TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 203, 1975

THE AUTOMORPHISM GROUP OF A COMPACT GROUP ACTION

BY

W. D. CURTIS

ABSTRACT. This paper contains results on the structure of the group, $\mathrm{Diff}^r_G(M)$, of equivariant C^r-diffeomorphisms of a free action of the compact Lie group G on M. $\mathrm{Diff}^r_G(M)$ is shown to be a locally trivial principal bundle over a submanifold of $\mathrm{Diff}^r(X)$, X the orbit manifold. The structural group of this bundle is $E^r(G, M)$, the set of equivariant C^r-diffeomorphisms which induce the identity on X. $E^r(G, M)$ is shown to be a submanifold of $\mathrm{Diff}^r(M)$ and in fact a Banach Lie group $(r < \infty)$.

0. **Introduction.** This paper studies the group of equivariant diffeomorphisms of a smooth action of a compact Lie group G on a compact manifold M. Specifically most of the paper deals with the case when G acts freely on M. In this case there is an orbit manifold X, and an equivariant C^r-diffeomorphism f of M induces a C^r-diffeomorphism $\bar{f}$ of X. This defines a homomorphism P: $\mathrm{Diff}^r_G(M) \longrightarrow \mathrm{Diff}^r(X)$ ($\mathrm{Diff}^r_G(M)$ is the group of equivariant diffeomorphisms on M of class C^r, $1 \leqslant r \leqslant \infty$). We obtain some results about the structure of P. We show P admits smooth local cross-sections (Theorem 3.5). The kernel of P is the group of C^r-equivariant diffeomorphisms of M which induce the identity on X. This group is the structural group of the locally trivial principal bundle determined by P. We show $\ker P$ is a smooth submanifold of $\mathrm{Diff}^r(M)$ and, that with respect to the induced differential structure, $\ker P$ is a Banach Lie group (Theorem 4.2). Recall $\mathrm{Diff}^r(M)$ is not a Banach Lie group as composition is not C^1 $(r < \infty)$.

The main technique introduced is the construction in §2 of G-lifts of sprays on X. This allows a precise connection between the manifold of maps differential structures on $\mathrm{Diff}^r_G(M)$ and $\mathrm{Diff}^r(X)$.

1. **Preliminaries.** M will always denote a compact, connected, C^∞-manifold, G a compact Lie group. Assume G acts on M (on the left). We denote by X the orbit space and we let $\pi\colon M \longrightarrow X$ be the orbit projection. If G acts freely and differentiably (C^∞) then X has a natural C^∞-structure such that π

Presented to the Society, January 26, 1973 under the title *The automorphism group of a compact Lie group action*; received by the editors March 22, 1973.

AMS (MOS) subject classifications (1970). Primary 58D05, 22E65; Secondary 55F10.

Key words and phrases. Diffeomorphism group, equivariant diffeomorphism, Banach Lie group, principal bundle, spray, free action.

is C^∞ and locally trivial (since G is compact).

If $g \in G$, $w \in M$ we write gw to denote the result of letting g act on w. We shall also write g to denote the diffeomorphism $w \longrightarrow gw$ (henceforth all manifolds, actions, etc. are assumed C^∞). An action of G on M induces an action of G on TM, the tangent bundle of M. If $g \in G$ we write $Tg(v)$ for the result of acting on $v \in TM$ by g. The resulting diffeomorphism of TM is Tg: $TM \longrightarrow TM$ and is just the tangent of g: $M \longrightarrow M$. Similarly G acts on $T^2M = T(TM)$ by "double tangents", $T^2g = T(Tg)$: $T^2M \longrightarrow T^2M$ being the diffeomorphism corresponding to the group element g.

DEFINITION 1.1. Let G act on M. An automorphism of class C^r is a diffeomorphism f: $M \longrightarrow M$ of class C^r which is G-equivariant. We denote by $\mathrm{Diff}_G^r(M)$ the group of all C^r-automorphisms of the action. (Here $1 \leqslant r \leqslant \infty$.)

Let $\mathrm{Diff}^r(M)$ be the group of all C^r-diffeomorphisms of M onto M. If G acts on M then $\mathrm{Diff}_G^r(M)$ is clearly a subgroup of $\mathrm{Diff}^r(M)$.

The differential structure on $\mathrm{Diff}^r(M)$: We review the standard manifold of maps construction as applied to make $\mathrm{Diff}^r(M)$ into a C^∞-manifold. We recall that $\mathrm{Diff}^r(M)$ is locally Banach if $1 \leqslant r < \infty$ while being locally Fréchet in the case $r = \infty$. Also in case $r = \infty$ the multiplication and inversion in $\mathrm{Diff}^r(M)$ are C^∞ so $\mathrm{Diff}^\infty(M)$ is a Fréchet Lie group [5].

Let ξ: $TM \longrightarrow T^2M$ be a C^∞-spray. There is an open neighborhood $\mathcal{O}$ of the 0-section in TM and an open neighborhood U of the "diagonal" in $M \times M$ and a C^∞-diffeomorphism Exp: $\mathcal{O} \longrightarrow U$ defined by $\mathrm{Exp}(v) = (\tau(v), \exp(v))$ where τ: $TM \longrightarrow M$ is projection and exp is the exponential of the spray ξ. We use Exp to construct "natural charts" around each $f \in \mathrm{Diff}^r(M)$. We define a neighborhood N_f of f by

$$N_f = \{h \in \mathrm{Diff}^r(M) | (f(x), h(x)) \in U \text{ for all } x \in M\}.$$

Letting $\Gamma^r(f^*TM)$ denote the space of all C^r-sections of the induced bundle f^*TM, with the C^r-topology, we define α_f: $N_f \longrightarrow \Gamma^r(f^*TM)$ by $\alpha_f(h)(x) = (x, \mathrm{Exp}^{-1}(f(x), h(x)))$. α_f maps N_f bijectively onto an open set $\Gamma^r(f^*TM)$ and the collection of all charts (N_f, α_f) for $f \in \mathrm{Diff}^r(M)$ is a C^∞-atlas. This C^∞-structure is independent of the choice of spray. For further discussion of this manifold of maps construction, we refer to [1], [3], [5].

For use later we now consider some properties of G-invariant sprays.

DEFINITION 1.2. If G acts on M we say a spray ξ: $TM \longrightarrow T^2M$ is G-invariant if, for all $g \in G$, we have $T^2g \circ \xi = \xi \circ Tg$.

The following proposition is easily proved.

PROPOSITION 1.3. *If* ξ *is a* G*-invariant spray, then the domain of* $\exp_\xi$ *is an invariant subset of* TM *and* $\exp_\xi$ *is* G*-equivariant.*

Using notation as before we have

LEMMA 1.4. *If* ξ *is a* G*-invariant spray then* $\mathcal{O}$ *may be chosen to be* G*-invariant.* Exp: $\mathcal{O} \longrightarrow M \times M$ *is then* G*-equivariant, where* G *acts on* $M \times M$ *by* $g(x_1, x_2) = (gx_1, gx_2)$. *It then follows that* $U = \operatorname{Exp}(\mathcal{O})$ *is an invariant set.*

PROOF. Let $\mathcal{O}_1$ be a neighborhood of the $\mathcal{O}$-section such that Exp is a diffeomorphism on $\mathcal{O}_1$. If we choose a G-invariant Riemannian metric on M, then for sufficiently small $\epsilon > 0$ we will have $W_\epsilon = \{v \in TM \mid \|v\| < \epsilon\} \subset \mathcal{O}_1$ (since M is compact). But W_ϵ is G-invariant since we have used an invariant metric. If $g \in G$, $v \in \mathcal{O} = W_\epsilon$ we have $\operatorname{Exp}(Tg \cdot v) = (\tau(Tg \cdot v), \exp(Tg \cdot v)) = (g\tau(v), g \exp(v)) = g \operatorname{Exp}(v)$. This proves the lemma.

The group $\operatorname{Diff}^r_G(M)$: If G acts freely on M then there is a homomorphism P: $\operatorname{Diff}^r_G(M) \longrightarrow \operatorname{Diff}^r(X)$. The main results of this paper are concerned with the structure of P. P is defined by $P(f) = f_1$ where f_1 makes the following diagram commute.

$$\begin{array}{ccc} M & \xrightarrow{\ f\ } & M \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ X & \xrightarrow{\ f_1\ } & X \end{array}$$

A preliminary result is

PROPOSITION 1.5. $\operatorname{Diff}^r_G(M)$ *is a submanifold of* $\operatorname{Diff}^r(M)$ *for any action of* G *on* M (*free or not*).

PROOF. Let ξ be an invariant spray. Such always exist, e.g. take ξ to be the geodesic spray of an invariant Riemann metric. We show that the natural chart at the identity map is a submanifold chart. We have

$$\alpha: N \longrightarrow \Gamma^r(TM), \quad \alpha(f)(x) = \operatorname{Exp}^{-1}(x, f(x)).$$

Let $\Gamma^r_G(TM) \subset \Gamma^r(TM)$ consist of all C^r-vectorfields ζ on M such that $Tg \circ \zeta = \zeta \circ g$ for all $g \in G$ (i.e., all invariant vectorfields). Using the fact that by Lemma 1.4 Exp is equivariant, one immediately sees that $\alpha(N \cap \operatorname{Diff}^r_G(M)) = \alpha(N) \cap \Gamma^r_G(TM)$. This proves the proposition.

ASSUMPTION. Henceforth we consider only free actions of G on M.

In order to investigate the homomorphism P defined above we shall, in the next section, construct a special type of spray on M. This spray will allow us to relate the differential structure on $\operatorname{Diff}^r_G(M)$ to that on $\operatorname{Diff}^r(X)$.

2. Construction of G**-lifts of sprays.** Suppose G acts freely on M, π: $M \longrightarrow X$ orbit projection. If η is a spray on X then a spray on M, say ξ, is called a lift of η if the following commutes.

$$\begin{array}{ccc} TM & \xrightarrow{\ \xi\ } & T^2M \\ {\scriptstyle T\pi}\downarrow & & \downarrow{\scriptstyle T^2\pi} \\ TX & \xrightarrow{\ \eta\ } & T^2X \end{array}$$

If, in addition, ξ is G-invariant we will say ξ is a G-lift of η. We shall prove

THEOREM 2.1. *If η is any spray on X then there exists a spray ξ on M which is a G-lift of η.*

The proof of Theorem 2.1 will be based on several lemmas.

Let $\{U_i\}_i$ (i in some index set) be an open cover for X. Let $M_i = \pi^{-1}(U_i)$ for each i. $\{M_i\}_i$ is an open cover of M and $\{TM_i = \tau^{-1}(M_i)\}_i$ is an open cover of TM.

LEMMA 2.2. *There is a C^∞-partition of unity $\{\varphi_i\}_i$, subordinate to $\{TM_i\}_i$ such that each φ_i is G-invariant, i.e., $\varphi_i \circ Tg = \varphi_i$ for all $g \in G$.*

PROOF. Choose a C^∞-partition of unity, $\{\psi_i\}_i$ subordinate to $\{U_i\}_i$. Let $\varphi_i = \psi_i \circ \pi \circ \tau$. $\{\varphi_i\}_i$ is easily seen to have the desired properties.

The next lemma is general in nature. Let Y, Z be C^∞-manifolds, ξ, η C^∞-sprays on Y, Z. There are vector bundle isomorphisms

$$\alpha\colon T(Y \times Z) \longrightarrow TY \times TZ, \qquad \alpha(v) = (T\pi_1(v), T\pi_2(v)),$$

$$\beta\colon T(TY \times TZ) \longrightarrow (T^2Y) \times (T^2Z), \qquad \beta(2) = (T\pi_1(w), T\pi_2(w)).$$

Thus we have a bundle isomorphism

$$\delta = \beta \circ T\alpha\colon T^2(Y \times Z) \longrightarrow (T^2Y) \times (T^2Z).$$

LEMMA 2.3. *Let γ be the unique map making the following diagram commute. Then γ is a spray on $Y \times Z$. (We shall write $\xi * \eta$ for γ.)*

$$\begin{array}{ccc} T^2(Y \times Z) & \xrightarrow{\ \delta\ } & (T^2Y) \times (T^2Z) \\ {\scriptstyle \gamma}\uparrow & & \uparrow{\scriptstyle \xi \times \eta} \\ T(Y \times Z) & \xrightarrow{\ \alpha\ } & (TY) \times (TZ) \end{array}$$

PROOF. If Y is modelled on E and Z is modelled on F, then the local version of the above diagram is

$$\begin{array}{ccc} U \times V \times E \times F \times E \times F \times E \times F & \xrightarrow{\ \delta\ } & (U \times E \times E \times E) \times (V \times F \times F \times F) \\ {\scriptstyle \gamma}\uparrow & & \uparrow{\scriptstyle \xi \times \eta} \\ U \times V \times E \times F & \xrightarrow{\ \alpha\ } & (U \times E) \times (V \times F) \end{array}$$

We have:

$$\alpha(x, y, u, v) = (x, u, y, v),$$

$$\delta(x, y, u, v, u', v', u'', v'') = (x, u, u', u'', y, v, v', v'').$$

If $\bar{\xi}, \bar{\eta}$ are the principal parts of ξ, η we easily see

$$\gamma(x, y, u, v) = (x, y, u, v, u, v, \bar{\xi}(x, u), \bar{\eta}(y, v)).$$

This local form shows γ is a spray.

Lemma 2.4. *Let notation be as above, $p\colon Y \times Z \longrightarrow Y$ projection. Then the following diagram commutes.*

$$\begin{array}{ccc} T^2(Y \times Z) & \xrightarrow{T^2 p} & T^2 Y \\ \uparrow{\scriptstyle \xi * \eta} & & \uparrow{\scriptstyle \xi} \\ T(Y \times Z) & \xrightarrow{Tp} & TY \end{array}$$

Proof. This is immediate from the local representation of Lemma 2.3, or an invariant proof can be given. We omit details.

Now return to $\pi\colon M \longrightarrow X$. Suppose $U \subset X$ is open and $\varphi\colon \pi^{-1}(U) \longrightarrow U \times G$ is an equivariant diffeomorphism such that $\pi_1 \circ \varphi = \pi$. ($G$ acts on $U \times G$ by $g(x, g') = (x, gg')$.) Recall that X is covered by such U. If η is a fixed spray on X, then η induces a spray on U which we denote η_U. Let $\mu\colon TG \longrightarrow T^2G$ be the canonical left-invariant spray of G [6, p. 222].

Then $\eta_U * \mu$ is a spray on $U \times G$.

Lemma 2.5. *Let G act on $U \times G$ as above. Then $\eta_U * \mu$ is G-invariant.*

Proof. Let $v \in T(U \times G)$, $\alpha(v) = (v_1, v_2)$. Now

$$\begin{aligned} T^2 g \circ (\eta_U * \mu)(v) &= T^2 g \circ (T\alpha)^{-1} \circ \beta^{-1} \circ (\eta_U \times \mu)(v_1, v_2) \\ &= T(Tg \circ \alpha^{-1}) \circ \beta^{-1}(\eta_U(v_1), \mu(v_2)) \\ &= T(\alpha^{-1} \circ (1 \times Tg)) \circ \beta^{-1}(\eta_U(v_1), \mu(v_2)) \\ &= T\alpha^{-1} \circ \beta^{-1} \circ (1 \times T^2 g)(\eta_U(v_1), \mu(v_2)) \\ &= T\alpha^{-1} \circ \beta^{-1}(\eta_U(v_1), T^2 g \circ \mu(v_2)) \\ &= T\alpha^{-1} \circ \beta^{-1}(\eta_U(v_1), \mu(Tg(v_2))) \\ &= T\alpha^{-1} \circ \beta^{-1} \circ (\eta_U \times \mu)(v_1, Tg(v_2)) \\ &= \eta_U * \mu(Tg(v)). \end{aligned}$$

Let ξ_U be the spray on $\pi^{-1}(U)$ such that the following diagram commutes.

$$\begin{array}{ccc} T^2(\pi^{-1}(U)) & \xrightarrow{T^2\varphi} & T^2(U \times G) \\ \xi_U \uparrow & & \uparrow \eta_U * \mu \\ T(\pi^{-1}(U)) & \xrightarrow{T\varphi} & T(U \times G) \end{array}$$

Since φ is an equivariant diffeomorphism, we see ξ_U is a G-invariant spray on $\pi^{-1}(U)$.

We can now prove Theorem 2.1.

PROOF OF THEOREM 2.1. Cover X by open sets U_i such that for each i there is an equivariant diffeomorphism $\varphi_i: \pi^{-1}(U_i) \longrightarrow U_i \times G$ as above. Let $\xi_i = \xi_{U_i}$ be as above. Let $\{\varphi_i\}_i$ be an invariant partition of unity subordinate to $\{TM_i\}_i$ as in Lemma 2.2. Let $\xi = \Sigma_i \varphi_i \xi_i: TM \longrightarrow T^2M$. ξ is a spray on M. Using the invariance of the φ_i we easily see ξ is invariant. It remains to show ξ is a lift of η. Let $v \in TM$.

$$T^2\pi \circ \xi(v) = T^2\pi\left(\sum_i \varphi_i(v)\xi_i(v)\right) = \sum_i \varphi_i(v) T^2\pi \circ \xi_i(v).$$

Suppose $T^2\pi \circ \xi_i(v) = \eta(T\pi(v))$ for each i with $v \in TM_i$. Then we will be done. If $v \in TM_i$ we compute as follows:

$$\begin{aligned} T^2\pi(\xi_i(v)) &= T^2\pi(T^2\varphi_i^{-1} \circ (\eta_U * \mu) \circ T\varphi_i(v)) \\ &= T^2(\pi \circ \varphi_i^{-1}) \circ (\eta_{U_i} * \mu) \circ T\varphi_i(v) = T_p^2 \circ \eta_{U_i} * \mu \circ T\varphi_i(v) \\ &= \eta_{U_i} \circ Tp(T\varphi_i(v)) = \eta_{U_i} \circ T\pi(v) = \eta(T\pi(v)). \end{aligned}$$

Here $p: U_i \times G \longrightarrow U_i$ is projection and we have used Lemma 2.4. This completes the proof of Theorem 2.1.

3. **The structure of P.** Henceforth η will be a fixed spray on X and ξ will be a fixed G-lift of η. Recall that the domain of $\exp_\xi$ is $\mathrm{Dom}(\exp_\xi) = \{v \in TM \mid 1 \in \mathrm{Dom}(\beta_v)\}$ where β_v is the maximal integral curve of ξ with initial condition v. Similarly $\mathrm{Dom}(\exp_\eta) = \{w \in TX \mid 1 \in \mathrm{Dom}(\gamma_w)\}$ where γ_w is the maximal integral curve of η with initial condition w. Since ξ is a lift of π we see ξ and η are $T\pi$-related, so $T\pi(\mathrm{Dom}(\exp_\xi)) \subset \mathrm{Dom}(\exp_\eta)$ and the following diagram commutes.

$$\begin{array}{ccc} \mathrm{Dom}(\exp_\xi) & \xrightarrow{T\pi} & \mathrm{Dom}(\exp_\eta) \\ \exp_\xi \downarrow & & \downarrow \exp_\eta \\ M & \xrightarrow{\pi} & X \end{array}$$

We shall use ξ and η to construct the differential structures on $\text{Diff}^r(M)$ and $\text{Diff}^r(X)$ respectively. Using the above properties of ξ and η we easily see we may choose open sets $\mathcal{O}, \bar{\mathcal{O}}$ containing the 0-sections of TM, TX, respectively, such that the following hold: (1) $\mathcal{O}$ is G-invariant; (2) $T\pi(\mathcal{O}) = \bar{\mathcal{O}}$; (3) Exp_ξ and Exp_η map $\mathcal{O}$ and $\bar{\mathcal{O}}$ diffeomorphically onto neighborhoods U and $\bar{U}$ of the respective diagonals; (4) $(\pi \times \pi)(U) = \bar{U}$. Also, we clearly have $(\pi \times \pi) \circ \text{Exp}_\xi = \text{Exp}_\eta \circ T\pi$.

As in §1 we get natural charts $(N, \alpha), (\bar{N}, \bar{\alpha})$ at the identities in $\text{Diff}^r(M)$ and $\text{Diff}^r(X)$. Since ξ is an invariant spray we see, as in Proposition 1.5, we get a submanifold chart at $1_M, (N^*, \alpha^*)$ where $N^* = N \cap \text{Diff}^r_G(M), \alpha^* = \alpha|N^*$: $N^* \longrightarrow \Gamma^r_G(TM)$.

Let $\text{Diff}^r_0(X)$ be the group of all C^r-diffeomorphisms, h, of X such that there is an equivariant C^r-diffeomorphism of M with $\pi \circ f = h \circ \pi$ (i.e., f covers h). It is easily seen that $\text{Diff}^r_0(X)$ is an open subgroup of $\text{Diff}^r(X)$ and hence a manifold. Clearly the map P defined earlier is a surjective homomorphism $P\colon \text{Diff}^r_G(M) \longrightarrow \text{Diff}^r_0(X)$. Let $\bar{N}^* = \bar{N} \cap \text{Diff}^r_0(X), \bar{\alpha}^* = \bar{\alpha}|\bar{N}^*$, so that $(\bar{N}^*, \bar{\alpha}^*)$ is a chart at the identity on $\text{Diff}^r_0(X)$.

LEMMA 3.1. $P(N^*) \subset \bar{N}^*$.

PROOF. Let $f \in N^*, P(f) = \bar{f}$. We must show $(x, \bar{f}(x)) \in \bar{U}$ for each $x \in X$. Let $x = \pi(y)$. Then $(x, \bar{f}(x)) = (\pi \times \pi)(y, f(y))$ as is easily seen. But $(y, f(y)) \in U$ and $(\pi \times \pi)(U) = \bar{U}$ so the lemma follows.

Let ρ be the unique map making the following diagram commute.

$$\begin{array}{ccc} N^* & \xrightarrow{\alpha^*} & \alpha^*(N^*) \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle \rho} \\ \bar{N}^* & \xrightarrow{\bar{\alpha}^*} & \alpha^*(\bar{N}^*) \end{array}$$

LEMMA 3.2. *There is a continuous linear map,* $\rho_1\colon \Gamma^r_G(TM) \longrightarrow \Gamma^r(TX)$, *such that* $\rho_1|\alpha^*(N^*) = \rho$.

PROOF. We first show that for $\zeta \in \Gamma^r_G(TM)$ there is a unique map, $\rho_1(\zeta)$ in $\Gamma^r(TX)$ such that $T\pi \circ \zeta = \rho_1(\zeta) \circ \pi$. If $\pi(u) = \pi(v)$ we must show $T\pi(\zeta(u)) = T\pi(\zeta(v))$. If $\pi(u) = \pi(v)$ then $v = gu$ for some $g \in G$. But then $T\pi(\zeta(v)) = T\pi(\zeta(gu)) = T\pi(Tg \circ \zeta(u)) = T\pi(\zeta(u))$. This defines ρ_1 and, working locally, one easily verifies that ρ_1 is continuous linear and, in particular, that $\rho_1(\zeta)$ is actually in $\Gamma^r(TX)$. Let $h \in N^*, v \in M$. Then

$$\begin{aligned} \rho_1(\alpha^*(h))(\pi(v)) &= T\pi(\alpha^*(h)(v)) = T\pi(\exp_\xi^{-1}(v, h(v))) \\ &= \text{Exp}_\eta^{-1}(\pi(v), P(h)(\pi(v))) = \bar{\alpha}^*(P(h))(\pi(v)). \end{aligned}$$

This proves $\rho_1|\alpha^*(N^*) = \rho$.

It follows from Lemma 3.2 that P is C^∞ in a neighborhood of the identity. Since P is a homomorphism and right translation is C^∞ it follows P is C^∞ everywhere.

PROPOSITION 3.3. *ρ_1 is a continuous linear surjection with split kernel.*

PROOF. Recall there is a bundle $VT(M)$ of vertical vectors. $VT(M) = \ker(T\pi) \subset TM$. $VT(M)$ is a G-invariant subbundle of TM. Let H be an invariant subbundle of TM which is complementary to $VT(M)$ (π is essentially a principal G-bundle so H is just an affine connection. See [4].) Let $\Gamma^r_G(VT(M))$ and $\Gamma^r_G(H)$ be the spaces of G-invariant C^r-sections of the two subbundles. We immediately have $\ker \rho_1 = \Gamma^r_G(VT(M))$. Using the G-invariance of the subbundle it is easy to see $\Gamma^r_G(TM) = \Gamma^r_G(VT(M)) \oplus \Gamma^r_G(H)$. So $\Gamma^r_G(H)$ is a complement for $\ker(\rho_1)$.

That ρ_1 is onto is just the well-known existence of the horizontal lift of a vectorfield with respect to an affine connection on a principal bundle. (See [4].)

COROLLARY 3.4. *$\ker P$ is a split submanifold of* $\mathrm{Diff}^r_G(M)$.

PROOF. $\alpha^*(N^* \cap \ker P) = \alpha^*(N^*) \cap \Gamma^r_G(VT(M))$, as is easily checked. The corollary follows.

THEOREM 3.5. *P admits smooth local cross-sections.*

PROOF. Let $\lambda_1\colon \Gamma^r(TX) \longrightarrow \Gamma^r_G(TM)$ be continuous linear with $\rho_1 \circ \lambda_1 =$ identity. Choose an open set W_1 containing 0 in $\Gamma^r(TX)$ such that $\lambda_1(W_1) \subset \alpha^*(N^*)$. Let $W = (\bar{\alpha}^*)^{-1}(W_1)$ and $\lambda(h) = (\alpha^*)^{-1} \circ \lambda_1 \circ \bar{\alpha}^*(h)$ for $h \in W$. Then h is a smooth section of P near the identity so the theorem is proved.

Let $E^r(G, M) = \ker P$. $E^r(G, M)$ is the group of self-equivalences of the bundle π of class C^r. In case $r = \infty$ we see $E^r(G, M)$ is a Fréchet Lie group since it is a subgroup and closed submanifold of $\mathrm{Diff}^\infty(M)$ which is a Fréchet Lie group [5].

If $r < \infty$ we cannot argue the same way since $\mathrm{Diff}^r(M)$ is not a Banach Lie group. In the next section we show that, nonetheless, $E^r(G, M)$ is a Banach Lie group. The following is immediate from Theorem 3.5 and Corollary 3.4.

THEOREM 3.6. *P: $\mathrm{Diff}^r_G(M) \longrightarrow \mathrm{Diff}^r(X)$ is a principal bundle with group $E^r(G, M)$. If $r = \infty$ this bundle is C^∞ while if $r < \infty$ we have a continuous bundle.*

The following problem is unsolved.

Determine conditions on π so that the bundle P is trivial, i.e., so that P admits a global cross-section. (If π is trivial, then so is P, as is easily seen.)

4. The group $E^r(G, M)$. In this section we assume $r < \infty$. We know by Corollary 3.4 that $E^r(G, M)$ is a C^∞-Banach manifold. We shall prove that

$E^r(G, M)$ is a Banach Lie group which is C^∞-anti-isomorphic to a Banach Lie subgroup of $C^r(M, G)$.

In [2] it was shown that $C^r(M, G)$ is a Banach Lie group. There is a Banach Lie subgroup E of $C^r(M, G)$ defined by

$$E = \{h \in C^r(M, G) | h(gv) = gh(v)g^{-1}, \text{ for all } g \in G, v \in M\}.$$

E acts on the right of M by $v * h = h(v)v$. (These things are discussed in [2] in more detail, the only difference being that in [2] G acts on the right and E on the left.) This action defines an injective antihomomorphism $\Phi\colon E \longrightarrow \mathrm{Diff}^r_G(M)$ and $\Phi(E) = E^r(G, M)$ as one easily sees.

THEOREM 4.1. *Φ maps diffeomorphically onto $E^r(G, M)$.*

PROOF. It is enough to show that Φ and Φ^{-1} are C^∞ in some neighborhoods of the respective identities. There is a map $\Phi^*\colon C^r(M, G) \longrightarrow C^r(M, M)$ defined by $\Phi^*(f)(v) = f(v)v = \mu(f(v), v)$ where $\mu\colon G \times M \longrightarrow M$ is the group action. This is a C^∞-map because there is a commutative diagram

$$\begin{array}{ccc} C^r(M, G) & \xrightarrow{\quad\Phi\quad} & C^r(M, M) \\ & {\scriptstyle\sigma}\searrow \quad \nearrow{\scriptstyle\Omega} & \\ & C^r(M, M \times G) & \end{array}$$

where $\sigma(f) = (1_M, f)$, $\Omega(k) = \mu \circ k$. Both σ and Ω are easily seen to be C^∞. $\Phi = \Phi^*|E$ so Φ is C^∞.

Now consider Φ^{-1}. Recall we have $\mathcal{O} \subset TM$. $\Gamma^r(VT(M) \cap \mathcal{O})$ is the set of C^r-vectorfields on M which take values in $VT(M) \cap \mathcal{O}$. We define ψ: $\Gamma^r(VT(M) \cap \mathcal{O}) \longrightarrow C^r(M, G)$. Let $M \times_X M$ be the submanifold of $M \times M$ consisting of all (v, w) such that $\pi(v) = \pi(w)$. There is a C^∞-map $\theta\colon M \times_X M \longrightarrow G$ by $\theta(v, w) = g$ if $w = gv$. Since G acts freely, this is well defined and θ is C^∞ as is seen by working locally. Now $\exp_\xi$ maps $VT(M) \cap \mathcal{O}$ into $M \times_X M$ as the following calculation shows. Let $v \in VT(M) \cap \mathcal{O}$. Then

$$\begin{aligned}(\pi \times \pi)(\mathrm{Exp}_\xi(v)) &= \mathrm{Exp}_\eta(T\pi(v)) = (\tau(T\pi(v)), \exp_\eta(T\pi(v))) \\ &= (\pi(\tau(v)), \exp_\eta(0)) = (\pi(\tau(v)), \pi(\tau(v)))\end{aligned}$$

where 0 is the zero vector in the tangent space to X at $\pi(\tau(v))$.

Thus we have a C^∞-map $\psi\colon \Gamma^r(VT(M) \cap \mathcal{O}) \longrightarrow C^r(M, G)$ by $\psi(\xi) = \theta \circ \mathrm{Exp}_\xi \circ \xi$. We show that ψ takes the subspace $\Gamma^r_G(VT(M) \cap \mathcal{O})$ into the subgroup E of $C^r(M, G)$. G acts on $M \times_X M$ by $g(v, w) = (gv, gw)$ and one easily sees $\theta(gv, gw) = g\theta(v, w)g^{-1}$. Thus if $\zeta \in \Gamma^r_G(VT(M) \cap \mathcal{O})$, we get

$$\psi(\zeta)(gw) = \theta(\mathrm{Exp}_\xi(\zeta(gw))) = \theta(\mathrm{Exp}_\xi(Tg(\zeta(w))))$$
$$= \theta(g\,\mathrm{Exp}_\xi(\zeta(w))) = g[\theta(\mathrm{Exp}_\xi(\zeta(w)))]g^{-1} = g[\psi(\zeta)(w)]g^{-1}.$$

Thus we have a C^∞-map $\psi\colon \Gamma^r_G(VT(M) \cap \mathcal{O}) \longrightarrow E$.

We show finally that $\psi \circ \alpha^* = \Phi^{-1}$ in a neighborhood of the identity. Choose an open set $V \subset E^r(G, M)$ such that $\alpha^*(V) \subset \Gamma^r_G(VT(M) \cap \mathcal{O})$. Let $h \in V$. Let $f = \Phi^{-1}(h)$. Then

$$\psi \circ \alpha^*(h) = \psi \circ \alpha^*(\Phi(f)) = \theta \circ \mathrm{Exp}_\xi \circ (\alpha^*\Phi(f)).$$

So

$$(\psi \circ \alpha^*(h))(w)\theta(\mathrm{Exp}_\xi([\alpha^*\Phi(f)](w))) = \theta(\mathrm{Exp}_\xi(\mathrm{Exp}_\xi^{-1}(w, f(w)w)))$$
$$= \theta(w, f(w)w) = f(w).$$

Therefore $\psi \circ \alpha^*(h) = f = \Phi^{-1}(h)$ as desired. This completes the proof of 4.1.

THEOREM 4.2. *$E^r(G, M)$ is a Banach Lie group* $(r < \infty)$.

PROOF. Φ is a diffeomorphism of the Lie group E onto $E^r(G, M)$ and Φ is an antihomomorphism so the multiplication and inversion in $E^r(G, M)$ are smooth.

REFERENCES

1. R. Abraham, *Lectures of Smale on differential topology*, Notes at Columbia University, 1962–63.
2. W. D. Curtis, Y.-L. Lee and F. Miller, *A class of infinite dimensional subgroups of* $\mathrm{Diff}^r(X)$ *which are Banach Lie groups*, Pacific J. Math. 47 (1973), 59–65.
3. H. I. Eliasson, *Geometry of manifolds of maps*, J. Differential Geometry 1 (1967), 169–194. MR 37 #2268.
4. S. Kobayashi and K. Nomizu, *Foundations of differential geometry*. Vol. 1, Interscience, New York, 1963. MR 27 #2945.
5. J. A. Leslie, *On a differential structure for the group of diffeomorphisms*, Topology 6 (1967), 263–271. MR 35 #1041.
6. S. Sternberg, *Lectures on differential geometry*, Prentice-Hall, Englewood Cliffs, N. J., 1964. MR 33 #1797.

DEPARTMENT OF MATHEMATICS, KANSAS STATE UNIVERSITY, MANHATTAN, KANSAS 66506