and 2 of [3]. For the sake of simpler notation I have written F_2, S_2 instead of something like $F_{2,p}, S_{2,p}$ which would indicate more explicitly that a priori there is no reason to believe the F_2 's and S_2 's for different p 's are related. By Proposition 3 of [3], π is surjective. By Westwick's theorem (p. 1127 of [6, Theorem and the preceding remark]), $\wedge^p GL(\wedge^2 V) = \text{Stab } D_p(\wedge^2 V)$ when $p < \dim \wedge^2 V/2$, and when $p = \dim \wedge^2 V/2$ then $(\text{Stab } D_p(\wedge^2 V) : \wedge^p GL(\wedge^2 V)) = 2$. The determination of F and S is thus nearly always the same as determining F_2 and S_2 . $S_2(F_2)$ is the group of linear transformations on $\wedge^2 V$ shuffling (fixing) $GL(V)$ -orbits of p -dimensional subspaces.

2. Determination of the orbit shuffling and fixing groups when $\dim V \neq 4$. Let $g = \dim V$ and suppose $g \neq 4$. We show first that $F_2 = \wedge^2 GL(V)$ relying heavily on Westwick's theorem [6, p. 1127] and the following algebro-geometric lemma whose proof was suggested to me by John Fogarty.

LEMMA 1. *Let W be an n -dimensional k -vector space and let $D \subset W$ be a d -dimensional homogeneous affine subset (not necessarily irreducible). For any*

$x \in W - D$ and any p satisfying $1 \leq p \leq n - d$, there is a p -dimensional subspace S containing x such that $S \cap D = \{0\}$.

PROOF. By the dimension of D we mean the maximal dimension of its irreducible components. Go by induction on n . Let H_x be the set of all hyperplanes of W containing x , a closed irreducible subset of the Grassmannian of all hyperplanes of W . Write $D = D_1 \cup \dots \cup D_m$ where the D_i are the irreducible components of D . For each i it is easy to find a hyperplane containing x but not all of D_i . Thus if we call $H_{x,i}$ the collection of all hyperplanes through x but not containing D_i , we see that $H_{x,i}$ is a nonempty open subset of H_x . Hence $\bigcap_i H_{x,i}$ is nonempty, that is, there is a hyperplane W_1 through x which contains none of the D_i . Then consider W_1 and $D' = D \cap W_1$, and observe that $\dim D' \leq d - 1$ since each component of D is reduced by the intersection with W_1 . Apply the induction hypothesis to complete the proof.

Since the decomposable 2-vectors in $\wedge^2 V$ are a homogeneous affine subvariety of $\wedge^2 V$ of dimension $2(g - 2) + 1$, we get the

COROLLARY 2. *Let x be a nondecomposable vector in $\wedge^2 V$ and suppose $\dim V = g$. For any p satisfying $1 \leq p \leq \binom{g}{2} - 2g + 3$ there is a p -dimensional subspace S containing x but no nonzero decomposable vectors.*

PROPOSITION 3. *If $1 \leq p \leq \binom{g}{2} - 2g + 3$ and $g \neq 4$ then $F_2 = \wedge^2 GL(V)$.*

PROOF. Since $g \neq 4$, by Westwick's theorem [6, p. 1127] $\wedge^2 GL(V)$ is the stabilizer in $GL(\wedge^2 V)$ of the set of all decomposable 2-vectors. Let $\rho \in F_2$, that is, ρ stabilizes each $GL(V)$ -orbit of p -dimensional subspaces. Suppose however that ρ does not stabilize the set of decomposable 2-vectors in $\wedge^2 V$. In particular, let x be a decomposable vector in $\wedge^2 V$ such that $\rho(x)$ is nondecomposable. By Corollary 2, we can find a p -dimensional subspace S containing $\rho(x)$ but no nonzero decomposable. Now $\rho^{-1}(S)$ is a p -dimensional subspace containing x and ρ stabilizes $GL(V)$ -orbits so there is a $\theta \in GL(V)$ with $\wedge^2(\theta)(\rho^{-1}(S)) = S$. This is clearly impossible since $\wedge^2(\theta)$ takes decomposables to decomposables and S has no nonzero decomposables. Hence ρ stabilizes the set of decomposable vectors; this stabilizer is $\wedge^2 GL(V)$.

Since we only need to pin down F_2 when $1 \leq p \leq \binom{g}{2}/2$, we need to check when $\binom{g}{2} - 2g + 3 \geq \lceil \binom{g}{2}/2 \rceil$ (i.e. greatest integer). An easy computation reveals this is the case whenever $g \geq 7$. Excluding $g = 4$, the values of g, p for which F_2 is not determined by Proposition 3 are: $g = 3, p = 1, g = 5, p = 4$ or 5 , and $g = 6, p = 7$.

I am forced to handle these remaining cases by a combination of ad hoc arguments which only serve to point out that if one was suitably good at Grassmannian geometry, a single proof for all cases could probably be given.

If $\dim V = g = 3$, then $\dim \wedge^2 V = 3$ and $GL(\wedge^2 V) = \wedge^2 GL(V)$. Thus $F_2 = \wedge^2 GL(V)$ when $g = 3, p = 1$.

Notice that if $\dim V = g$, there are subspaces S of $\wedge^2 V$ of dimension up to $(g - 1)$ which consist entirely of decomposable vectors. Namely, they are subspaces of a subspace of the type $v \wedge V$ where v runs over V . A subspace $v \wedge V$ will be called an M_δ subspace.

So suppose $g = 5$ and $p = 4$, and let $\rho \in F_2$. If ρ does not preserve decomposables there is an $x \wedge y$ such that $\rho(x \wedge y)$ is not decomposable. By the paragraph above we can pick an M_δ subspace S containing $x \wedge y$ and $\dim S = 4$. Since $\rho \in F_2$ there is a $\theta \in GL(V)$ with $\wedge^2(\theta)(S) = \rho(S)$. This is clearly impossible since $\wedge^2(\theta)$ takes M_δ 's to M_δ 's and $\rho(S)$ is obviously not an M_δ . Hence ρ preserves decomposables. As before, this forces $F_2 = \wedge^2 GL(V)$.

The remaining cases are handled using the following result.

LEMMA 4. *Let $S \subset \wedge^2 V$ be an s -dimensional subspace of codimension at least two and containing no M_δ subspace. If $2(g - 2) - s \geq 2$ where $g = \dim V$, then there is an $(s + 1)$ -dimensional subspace T containing S , and having no M_δ subspace.*

PROOF. An M_δ is $(g - 1)$ -dimensional. Let v be any vector in $\wedge^2 V - S$. If $S + \langle v \rangle$ has no M_δ subspace we are done. So suppose $S + \langle v \rangle \supseteq z \wedge V$ for some $z \in V$. Since $z \wedge V \not\subseteq S$ we know that $(z \wedge V) \cap S$ is $(g - 2)$ -dimensional. Let $w \in V$ be such that $z \wedge w \notin S$.

Now let T be any other $(s + 1)$ -dimensional space containing S and suppose it has an M_δ subspace $y \wedge V$. Then $(y \wedge V) \cap S$ is again $(g - 2)$ -dimensional, and since $2(g - 2) - s \geq 2$, we have $(y \wedge V) \cap (z \wedge V) \supseteq (y \wedge V) \cap (z \wedge V) \cap S$ is at least 2-dimensional. That is, for some independent sets $\{z', z''\}, \{y', y''\}$ in V , we have $z \wedge z' = y \wedge y', z \wedge z'' = y \wedge y''$. In terms of subspaces of V these equations read $\langle z, z' \rangle = \langle y, y' \rangle, \langle z, z'' \rangle = \langle y, y'' \rangle$. Intersecting these results, we see that $\langle z \rangle = \langle y \rangle$ and $z \wedge V = y \wedge V$. In other words, the only M_δ in any $(s + 1)$ -dimensional subspace T containing S , is $z \wedge V$. Since $\text{cod } S \geq 2$ we can enlarge S to an $(s + 1)$ -dimensional space T such that $z \wedge w \notin T$. Hence $z \wedge V \not\subseteq T$, and by preceding remarks this is sufficient to guarantee that T has no M_δ subspace.

Now suppose $g = 5, p = 5$, or $g = 6, p = 7$. Let $\rho \in F_2$. If ρ does not preserve decomposables there is an M_δ subspace $z \wedge V$ such that $\rho(z \wedge V)$ is not an M_δ subspace (in fact, since $\rho(z \wedge V)$ is $(g - 1)$ -dimensional, it contains no M_δ subspace). By one or two applications of Lemma 4 we can enlarge $\rho(z \wedge V)$ to a p -dimensional subspace T containing no M_δ subspace. Set

$S = \rho^{-1}(T)$ and note that $z \wedge V \subset S$. Since $\rho \in F_2$, there is a $\theta \in GL(V)$ satisfying $\wedge^2(\theta)(S) = T$. But $\wedge^2(\theta)$ takes M_δ 's to M_δ 's and T contains no M_δ 's. This contradiction shows ρ stabilizes the set of decomposable vectors and as before $F_2 = \wedge^2 GL(V)$. We have thus proven part of

PROPOSITION 5. *When $g \neq 4$, $F_2 = \wedge^2 GL(V) = S_2$.*

PROOF. The preceding lemma and discussion show that $F_2 = \wedge^2 GL(V)$. Every element of S_2 normalizes $F_2 = \wedge^2 GL(V)$. Considering the action of $\wedge^2 GL(V)$ on $\wedge^2 V$ this says that S_2 shuffles $\wedge^2 GL(V)$ -orbits. There are exactly $[g/2] + 1$ such orbits (see §5 of [2]); $\{0\}, O_1, \dots, O_{[g/2]}$, and O_k consists of those vectors which can be written as a sum of k -decomposables but not less than k . (Under the isomorphism $\wedge^2 V \simeq \text{Alt}(V^*)$ (§4 of [2]) the elements of O_k correspond to forms of rank $2k$ on V^* .) For each $i < j$, $O_i \subset \text{Cl}(O_j)$ where Cl denotes Zariski-closure. Thus, as varieties, the various O_i have different dimensions. So if $\sigma \in S_2$ shuffles orbits, by a dimension argument it fixes them. In particular it fixes O_1 —the set of decomposable vectors. Westwick's result then forces $S_2 = \wedge^2 GL(V)$.

We are now prepared to prove

THEOREM 6. *Suppose $g \neq 4$. If $p < \dim \wedge^2 V/2$ then $S = F = \pi \wedge^p \wedge^2(GL(V))$. If $p = \dim \wedge^2 V/2$ then $(S : \pi \wedge^p \wedge^2(GL(V))) \leq 2$, hence $(S : F) \leq 2$.*

PROOF. If $p < \dim \wedge^2 V/2$ then both π and \wedge^p in the diagram at the end of §1 are surjective. So the result follows from Proposition 5. If $\dim \wedge^2 V/2 = p$, then $(\text{Stab } D_p(\wedge^2 V) : \wedge^p GL(\wedge^2 V)) = 2$ by Westwick's result. Hence, since $\wedge^p(S_2) = S_1 \cap (\wedge^p GL(\wedge^2 V))$, we get $(S_1 : \wedge^p S_2) \leq 2$. But $\wedge^p(S_2) = \wedge^p \wedge^2 GL(V)$, so $(S_1 : \wedge^p \wedge^2 GL(V)) \leq 2$. Now π is always surjective, so in addition $(S : \pi \wedge^p \wedge^2 GL(V)) \leq 2$, and obviously $(S : F) \leq 2$.

3. Determination of the orbit shuffling and fixing groups when $\dim V = 4$. When $\dim V = 4$, $\dim \wedge^2 V = 6$ and $p = 1, 2$, or 3 .

For the cases $p = 1$ and $p = 2$ we rely on the classification of the algebras N such that $\text{cod } N^2 = 4$ and $\dim N^2 = 5$ and 4 respectively (§5 and Theorem 7.12 of [2]).

There are exactly two orbits in $G_1(\wedge^2 V)$ when $\dim V = 4$, the orbit O_1 of $\langle x_1 \wedge x_2 \rangle$ and the orbit O_2 of $\langle x_1 \wedge x_2 + x_3 \wedge x_4 \rangle$ where x_1, \dots, x_4 is any basis of V . Now $O_1 \subset \text{Cl}(O_2)$ since the nondecomposable vectors in $\wedge^2 V$ are open. Thus $\dim O_1 < \dim O_2$. (The closure of an orbit O is the union of O and orbits of strictly smaller dimension.) Hence any automorphism of $G_1(\wedge^2 V)$ which shuffles orbits must fix them.

There are exactly three orbits in $G_2(\wedge^2 V)$ when $\dim V = 4$, the orbit O_1 of $\langle x_1 \wedge x_2, x_3 \wedge x_4 \rangle$, the orbit O_2 of $\langle x_1 \wedge x_4 + x_2 \wedge x_3, x_2 \wedge x_4 \rangle$ and the orbit O_3 of $\langle x_2 \wedge x_4, x_3 \wedge x_4 \rangle$ where x_1, \dots, x_4 is any basis of V . The dimensions of these three orbits are pairwise distinct. This can be shown by computing the ranks of the Jacobian at $1_{GL(V)}$ of the maps $\rho_i: GL(V) \rightarrow \wedge^2 \wedge^2 V$ where $\rho_i(g) = \wedge^2(\wedge^2(g))(v_i)$ and $v_1 = (x_1 \wedge x_2) \wedge (x_3 \wedge x_4)$, $v_2 = (x_1 \wedge x_4 + x_2 \wedge x_3) \wedge (x_2 \wedge x_4)$, $v_3 = (x_2 \wedge x_4) \wedge (x_3 \wedge x_4)$ since $\dim O_i = (\text{rank } J\rho_i(1)) - 1$. Since these computations are tiresome at best, I will indicate here a proof which is satisfactory when $k = \mathbb{C}$ —the field of complex numbers. In this case any variety is also endowed with a strong topology (Chapter 1, §10 of [4]) and for any constructible subset of a variety, its closure in the strong topology is the same as its Zariski-closure. In particular, this holds for the orbits O_1, O_2, O_3 since they are locally closed subvarieties [2, Lemma 7.7] of $G_2(\wedge^2 V)$. Holding x_2, x_3, x_4 fixed and letting x_1 approach x_4 , we see that O_3 is contained in the strong closure of O_1 hence $O_3 \subset \text{Cl}(O_1)$ where Cl is Zariski-closure. Similarly, since $\langle x_2 \wedge x_3, x_2 \wedge x_4 \rangle$ is a representative of O_3 , fixing x_2, x_3, x_4 and letting x_1 approach x_4 we see that O_3 is contained in the strong closure of O_2 , hence $O_3 \subset \text{Cl}(O_2)$. Thus $\dim O_3 < \dim O_1, \dim O_2$. Also $G_2(\wedge^2 V) = O_1 \cup O_2 \cup O_3 = \text{Cl}(O_1) \cup \text{Cl}(O_2)$ is irreducible in the Zariski topology, so $G_2(\wedge^2 V) = \text{Cl}(O_1)$ or $\text{Cl}(O_2)$. That is, either $O_1 \subset \text{Cl}(O_2)$ or $O_2 \subset \text{Cl}(O_1)$. In either case $\dim O_2 \neq \dim O_1$. Thus, since the orbits O_1, O_2, O_3 have pairwise distinct dimensions, any automorphism of $G_2(\wedge^2 V)$ which shuffles orbits must fix them.

We have thus seen that for $g = 4$ and $p = 1$ or 2 the groups S and F are identical. For the case $g = 4, p = 3$ we consider again the diagram of groups at the end of §1.

Since $\dim V = 4, \wedge^2 GL(V)$ is of index two in the stabilizer of the set of decomposable two-vectors— $\text{Stab } D_2$. So suppose $\rho \in F_2$ and ρ does not preserve decomposables. Repeating the arguments of the case $g = 5, p = 4$ of §2 we arrive at a contradiction. So ρ does stabilize decomposables. Hence $(F_2 : \wedge^2 GL(V)) \leq 2$. But $\wedge^2 GL(V)$ is connected, so it is the identity component of F_2 . Now S_2 normalizes F_2 , hence its identity component $\wedge^2 GL(V)$. Repeating the arguments of Proposition 5, we see that S_2 preserves decomposables, thus $(S_2 : \wedge^2 GL(V)) \leq 2$. Since $3 = \binom{4}{2}/2, \wedge^3 GL(\wedge^2 V)$ is of index 2 in $\text{Stab } D_3(\wedge^2 V)$, and thus $(S_1 : \wedge^p S_2) \leq 2$. Hence

$$\begin{aligned} (S_1 : \wedge^p \wedge^2 GL(V)) &= (S_1 : \wedge^p S_2)(\wedge^p S_2 : \wedge^p \wedge^2 GL(V)) \\ &\leq (S_1 : \wedge^p S_2)(S_2 : \wedge^2 GL(V)) \leq 2 \cdot 2 = 4. \end{aligned}$$

Now π is always surjective so $(S : \pi \wedge^p \wedge^2 GL(V)) \leq 4$. By the first and second main theorems of [3], however, $(F : \pi \wedge^p \wedge^2 GL(V)) \geq 2$. That is, the automorphism ρ of the first main theorem when $\dim V = 4$, $p = 3$ centralizes $\pi \wedge^p \wedge^2 GL(V)$, hence shuffles orbits. But in the second main theorem it is shown ρ fixes orbits. Thus $\rho \in F - \pi \wedge^p \wedge^2 GL(V)$. Hence $(S : F) \leq 2$. We have thus proved

THEOREM 7. *Suppose $g = \dim V = 4$. When $p = 1$ or 2 then $S = F$, and when $p = 3 = (\frac{g}{2})/2$ then $(S : F) < 2$.*

4. The uniqueness theorem. To establish the uniqueness it remains only to show $S = F$ when $p = (\frac{g}{2})/2$. We know $(S : F) \leq 2$ in these cases, and in fact $(S : F) = (S_1 : F_1)$ since π is surjective and its kernel belongs to F_1 . Now S_1 normalizes F_1 , hence its identity component $\wedge^p \wedge^2 GL(V)$. Likewise, since $\wedge^p \wedge^2 SL(V)$ is the commutator subgroup of $\wedge^p \wedge^2 GL(V)$, S_1 normalizes it as well.

Suppose that $S_1 \neq F_1$ and pick any s in S_1 . Now s stabilizes $\wedge^p \wedge^2 SL(V)$ by conjugation, so it stabilizes the corresponding Lie algebra $d(\wedge^p \wedge^2)sl(V)$ by conjugation ($d(\wedge^p \wedge^2)$ is the differential of $\wedge^p \wedge^2$), inducing on it an automorphism. Now $d(\wedge^p \wedge^2)$ is an isomorphism of $sl(V)$ onto $d(\wedge^p \wedge^2)sl(V)$, and $sl(V)$ has basically two types of automorphisms. The first are conjugations by elements of $GL(V)$, the second are compositions of a conjugation with the operation of taking a matrix to its negative transpose.

If the conjugation of s on $d(\wedge^p \wedge^2)sl(V)$ is an automorphism of the first type, then multiplying s by a suitable element of $\wedge^p \wedge^2 GL(V)$ (only its coset matters) we can presume it centralizes $d(\wedge^p \wedge^2)sl(V)$, hence $d(\wedge^p \wedge^2)gl(V)$. The associative algebra spanned by $d(\wedge^p \wedge^2)gl(V)$ contains $\wedge^p \wedge^2 GL(V)$, so s centralizes it. By the first main theorem of [3] (second main theorem), when $g \neq 4$ ($g = 4$), this contradicts $(S_1 : F_1) = 2$.

So suppose the conjugation of s on $d(\wedge^p \wedge^2)sl(V)$ is of the second type. Multiplying s by a suitable element of $\wedge^p \wedge^2 GL(V)$, and identifying $d(\wedge^p \wedge^2)sl(V)$ with $sl(V)$, we can assume conjugation by s induces $-\text{Id}$ on the Cartan subalgebra H of diagonal matrices of trace zero. Thus, as a linear transformation on $\wedge^p \wedge^2 V$, s must take any H -weight space to an H -weight space whose weight is the negative of the original one. Since the groups S_1 and $\wedge^p \wedge^2 GL(V)$ have identical Lie algebras, and since the groups and their algebras stabilize the same subspaces of $\wedge^p \wedge^2 V$, s stabilizes the simple $sl(V)$ -submodules. By the preceding remark, each of these simple submodules would have to be self-contragredient. If $\lambda_1, \dots, \lambda_l$ are the fundamental weights of $sl(V)$ ($\dim V = l + 1$), then the module contragredient to the simple $sl(V)$ -module of highest weight $\sum_i m_i \lambda_i$ is the one with highest weight $\sum_i m_{l+1-i} \lambda_i$. Now applying the results

of §3 of [3], one can pick out simple submodules of $\bigwedge^p \bigwedge^2 V$ which are not self-contragredient. Thus this second type of automorphism cannot occur.

As a result $S_1 = F_1$ for all g, p and we have completed the proof of the

MAIN THEOREM. *There is only one algebraic duality theory.*

REFERENCES

1. J. Fogarty, *Invariant theory*, Benjamin, New York, 1969. MR 39 #1458.
2. M. Gauger, *On the classification of metabelian Lie algebras*, Trans. Amer. Math. Soc. 179 (1973), 293–329.
3. ———, *Duality theories for metabelian Lie algebras*, Trans. Amer. Math. Soc. 187 (1974), 89–102.
4. D. Mumford, *Introduction to algebraic geometry*, preliminary version, Harvard Univ. Press, Cambridge, Mass.
5. J. Scheuneman, *Two-step nilpotent Lie algebras*, J. Algebra 7 (1967), 152–159. MR 36 #225.
6. R. Westwick, *Linear transformations on Grassmann spaces*, Pacific J. Math. 14 (1964), 1123–1127. MR 29 #4766.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MASSACHUSETTS, AMHERST,
MASSACHUSETTS 01002