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SYMMETRIES OF SPHERICAL HARMONICS[1]

BY

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ABSTRACT. Let G be a group of linear transformations of R^n and $H_k(G)$ the vector space of spherical harmonics invariant under G. The Pólya function is the formal power series $\Sigma_{k\geqslant 0} t^k \dim H_k(G)$. In this paper, after classifying all closed subgroups of $O(4)$, we compute the Pólya functions for these groups. These functions have recently proved to be of interest in quantum mechanics and elementary particle physics.

1. Pólya functions. We denote by S^{n-1} the unit sphere in euclidean space R^n, by $O(n)$ the orthogonal group of R^n, by $SO(n)$ the subgroup of $O(n)$ consisting of those $\sigma \in O(n)$ such that $\det \sigma = 1$. G denotes a fixed closed subgroup of $O(n)$.

DEFINITION. A spherical harmonic f_k of degree k is the restriction to S^{n-1} of a homogeneous polynomial $F_k(x)$ of degree k which is harmonic.

We have $x = (x_1, \cdots, x_n) \in R^n$, $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$ and $F_k(x) = r^k f_k(x/r)$ for $r = |x| \neq 0$.

We denote by H_k the vector space of spherical harmonics of degree k, and by $H_k(G)$ the vector subspace of those invariant under G. The Pólya function is defined by $F_G(t) = \Sigma_{k\geqslant 0} t^k \dim H_k(G)$. The classical Molien formula is

$$F_G(t) = \int_G \frac{1-t^2}{\det(1-t\sigma)}\, d\sigma$$

where $d\sigma$ is the normalized Haar measure on the group G. If G is a closed subgroup of $SO(4)$, we have

$$F_G(t) = \int_G \frac{1-t^2}{(1-2t\cos\alpha_\sigma + t^2)(1-2t\cos\beta_\sigma + t^2)}\, d\sigma$$

where $\alpha_\sigma, \beta_\sigma$ are the angles of the rotation σ.

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Remark. Pólya and Meyer computed the generating function $F_G(t)$ for the finite subgroups of $O(3)$. Their results are given in [11].

We want to compute the function $F_G(t)$ for the closed subgroups of $O(4)$. Then our first task is to determine these subgroups.

2. **Subgroups of $SO(4)$.** Any rotation in R^4 can be written as a mapping $q \longrightarrow lqr^{-1}$ where $q = (w, x, y, z) = w + xi + yj + zk$ is a quaternion and l, r are unit quaternions. Let $\varphi\colon S^3 \times S^3 \longrightarrow SO(4)$ be the 2-1 homomorphism defined by $\varphi(l, r)(q) = lqr^{-1}$. Let $G \subset SO(4)$ be a closed subgroup and $A = \varphi^{-1}(G)$. Let $\mathrm{pr}_i\colon S^3 \times S^3 \longrightarrow S^3$, $i = 1, 2$, denote the projection onto the ith factor. Set $L = \mathrm{pr}_1 A$, $R = \mathrm{pr}_2 A$; L and R are closed subgroups of S^3. Set $L_K = \{l \in L; (l, 1) \in A\}$, $R_K = \{r \in R; (1, r) \in A\}$; L_K (resp. R_K) is a closed normal subgroup of L (resp. R). We have an isomorphism of Lie groups, Φ: $L/L_K \longrightarrow R/R_K$, given by $\Phi(lL_K) = rR_K$, where $(l, r) \in A$.

Conversely, given L, R, L_K, R_K, Φ as above we define the group G by $G = \{g \in SO(4);\ g(q) = lqr^{-1}$ with $l \in L, r \in R, \Phi(lL_K) = rR_K\}$.

If $A = \varphi^{-1}(G)$ then $\mathrm{pr}_1 A = L$, $\mathrm{pr}_2 A = R$, $\{l \in L; (l, 1) \in A\} = L_K$, $\{r \in R; (1, r) \in A\} = R_K$. We use the notation $G = (L/L_K; R/R_K)_\Phi$ or $G = (L/L_K; R/R_K)$ when Φ is clearly understood.

Proposition. *Let $G = (L/L_K; R/R_K)_\Phi$ and $H = (M/M_K; N/N_K)_\Psi$ be two subgroups of $SO(4)$. Then*

(i) *there exists $\gamma \in SO(4)$ such that $G = \gamma H \gamma^{-1}$ if and only if there exist isomorphisms of Lie groups $\alpha\colon L/L_K \longrightarrow M/M_K$, $\beta\colon R/R_K \longrightarrow N/N_K$ of the form $\alpha(lL_K) = a^{-1}laM_K$, $\beta(rR_K) = b^{-1}rbN_K$, where $a, b \in S^3$ and $\Psi\alpha = \beta\Phi$. Moreover, a and b are such that $L = aMa^{-1}$, $R = bNb^{-1}$;*

(ii) *there exists $\gamma \in O(4) - SO(4)$ such that $G = \gamma H \gamma^{-1}$ if and only if there exist isomorphisms $\alpha\colon N/N_K \longrightarrow L/L_K$, $\beta\colon R/R_K \longrightarrow M/M_K$ of the form $\alpha(nN_K) = ana^{-1}L_K$, $\beta(rR_K) = b^{-1}rbM_K$ where $a, b \in S^3$ and $\beta\Phi\alpha\Psi = 1$. Moreover, a and b are such that $L = aNa^{-1}$, $R = bMb^{-1}$.*

Proof. (i) If $\gamma(q) = aqb^{-1}$ is such that $G = \gamma H \gamma^{-1}$, then the elements of G are of the form $g(q) = (ama^{-1})q(bnb^{-1})^{-1}$ where $m \in M$, $n \in N$, $(mM_K) = nN_K$. Then $L = aMa^{-1}$, $R = bNb^{-1}$. Define $\alpha(lL_K) = a^{-1}laM_K$, $\beta(rR_K) = b^{-1}rbN_K$. Then α, β are isomorphisms and $\Phi(ama^{-1}L_K) = bnb^{-1}R_K$ implies that $\Psi\alpha = \beta\Phi$.

Conversely, if there exist isomorphisms α, β as above then define $\gamma(q) = aqb^{-1}$. It is easy to see that $G = \gamma H \gamma^{-1}$.

The proof of (ii) is similar.

Subgroups of S^3. The proper closed subgroups of $SO(3)$ are (up to conjugacy):

$\mathcal{T}$ = tetrahedral group of order 12;

$\mathcal{O}$ = octahedral group of order 24;

$\mathcal{I}$ = icosahedral group of order 60;

$\mathcal{C}_n$ = cyclic group of order n;

$\mathcal{D}_n$ = dihedral group of order $2n$;

$\mathcal{H}_1$ = group of all rotations about a given fixed axis;

$\mathcal{H}_2$ = group of all rotations about a given fixed axis together with rotations of π about all axes perpendicular to the given axis.

We have the 2-1 homomorphism $\pi: S^3 \longrightarrow SO(3)$ given by $\pi(p)(q) = pqp^{-1}$. Then we obtain all the closed subgroups of S^3 taking $G = \pi^{-1}(\mathcal{G})$, where $\mathcal{G}$ is a closed subgroup of $SO(3)$. We obtain the following proper subgroups of S^3:

$T = \pi^{-1}(\mathcal{T})$ of order 24;

$O = \pi^{-1}(\mathcal{O})$ of order 48;

$I = \pi^{-1}(\mathcal{I})$ of order 120;

$C_n = \pi^{-1}(\mathcal{C}_n)$ if n is odd; $\pi^{-1}(\mathcal{C}_{n/2})$ if n is even, of order n;

$D_n = \pi^{-1}(\mathcal{D}_n)$ of order $4n$;

$H_1 = \pi^{-1}(\mathcal{H}_1)$. H_1 is connected and $H_1 = (\cos\theta, 0, 0, \sin\theta); 0 \leqslant \theta \leqslant 2\pi$,

$H_2 = \pi^{-1}(\mathcal{H}_2)$. $H_2 = H_1 \cup H_1 i$.

With these subgroups we can now determine all the closed subgroups of $SO(4)$. The complete list appears in the table at the end of the paper. For the finite subgroups of $SO(4)$ the list is given in [7].

REMARK. There is only one group (up to conjugacy) of the form $(S^3/L_K; S^3/R_K)_\Phi$, $(H_1/C_n; H_1/C_n)_\Phi$, $(H_2/C_n; H_2/C_n)_\Phi$.

3. **Subgroups of $O(4)$.** If $q = (w, x, y, z)$ then $\bar{q} = (w, -x, -y, -z)$ is the conjugate of q. The mapping $q \longrightarrow \bar{q}$ is a reflection in R^4, and the resultant of this reflection with a general rotation $q - lqr^{-1}$ is the reflection $q \longrightarrow l\bar{q}r^{-1}$, and any reflection can be put in this form, i.e., the general reflection in R^4 has the form $q \longrightarrow a\bar{q}b$ with $|a| = |b| = 1$. If $B(q) = a\bar{q}b$ and $g(q) = lqr^{-1}$ then $BgB^{-1}(q) = (ara^{-1})q(b^{-1}lb)^{-1}$.

Let G^* be a subgroup of $O(4)$ which is closed and has $G = (L/L_K; R/R_K)_\Phi$ as its pure subgroup. If $B \in G^* - G$, $B(q) = a\bar{q}b$, then the mapping $g \in G \longrightarrow BgB^{-1} \in G$ is an automorphism of G, and we have $L = aRa^{-1}$, $L_K = aR_Ka^{-1}$, $R = b^{-1}Lb$, $R_K = b^{-1}L_Kb$, that is, L and R are conjugate in S^3, and so are L_K and R_K. It is easy to see that there exists an automorphism $\Psi: L/L_K \longrightarrow L/L_K$ such that G and $H = (L/L_K; R/R_K)_\Psi$ are conjugate. Since we are interested in G^* only up to conjugacy, it follows that we can take $L = R$ and $L_K = R_K$, and we have $L = aLa^{-1} = bLb^{-1}$, $L_K = aL_Ka^{-1} = bL_Kb^{-1}$, showing that a and b must belong to the normalizers of L and L_K in S^3. If $B \in G^*$, $B(q) = a\bar{q}b$, then its square is $q \longrightarrow (ab^{-1})q(b^{-1}a)$ and belongs to G, whence $ab^{-1} \in L$, which shows that a, b belong to the same coset of L in the intersection of the normalizers of L and L_K in S^3.

If G^* has $G = (L/L_K; L/L_K)_\Phi$ as its pure subgroup then define automorphisms $\alpha(lL_K) = ala^{-1}L_K$, $\beta(lL_K) = blb^{-1}L_K$, where $B(q) = a\bar{q}b$ and $B \in G^*$; if $\Phi(L_K) = rL_K$ then $\Phi(ara^{-1}L_K) = b^{-1}lbL_K$ and $\beta\Phi\alpha\Phi = 1$.

Accordingly we have the following cases of subgroups of $O(4)$, not in $SO(4)$, with dimension greater than zero:

(a) $G = (S^3/S^3; S^3/S^3) = SO(4)$. We have $G^* = O(4)$. The reflections are $q \longrightarrow a\bar{q}b$ with $a, b \in S^3$.

(b) $G = (S^3/C_2; S^3/C_2)$. The reflections of G^* are $q \longrightarrow a\bar{q}b$ with $a, b \in S^3$ and $ab^{-1} = \pm 1$.

(c) $G = (S^3/C_1; S^3/C_1)$. We have two subgroups $G_1^* = G \cup g_1^*G$ and $G_2^* = G \cup g_2^*G$ containing G as pure subgroup, where $g_1^*(q) = \bar{q}$ and $g_2^*(q) = -\bar{q}$. G_1^*, G_2^* are not conjugate.

(d) $G = (H_2/H_2; H_2/H_2)$. The reflections of G^* are $q \longrightarrow a\bar{q}b$ with $a, b \in H_2$.

(e) $G = (H_2/H_1; H_2/H_1)$. We have two groups, G_1^* and G_2^*, having G as pure subgroup, the reflections of G_1^* (resp. G_2^*) are $q \longrightarrow a\bar{q}b$ with $a, b \in H_2$, $ab \in H_1$ (resp. $q \longrightarrow a\bar{q}b$ with $a, b \in H_1 i$).

(f) $G = (H_1/H_1; H_1/H_1)$. The reflections of G^* are $q \longrightarrow a\bar{q}b$ with $a, b \in H_1$.

(g) $G = (H_1/C_n; H_1/C_n)$. We have three groups having G as pure subgroups:

G_1^*—with reflections $q \longrightarrow l\bar{q}r^{-1}$;

G_2^*—with reflections $q \longrightarrow l\rho^{1/2}i^{-1}\bar{q}ir^{-1}$;

G_3^*—with reflections $q \longrightarrow l\rho i^{-1}\bar{q}ir^{-1}$, where $l, r \in H_1$, $lr^{-1} \in C_n$ and $\rho^{1/2} = (\cos - \pi/n -, 0, 0, \sin - \pi/n)$.

(h) $G = (H_2/C_n; H_2/C_n)$. We have two groups, G_1^* and G_2^*, having G as pure subgroup. The reflections of G_1^* (resp. G_2^*) are of the form $q \longrightarrow l\rho^{1/2}\bar{q}r^{-1}$ (resp. $q \longrightarrow l\rho\bar{q}r^{-1}$) where $l, r \in H_2$ and $lr^{-1} \in C_n$.

4. Examples of computation of the Pólya function. If G is a closed subgroup of $SO(4)$ then we have

$$F_G(t) = \int_A \frac{1 - t^2}{(1 - 2t\cos[\alpha(a) - \beta(a)] + t^2)(1 - 2t\cos[\alpha(a) + \beta(a)] + t^2)}\, da,$$

where $l = (\cos\alpha, b\sin\alpha, c\sin\alpha, d\sin\alpha)$, $r = (\cos\beta, m\sin\beta, n\sin\beta, p\sin\beta)$, $b^2 + c^2 + d^2 = m^2 + n^2 + p^2 = 1$ and $g = \varphi(l, r)$, α and β being functions of $a \in A$, and $\int_A da = 1$.

EXAMPLE 1. $G = (H_1/H_1; H_1/H_1)$. G is a connected subgroup of $SO(4)$ of dimension two. Here, $A = \varphi^{-1}(G) = H_1 \times H_1$ and $da = (1/4\pi^2)\,d\alpha\, d\beta$. Then,

$$F_G(t) = \frac{1-t^2}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{d\alpha\, d\beta}{(1-2t\cos(\alpha-\beta)+t^2)(1-2t\cos(\alpha+\beta)+t^2)} = \frac{1}{1-t^2}.$$

In order to treat the next example we need the lemma that follows. I am grateful to Professor T. Ganelius for showing me the technique used in its proof.

LEMMA.

$$\sum_{k=0}^{n-1}\frac{1}{1-2t\cos(2k/n)\pi+t^2} = n\frac{1+t^n}{(1-t^2)(1-t^n)}.$$

PROOF. The sum of all the residues (in the extended plane) of a rational function is zero. We get the result considering the function

$$f(z) = nz^n/(z-t)(1-tz)(z^n-1).$$

EXAMPLE 2. $G = (H_1/H_1; C_{2n}/C_{2n})$. G has dimension one and n components. We have $A = H_1 \times C_{2n}$, $A_0 = H_1 \times \{1\}$, where A_0 is the component of the identity of A. Then,

$$F_G(t) = \frac{1}{2n}\int_{A_0} F(\varphi(l, 1))\, da_0 + \cdots + \frac{1}{2n}\int_{A_0} F(\varphi(l, \rho^{2n-1}))\, da_0,$$

where $l = (\cos\alpha, 0, 0, \sin\alpha) \in H_1$, $\rho = (\cos(\pi/n), 0, 0, \sin(\pi/n))$ and $da_0 = d\alpha/2\pi$. Now, $\varphi(l, \rho^k)$ = rotation of angles $(\alpha - k\pi/n)$, $(\alpha + k\pi/n)$. Therefore,

$$F_G(t) = \frac{1}{2n}\sum_{k=0}^{2n-1}\frac{1}{2\pi}\int_0^{2\pi}\frac{1-t^2}{[1-2t\cos(\alpha-k\pi/n)+t^2]\,[1-2t\cos(\alpha+k\pi/n)+t^2]}\,d\alpha$$

$$= \frac{1}{2n}\sum_{k=0}^{2n-1}\frac{1+t^2}{1-2t^2\cos(2k\pi/n)+t^4} = \frac{1+t^{2n}}{(1-t^2)(1-t^{2n})}.$$

EXAMPLE 3. $G = (T/V; T/V)$. G has 96 elements.

We have $T = V \cup V_1 \cup V_2$, where $V = \{\pm 1, \pm i, \pm j, \pm k\}$, $V_1 = tV$, $V_2 = t^2 V$, $t = ½(1, 1, 1, 1)$. The rotations of G are schematically VqV, $V_1 q V_2$,

$V_2 q V_1$. If θ_1, θ_2 are the angles of $q \longrightarrow lqr^{-1}$ then we have the following table for the elements of T:

$\cos\theta_1$	$\cos\theta_2$	#
1	1	1
0	0	12
−1	−1	1
1	−1	18
½	−1	32
1	−½	32

We have

$$96F_G(t) = \sum_G \frac{1 - t^2}{(1 - 2t\cos\theta_1 + t^2)(1 - 2t\cos\theta_2 + t^2)}$$

and we get $F_G(t) = (1 + t^{12})/(1 - t^4)^2(1 - t^6)$.

EXAMPLE 4. $G^* = (T/V; T/V)^*$. The reflections are of the form $q \longrightarrow a\bar{q}b$ with a, b both in V or one each in V_1, V_2. If the angle between a and b is θ then $g^*: q \longrightarrow a\bar{q}b$ is conjugate to $h^*: q \longrightarrow p\bar{q}p^{-1}$ where $p = (\cos(\theta/2), 0, 0, \sin(\theta/2))$. Then, $\det(1 - tg^*) = \det(1 - th^*) = (1 - t^2)(1 + 2t\cos\theta + t^2)$ and we have

$$F_{G^*}(t) = \frac{1}{2}F_G(t) + \frac{1}{2|G|}\sum_{G^*-G} \frac{1}{(1 + 2t\cos\theta + t^2)}.$$

In our case we have the following values for $\cos\theta$:

$\cos\theta$	#
1	4
−1	4
0	24
½	32
−½	32

Therefore,

$$F_{G^*}(t) = \frac{1 + t^{12}}{(1 - t^4)(1 - t^6)(1 - t^8)}.$$

EXAMPLE 5. $G = (S^3/C_2; S^3/C_2) \cong O(3)$. G has dimension 3 and 2 components. Here, $A = A_0 \cup A_1$ where $A_0 = \{(l, l); l \in S^3\}$, $A_1 = \{(l, -l)$;

$l \in S^3\}$. An element of S^3 is given by $l = (\cos\theta_1, \sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2 \cdot \sin\theta_3, \sin\theta_1\sin\theta_2\cos\theta_3)$ and the volume element of S^3 is $\omega = (1/2\pi^2)\cdot \sin^2\theta_1 \sin\theta_2\, d\theta_1\, d\theta_2\, d\theta_3$, where $0 \leqslant \theta_1, \theta_2 \leqslant \pi, 0 \leqslant \theta_3 \leqslant 2\pi$.

If $g(q) = lql^{-1}$ then $\det(1 - tg) = (1-t)^2(1 - 2t\cos 2\theta_1 + t^2)$. If $g(q) = -lql^{-1}$ then $\det(1 - tg) = (1+t)^2(1 + 2t\cos 2\theta_1 + t^2)$. Therefore,

$$\int_{A_0} F(\varphi(l, r))\, da_0 = \int_{S^3} \frac{(1-t^2)\omega}{(1-t)^2(1 - 2t\cos 2\theta_1 + t^2)} = \frac{1}{1-t},$$

$$\int_{A_1} F(\varphi(l, r))\, da_1 = \int_{S^3} \frac{(1-t^2)\omega}{(1+t)^2(1 + 2t\cos 2\theta_1 + t^2)} = \frac{1}{1+t}.$$

Therefore, $F_G(t) = 1/(1-t^2)$.

5. Remarks about the Pólya function.

(1) G acts on S^3 by $(g, x) \longrightarrow g(x)$. We have:

PROPOSITION. $F_G(t) = 1$ *if and only if* G *acts transitively on* S^3.

PROOF. If G acts transitively on S^3 and $f: S^3 \longrightarrow R$ is invariant under G then f is constant. Therefore

$$\dim H_k(G) = 0 \quad \text{if } k = 0, \qquad = 1 \quad \text{if } k \neq 0,$$

and then $F_G(t) = 1$.

Conversely, suppose that $F_G(t) = 1$. Let $\bar{x}, \bar{y} \in S^3$ be arbitrary and $O_{\bar{x}}, O_{\bar{y}}$ the corresponding orbits. If $O_{\bar{x}} \cap O_{\bar{y}} = \varnothing$ then there exists a continuous function $f: S^3 \longrightarrow R$ such that $f(O_{\bar{x}}) = 1$ and $f(O_{\bar{y}}) = 0$. If $H(x) = \int_G f(gx)\, dg$ then H is invariant under G and $H(\bar{x}) = 1, H(\bar{y}) = 0$. Expand H in spherical harmonics: $H = \Sigma_{k \geqslant 0} f_k$. Then each f_k is invariant under G. $F_G(t) = 1$ implies that $f_0 =$ constant, $f_k = 0$ for $k > 0$, and then H is constant, a contradiction.

REMARK. It is clear that $(S^3/C_2; S^3/C_2)$ and $(S^3/C_1; S^3/C_1)$ cannot act transitively on S^3. The other possible groups are of the form $G = (S^3/S^3; M/M)$ and $O(4)$, and they indeed act transitively on S^3.

(2) A reflection $t: R^n \longrightarrow R^n$ is called a *simple* reflection if t leaves a hyperplane pointwise fixed; we have then $t^2 = 1$. A group G of linear mappings of R^n is a *finite reflection group* if it is finite and generated by simple reflections. Chevalley [4] proved that, for a finite reflection group $G, S(G) = R[I_1, \cdots, I_n]$, where $S(G)$ denotes the algebra of polynomials invariant

under G and I_i is a homogeneous invariant polynomial of degree $(m_i + 1)$. It follows that the Pólya function is given by

$$F_G(t) = \frac{1 - t^2}{(1 - t^{m_1+1}) \cdots (1 - t^{m_n+1})}.$$

The groups $(O/T; O/T)^*$, $(O/V; O/V)^*$, $(I/I; I/I)^*$, $(I^+/C_1; I/C_1)^{+*}$, $(T/C_1; T/C_1)^*$, $(O/C_1; O/C_1)^*$, $(I/C_1; I/C_1)^*$ and $(C_{2r}/C_1; C_{2r}/C_1)^*_{s=2r-1;h=r}$ are finite reflection groups and the exponents m_i are given in [5, p. 141].

About the pure subgroup of a finite reflection group we have the following result:

THEOREM. *Let G^* be a finite reflection group and $G = G^* \cap SO(4)$. Then $F_G(t) = (1 + t^d)F_{G^*}(t)$, with $d = \Sigma_{i=1}^n m_i$.*

PROOF. Let $S = R[x_1, \cdots, x_n]$, $S(G)$ be the algebra of polynomials invariant under G, $S(G^*) = R[I_1, \cdots, I_n]$, with degree of I_1 equal to $(m_i + 1)$. Let $\overline{S}$, $\overline{S(G)}$, $\overline{S(G^*)}$, denote the corresponding quotient fields. Then S is integral over $S(G^*)$, $\overline{S(G^*)}$ is integrally closed, $\overline{S(G)} = \overline{S}(G)$; Then $\overline{S} \supset \overline{S(G)}$ is a Galois extension. We have $\overline{S}(G) = \overline{S}(G^*)[J]$ where $J \in S(G) - S(G^*)$, $J^2 \in S(G^*)$ and we can take J^2 square free and homogeneous.

Suppose $(P + QJ)/R \in S(G)$, where $P, Q, R \in S(G^*)$. Then the norm and the trace of this element belong to $S(G^*)$, and we conclude that R divides P and R divides Q. Therefore $S(G) = S(G^*) \oplus S(G^*)J$. If the degree of J is d then we get the result. We have $d = \Sigma m_i$ because $F_G(1/t) = (-1)^n t^n F_G(t)$.

(3) In the table of Pólya functions that follows we use the following notations and results:

(a) $$S(m, \alpha) = \sum_{k=0}^{m-1} \frac{1 - t^2}{m[1 - 2t\cos(2k\pi/m - \alpha) + t^2][1 - 2t\cos(2k\pi/m + \alpha) + t^2]},$$

(b) $$T(m, \beta) = \sum_{k=0}^{m-1} \frac{1 - t^2}{m[1 + 2t\cos(2k\pi/m + \beta) + t^2]}.$$

By the same technique used in the proof of the lemma on p. 8 we can prove that

$$S(m, \alpha) = \frac{2t^m(1 - t^2)\cos\alpha \sin m\alpha + \sin\alpha(1 + t^2)(1 - t^{2m})}{\sin\alpha(1 - 2t^2\cos 2\alpha + t^4)(1 - 2t^m\cos m\alpha + t^{2m})};$$

$$T(m, \beta) = \frac{1 - t^{2m}}{1 - 2t^m\cos m\beta + t^{2m}} \quad \text{if } m \text{ is even,}$$

$$= \frac{1 - t^{2m}}{1 + 2t^m\cos m\beta + t^{2m}} \quad \text{if } m \text{ is odd.}$$

(c) $$A_s(2m, 2n, r) = \sum_{k=0}^{2m-1} \sum_{h=0}^{2n-1} \sum_{j=0}^{r-1} \frac{1-t^2}{4mnr(1-2t\cos\alpha+t^2)(1-2t\cos\beta+t^2)},$$

where

$$\alpha = \frac{(n-ms)j + r(nk-mh)}{mnr}\pi, \quad \beta = \frac{(n+ms)j + r(nk+mh)}{mnr}\pi.$$

$$B(2m, 2n) = A_s(2m, 2n, 1)$$

(d) $$= \sum_{k=0}^{2m-1} \sum_{h=0}^{2n-1} \frac{1-t^2}{4mn(1-2t\cos\alpha+t^2)(1-2t\cos\beta+t^2)}$$

with $\alpha = (nk - mh)\pi/mn$, $\beta = (nk + mh)\pi/mn$. In particular, we have

$$B(2m, 2m) = (1 + 6t^{2m} + t^{4m})/(1-t^2)(1-t^{2m})^2.$$

$$C(2m, 2n) = 2A_1(2m, 2n, 2) - B(2m, 2n)$$

(e) $$= \sum_{k=1}^{2m-1} \sum_{h=0}^{2n-1} \frac{1-t^2}{4mn\left[1-2t\cos\left(\frac{2k+1}{2m}\pi - \frac{2h+1}{2n}\pi\right)+t^2\right]\left[1-2t\cos\left(\frac{2k+1}{2m}\pi + \frac{2h+1}{2m}\pi\right)+t^2\right]}.$$

In particular, $C(2m, 2m) = 1/(1-t^2)$.

$$D(n, 2r, s, h, k)$$

(f) $$= \sum_{\alpha=0}^{n-1} \sum_{\beta=0}^{n-1} \sum_{j=0}^{2r-1} \left(2n^2r\left[1 + 2t\cos\frac{2r(\alpha+\beta) + \frac{1}{2}(h-k) + (1+s)j}{nr}\pi + t^2\right]\right)^{-1}.$$

(g) $$E_s(m, n, 2r) = \sum_{k=0}^{m-1} \sum_{h=0}^{n-1} \sum_{j=0}^{2r-1} \frac{1-t^2}{2mnr(1-2t\cos\alpha+t^2)(1-2t\cos\beta+t^2)},$$

where

$$\alpha = \frac{(n-ms)j + 2r(kn-mh)}{mnr}\pi, \quad \beta = \frac{(n+ms)j + 2r(kn+mh)}{mnr}\pi.$$

$$H(n, 2r, s, h, k)$$

(h) $$= \sum_{\alpha=0}^{n-1} \sum_{\beta=0}^{n-1} \sum_{j=0}^{2r-1} \left(2n^2r\left[1 - 2t\cos\frac{(1-s)j + 2r(\alpha-\beta) - \frac{1}{2}(h+k)}{nr}\pi + t^2\right]\right)^{-1}.$$

(4) If $F_G(t) = \Sigma_{k\geq 0} a_k t^k$ is such that $a_k = 0$ for k odd, then it follows from the results of [8] that G cannot be the group of symmetries of a hypersurface of constant width. An example is $(I/I; I/I)$ in contrast with the icosahedral group in $\mathbf{R}^3$.

6. Pólya functions of the closed subgroups of $O(4)$.

	Group	Pólya function
(1)	$(S^3/S^3; S^3/S^3) = SO(4)$	1
(2)	$(S^3/C_2; S^3/C_2)$	$\frac{1}{1-t^2}$
(3)	$(S^3/C_1; S^3/C_1)$	$\frac{1}{1-t}$
(4)	$(S^3/S^3; H_2/H_2)$	1
(5)	$(S^3/S^3; H_1/H_1)$	1
(6)	$(S^3/S^3; I/I)$	1
(7)	$(S^3/S^3; O/O)$	1
(8)	$(S^3/S^3; T/T)$	1
(9)	$(S^3/S^3; D_n/D_n)$	1
(10)	$(S^3/S^3; C_{2n}/C_{2n})$	1
(11)	$(H_2/H_2; H_2/H_2)$	$\frac{1}{1-t^4}$
(12)	$(H_2/H_1; H_2/H_1)$	$\frac{1}{1-t^2}$
(13)	$(H_2/H_2; H_1/H_1)$	$\frac{1}{1-t^4}$
(14)	$(H_2/H_1; O/T)$	$\frac{1}{(1-t^6)(1-t^8)}$
(15)	$(H_2/H_1; D_n/C_{2n})$	$\frac{1}{(1-t^2)(1-t^{2n})}$
(16)	$(H_2/H_1; D_{2n}/D_n)$	$\begin{cases} \frac{1}{(1-t^4)(1-t^{2n})} & \text{if } n \text{ is odd} \\ \frac{1+t^{2n+2}}{(1-t^4)(1-t^{4n})} & \text{if } n \text{ is even} \end{cases}$

	Group	Pólya function
(17)	$(H_2/H_1; C_{2n}/C_n)$	$\begin{cases} \dfrac{1+2t^{n+2}+t^{2n}}{(1-t^4)(1-t^{2n})} & \text{if } n \equiv 0 \pmod 4 \\ \dfrac{1+2t^{2n+2}+t^{4n}}{(1-t^4)(1-t^{4n})} & \text{if } n \equiv 1 \text{ or } 3 \pmod 4 \\ \dfrac{1+t^n}{(1-t^2)(1-t^n)} & \text{if } n \equiv 2 \pmod 4 \end{cases}$
(18)	$(H_2/H_2; I/I)$	$\dfrac{1}{(1-t^{12})(1-t^{20})}$
(19)	$(H_2/H_2; O/O)$,	$\dfrac{1}{(1-t^8)(1-t^{12})}$
(20)	$(H_2/H_2; T/T)$,	$\dfrac{1+t^{12}}{(1-t^8)(1-t^{12})}$
(21)	$(H_2/H_2; D_n/D_n)$	$\begin{cases} \dfrac{1+t^{2n+2}}{(1-t^4)(1-t^{4n})} & \text{if } n \text{ is odd} \\ \dfrac{1}{(1-t^4)(1-t^{2n})} & \text{if } n \text{ is even} \end{cases}$
(22)	$(H_2/H; C_{2n}/C_{2n})$	$\dfrac{1+t^{2n}}{(1-t^4)(1-t^{2n})}$
(23)	$(H_2/C_n; H_2/C_n)$	$\dfrac{1}{(1-t^2)(1-t^n)}$
(24)	$(H_1/H_1; H_1/H_1)$	$\dfrac{1}{1-t^2}$
(25)	$(H_1/C_n; H_1/C_n)$	$\dfrac{1+t^n}{(1-t^2)(1-t^n)}$
(26)	$(H_1/H_1; I/I)$	$\dfrac{1-2t^4-t^6+t^8+t^{10}+t^{12}+t^{14}-t^{16}-2t^{18}+t^{22}}{(1-t^4)^2(1-t^6)(1-t^{10})}$
(27)	$(H_1/H_1; O/O)$	$\dfrac{1+t^{18}}{(1-t^8)(1-t^{12})}$
(28)	$(H_1/H_1; T/T)$	$\dfrac{1+t^{12}}{(1-t^6)(1-t^8)}$
(29)	$(H_1/H_1; D_n/D_n)$	$\dfrac{1+t^{2n+2}}{(1-t^4)(1-t^{2n})}$

	Group	Pólya function
(30)	$(H_1/H_1; C_{2n}/C_{2n})$	$\dfrac{1+t^{2n}}{(1-t^2)(1-t^{2n})}$
(31)	$(C_{2mr}/C_{2m}; C_{2nr}/C_{2n})_s$	$A_s(2m, 2n, r)$
	$(C_{2m}/C_{2m}; C_{2m}/C_{2m})_s$	$\dfrac{1+6t^{2m}+t^{4m}}{(1-t^2)(1-t^{2m})^2}$
(32)	$(C_{2m}/C_{2m}; D_n/D_n)$	$\frac{1}{2}B(2m, 2n) + \frac{1}{2}S(2m, \pi/2)$
	$(C_{2m}/C_{2m}; D_m/D_m)$	$\begin{cases} \dfrac{1+3t^{2m}+3t^{2m+2}+t^{4m+2}}{(1-t^4)(1-t^{2m})^2} & \text{if } m \text{ is even} \\ \dfrac{1+2t^{2m}+5t^{2m+2}+5t^{4m}+2t^{4m+2}+t^{6m+2}}{(1-t^4)(1-t^{2m})(1-t^{4m})} & \text{if } m \text{ is odd} \end{cases}$
(33)	$(C_{4m}/C_{2m}; D_n/C_{2n})$	$\frac{1}{2}B(2m, 2n) + S(4m, \pi/2) - \frac{1}{2}S(2m, \pi/2)$
	$(C_{4m}/C_{2m}; D_m/C_{2m})$	$\begin{cases} \dfrac{1+2t^{2m}+5t^{2m+2}+5t^{4m}+2t^{4m+2}+t^{6m+2}}{(1-t^4)(1-t^{2m})(1-t^{4m})} & \text{if } m \text{ is even} \\ \dfrac{1+4t^{2m}+3t^{2m+2}+3t^{4m}+4t^{4m+2}+t^{6m+2}}{(1-t^4)(1-t^{2m})(1-t^{4m})} & \text{if } m \text{ is odd} \end{cases}$
(34)	$(C_{4m}/C_{2m}; D_{2n}/D_n)$	$\frac{1}{4}B(2m, 2n) + \frac{1}{4}C(2m, 2n) + \frac{1}{2}S(4m, \pi/2)$
	$(C_{4m}/C_{2m}; D_{2m}/D_m)$	$\dfrac{1+t^{2m}+2t^{2m+2}+2t^{4m}+t^{4m+2}+t^{6m+2}}{(1-t^4)(1-t^{2m})(1-t^{4m})}$
(35)	$(C_{2m}/C_{2m}; T/T)$	$\frac{1}{12}S(2m, 0) + \frac{2}{3}S\left(2m, \frac{\pi}{3}\right) + \frac{1}{4}S\left(2m, \frac{\pi}{2}\right)$
(36)	$(C_{6m}/C_{2m}; T/V)$	$\frac{1}{12}S(2m, 0) + \frac{1}{4}S\left(2m, \frac{\pi}{2}\right) + S\left(6m, \frac{\pi}{3}\right) - \frac{1}{3}S\left(2m, \frac{\pi}{3}\right)$
(37)	$(C_{2m}/C_{2m}; O/O)$	$\frac{3}{8}S\left(2m, \frac{\pi}{2}\right) + \frac{1}{4}S\left(2m, \frac{\pi}{4}\right) + \frac{1}{24}S(2m, 0) + \frac{1}{3}S\left(2m, \frac{\pi}{3}\right)$
(38)	$(C_{4m}/C_{2m}; O/T)$	$\frac{1}{2}S\left(4m, \frac{\pi}{2}\right) - \frac{1}{8}S\left(2m, \frac{\pi}{2}\right) + \frac{1}{2}S\left(4m, \frac{\pi}{4}\right) - \frac{1}{4}S\left(2m, \frac{\pi}{4}\right) + \frac{1}{24}S(2m, 0) + \frac{1}{3}S\left(2m, \frac{\pi}{3}\right)$
(39)	$(C_{2m}/C_{2m}; I/I)$	$\frac{1}{60}S(2m, 0) + \frac{1}{10}S\left(2m, \frac{2\pi}{5}\right) + \frac{1}{10}S\left(2m, \frac{3\pi}{5}\right) + \frac{1}{3}S\left(2m, \frac{\pi}{3}\right) + \frac{1}{10}S\left(2m, \frac{\pi}{5}\right) + \frac{1}{10}S\left(2m, \frac{4\pi}{5}\right) + \frac{1}{4}S\left(2m, \frac{\pi}{2}\right)$

	Group	Pólya function
(40)	$(D_m/D_m; D_n/D_n)$	$\frac{1}{4}B(2m, 2n) + \frac{1}{4}S\left(2m, \frac{\pi}{2}\right) + \frac{1}{4}S\left(2n, \frac{\pi}{2}\right) + \frac{1}{4}\frac{1}{1-t^2}$
	$(D_m/D_m; D_m/D_m)$	$\begin{cases} \dfrac{1 + t^{2m} + t^{2m+2} + t^{4m+2}}{(1-t^4)(1-t^{2m})^2} & \text{if } m \text{ is even} \\ \dfrac{1 + 3t^{2m+2} + 3t^{4m} + t^{6m+2}}{(1-t^4)(1-t^{2m})(1-t^{4m})} & \text{if } m \text{ is odd} \end{cases}$
(41)	$(D_{mr}/C_{2m}; D_{nr}/C_{2n})_s$	$\frac{1}{2}A_s(2m, 2n, r) + \frac{1}{2}\frac{1}{1-t^2}$
	$(D_m/C_{2m}; D_m/C_{2m})_s$	$\frac{(1+t^{2m})^2}{(1-t^2)(1-t^{2m})^2}$
(42)	$(D_{2m}/D_m; D_{2n}/D_n)$	$\frac{1}{8}B(2m, 2n) + \frac{1}{8}C(2m, 2n) + \frac{1}{4}S\left(4m, \frac{\pi}{2}\right) + \frac{1}{4}S\left(4n, \frac{\pi}{2}\right) + \frac{1}{4}\frac{1}{1-t^2}$
	$(D_{2m}/D_m; D_{2m}/D_m)$	$\frac{(1+t^{2m+2})(1+t^{4m})}{(1-t^4)(1-t^{2m})(1-t^{4m})}$
(43)	$(D_{2m}/D_m; D_n/C_{2n})$	$\frac{1}{4}B(2m, 2n) + \frac{1}{4}S\left(2n, \frac{\pi}{2}\right) + \frac{1}{2}S\left(4m, \frac{\pi}{2}\right) - \frac{1}{4}S\left(2m, \frac{\pi}{2}\right) + \frac{1}{4}\frac{1}{1-t^2}$
	$(D_{2m}/D_m; D_m/C_{2m})$	$\frac{1 + t^{2m} + 2t^{2m+2} + 2t^{4m} + t^{4m+2} + t^{6m+2}}{(1-t^4)(1-t^{2m})(1-t^{4m})}$
(44)	$(D_m/D_m, T/T)$	$\frac{1}{24}S(2m, 0) + \frac{1}{3}S\left(2m, \frac{\pi}{3}\right) + \frac{1}{8}S\left(2m, \frac{\pi}{2}\right) + \frac{1}{2}\frac{1+t^{12}}{(1+t^6)(1-t^8)}$
(45)	$(D_m/D_m; O/O)$	$\frac{1}{48}S(2m, 0) + \frac{1}{6}S\left(2m, \frac{\pi}{3}\right) + \frac{3}{16}S\left(2m, \frac{\pi}{2}\right) + \frac{1}{8}S\left(2m, \frac{\pi}{4}\right) + \frac{1}{2}\left(\frac{1-t^{18}}{(1-t^8)(1-t^{12})}\right)$
(46)	$(D_m/C_{2m}; O/T)$	$\frac{1}{24}S(2m, 0) + \frac{1}{3}\left(2m, \frac{\pi}{3}\right) + \frac{1}{8}S\left(2m, \frac{\pi}{2}\right) + \frac{1}{2}\frac{1+t^6}{1-t^8}$
(47)	$(D_{2m}/D_m; O/T)$	$\frac{1}{48}S(2m, 0) + \frac{1}{4}S\left(4m, \frac{\pi}{2}\right) - \frac{1}{16}S\left(2m, \frac{\pi}{2}\right) + \frac{1}{6}S\left(2m, \frac{\pi}{3}\right) + \frac{1}{4}S\left(4m, \frac{\pi}{4}\right) - \frac{1}{8}S\left(2m, \frac{\pi}{4}\right) + \frac{1}{2}\frac{1-t^{18}}{(1-t^8)(1-t^{12})}$

	Group	Pólya function
(48)	$(D_{3m}/C_{2m}; O/V)$	$\frac{1}{24}S(2m, 0) + \frac{1}{8}S\left(2m, \frac{\pi}{2}\right) + \frac{1}{2}S\left(6m, \frac{\pi}{3}\right) - \frac{1}{6}S\left(2m, \frac{\pi}{3}\right) + \frac{1}{2}\ \frac{1+t^6}{1-t^8}$
(49)	$(D_m/D_m; I/I)$	$\frac{1}{120}S(2m, 0) + \frac{1}{8}S\left(2m, \frac{\pi}{2}\right) + \frac{1}{6}S\left(2m, \frac{\pi}{3}\right) + \frac{1}{20}S\left(2m, \frac{\pi}{5}\right) + \frac{1}{20}S\left(2m, \frac{2\pi}{5}\right) + \frac{1}{20}S\left(2m, \frac{3\pi}{5}\right) + \frac{1}{20}S\left(2m, \frac{4\pi}{5}\right) + \frac{1}{2}\ \frac{1-t^{30}}{(1-t^{12})(1-t^{20})}$
(50)	$(T/T; T/T)$	$\frac{(1-t^4+t^8)(1+t^{12})}{(1-t^4)(1-t^6)(1-t^{12})}$
(51)	$(T/C_2; T/C_2)$	$\frac{(1+t^4)(1-t^2+t^4)}{(1-t^2)^2(1-t^6)}$
(52)	$(T/V; T/V)$	$\frac{1+t^{12}}{(1-t^4)^2(1-t^6)}$
(53)	$(T/T; O/O)$	$\frac{1-t^6-t^8+2t^{12}+t^{14}-2t^{20}+2t^{24}-t^{30}-2t^{32}+t^{36}+t^{38}-t^{44}}{(1-t^6)(1-t^8)^2(1-t^{24})}$
(54)	$(T/T; I/I)$	$\frac{1+t^{12}+t^{20}+t^{24}+2t^{30}+t^{36}+t^{40}+t^{48}+t^{60}}{(1-t^{12})(1-t^{20})(1-t^{30})}$
(55)	$(O/O; O/O)$	$\frac{1-t^6+t^{12}+t^{24}-t^{30}+t^{36}}{(1-t^6)(1-t^8)(1-t^{24})}$
(56)	$(O/C_2; O/C_2)$	$\frac{1+t^{10}}{(1-t^2)(1-t^4)(1-t^6)}$
(57)	$(O/V; O/V)$	$\frac{1+t^{16}}{(1-t^4)(1-t^6)(1-t^8)}$
(58)	$(O/T; O/T)$	$\frac{1+t^{24}}{(1-t^6)(1-t^8)(1-t^{12})}$
(59)	$(O/O; I/I)$	$\frac{1+t^{12}+t^{20}+t^{24}+2t^{32}+t^{36}+t^{40}+t^{42}+t^{44}+t^{48}+t^{50}+t^{52}+t^{56}+2t^{60}+t^{68}+t^{72}+t^{80}+t^{92}}{(1-t^{24})(1-t^{30})(1-t^{40})}$
(60)	$(I/I; I/I)$	$\frac{1+t^{60}}{(1-t^{12})(1-t^{20})(1-t^{30})}$
(61)	$(I/C_2; I/C_2)$	$\frac{1+t^{16}}{(1-t^2)(1-t^6)(1-t^{10})}$

	Group	Pólya function
(62)	$(I^+/C_2; I/C_2)^+$	$\dfrac{(1+t^8)(1+t^{10})}{(1-t^4)(1-t^6)(1-t^{10})}$
(63)	$(C_{2mr}/C_m; C_{2nr}/C_n)_s$ $m \equiv n \equiv 1 \pmod 2$	$E_s(m, n, 2r)$
	$(C_{2r}/C_1; C_{2r}/C_1)_{s=2r-1}$	$\dfrac{1+t^r}{(1-t)^2(1-t^r)}$
(64)	$(D_{mr}/C_m; D_{nr}/C_n)_s$	$\frac{1}{2}E_s(m, n, 2r) + \frac{1}{2}\frac{1}{1-t^2}$
	$(D_r/C_1; D_r/C_1)_{s=2r-1}$	$\dfrac{1+t^{r+1}}{(1-t)(1-t^2)(1-t^r)}$
(65)	$(T/C_1; T/C_1)$	$\dfrac{1+t^6}{(1-t)(1-t^3)(1-t^4)}$
(66)	$(O/C_1; O/C_1)_1$	$\dfrac{1+t^9}{(1-t)(1-t^4)(1-t^6)}$
(67)	$(O/C_1; O/C_1)_2$	$\dfrac{1+t^7}{(1-t^2)(1-t^3)(1-t^4)}$
(68)	$(I/C_1 : I/C_1)$	$\dfrac{1+t^{15}}{(1-t)(1-t^6)(1-t^{10})}$
(69)	$(I^+/C_1; I/C_1)^+$	$\dfrac{1+t^{10}}{(1-t^3)(1-t^4)(1-t^5)}$
(70)	$(S^3/S^3; S^3/S^3)^* = O(4)$	1
(71)	$(S^3/C_2; S^3/C_2)^*$	$\dfrac{1}{1-t^2}$
(72)	$(S^3/C_1; S^3/C_1)^*_1$	$\dfrac{1}{1-t}$
(73)	$(S^3/C_1; S^3/C_1)^*_2$	$\dfrac{1}{1-t^2}$
(74)	$(H_2/H_2; H_2/H_2)^*$	$\dfrac{1}{1-t^4}$
(75)	$(H_2/H_1; H_2/H_1)^*_1$	$\dfrac{1}{1-t^2}$
(76)	$(H_2/H_1; H_2/H_1)^*_2$	$\dfrac{1}{1-t^4}$

	Group	Pólya function
(77)	$(H_2/C_n; H_2/C_n)_1^*$	$\dfrac{1}{(1-t^2)(1-t^{2n})}$
(78)	$(H_2/C_n; H_2/C_n)_2^*$	$\dfrac{1}{(1-t^2)(1-t^{n})}$
(79)	$(H_1/H_1; H_1/H_1)^*$	$\dfrac{1}{1-t^2}$
(80)	$(H_1/C_n; H_1/C_n)_1^*$	$\dfrac{1}{(1-t^2)(1-t^{n})}$
(81)	$(H_1/C_n; H_1/C_n)_2^*$	$\dfrac{1+t^{2n}}{(1-t^2)(1-t^{2n})}$
(82)	$(H_1/C_n; H_1/C_n)_3^*$	$\dfrac{1+t^{n}}{(1-t^2)(1-t^{n})}$
(83)	$(C_{2nr}/C_n; C_{2nr}/C_n)_{s,h}^*$	$\tfrac{1}{2}E_s(n, n, 2r) + \tfrac{1}{2}D(n, 2r, s, 2h, 0)$
	$(C_{2r}/C_1; C_{2r}/C_1)_{s=2r-1;h=r}$	$\dfrac{1}{(1-t^2)(1-t^{r})}$
(84)	$(D_n/D_n; D_n/D_n)^*$	$\begin{cases} \dfrac{1+t^{2n+2}}{(1-t^4)(1-t^{2n})^2} & \text{if } n \text{ is even} \\ \dfrac{1+2t^{2n+2}+t^{4n}}{(1-t^4)(1-t^{2n})(1-t^{4n})} & \text{if } n \text{ is odd} \end{cases}$
(85)	$(D_{nr}/C_n; D_{nr}/C_n)_{s,h,k}^*$	$\dfrac{1}{2}D(n, r, s, h, k) + \dfrac{1}{4}E_s(n, n, 2r) + \dfrac{1}{4}\dfrac{1}{1-t^2} + \dfrac{1}{4}H(n, 2r, s, h, k)$
	$(D_r/C_1; D_r/C_1)_{s=1;h=k=r}^*$	$\begin{cases} \dfrac{1+t^{r+1}}{(1-t^2)^2(1-t^{r})} & \text{if } r \text{ is even} \\ \dfrac{1+t^{r+1}+t^{r+2}+t^{2r+1}}{(1-t^2)^2(1-t^{2r})} & \text{if } r \text{ is odd} \end{cases}$
(86)	$(D_{nr}/C_n; D_{nr}/C_n)_{s,h,k,-}^*$	$\dfrac{1}{4}E_s(n, n, 2r) + \dfrac{3-t^2}{4(1-t^4)}$
(87)	$(D_{2n}/D_n; D_{2n}/D_n)^*$	$\dfrac{1+t^{2n+2}}{(1-t^4)(1-t^{2n})(1-t^{4n})}$
(88)	$(D_{2n}/D_n; D_{2n}/D_n)_-^*$	$\dfrac{1-t^{2n}+t^{4n}+t^{4n+2}}{(1-t^4)(1-t^{2n})(1-t^{4n})}$

	Group	Pólya function
(89)	$(T/C_2; T/C_2)^*_c$	$\dfrac{1+t^6}{(1-t^2)(1-t^4)(1-t^6)}$
(90)	$(T/C_2; T/C_2)^*$	$\dfrac{1+t^4}{(1-t^2)(1-t^4)(1-t^6)}$
(91)	$(T/V; T/V)^*$	$\dfrac{1+t^{12}}{(1-t^4)(1-t^6)(1-t^8)}$
(92)	$(T/V; T/V)^*_-$	$\dfrac{1+t^4}{(1-t^4)(1-t^6)(1-t^8)}$
(93)	$(T/T; T/T)^*$	$\dfrac{1+t^{12}}{(1-t^6)(1-t^8)(1-t^{12})}$
(94)	$(O/C_2; O/C_2)^*$	$\dfrac{1}{(1-t^2)(1-t^4)(1-t^6)}$
(95)	$(O/T; O/T)^*$	$\dfrac{1}{(1-t^6)(1-t^8)(1-t^{12})}$
(96)	$(O/T; O/T)^*_-$	$\dfrac{1-t^6+t^{18}}{(1-t^6)(1-t^8)(1-t^{12})}$
(97)	$(O/V; O/V)^*$	$\dfrac{1}{(1-t^4)(1-t^6)(1-t^8)}$
(98)	$(O/O; O/O)^*$	$\dfrac{1+t^{18}}{(1-t^8)(1-t^{12})(1-t^{24})}$
(99)	$(I/C_2; I/C_2)^*$	$\dfrac{1}{(1-t^2)(1-t^6)(1-t^{10})}$
(100)	$(I/I; I/I)^*$	$\dfrac{1}{(1-t^{12})(1-t^{20})(1-t^{30})}$
(101)	$(I^+/C_2; I/C_2)^{+*}$	$\dfrac{1+t^8}{(1-t^4)(1-t^6)(1-t^{10})}$
(102)	$(T/C_1; T/C_1)^*_c$	$\dfrac{1+t^6}{(1-t)(1-t^4)(1-t^6)}$
(103)	$(T/C_1; T/C_1)^*_{c-}$	$\dfrac{1+t^6}{(1-t^2)(1-t^4)(1-t^3)}$
(104)	$(T/C_1; T/C_1)^*$	$\dfrac{1}{(1-t)(1-t^3)(1-t^4)}$

Group	Pólya function
(105) $(T/C_1; T/C_1)^*_-$	$\frac{1-t^3+t^4+t^6}{(1-t^2)(1-t^3)(1-t^4)}$
(106) $(O/C_1; O/C_1)^*$	$\frac{1}{(1-t)(1-t^4)(1-t^6)}$
(107) $(O/C_1; O/C_1)^*_-$	$\frac{1+t^9}{(1-t^2)(1-t^4)(1-t^6)}$
(108) $(I/C_1; I/C_1)^*$	$\frac{1}{(1-t)(1-t^6)(1-t^{10})}$
(109) $(I/C_1; I/C_1)^*_-$	$\frac{1+t^{15}}{(1-t^2)(1-t^6)(1-t^{10})}$
(110)- $(I^+/C_1; I/C_1)^{+*}$	$\frac{1}{(1-t^3)(1-t^4)(1-t^5)}$
(111) $(I^+/C_1; I/C_1)^{+*}_-$	$\frac{1-t^5+t^8+t^{10}}{(1-t^4)(1-t^5)(1-t^6)}$.

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