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TOPOLOGICAL DYNAMICS ON C^*-ALGEBRAS

BY

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ABSTRACT. Dynamical properties of a group of homeomorphisms of a compact Hausdorff space X can be interpreted in terms of the commutative C^*-algebra $C(X)$. We investigate a noncommutative topological dynamics extending dynamical concepts to the context of a group of automorphisms on a general C^*-algebra with unit. Such concepts as minimality, almost periodicity, and pointwise almost periodicity are extended to this situation. Theorems are obtained extending commutative dynamical results and relating the noncommutative dynamics to the transformation groups induced on the state space and the weak* closure of the pure states. We show, for example, that the group acts almost periodically on the C^*-algebra if and only if each of these induced transformation groups is almost periodic.

Introduction. One standard setting for topological dynamics is that of a group of homeomorphisms on a compact Hausdorff space. The group induces a group of *-automorphisms on the C^*-algebra of complex-valued continuous functions on the space, while any group of *-automorphisms on a commutative C^*-algebra with unit is induced in this way by a group of homeomorphisms on its spectrum space. Topological dynamics, in the above sense, can thus be faithfully translated into the language of groups of automorphisms on C^*-algebras with unit. In this paper we initiate a "noncommutative" topological dynamics, by extending such dynamical concepts as almost periodicity, pointwise almost periodicity, and minimality to the context of groups of automorphisms on a general C^*-algebra with unit, and investigating their properties. For general notions of topological dynamics, see [2]; for those of operator theory, see [3].

1. **Definitions.** By a C^* *flow* we shall mean a pair (A, G), where A is a C^*-algebra with unit, and G is a group of *-automorphisms of A.

We say (B, G) is a *subflow* of the C^* flow (A, G) if B is a sub-C^*-algebra of A containing the unit of A which is invariant under each of the elements of G.

A *homomorphism* of the C^* flow (A, G) to the C^* flow (B, G) is a

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*-algebra homomorphism $\pi\colon A \longrightarrow B$ such that for every $a \in A$ and $g \in G$, $\pi(g(a)) = g\pi(a)$. (We shall not distinguish notationally between the actions of the group G on distinct C^*-algebras.) An isomorphism of C^* flows is an invertible homomorphism.

If $\{(A_i, G)\colon i \in I\}$ is any family of C^* flows, the *direct sum* $\Sigma_{i\in I} \bigoplus (A_i, G)$ is the C^* flow obtained by letting G act coordinatewise on the direct sum C^*-algebra.

If $\{(A_i, G)\colon i = 1, \cdots, n\}$ are C^* flows, the *tensor product* flow $(A_1, G) \otimes \cdots \otimes (A_n, G)$ is the flow $(\bigotimes_{i=1}^n A_i, G)$ where G acts coordinatewise, i.e, $g \in G$ acts on $\bigotimes_{i=1}^n A_i$ as the unique *-automorphism with $g(a_1 \otimes \cdots \otimes a_n) = g(a_1) \otimes \cdots \otimes g(a_n)$ for $a_i \in A_i$, $i = 1, \cdots, n$.

Let $\{A_\lambda, \varphi_{\lambda\mu}, \Lambda\}$ be an inductive system of C^*-algebras with unit, with inductive limit A, and let $\varphi_\lambda\colon A_\lambda \longrightarrow A$ be the embedding map for $\lambda \in \Lambda$. Suppose (A_λ, G) is a C^* flow for each $\lambda \in \Lambda$, and for $\lambda, \mu \in \Lambda$, $\lambda < \mu$, the map $\varphi_{\lambda\mu}\colon A_\lambda \longrightarrow A_\mu$ defines an isomorphism (into) of the C^* flows. Then there is a unique action of G on A such that (A, G) is a C^* flow and $\varphi_\lambda g(a_\lambda) = g\varphi_\lambda(a_\lambda)$ for $\lambda \in \Lambda$, $a_\lambda \in A_\lambda$, $g \in G$. We say (A, G) is the *inductive limit* of the *inductive system* of C^* flows $\{(A_\lambda, G), \varphi_{\lambda\mu}, \Lambda\}$. In particular, the infinite tensor product of C^* flows $\bigotimes_{i\in I}(A_i, G)$ is defined as the inductive limit of the finite tensor products.

If T is a group of homeomorphisms on a compact Hausdorff space X, we denote by G_T the induced group of automorphisms on the C^*-algebra $C(X)$ of continuous complex-valued functions on X; that is, if $t \in T$, $a \in C(X)$, $x \in X$, then $(g_t(a))(x) = a(t(x))$.

We note that subflows, homomorphs, direct sums, tensor products, and inductive limits of commutative C^* flows correspond, respectively, to homomorphs, closed invariant sets, disjoint unions, cartesian products, and inverse limits of transformation groups in the sense that:

(i) A subflow of $(C(X), G_T)$ is isomorphic to $(C(Y), G_T)$ for (Y, T) a homomorph of (X, T).

(ii) A homomorph of $(C(X), G_T)$ is isomorphic to $(C(Y), G_T)$ for Y a closed invariant subset of X.

(iii) $\Sigma_{i\in I} \bigoplus (C(X_i), G_T) = (C(\Sigma_{i\in I} X_i), G_T)$ (where $\Sigma_{i\in I} X_i$ is the disjoint union of the spaces $\{X_i\colon i \in I\}$, with the obvious action by T).

(iv) $\bigotimes_{i=1}^n (C(X_i), G_T) = (C(\Pi_{i=1}^n X_i), G_T)$.

(v) If $((C(X_\lambda)G_T), \varphi_{\lambda\mu}, \Lambda)$ is an inductive system of C^* flows, then $((X_\lambda, T), \psi_{\mu\lambda}, \Lambda)$ is an inverse system of transformation groups with respect to maps $\psi_{\mu\lambda}$ such that $\varphi_{\lambda\mu}(f_\lambda)(x_\mu) = f_\lambda(\psi_{\mu\lambda}(x_\mu))$ for $\lambda < \mu$, $f_\lambda \in C(X_\lambda)$, $x_\mu \in X_\mu$; and

$$\varinjlim((C(X_\lambda), G_T), \varphi_{\lambda\mu}, \Lambda) = C(\varprojlim(X_\lambda, \psi_{\mu\lambda}, \Lambda), G_T).$$

(vi) In particular,

$$\bigotimes_{i\in I} (C(X_i), G_T) = \left(C\left(\prod_{i\in I} X_i\right), G_T\right).$$

2. **Almost periodicity.**

DEFINITION. Let (A, G) be a C^* flow. Then $a \in A$ is *almost periodic* if $G(a) = \{g(a): g \in G\}$ has (norm) compact closure in A. The C^* flow (A, G) is *almost periodic* if every $a \in A$ is almost periodic.

REMARK. A commutative C^* flow $(C(X), G_T)$ is almost periodic if and only if (X, T) is almost periodic [1].

EXAMPLE 1. Let (A, G) be a C^* flow with A finite dimensional. Since each $g \in G$ is an isometry, $G(a)$ is norm bounded; the norm topology on A is the euclidean topology, so $G(a)^-$ is compact. Thus, (A, G) is almost periodic.

LEMMA. *Let* (A, G) *be a* C^**-algebra. Then* $\{a \in A$: a *is almost periodic*$\}$ *is a closed* G*-invariant* **-subalgebra of* A.

PROOF. Straightforward computations show that the set of almost periodic points is an invariant *-subalgebra; a diagonalization argument shows it is closed.

THEOREM 1. (i) *Subflows of an almost periodic flow are almost periodic.*

(ii) *Homomorphs of an almost periodic flow are almost periodic.*

(iii) $\bigotimes_{i\in I}(A_i, G)$ *is almost periodic if and only if* (A_i, G) *is almost periodic for each* $i \in I$.

(iv) $\bigoplus_{i\in I}(A_i, G)$ *is almost periodic if and only if* (A_i, G) *is almost periodic for each* $i \in I$.

PROOF. (i) is clear.

(ii) Let $\pi: (A, G) \longrightarrow (B, G)$ be a homomorphism, $b \in B$, and $\{g_n(b)\}$ a sequence in $G(b)$. Let $a \in \pi^{-1}(b)$. If (A, G) is almost periodic, then $\{g_n(a)\}$ has a subsequence converging to some a_0, and the corresponding subsequence of $\{g_n(b)\}$ converges to $\pi(a_0)$.

(iii) Each (A_i, G) is isomorphic to a subflow of $\bigotimes_{i\in I}(A_i, G)$, so the "only if" assertion follows by (i). Conversely, if each (A_i, G) is almost periodic then the image of (A_i, G) in $\bigotimes_{i\in I}(A_i, G)$ under the natural embedding map is almost periodic; the smallest C^*-algebra containing the embedded copies of $\{A_i\}_{i\in I}$ is $\bigotimes_{i\in I}A_i$; $\bigotimes_{i\in I}(A_i, G)$ is thus almost periodic by the lemma.

(iv) The proof is similar to that of (iii).

EXAMPLE 2. Let A be an approximately finite algebra, i.e. the infinite tensor product of finite dimensional algebras $\{A_i\}_{i\in I}$. Let G be a group of automorphisms with $g(A_i) \subset A_i$ for each $g \in G$ (where we identify A_i with its copy in $\bigotimes_{i \in I} A_i$). Then, by Example 1 and Theorem 1(iii), (A, G) is almost periodic.

DEFINITION. If A is a C^*-algebra, we denote by S the states of A, with the $\sigma(A^*, A)$ topology, and by S_E the pure states of A. If φ is a state, and g is a *-automorphism of A, then $\varphi \cdot g^{-1}$ is a state, which we denote by $\hat{g}(\varphi)$; if φ is pure, so is $\hat{g}(\varphi)$, and the map $\hat{g}: S \longrightarrow S$ is a $\sigma(A^*, A)$ homeomorphism. If (A, G) is a C^* flow, we denote by $\hat{G}$ the group of homeomorphisms on S induced in this way. We prove an Ascoli type theorem:

THEOREM 2. *Let A be a C^* flow algebra, F a norm bounded subset of A. Then the following are equivalent*:

(1) *F^- is norm compact.*

(2) *F is equicontinuous on S_E^- (where elements of A act on A^* in the usual manner).*

(3) *F is equicontinuous on S.*

PROOF. (1) $\longrightarrow$ (2): Suppose F, considered as maps on S_E^-, is not equicontinuous at $\varphi \in S_E^-$. There is an $\epsilon > 0$, a net $\{\varphi_\alpha\}$ in S_E^-, and a corresponding net $\{a_\alpha\}$ in F such that $\varphi_\alpha \longrightarrow \varphi \sigma(A^*, A)$, and $|\varphi_\alpha(a_\alpha) - \varphi(a_\alpha)| > \epsilon$. The net $\{a_\alpha\}$ has a subnet $\{a_\beta\}$ converging to some a_0. Then

$$|\varphi_\beta(a_\beta) - \varphi(a_\beta)| \leqslant |\varphi_\beta(a_\beta) - \varphi_\beta(a_0)| + |\varphi_\beta(a_0) - \varphi(a_0)| + |\varphi(a_0) - \varphi(a_\beta)|$$

$$\leqslant 2\|a_\beta - a_0\| + |\varphi_\beta(a_0) - \varphi(a_0)| \longrightarrow 0;$$

which is a contradiction.

(2) $\longrightarrow$ (3): Suppose F is equicontinuous on S_E^-. By the usual form of the Ascoli theorem, F, considered as embedded in $C(S_E^-)$, is relatively compact in the uniform topology. Let $\{a_n\} \subset F$ converge to a_0 in $C(S_E^-)$, and let $\epsilon > 0$. There is a positive integer n_0 such that for all $n > n_0$ and all $\varphi \in S_E^-$, $|\varphi(a_n) - \varphi(a_0)| < \epsilon$. For any $\psi \in S$ there is, by Choquet's theorem, a normalized measure μ_ψ on S_E such that, for every $a \in A$,

$$\psi(a) = \int_{S_E} \varphi(a)\, d\mu_\psi(\varphi).$$

Thus, if $n > n_0$,

$$|\psi(a_n) - \psi(a_0)| \leqslant \int_{S_E} |\varphi(a_n) - \varphi(a_0)|\, d\mu_\psi(\varphi) < \epsilon,$$

so that $\{a_n\}$ converges to a_0 in $C(S)$. It follows that F is relatively compact in $C(S)$, and hence equicontinuous on S.

(3) $\longrightarrow$ (1): Assume F is equicontinuous on S. Let $F_1 = \{\mathrm{Re}(a): a \in F\} \cup \{\mathrm{Im}(a): a \in F\}$. A straightforward computation shows that F_1 is equicontinuous on S, and hence relatively compact when embedded in $C(S)$. Since the embedding of A into $C(S)$ is an isometry on selfadjoint elements, F_1 is relatively compact and, since $F \subset F_1 + iF_1$, F is relatively compact.

COROLLARY. *Let* (A, G) *be a* C^* *flow,* $(S_E^-, \hat{G})$ *and* $(S, \hat{G})$ *the induced transformation groups on* S_E^- *and* S. *Then the following are equivalent*:

(1) (A, G) *is almost periodic.*

(2) $(S_E^-, \hat{G})$ *is almost periodic.*

(3) $(S, \hat{G})$ *is almost periodic.*

3. **Minimality.**

DEFINITION. A C^* flow (A, G) is *minimal* if A has no proper G-invariant left ideals.

REMARKS. 1. Note that, since A has a unit, the absence of proper G-invariant left ideals is equivalent to the absence of proper G-invariant closed left ideals.

2. If $(C(X), G_T)$ is a commutative C^* flow, then a proper G-invariant closed left ideal of $C(X)$ consists of functions vanishing on a closed nonempty proper T-invariant subset of X. Thus, $(C(X), G_T)$ is minimal if and only if (X, T) is minimal.

3. Results which follow will justify the choice of left ideals in the above definition as the appropriate extension of the concept of minimality, rather than two-sided ideals. We note here that a C^* flow (A, G) may fail to have G-invariant two-sided ideals in a way which does not reflect the dynamical properties: e.g., if A is simple.

LEMMA. *Let* (A, G) *be a* C^* *flow. For a* G*-invariant state* φ *on* A, *let* $N_\varphi = \{x \in A: \varphi(x^*x) = 0\}$.

(1) N_φ *is a* G*-invariant closed left ideal.*

(2) *If* φ *is ergodic—that is, extreme in the* G*-invariant states—then* N_φ *is maximal among* G*-invariant left ideals.*

(3) *If* G *is amenable, then every maximal* G*-invariant left ideal is of the form* N_φ *for some ergodic* φ.

PROOF. (1) is clear.

(2) Let $\pi_\varphi(A)$ be the state representation on H_φ, $u_\varphi(G)$ the induced group of unitaries acting on H_φ, and $\mathcal{D}$ the C^*-algebra in $\mathcal{B}(H_\varphi)$ generated by $\pi_\varphi(A)$ and $u_\varphi(G)$. Then, since φ is ergodic, $\mathcal{D}'' = \mathcal{B}(H_\varphi)$ and so $\mathcal{D}$ acts irreducibly on H_φ, and hence algebraically irreducibly. Thus, $H_\varphi = A/N_\varphi$. If

M were a G-invariant left ideal of $\mathcal{A}$ containing I, then M/I would be a linear subspace of H_φ left invariant by $\mathcal{D}$.

(3) Suppose G is amenable, and M is a maximal G-invariant left ideal. Considering $\mathcal{A}$ in its universal representation, G acts as outer automorphisms, and so extends to a W^* flow $(\mathcal{A}^{**}, G)$. Let $M^{\circ\circ}$ be the bipolar of M in $\mathcal{A}^{**}$. Then $M^{\circ\circ}$ is a G-invariant weakly closed left ideal of $\mathcal{A}^{**}$, so $M^{\circ\circ} = \mathcal{A}^{**}p$ for some projection $p < 1$. The projection p, and hence $1 - p$, is fixed under G. Indeed, for any $g \in G$, $g(p)p = g(p)$, that is $g(p) \leqslant p$; while since $g^{-1}(p) \leqslant p$, $p \leqslant g(p)$. Thus, G acts as a group of automorphisms on $(1 - p)\mathcal{A}^{**}(1 - p)$. Since G is amenable, and the states on $(1 - p)\mathcal{A}^{**}(1 - p)$ are convex and $\sigma(((1 - p)\mathcal{A}^{**}(1 - p))^*, (1 - p)\mathcal{A}^{**}(1 - p))$ compact, there is a G-invariant state $\tilde{\varphi}$ on $(1 - p)\mathcal{A}^{**}(1 - p)$. Define φ on $\mathcal{A}$ by:

$$\varphi(a) = \tilde{\varphi}((1 - p)a(1 - p)).$$

Then φ is a G-invariant state on $\mathcal{A}$, and $M \subset N_\varphi$. By the maximality of M, $M = N_\varphi$.

It remains to show there exists an ergodic φ with $M = N_\varphi$. Let $\Phi = \{\varphi\colon \varphi$ is a G-invariant state on $\mathcal{A}$, and $M = N_\varphi\}$. Then Φ is convex and $\sigma(\mathcal{A}^*, \mathcal{A})$ compact. Let ψ be an extreme point of Φ. Suppose $\psi = \lambda\varphi_1 + (1 - \lambda)\varphi_2$, where φ_1 and φ_2 are G-invariant states and $0 < \lambda < 1$. Then, for $x \in I$,

$$0 = \psi(x^*x) = \lambda\varphi_1(x^*x) + (1 - \lambda)\varphi_2(x^*x) \geqslant 0.$$

Thus φ_1 and φ_2 are in Φ and $\psi = \varphi_1 = \varphi_2$, i.e. φ is ergodic.

The following theorem, extending a known result about invariant measures on transformation groups, is an immediate consequence of the lemma.

THEOREM 3. *Let* $(\mathcal{A}, G)$ *be a* C^* *flow. Consider the statements*:

(1) $(\mathcal{A}, G)$ *is minimal.*

(2) *Every* G*-invariant state on* $\mathcal{A}$ *is strictly positive.*

(3) *Every ergodic state on* $\mathcal{A}$ *is strictly positive.*

Then (1) $\Rightarrow$ (2) $\Longleftrightarrow$ (3) *and if* G *is amenable they are equivalent.*

COROLLARY. *Let* $(\mathcal{A}, G)$ *be a minimal* C^* *flow with* G *amenable, and* $(\mathcal{B}, G)$ *a subflow. Then* $(\mathcal{B}, G)$ *is minimal.*

PROOF. Let φ be a G-invariant state on $\mathcal{B}$. The set of states on $\mathcal{A}$ which extend φ is a convex, $\sigma(\mathcal{A}^*, \mathcal{A})$ compact set which is invariant under the induced action of G, and hence contains a G-invariant state ψ, which is strictly positive.

REMARK. The inductive limit of minimal C^* flows is minimal. Indeed, if (A, G) is the inductive limit of minimal subflows (A_λ, G), a nonzero closed invariant left ideal of A contains all the A_λ from some point on.

THEOREM 4. *Let (A, G) be a C^* flow with G amenable, and $(S_E^-, \hat{G})$ the induced transformation group. Then $(S_E^-, \hat{G})$ minimal implies (A, G) is minimal.*

PROOF. Suppose (A, G) is not minimal. Then, by Theorem 3, there is a G-invariant state φ on A, and a positive element $h \in A$ with $\varphi(h) = 0$. By the Choquet-Bishop-DeLeeuw Theory, there is a normalized regular Borel measure μ_φ on S_E^- such that $\varphi(a) = \int_{S_E^-} \nu(a)\, d\mu_\varphi(\nu)$ for every $a \in A$. Let $T = \{\psi \in S_E^-: \hat{g}(\psi)(h) = 0 \text{ for all } g \in G\}$. If ν_0 is in the support of the measure μ_φ, then

$$0 = \varphi(h) = \varphi(g^{-1}(h)) = \int_{S_E^-} \nu(g^{-1}(h))\, d\mu_\varphi(\nu),$$

so $\nu_0 \in T$, and T is nonempty. T is a $\sigma(A^*, A)$ closed proper G-invariant subset of S_E^-.

EXAMPLE 3. Let $A = M_n(C)$, all $n \times n$ complex matrices, and G the group of all inner automorphisms. We observe that G is not amenable for $n > 2$. We show that (A, G) and (S_E^-, G) are both minimal.

(i) (A, G) is minimal. Any ideal of $M_n(C)$ is, with respect to some basis of C^n, of the form $\{(a_{ij}) \in M_n(C): a_{ij} = 0 \text{ for } k \leqslant i \leqslant n \text{ and } 1 \leqslant j \leqslant n\}$ for some k, and is thus not G-invariant.

(ii) (S_E^-, G) is minimal. For a unitary matrix U let g_U be the automorphism defined by $g_U(A) = U^*AU$, and let $\hat{g}_U$ be the induced map on states. A state on $M_n(C)$ is of the form $\varphi_B(A) = \text{trace}(BA)$ for some positive matrix B of trace 1; φ_B is pure if and only if B is a minimal projection. Note that the $\sigma(A^*, A)$ topology on A^* is the norm topology on $M_n(C)$; $S_E(A)$ considered as matrices is all idempotent selfadjoint matrices of trace 1, and hence is closed. If P is any minimal projection, $\{\hat{g}_U(\varphi_P): U \text{ unitary}\} = \{\varphi_{U^*PU}\} = S_E$, i.e. the orbit of every point in S_E is S_E.

EXAMPLE 4. Let $A = B(H)$, where H is any infinite dimensional Hilbert space, and G any group of inner automorphisms. Then compact operators form a G-invariant ideal, so $(B(H), G)$ is not minimal.

4. Semiminimality.

DEFINITION. A C^* flow (A, G) is *semiminimal* if the intersection of all maximal G-invariant left ideals is 0.

REMARKS. 1. I is a maximal G-invariant left ideal if and only if for every maximal left ideal M containing I, $\bigcap\{g(M): g \in G\} = I$.

2. A commutative C^* flow $(C(X), G_T)$ is semiminimal if and only if (X, T) is pointwise almost periodic. Indeed, a G-invariant maximal (left) ideal in $C(X)$ consists of all functions vanishing on a minimal set. The intersection of such ideals is 0 if and only if every point is in a minimal set.

3. If $(A_i, G)_{i\in I}$ is a family of C^* flows, then $\Sigma_{i\in I} \bigoplus (A_i, G)$ is semiminimal if and only if each (A_i, G) is semiminimal. Indeed, suppose each (A_i, G) is minimal, and let $x \in \Sigma \bigoplus A_i$, $x \neq 0$. If $\pi_i: \Sigma \bigoplus A_i \longrightarrow A_i$ is the projection, then $\pi_i(x) \neq 0$ for some i; there is some M_i maximal G-invariant left ideal of A_i which does not contain $\pi_i(x)$, and $\pi_i^{-1}(M_i)$ is a maximal G-invariant left ideal not containing x. Conversely, suppose $\Sigma \bigoplus (A_i, G)$ is minimal, and let $x_i \in A_i$. If x is the point in $\Sigma \bigoplus A_i$ with ith coordinate x_i and all other coordinates 0, and if M is a G-invariant maximal left ideal which excludes x, then $\pi_i(M)$ is a G-invariant maximal left ideal of A_i which excludes x_i.

THEOREM 5. *Let (A, G) be a C^* flow. If A has a strictly positive family of G-invariant states, then (A, G) is semiminimal; if G is amenable the converse is true.*

PROOF. If A has a strictly positive family of G-invariant states it has a strictly positive family of ergodic states. The left kernels of these are, by the lemma before Theorem 3, maximal G-invariant ideals, and have 0 intersection. If G is amenable, every maximal left ideal is the left kernel of a G-invariant state; if the ideals have 0 intersection the states are strictly positive.

COROLLARY. *Let (A, G) be a semiminimal C^* flow with G amenable. Then subflows of (A, G) are semiminimal.*

PROOF. Restrictions of a strictly positive family of G-invariant states of A to B form a strictly positive family of G-invariant states of B.

THEOREM 6. *Let (A, G) be a C^* flow with G amenable. Then if $(S_E^-, \hat{G})$ is pointwise almost periodic, (A, G) is semiminimal.*

PROOF. Suppose (A, G) is not semiminimal. Then there is a positive $h \in A$ such that $\varphi(h) = 0$ for every G-invariant state φ. If $(S_E^-, \hat{G})$ is pointwise almost periodic, then for every $\varphi \in S_E^-$, $\hat{G}(\varphi)^-$ is minimal. There is a $\hat{G}$-invariant regular Borel measure μ_φ on $\hat{G}(\varphi)^-$, which necessarily has support $\hat{G}(\varphi)^-$. Define

$$\xi_\varphi(a) = \int_{\hat{G}(\varphi)^-} \psi(a)\, d\mu_\varphi(\psi) \qquad (a \in A).$$

Then ξ_φ is linear, positive, and $\xi_\varphi(1) = 1$, so ξ_φ is a state. For $g \in G$,

$$\xi_\varphi(g(a)) = \int_{\hat{G}(\varphi)^-} \psi(g(a))\, d\mu_\varphi(\psi)$$
$$= \int_{\hat{G}(\varphi)^-} \psi(a)\, d\mu_\varphi(\psi) = \xi_\varphi(a),$$

since μ_φ is invariant. Thus, $\xi_\varphi(h) = 0$. But $\psi(h)$ is a nonnegative continuous function of ψ and μ_φ has support $\hat{G}(\varphi)^-$. Thus $\psi(h) = 0$ for every $\psi \in \hat{G}(\varphi)^-$. Since this holds for every φ, $\psi(h) = 0$ for every $\psi \in S_E(A)^-$ and thus for every $\psi \in S(A)$, a contradiction.

5. **Concluding remarks.** We can define in this context such additional dynamical concepts as distality (by considering the induced transformation group (S_E^-, G)), topological ergodicity, and weak and strong mixing. Note that the (doubly) infinite tensor product of matrix algebras yields a noncommutative analog of Bernoulli shifts. We shall deal with these concepts in subsequent papers.

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