SURGERY ON A CURVE IN A SOLID TORUS

BY

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ABSTRACT. We consider the following surgery question: If a regular neighborhood of a polyhedral knot in a solid torus is removed and then sewn back differently, what manifold results? We consider two classes of knots, torus knots and what we call doubly twisted knots. We obtain some related results on surgery on knots in $S^3$.

1. Introduction. In this paper we consider surgery on a simple closed curve in a solid torus. That is, we consider the following question: If a regular neighborhood of a knot in a solid torus is removed and then sewn back differently, what manifold results? In particular, when is the manifold a solid torus?

A similar kind of surgery, surgery on knots in the 3-sphere, has been extensively studied. For a discussion of the history and importance of this problem, and for other references, see [7], [2], or [8]. Here we consider surgery on two kinds of knots, torus knots and what we call "doubly twisted knots." We also obtain some related results concerning surgery on knots in the 3-sphere.

2. Definitions and other preliminaries. In this paper, all manifolds, embeddings, regular neighborhoods, etc., will be piecewise linear. A solid torus is a 3-manifold-with-boundary homeomorphic to the product $S^1 \times D^2$ of a circle and a disk. A torus is a 2-manifold-without-boundary homeomorphic to the boundary of a solid torus. A homotopy 3-cell is a compact simply connected 3-manifold whose boundary is a 2-sphere. A fake cube is a homotopy 3-cell which is not a 3-cell. A homotopy solid torus is an orientable 3-manifold-with-boundary homeomorphic to a homotopy 3-cell with a pair of disks on its boundary identified. Note that the boundary of a homotopy solid torus is a torus and that if the homotopy 3-cell is a real 3-cell then the homotopy solid torus is a real solid torus. If $T$ is a homotopy solid torus, a meridian of $T$ is a simple closed curve on $\text{Bd} \ T$ that bounds a disk in $T$ but is not null-homologous in $\text{Bd} \ T$. A longitude of $T$ is a simple closed curve on $\text{Bd} \ T$ which is transverse to a meridian. A core of a solid torus $T$ is a simple closed curve $h(S^1 \times \{a\})$ where $h$ is a homeomorphism of $S^1 \times D^2$ onto $T$ and $a \in \text{Int} \ D^2$.  

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Let $T$ be an unknotted solid torus in $S^3$. Let $(M, L)$ be a meridian-longitude pair for $T$ such that $L$ bounds a disk in $S^3 - \text{Int } T$. Suppose also $K$ is a polyhedral simple closed curve in $\text{Int } T$, $T'$ is a regular neighborhood of $K$ in $\text{Int } T$ and $(c, g)$ is a meridian-longitude pair for $T'$. If $r$ and $s$ are relatively prime nonzero integers, we shall let $T_g(K; r, s)$ denote the orientable 3-manifold-with-boundary obtained from $T$ by $(r, s)$-surgery on $T'$. That is, $T_g(K; r, s)$ is obtained from $T$ by removing $\text{Int } T'$ and then sewing a solid torus $T''$ to $T - \text{Int } T'$ by a homeomorphism of $\partial T''$ onto $\partial T'$ which takes a meridian of $T''$ to a curve on $\partial T'$ homologous to $rc + sg$. If $g$ is homologous in $T - \text{Int } T'$ to a multiple of $L$ then we abbreviate $T_g(K; r, s)$ to $T(K; r, s)$. Note that, for a particular knot $K$ in $\text{Int } T$, the family of 3-manifolds

$$\{T_g(K; r, s); (r, s) = 1\}$$

is the same no matter what longitude $g$ is used. For, if $g$ and $g'$ are longitudes transverse to the meridian $c$, then $g \sim kc + g'$ ($\sim$ means homologous with integer coefficients) for some integer $k$. Hence, $rc + sg \sim (r + sk)c + sg'$ so that $T_g(K; r, s)$ is topologically equivalent to $T_{g'}(K; r + sk, s)$. We shall use $G(K; r, s)$ to denote the group $\pi_1(T(K; r, s))$. By van Kampen’s theorem,

$$G(K; r, s) \cong \pi_1(T - K)/c^g$$

where the right side denotes $\pi_1(T - K)$ modulo the smallest normal subgroup containing $c^g$.

Suppose $K$ is a knot in $S^3$, $(c, g)$ is a meridian-longitude pair for $K$ and $(r, s)$ is a pair of relatively prime integers. We shall let $M_g(K; r, s)$ denote the manifold obtained by removing a regular neighborhood of $K$ from $S^3$ and sewing in a solid torus whose meridian is sewn to a simple closed curve homologous to $rc + sg$. If $g \sim 0$ in $S^3 - K$ then we write $M^2(K; r, s)$ instead of $M_g(K; r, s)$. We say $K$ has property $P$ if and only if $\pi_1(M^2(K; r, s))$ is trivial only in the case $(r, s) = (\pm 1, 0)$. It is known that a nontrivial knot $K$ has property $P$ if and only if no counterexample to the Poincaré conjecture can be obtained by surgery on $K$ and the complement of $K$ in $S^3$ is unique among knot complements. See [2] or [8].

In [2], Bing and Martin considered surgery on certain knots in solid tori and we include their result for completeness.

**Theorem 1.** If $K$ is a knot in the solid torus $T$ such that $K$ is not a core of $T$ and $T$ has as meridional disk which intersects $K$ exactly once, then the surgery manifold $T(K; r, s)$ is not a homotopy solid torus except in the trivial case $(r, s) = (\pm 1, 0)$. 
We conclude this section with a lemma that characterizes homotopy solid tori algebraically. Before stating our lemma, we note the following fact: If \( m \) and \( n \) are relatively prime, then the group \( G = \{ a, b : ab = ba, a^m b^n = 1 \} \) is infinite cyclic. To show this, let \( \{ c \} \) be the free group on one generator and define homomorphisms \( f : \{ c \} \to G \) and \( g : G \to \{ c \} \) by \( g(a^m b^n) = c^{n-m} \) and \( f(c) = a^m b^{-n} \) where \( am + bn = 1 \). Then \( fg \) and \( gf \) are both the identity mappings.

**Lemma 1.** Suppose \( T \) is an orientable 3-manifold such that \( \partial T \) is a torus and \( \pi_1(T) \) is infinite cyclic. Then \( T \) is a homotopy solid torus.

**Proof.** By the loop theorem, there is a simple closed curve \( K \) on \( \partial T \) such that \( K \) is homotopically nontrivial in \( \partial T \) and such that \( K \) bounds a disk \( D \) whose interior lies in \( \text{Int} \ T \). Let \( N_1 \) be a regular neighborhood (in \( T \)) of \( D \) such that \( N_1 \cap \partial T \) is an annulus. Let \( C_1 = \text{Cl}(T - N_1) \). Then \( \partial C_1 \) is a 2-sphere. Let \( N_2 \) be a regular neighborhood (in \( C_1 \)) of \( \partial C_1 \). Let \( C = \text{Cl}(C_1 - N_2) \). Then \( \partial C \) is a 2-sphere and we wish to show \( C \) is simply connected. Now \( T = C \cup (N_1 \cup N_2) \), \( C \cap (N_1 \cup N_2) = \partial D = \text{Bd}(N_1 \cup N_2) \), and \( N_1 \cup N_2 \) is a regular neighborhood of \( D \cup \partial T \). Also, \( \pi_1(\partial T) = \{ a, b : ab = ba \} \); therefore,

\[
\pi_1(D \cup \partial T) = \{ a, b : ab = ba, a^m b^n = 1 \}
\]

where \( K \) represents the element \( d^m b^n \) of \( \pi_1(\partial T) \). Since \( K \) is a simple closed curve, \( m \) and \( n \) are relatively prime; hence \( \pi_1(N_1 \cup N_2) \cong \pi_1(D \cup \partial T) \) is infinite cyclic. But \( (N_1 \cup N_2) \cap C \) is simply connected; therefore \( \pi_1(T) \cong \pi_1(N_1 \cup N_2) \ast \pi_1(C) \), that is, the free product. But \( \pi_1(T) \) and \( \pi_1(N_1 \cup N_2) \) are both infinite cyclic; hence \( \pi_1(C) \) must be trivial. Therefore, \( C \) is a homotopy 3-cell and \( T \) is a homotopy solid torus.

3. Surgery on a torus knot in a solid torus. In this section we begin consideration of surgery on curves in solid tori. Our first case is that of a nicely embedded nontrivial torus knot. Throughout the rest of this paper, \( T \) will refer to the standard unknotted solid torus in \( S^3 \). In this section, \( K(p, q) \) will refer to a \( (p, q) \) torus knot nicely embedded in \( T \). That is, we assume there is a polyhedral annulus \( A \) in \( T \) such that one boundary component of \( A \) is \( K(p, q) \) and the other boundary component is a simple closed curve \( K_1 \) on \( \partial T \) which is homologous to \( pM + qL \), where \( M \) is a meridian of \( T \) and \( L \) is a longitude of \( T \) which is null-homologous in \( S^3 - \text{Int} \ T \). We may assume that \( q > 0 \). By nontriviality, we mean that \( K \) is not a core of \( T \), hence \( q \neq 1 \). In the next theorem we will use the following construction: Let \( T' \) be a regular neighborhood of \( K \) in \( \text{Int} \ T \). We will assume that \( T' \) is constructed carefully enough
so that \( A_1 = A \cap (T - \text{Int } T') \) is an annulus, one of whose boundary components is \( K_1 \) and whose other boundary component is \( A_1 \cap T' = A_1 \cap \text{Bd } T' = A \cap \text{Bd } T' \). We shall also assume there is a regular neighborhood \( R \) of \( A_1 \) in \( T - \text{Int } T' \) such that \( A_1 \cap \text{Bd } R = \text{Bd } A_1, R \cap \text{Bd } T = \text{Bd } R \cap \text{Bd } T' \) is an annulus with \( K_1 \) as a centerline, and \( R \cap T' = \text{Bd } R \cap \text{Bd } T' \) is an annulus with \( A_1 \cap T' \) as a centerline. In the following theorem we will use the longitude \( d \) of \( T' \) where \( d \) is a boundary component of the annulus \( R \cap T' \). Note that \( d \sim A_1 \cap T' \) on \( \text{Bd } T' \) and \( d \sim K_1 \) in \( T - \text{Int } T' \). It should also be noted that \( d \) is, in general, not homologous (in \( T - \text{Int } T' \)) to any multiple of \( L \) so that \( T_d(K: r, s) \neq T(K: r, s) \). We shall assume, unless stated otherwise, that the surgery is nontrivial, that is, \( (r, s) \neq (\pm 1, 0) \).

**Theorem 2.** If \( K \) is a \((p, q)\) torus knot, \( q > 1 \), embedded in \( T \) as described above, then \( T_d(K: r, s) \) is a union of two solid tori \( T_1 \) and \( T_2 \) such that \( T_1 \cap T_2 \) is an annulus \( B \) on the boundary of each and \( B \) circles \( \text{Bd } T_1 \) \( q \) times longitudinally and circles \( \text{Bd } T_2 \) \( r \) times longitudinally.

**Proof.** First we recall that \( T_d(K: r, s) = (T - \text{Int } T') \cup T'' \) where \( T'' \cap (T - \text{Int } T') = \text{Bd } T'' \cap \text{Bd } (T - \text{Int } T') = \text{Bd } T' \) and a meridian of \( T'' \) is sewn to a curve homologous to \( rc + sd \). \((c \) is a meridian of \( T' \).) Let \( T_1 = \text{Cl}[T - (R \cup T')] \) and \( T_2 = R \cup T'' \). \( T_1 \) is a solid torus since \( R \cup T' \) is a regular neighborhood of the curve \( K_1 \) on \( \text{Bd } T \). Next we show \( T_2 \) is a solid torus. Let \( B_1 = R \cap T'' \). \( B_1 \) is an annulus with \( A_1 \cap T'' \) as a centerline. Now \( B_1 \) circles \( \text{Bd } R \) exactly once longitudinally so \( R \) is a regular neighborhood of the simple closed curve \( A_1 \cap T'' \). Therefore \( T_2 \) is \( T'' \) plus a regular neighborhood of a curve on \( \text{Bd } T'' \). Hence \( T_2 \) is homeomorphic to \( T'' \).

It remains to show that the intersection of \( T_1 \) and \( T_2 \) is correct. Let \( B = T_1 \cap T_2 \). Now \( B = [R \cup T'] \cap \text{Cl}[T - (R \cup T')] = \text{Cl}[(\text{Bd}(R \cup T') - \text{Bd } T)] \) which is an annulus since \( \text{Bd } T \cap \text{Bd}(R \cup T') = \text{Bd } T \cap \text{Bd } R \) is an annulus. Now \( d \) is a centerline of \( B \) and \( I \) circles \( T \) \( q \) times longitudinally. Therefore \( d \) (and hence \( B \)) circles \( T_1 \) \( q \) times longitudinally, since \( T_1 \) is a strong deformation retract of \( T \). Also, there is a homeomorphism of \( T_2 = T'' \cup R \) onto \( T'' \) which is the identity on \( \text{Cl}[(\text{Bd } T'' - R)] \) and which takes \( \text{Cl}[(\text{Bd } R - T'')] \) onto \( R \cap T'' \). Since \( d \) is a centerline of \( B \) and the homeomorphism is the identity on \( d \), it suffices to find the number of times \( d \) circles \( T'' \) longitudinally. To do this, it suffices to find the (algebraic) intersection number of \( d \) and a meridian of \( T'' \). Since a meridian of \( T'' \) is homologous to \( rc + sd \), that intersection number is \( r \). Hence \( d \) circles \( T'' \) \( r \) times longitudinally and so \( B \) circles \( T_2 \) \( r \) times longitudinally. This finishes the proof of Theorem 2.

As previously mentioned, \( d \sim K_1 \sim pM + qL \) in \( T - \text{Int } T' \). Also,
\( M \sim qc \). Therefore \((- pq)c + d \sim -qL\). Hence, if \( g \) denotes a longitude of \( T' \) which is homologous in \( T' - \text{Int } T' \) to a multiple of \( L, g \sim (- pq)c + d \) or \( d \sim pqc + g \). Therefore, \( rc + sd \sim rc + spqc + sg \). Hence \( T_d(K; r, s) = T(K; r + spq, s) \) and \( T(K; r', s') = T_d(K; r' - s'pq, s') \).

**Corollary 1.** If \( K = K(p, q) \) then

(i) \( \pi_1(T_d(K; r, s)) = \{w, e: w' = e^q\} \) and

(ii) \( G(K; r, s) = \{w, e: w^{-spq} = e^q\} \).

This corollary follows directly from Theorem 2 by an application of van Kampen's theorem.

**Corollary 2.** \( T_d(K; r, s) \) is a homotopy solid torus if and only if \( r = \pm1 \). \( T(K; r, s) \) is a homotopy solid torus if and only if \( r - spq = \pm1 \).

**Proof.** This corollary follows from Lemma 1 and the fact that \( \pi_1(T_d(K; r, s)) \) is infinite cyclic if \( r = \pm1 \) and is nonabelian if \( |r| > 1 \).

**Corollary 3.** If \( T_d(K; r, s) \) or \( T(K; r, s) \) is a homotopy solid torus then it is a real torus.

**Proof.** If \( T_d(K; r, s) \) is a homotopy solid torus then \( r = \pm1 \). Thus \( T_d(K; r, s) = T_1 \cup T_2 \) where \( T_1 \cap T_2 \) is an annulus which circles \( \text{Bd } T_2 \) exactly once longitudinally. Hence \( T_2 \) is a regular neighborhood of \( T_1 \cap T_2 \), so \( T_1 \cup T_2 \) is homeomorphic to \( T_1 \).

**Corollary 4.** If \( p \) is congruent to \( r \pmod{q} \) and \( n \) is congruent to \( q \pmod{r} \) where \( m \) and \( n \) are integers such that \( mr + ns = 1 \), then \( T_d(K; r, s) \) is homeomorphic to a cube with an \((r, q)\) torus knotted hole and \( T(K; r, s) \) is homeomorphic to a cube with an \((r - spq, q)\) knotted hole.

A cube with an \((r, q)\) torus knotted hole is \( \text{Cl}[S^3 - N(r, q)] \) where \( N(r, q) \) is a regular neighborhood of an \((r, q)\) torus knot. Since the knot lies on the boundary of an unknotted solid torus, the cube with a hole may be written as \( T_1^* \cup T_2^* \) where \( B^* = T_1^* \cap T_2^* = \text{Bd } T_1^* \cap \text{Bd } T_2^* \) is an annulus which circles \( \text{Bd } T_1^* \) \( q \) times longitudinally and \( r \) times meridionally and which circles \( \text{Bd } T_2^* \) \( r \) times longitudinally and \( q \) times meridionally.

**Proof of Corollary 4.** By Theorem 2, \( T_d(K; r, s) = T_1 \cup T_2 \) where \( T_1 \cap T_2 \) is the annulus \( B \) with \( d \) as a centerline and \( B \) circles \( \text{Bd } T_1 \) \( q \) times longitudinally and circles \( \text{Bd } T_2 \) \( r \) times longitudinally. Since \( T_1 \) is a solid torus contained in \( T \) and a centerline of \( T_1 \) is also a centerline of \( T \), there is a meridian-longitude pair \((M_1, L_1)\) for \( T_1 \) such that \( d \sim pM_1 + qL_1 \).

Next we wish to find a meridian-longitude pair for \( T_2 \). Let \( c_2 \) be a
meridian of the solid torus $T' \cup R$ which bounds a disk $b_2$ in $T' \cup R$ such that $b_2 \cap \text{Bd } T'$ is the meridian $c$ of $T'$. Also, $d$ is a longitude of $T' \cup R$. Let $M''$ be a simple closed curve on $\text{Bd } T''$ homologous to $rc + sd$. By construction, $M''$ is a meridian of $T''$; hence a simple closed curve $M_2$ on $\text{Bd } T_2$ which is homologous to $rc_2 + sd$ is a meridian of $T_2$.

Now there are integers $m$ and $n$ such that $mr + ns = 1$. We claim a simple closed curve $L_2$ on $\text{Bd } T_2$ such that $L_2 \sim -nc_2 + md$ is a longitude for $T_2$. First $d \sim nM_2 + rL_2$ and $c_2 \sim mM_2 - sL_2$, so $M_2$ and $L_2$ generate $H_1(\text{Bd } T_2)$. Hence it suffices to show that the algebraic intersection number of $M_2$ and $L_2$ is 1. Let $(J_1, J_2)$ stand for the algebraic intersection number of two oriented curves $J_1$ and $J_2$ on a torus. Then, since $M_2 \sim rc_2 + sd$, 
\[ \#(c_2, M_2) = -s \text{ (with appropriate orientations).} \]
Hence 
\[ \#(L_2, M_2) = \#(M_2, M_2) = -n(-s) + mr = 1. \]

Now we construct a homeomorphism which takes $T_d(K; r, s)$ onto $T^*_1 \cup T^*_2$. By hypothesis, there is an integer $n_1$ such that $r = p + n_1q$. Let $L_1'$ be a simple closed curve on $\text{Bd } T_1$ such that $L_1' \sim n_1M_1 + L_1$. Then $L_1'$ is a longitude of $T_1$ and $d \sim rM_1 + qL_1'$. Therefore, since $d$ is a centerline of $B$, there is a homeomorphism $f_1$ of $T_1$ onto $T^*_1$ such that $(f_1(M_1), f_1(L_1'))$ is a meridian-longitude pair for $T^*_1$ and $f_1(B) = B^*$. Similarly, there is an integer $n_2$ such that $n = q + n_2r$. Let $L_2' = n_2M_1 + L_2$. Then $d \sim qM_1 + rL_2'$. Therefore, there is a homeomorphism $f_2$ of $T_2$ onto $T^*_2$ such that $(f_2(M_2), f_2(L_2'))$ is a meridian-longitude pair for $T^*_2$ and $f_2$ agrees with $f_1$ on $B$. This concludes the proof of Corollary 4.

**Lemma 2(a).** If $r - spq = 1$ (and hence $T(K(p, q); r, s)$ is a solid torus) then
\[ G(K(p, q); r, s) = \{ b, x : bx = xb, x^r b^{sq^2} = 1 \}, \]
where $b$ is the element of $\pi_1(T(K; r, s))$ generated by the longitude $L$ and $x$ is the element generated by the meridian $M$. In addition, a simple closed curve homologous to $rM + sq^2L$ is a meridian of $T(K; r, s)$. If $r - spq = -1$ then the conclusion is the same with $s$ replaced by $-s$.

**Proof.** Our first task is to construct a presentation for $\pi_1(T - K(p, q))$ in terms of $b, x$ and $y$ where $y$ is the element of the fundamental group generated by a core of $T$. Since $K$ is nicely embedded in $T$, there is a solid torus $T^*$ in Int $T$ such that $K$ lies on $\text{Bd } T^*$ and such that a core of $T^*$ is also a core of $T$.

Let $S = \text{Cl}(T - T^*)$. Then $S$ is homeomorphic to the product of a torus and an interval. Let $T'$ be the regular neighborhood of $K$ as described at the
beginning of this section. We shall assume that $T' \cap T^*, T' \cap S$ and $T' \cap \text{Bd } T^*$ are regular neighborhoods of $K$ in $T^*, S$ and $\text{Bd } T^*$ respectively. We will find generators and relations for $\pi_1(T - K) \cong \pi_1(\text{Cl}(T - T'))$ from $\pi_1(\text{Cl}(T^* - T'))$ and $\pi_1(\text{Cl}(R - T'))$ using van Kampen’s theorem.

Now $\text{Cl}(T^* - T')$ is a solid torus. Let $y$ be the element of $\pi_1(\text{Cl}(T^* - T'))$ generated by a core of $T^*$ which misses $T'$. Then $\pi_1(\text{Cl}(T^* - T'))$ is a free group with generator $y$. Also, $\pi_1(\text{Cl}(R - T')) \cong \{x, b: bx = xb\}$. Now

$$\text{Cl}(R - T') \cap \text{Cl}(T^* - T') = \text{Cl}(\text{Bd } T^* - T')$$

is an annulus which circles $T^*$ $p$ times meridionally and $q$ times longitudinally. Let $z$ be a generator of $\pi_1(\text{Cl}(\text{Bd } T^* - T'))$. In $\pi_1(\text{Cl}(T^* - T'))$, $z = y^q$ and in $\pi_1(\text{Cl}(R - T'))$, $z = b^q x^p$. Therefore, by van Kampen’s theorem,

$$\pi_1(T - K(p, q)) \cong \pi_1(\text{Cl}(T - T')) \cong \{b, x, y: bx = xb, y^q = b^q x^p\}.$$

Our next task is to find a meridian-longitude pair for $K(p, q)$ in terms of $x$, $y$ and $b$. Our method is to use the overcrossing-undercrossing presentation of $T - K(p, q)$ in essentially the same way as J. Hempel did in [6]. Instead of using $\pi_1(T - K(p, q))$ directly, we will use the link group $\pi_1(S^3 - (M^* \cup K(p, q)))$ where $M^*$ is a curve in $S^3 - T$ which is parallel to $M$.

Now, since $p$ and $q$ are relatively prime, there are integers $\alpha$ and $\beta$ such that $\alpha p + \beta q = 1$ and $\beta > 0$. Then $\pi_1(S^3 - (M^* \cup K(p, q)))$ has generators $a_1, \ldots, a_{p+q}$ (see Figure 1) and the following relations:
(1) for \( 1 \leq k \leq p \), \( a_k \cdots a_{k+q-1} = a_{k+1} \cdots a_{k+q} \), and
(2) if \( k > p \) and \( k = np + j \) with \( 0 \leq j < p \) then \( a_k = b^n a_{p+j} b^{-n} \)
where we define \( a_0 = a_p \).

Since \( y \) is represented by a core of \( T^* \), \( y = a_1 \cdots a_p b \) and, since \( x \) is represented by a curve parallel to \( M \), \( x = a_1 \cdots a_q \). Now the relations in (2) imply

(3) if \( 1 \leq k \leq p \) and \( np + k \leq p + q \) then \( a_{np+1} \cdots a_{np+k} = b^n a_1 \cdots a_k b^{-n} \). Define \( x_k = a_k \cdots a_{k+q-1} \) for \( 1 \leq k \leq p + 1 \). Then the relations in (1) are \( x_1 = x_2 = \cdots = x_{p+1} \). We wish to write each \( x_k \) in terms of \( a_1, \cdots, a_p \) and \( b \). From this point, we proceed in two cases.

Case 1. \( p < q \). Let \( q = jp + \delta \) where \( 0 < \delta < p \). Using the relations in (3),

\[
x = x_1 = a_1 \cdots a_q
\]
\[
= (a_1 \cdots a_p)(a_{p+1} \cdots a_{2p}) \cdots (a_{(j-1)p+1} \cdots a_{jp})(a_{jp+1} \cdots a_{jp+\delta})
\]
\[
= (a_1 \cdots a_p)(b a_1 \cdots a_p b^{-1}) \cdots (b^j a_1 \cdots a_p b^{-j})(b^j a_1 \cdots a_p b^{-j})
\]
\[
= (a_1 \cdots a_p b)(a_1 \cdots a_p b^{-1})(a_1 \cdots a_\delta b^{-\delta}).
\]

Similarly,

\[
x_2 = a_2 \cdots a_{q+1}
\]
\[
= (a_2 \cdots a_p)(a_{p+1} \cdots a_{2p}) \cdots (a_{(j-1)p+1} \cdots a_{jp})(a_{jp+1} \cdots a_{jp+\delta+1})
\]
\[
= (a_2 \cdots a_p b)(a_1 \cdots a_p b^{-1})(a_1 \cdots a_\delta b^{-\delta}).
\]

In general, for \( 1 \leq k \leq p \),

\[
x_k = (a_k \cdots a_p b)(a_1 \cdots a_p b^{-1})(a_1 \cdots a_\delta b^{-\delta})
\]
\[
\text{if } \delta + k - 1 < p,
\]
\[
x_k = (a_k \cdots a_p b)(a_1 \cdots a_p b^{-1})(a_1 \cdots a_{\delta+k-1-p} b^{-\delta-1})
\]
\[
\text{if } \delta + k - 1 > p;
\]
\[
x_{p+1} = b(a_1 \cdots a_p b)(a_1 \cdots a_\delta b^{-\delta}).
\]

Now we are ready to find \( a_1 \), a meridian of \( K(p, q) \), in terms of \( x, y \) and \( b \).
We write \( x^\theta = z_1 z_2 \cdots z_\beta \) where each \( z_i = a_1 \cdots a_q \) and we make a replacement for each \( z_i \) to find a new representation for \( x^\theta \). Replace \( z_1 \) by

\[
v_1 = x_1 = (a_1 \cdots a_p b)(a_1 \cdots a_\delta b^{-\delta}).
\]

Replace \( z_2 \) by \( v_2 = x_{\delta+1} \) where \( x_{\delta+1} \) is written as in (4). Note that the last "a" letter in \( v_1 \) is \( a_\delta \) and the first "a" letter in \( x_{\delta+1} \) is \( a_{\delta+1} \). In general,
(a) if \( z_i \) is replaced by
\[
v_i = x_k = a_k \cdots a_p b(a_1 \cdots a_p b)^{y-1}(a_1 \cdots a_{\delta+k-1})b^{-1}
\]
then \( z_{i+1} \) is replaced by \( x_{\delta+k} \).

(b) if \( z_i \) is replaced by
\[
v_i = a_k \cdots a_p b(a_1 \cdots a_p b)(a_1 \cdots a_{\delta+k-1-p})b^{-1}
\]
then \( z_{i+1} \) is replaced by \( v_{i+1} = x_{\delta+k-p} \), and

(c) if \( z_i \) is replaced by
\[
v_i = b(a_1 \cdots a_p b)(a_1 \cdots a_{\delta-1})b^{-1}
\]
then \( z_{i+1} \) is replaced by \( v_{i+1} = x_\delta \), where each of the \( x_i \)'s is written as in (4). Note that the \( v_i \) terms are chosen so that the “\( a \)” terms “match up,” that is, if the last “\( a \)” letter in \( v_i \) is \( a_l \) then the first “\( a \)” letter in \( v_{i+1} \) is \( a_{l+1} \).

A general formula is \( v_i = x_{i(i-1)q+1} \) where, for \( m \) an integer, \( [m] \) is defined as follows:

\[
[m] = \mu \quad \text{if} \quad m = op + \mu, \quad 0 < \mu < p,
\]
\[
[m] = p \quad \text{if} \quad m = op.
\]

Let us examine the product \( v_1 v_2 \cdots v_{\beta} \). If all the \( b \)'s were deleted it would be
\[
(a_1 \cdots a_p)(a_1 \cdots a_p) \cdots (a_1 \cdots a_p)a_1
\]
with the group \( (a_1 \cdots a_p) \) appearing \((-\alpha)\) times. The last letter is \( a_1 \) since the product \( x^\beta = (a_1 \cdots a_q)^\beta \) contains \( \beta q \) “\( a \)” letters which we have divided into \((-\alpha)\) groups of \( p \) plus one. Now let us consider where the various \( b \) terms lie in \( v_1 \cdots v_{\beta} \). First, there is a \( b \) (with no exponent) after each \( a_p \).

Also, there are the \( b^{-1} \) and \( b^{-1-1} \) terms. Now \( b \) and \( x \) commute, \( v_i = x \) in \( \pi_1(T-K(p, q)) \), and the terms \( b^{-1} \) and \( b^{-1-1} \) appear at the end of each \( v_i \). Hence, all terms of this form may be commuted to the right-hand end of the product. Hence,
\[
x^\beta = v_1 v_2 \cdots v_{\beta} = (a_1 \cdots a_p b)^{-\alpha} a_1 b^\mu
\]
where \( b^\mu \) is the product of all the terms of the form \( b^{-1} \) or \( b^{-1-1} \).

Next we show \( \mu = -\alpha \). Now \( x^\beta \) is homologous in \( T-K(p, q) \) to \( a_1^{\alpha q} = a_1^{-\alpha p} \). But if we look at \( x^\beta = (a_1 \cdots a_p b)^{-\alpha} a_1 b^\mu \) as a word in \( \pi_1(T-K(p, q)) \) we get \( x^\beta = a_1^{\alpha p+1} b^{-\alpha+\mu} \); hence \( -\alpha + \mu = 0 \). Therefore,
\[
x^\beta = (a_1 \cdots a_p b)^{-\alpha} a_1 b^\alpha = y^{-\alpha} a_1 b^\alpha.
\]
Hence, \( a_i = y^\alpha x^\beta b^{-\alpha} \). Therefore, in case \( p < q \), a meridian of \( K(p, q) \) is \( y^\alpha x^\beta b^{-\alpha} \).

**Case 2.** \( p > q \). As in Case 1, we write each \( x_k \) in terms of \( a_1, \ldots, a_p \), and \( b \):

\[
x_k = a_k \cdots a_{k+q-1} \quad \text{if} \quad k + q - 1 \leq p,
\]

(5) \[
x_k = a_k \cdots a_p b a_1 \cdots a_{k+q-1-p} b^{-1} \quad \text{if} \quad p < k + q - 1 < p + q;
\]

\[
x_{p+1} = b a_1 \cdots a_q b^{-1}.
\]

As in Case 1, we write \( x^\beta = z_1 \cdots z_\beta \) where each \( z_i = a_1 \cdots a_q \). Again we replace each \( z_i \) in this product. \( z_1 \) is replaced by \( v_1 = x_1 = a_1 \cdots a_q \). In general,

\begin{enumerate}
\item[(a)] if \( z_i \) is replaced by \( v_i = x_k = a_k \cdots a_{k+q-1} \) with \( k + q - 1 \leq p \) then \( z_{i+1} \) is replaced by \( v_{i+1} = x_{k+q} \), written as in (5),
\item[(b)] if \( z_i \) is replaced by \( v_i = a_k \cdots a_p b a_1 \cdots a_{k+q-1-p} b^{-1} \) then \( z_{i+1} \) is replaced by \( v_{i+1} = x_{k+q-p} \), and
\item[(c)] if \( z_i \) is replaced by \( v_i = b a_1 \cdots a_q b^{-1} \) then \( z_{i+1} \) is replaced by \( v_{i+1} = x_{q+1} \).
\end{enumerate}

Then \( x^\beta = v_1 v_2 \cdots v_\beta \) and if all the \( b \)'s were removed, the product would be \((a_1 \cdots a_p)^{\alpha} a_1\). Now there is a \( b \) after every \( a_p \) and there is a \( b^{-1} \) at the end of each \( v_i \). As in Case 1, the \( b^{-1} \) terms may be commuted to the right-hand end of the product and we have

\[
x^\beta = (a_1 \cdots a_p b)^{-\alpha} a_1 b^\alpha = y^{-\alpha} a_1 b^\alpha.
\]

Hence, in either case, a meridian of \( K(p, q) \) is \( c = a_1 = y^\alpha x^\beta b^{-\alpha} \). Now \( x^p b^q = y^q \) is a longitude of \( K(p, q) \) and \( y^q \sim a_i^q b^q \) in \( T - K(p, q) \). Let \( g = c^{-p} y^q \). Then \( g \sim b^q \), so \( g \) is homologous, in \( T - K(p, q) \), to a multiple of \( L \).

Now \( G(K: r, s) \) is obtained from \( \pi_1(T - K(p, q)) \) by adding the relation \( c^r g^s = 1 \), that is, \( c^r (c^{-p} y^q)^s = 1 \). Now \( c \) is a meridian of \( K \) and \( y^q \) is a longitude; hence they commute in \( \pi_1(T - K(p, q)) \). Therefore, the added relation is \( c^r - spq y^q s = 1 \) or \( (y^\alpha x^\beta b^{-\alpha}) y^{-spq} y^q s = 1 \). In this lemma we are assuming \( r - spq = 1 \); hence \( G(K: r, s) \) has the following presentation:

\[
\{b, x, y: bx = xb, y^q = b^q x^p, y^\alpha x^\beta b^{-\alpha} y^q s = 1\}.
\]

Our next step is to eliminate \( y \) from this presentation.

The last relation is \( y^\alpha = y^{-qs} b^\alpha x^{-\beta} \) or

\[
y^\alpha = (y^q)^{-s} b^\alpha x^{-\beta} = (b^q x^p)^{-s} b^\alpha x^{-\beta} = b^{-sp} x^{-sp} b^\alpha x^{-\beta}.
\]
Hence, \( y = y^{\alpha p + \beta q} = (y^{\alpha})^p (y^{\beta})^q = (b^{\alpha - \frac{s q}{p}} x^{\frac{sp}{p}})^{\beta} (b^{\frac{sp}{q}} x^{\beta})^p = b^{1 - spq} x^{sp - p^2}. \)

Substituting for \( y \) in the second relation in (6), we obtain
\[
(b^{1 - spq} x^{sp - p^2})^{\beta} = b^{sp} x^p \quad \text{or} \quad b^{-spq}^2 = x^p(1 + spq) = x^{p r}.
\]

Substituting for \( y \) in \( y^{\alpha} = b^{\alpha - \frac{s q}{p}} x^{\frac{sp}{p} - \beta} \), which is the same as the last relation in (4), we obtain \( (b^{1 - spq} x^{sp^2})^{\alpha} = b^{\alpha - \frac{s q}{p} x^{sp - \beta}} \), which reduces to
\[
x^{\beta + sp(1 - ap)} = b^{spq (ap - 1)}, \quad \text{or} \quad x^{\beta + spq} = b^{-b_{sq}^2}, \quad \text{or} \quad x^{\beta r} = b^{-b_{sq}^2}.
\]

Hence \( G(K: r, s) \) is isomorphic to
\[
(7) \quad \{ b, x: bx = xb, x^{p r} = b^{-spq^2}, x^{\beta r} = b^{-b_{sq}^2} \}.
\]
The last two relations imply
\[
x^{r} = (x^{pr})^{\alpha} (x^{\beta r})^{q} = b^{-\alpha spq^2 - b_{sq}^3} = b^{-s q^2 + b_{spq} + b_{sq}^2} = b^{-sq^2}.
\]
But the relation \( x^{r} = b^{-sq^2} \) implies the last two relations in (5). Hence the group in (5) is isomorphic to \( \{ b, x: bx = xb, x^{p r} = b^{-spq^2} \} \).

Now if \( C \) is a simple closed curve on \( Bd T \) which is homologous (on \( Bd T \)) to \( r M + sq^2 L \) then \( C \) is homotopically nontrivial on \( Bd T \) but the element of \( G(K: r, s) \) generated by \( C \) is \( x^{r} b^{sq^2} \). Hence \( C \) is null-homotopic in \( T(K: r, s) \) so \( C \) is a meridian of \( T(K: r, s) \).

**Lemma 2(b).** If \( r = 1 \) then
\[
G_d(K(p, q): r, s) = \{ b, x: bx = xb, x^{r + spq} b^{sq^2} = 1 \}
\]
and a curve homologous to \( (r + spq)M + sq^2 L \) is a meridian of the solid torus \( T_d(K: r, s) \), where \( x \) and \( b \) are as in Lemma 2(a). If \( r = -1 \) then the conclusion is the same with \( s \) replaced by \(-s\).

This lemma follows from Lemma 2(a) and the remarks which precede Corollary 1.

**Corollary 5.** If \( K \) is a \( (p, q) \) torus knot in \( S^3 \) and \( |r - spq| = 1 \) then \( M^3(K: r, s) \) is an \( L(sq^2, r) \) lens space.

**Proof.** From [1, p. 108], a lens space \( L(sq^2, r) \) is homeomorphic to a union of two solid tori \( T_1 \) and \( T_2 \) where \( T_1 \cap T_2 = Bd T_1 = Bd T_2 \) and such that a meridian of \( T_1 \) is sewn to a curve \( C \) on \( Bd T_2 \) with \( C \sim sq^2 M_2 + rL_2 \) where \( (M_2, L_2) \) is a meridian-longitude pair for \( T_2 \).

Now the manifold \( M^3(K: r, s) \) is obtained from \( S^3 \) by removing the
interior of a regular neighborhood $T'$ of $K$ and sewing in a solid torus $T''$ such that a meridian of $T''$ is sewn to a curve on $\partial T'$ homologous to $rc + sq$ where $(c, g)$ is a meridian-longitude pair for $T'$ with $g \sim 0$ in $S^3 - K$.

Let $T$ be an unknotted solid torus in $S^3$ such that $K \subset \text{Int } T$ and $K$ co-bounds an annulus with a curve $K_1$ on $\partial T$ such that $K_1 \sim rc + sg$ on $\partial T$. Let $T_2 = S^3 - \text{Int } T$. Then the embedding of $K$ in $T$ is of the kind considered in Theorem 2; hence $M^3(K: r, s) = T(K: r, s) \cup T_2$. Since $|r - spq| = 1$, $T(K: r, s)$ is a solid torus. Also, by Lemma 2(a), a meridian of $T(K: r, s)$ is a curve on $\partial T$ homologous to $rM \pm sq^2L$ where $(M, L)$ is a meridian-longitude pair for $T(K: r, s)$.

Therefore, by the remarks of the preceding paragraph, $M^3(K: r, s)$ is the lens space $L(sq^2, r)$.

We conclude this section with two remarks concerning some results of J. Simon [7].

**Remark 1.** In Theorem 3 of [7], Simon noted the following: If $K$ is a knot in $S^3$ and $J$ is a $(1, 2)$ cable about $K$ then $\pi_1(M^3(J: 1, 1)) \cong \pi_1(M^3(K: 1, 4))$.

The results of this section may be used to conclude the following: If $T$ is a regular neighborhood of $K$ in $S^3$ such that $J \subset \text{Int } T$ and $J$ co-bounds an annulus with a $(1, 2)$ curve on $\partial T$, then

$$M^3(J: 1, 1) = T(J(1, 1)) \cup (S^3 - \text{Int } T).$$

Now in this case, $r - spq = -1$; hence $T(J: 1, 1)$ is a solid torus by Corollary 3. By Lemma 2, a meridian of $T(J: 1, 1)$ is a curve on $\partial T$ homologous to $M + 4L$ where $(M, L)$ is a meridian-longitude pair for $T$ such that $L \sim 0$ in $S^3 - \text{Int } T$. Therefore, we may conclude that the two fundamental groups mentioned above are isomorphic because the two surgery manifolds $M^3(J: 1, 1)$ and $M^3(K: 1, 4)$ are homeomorphic.

**Remark 2.** In Theorem 4 of [7], Simon noted that a simply connected 3-manifold can be obtained by nontrivial surgery on a link $K \cup J$ in $S^3$ where $K$ is a nontrivial knot and $J$ is a cable about $K$. The results of this section can be used to conclude two things in this regard: (1) Any homotopy 3-sphere obtained by this kind of surgery can also be obtained by surgery on $K$, and (2) $S^3$ can be obtained by nontrivial surgery on such a link.

First, we describe the construction in detail. Let $K$ be a knot in $S^3$ and suppose $J$ is a $(p, q)$ cable about $K$. This means there is a solid torus $T^*$ in $S^3$ with a meridian-longitude pair $(M^*, L^*)$ such that:

(i) there is a simple closed curve $J^*$ in $\text{Int } T^*$ which co-bounds an annulus with a curve $J^*_1$ on $\partial T^*$ such that $J^*_1 \sim pM^* + qL^*$, and
(ii) there is a homeomorphism \( f \) of \( T^* \) into \( S^3 \) such that \( f \) takes a core of \( T^* \) onto \( K \), \( f(J^*) = J \) and \( f(L^*) \sim 0 \) in \( S^3 - \text{Int} \, f(T^*) \).

Let \( T = f(T^*) \), \( M = f(M^*) \), \( L = f(L^*) \) and \( J_1 = f(J^*) \). Let \( T_K \) be a regular neighborhood of \( K \) in \( \text{Int} \, T \) such that \( T_K \cap J = \emptyset \). Let \( (c_K, g_K) \) be a meridian-longitude pair for \( T_K \) such that \( g_K \sim f(L) \) in \( T - K \). Let \( T_J \) be a regular neighborhood of \( J \) in \( \text{Int} \, T \) such that \( T_J \cap T_K = \emptyset \). Let \( (c_J, g_J) \) be a meridian-longitude pair for \( T_J \) such that \( g_J \sim J_1 \) in \( T - (K \cup J) \).

Now suppose \((r_1, s_1)\) and \((r_2, s_2)\) are pairs of relatively prime integers. Let \( M^3 = M^3(K; \, r_1, s_1; \, r_2, s_2) \) denote the manifold obtained by \((r_1, s_1)\)-surgery on \( K \) and \((r_2, s_2)\)-surgery on \( J \). That is \( M^3 \) is obtained from \( S^3 \) by removing \( T_K \) and \( T_J \) and sewing in solid tori \( T''_K \) and \( T''_J \) where a meridian of \( T''_K \) is sewn to a curve homologous to \( r_1 c_K + s_1 g_K \) and a meridian of \( T''_J \) is sewn to a curve homologous to \( r_2 c_J + s_2 g_J \). We consider the surgery on \( K \) first. Let \( T_1 = (T - \text{Int} \, T_K) \cup T''_K \). \( T_1 \) is a solid torus since it is a union of a solid torus, namely \( T''_K \), and a boundary collar. Also a curve \( M_1 \) on \( \text{Bd} \, T \) with \( M_1 \sim r_1 M + s_1 L \) is a meridian of \( T_1 \) and \( J \) is a nicely embedded torus knot in \( T_1 \).

Now \( r_1 \) and \( s_1 \) are relatively prime; hence there are integers \( \alpha \) and \( \beta \) such that \( \alpha r_1 + \beta s_1 = 1 \). Let \( L_1 \) be a simple closed curve on \( \text{Bd} \, T_1 = \text{Bd} \, T \) such that \( L_1 \sim \beta M - \alpha L \). We claim that \( L_1 \) is a longitude of \( T_1 \). First, \( M \sim \alpha M_1 + s_1 L_1 \) and \( L \sim \beta M_1 - r_1 L_1 \) so \( M_1 \) and \( L_1 \) generate the homology group \( H_1(\text{Bd} \, T_1) \). It remains to show that the algebraic intersection number \(#(M_1, L_1) = \pm 1 \). But \(#(M, M_1) = s_1 \) and \(#(L, M_1) = -r_1 \); hence \(#(L, M_1) = #(\beta M - \alpha L, M_1) = \beta s_1 - \alpha (-r_1) = 1 \). Hence \((M_1, L_1)\) is a meridian-longitude pair for the solid torus \( T_1 \).

Now, \( J_1 \sim pM + qL \sim (\alpha p + \beta q) M_1 + (ps_1 - qr_1) L_1 \); hence \( J \) is an \((\alpha p + \beta q, ps_1 - qr_1) \) torus knot in \( T \). Let \( p' = \alpha p + \beta q \) and \( q' = ps_1 - qr_1 \). Let \( T_2 \) denote the result of \((r_2, s_2)\) on the knot \( J \) in \( T_1 \). That is, \( T_2 = (T_1 - \text{Int} \, T_2) \cup T''_J \).

Case 1. If \( r_2 \neq \pm 1 \), then \( M^3 \) is not a homotopy 3-sphere. If \( M^3 \) were simply connected then, by Dehn's lemma, once closed complementary domain of the torus \( \text{Bd} \, T \) would be a homotopy solid torus. But one complementary domain is \( S^3 - \text{Int} \, T \) and the other is \( T_2 \), neither of which is a homotopy solid torus.

Case 2. \( r_2 = \pm 1 \). Then \( T_2 \) is a solid torus and, by Lemma 2(b), a meridian of \( T_2 \) is a curve \( M_2 \) on \( \text{Bd} \, T = \text{Bd} \, T_2 \) such that \( M_2 \sim (r_2 + s_2 p' q') M_1 + s_2 (q')^2 L_1 \). Letting \( p' = \alpha p + \beta q \) and \( q' = ps_1 - qr_1 \), \( M_1 = r_1 M + s_1 L \) and \( L_1 = \beta M - \alpha L \) we obtain, using the equation \( \alpha r_1 + \beta s_1 = 1 \),

\[
M_2 \sim (r_1 r_2 + p^2 s_1 s_2 - p q r_1 s_2) M + (r_2 s_1 + p q s_1 s_2 - q^2 r_1 s_2) L.
\]
Therefore, if \( r_2 = \pm 1 \), the surgery manifold \( M^3 \) is homeomorphic to
\[
M^3(K: r_1 r_2 + p^2 s_1 s_2 - p q r_1 s_2, r_2 s_1 + p q s_1 s_2 - q^2 r_1 s_2).
\]

Hence conclusion (1) follows from Case 1 and Case 2. Also it follows from
Case 2 that \( M^3 \approx S^3 \) if
\[
\begin{align*}
& r_2 = \pm 1, \\
& r_1 r_2 + p^2 s_1 s_2 - p q r_1 s_2 = \pm 1, \quad \text{and} \\
& r_2 s_1 + p q s_1 s_2 - q^2 r_1 s_2 = 0.
\end{align*}
\]

4. Doubly twisted knots. Before defining doubly twisted knots and con-
sidering surgery on them, we describe a method of presenting the group of knots
"with twists." Suppose \( C \) is the cube \([0, 1] \times [0, 1] \times [0, 1]\), \( A \) and \( B \)
are parallel rectilinear spanning segments of \( C \) and \( T \) is an unknotted solid
torus in \( C - (A \cup B) \). See Figure 2a. Suppose \( T \) has a longitude \( g \) which
bounds a disk \( D \) in \( C - \text{Int } T \) such that \( D \) intersects each of \( A \) and \( B \) in
a single interior point. Let \( c \) be a meridian of \( T \). Suppose \( \text{Int } T \) is removed
from \( C \) and a solid torus \( T_1 \) is sewn in with a meridian of \( T_1 \) sewn to a
simple closed curve homologous to \( c + ng \). Then there is a homeomorphism of
\( (C - \text{Int } T) \cup T_1 \) onto \( C \) which is the identity on \( \text{Bd } C \) and which takes \( A \)
and \( B \) onto arcs \( A' \) and \( B' \), as in Figure 2b, where \( A' \) and \( B' \) have 2n
crossings. We construct this homeomorphism as follows:

\begin{figure}
\centering
\includegraphics{figure2a}
\caption{Figure 2a}
\end{figure}

\begin{figure}
\centering
\includegraphics{figure2b}
\caption{Figure 2b}
\end{figure}

Let \( h \) be a homeomorphism of \( D \times [-1, 1] \) into \( C - \text{Int } T \) such that
\( h(D \times \{0\}) = D \) and \( h(\text{Bd } D \times [-1, 1]) \subset \text{Bd } T = \text{Bd } T_1 \). Let \( f \) be a
homeomorphism of \( h(D \times [-1, 1]) \) onto itself which is the identity on \( D \times \{-1, 1\} \) and which gives one end of \( h(D \times [-1, 1]) \) \( n \) full twists in the +\( g \)
direction. Let \( f \) be the identity on \( C - (h(D \times [-1, 1]) \cup \text{Int } T) \). Since \( f \)
takes a meridian of $T$ onto a curve homologous to $c + ng$, $f$ may be extended to take $C$ onto $(C - \text{Int } T) \cup T_1$. Then $f^{-1}$ is the required homeomorphism.

The action of $f^{-1}$ on the pair $(A, B)$ may be visualized as follows: $f^{-1}$ cuts the pair in the middle, gives them $n$ full twists in the $-g$ direction and pastes them back together.

The twisting homeomorphism $f^{-1}$ provides a connection between certain knot groups and groups obtained by adding a surgery relation to an appropriate link group. For example, the group of the $2n$-twist knot, Figure 3a, may be presented as the group of the link $K \cup J$ in Figure 3b with the added relation $a(y^{-1}x)^{-n} = 1$.

Theorem 3. For $(m, n) \neq (\text{even}, \pm 1)$, $T(T(m, n); r, s)$ is a homotopy solid torus only in the trivial case $(r, s) = (\pm 1, 0)$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{Figure 3a}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{Figure 3b}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{Figure 4}
\end{figure}
Note that in case \((m, n) = (\text{even}, \pm 1)\) the knot \(T(m, n)\) is a torus knot as considered in §3.

**Proof of Theorem 3.** We consider three cases: (1) \((m, n) = (\text{even}, \text{even})\),
(2) \((m, n) = (\text{odd}, \text{even})\), and (3) \((m, n) = (\text{even}, \text{odd})\) with \(n \neq \pm 1\). In each case we show that the fundamental group of \(T(T(m, n) : r, s)\) is not infinite cyclic.

**Case 1.** Both \(m\) and \(n\) are even. For a fixed even integer \(n\), the manifolds \(T(m, n), m\) even, are equivalently embedded in \(T\). Hence we consider only the case \(m = 0\). To calculate \(\pi_1(T - T(m, n))\) we consider the link in Figure 5. The group \(\pi_1(T - T(m, n))\) will be calculated as \(\pi_1(T - (K_1 \cup K_2 \cup K_3))\) with the added relation \(b(yx^{-1})^{-k} = 1\), where \(n = 2k\). The added relation comes from surgery on \(K_3\) and gives the knot the \(2k\) crossings. Hence \(\pi_1(T - T(0, n))\) has a presentation with generators \(a, b, w, x\) and \(y\) and relations:

\[
\begin{align*}
(1) \quad a^{-1}wa &= b^{-1}yb, \\
(2) \quad xax^{-1} &= waw^{-1}, \\
(3) \quad y^{-1}by &= x^{-1}bx, \\
(4) \quad y &= a^{-1}xa, \\
(5) \quad w &= b^{-1}xb, \text{ and } \\
(6) \quad b &= (yx^{-1})^k.
\end{align*}
\]

Now (3) is a consequence of (6). We add a generator \(z\) and the relation \(z = yx^{-1}\) and then we eliminate \(y\). Then, using (6), we eliminate \(b\). We obtain a presentation with generators \(a, w, x\) and \(z\) and relations:

\[
\begin{align*}
(1') \quad a^{-1}wa &= z^{1-k}xz^k, \\
(2') \quad xax^{-1} &= waw^{-1}, \\
(4') \quad zx &= a^{-1}xa, \\
(5') \quad w &= z^{-k}xz^k.
\end{align*}
\]

Now (5'), (4') and (2') imply (1'); hence \(\pi_1(T - T(0, n))\) has the presentation

\[
\{a, w, x, z : z = a^{-1}xax^{-1} = a^{-1}waw^{-1}, w = z^{-k}xz^k\}.
\]

Now a meridian of \(K_3 = T(0, n)\) is \(c = w\) and a longitude is \(g = ab^{-1}a^{-1}b = az^{-k}a^{-1}z^k\). Note that \(g \sim 0\) in \(T - T(0, n)\). Then the surgery manifold
\( T(T(0, n): r, s) \) has fundamental group \( G(T(0, n): r, s) \) which is obtained from (7) by adding the relation \( c^g = 1 \) or \( w'(az^{-k}a^{-1}z^k)^s = 1 \). If this presentation of \( G(T(m, n): r, s) \) is abelianized, we obtain the group \( \{a, w: aw = wa, w^r = 1\} \) which is not infinite cyclic unless \( r = \pm 1 \). Hence we assume \( r = -1 \). (The case \( r = 1 \) is essentially the same.) Hence we have the group

(8) \( \{a, w, x, z: z = a^{-1}xax^{-1} = a^{-1}waw^{-1}, w = z^{-k}xz^k, w = (az^{-k}a^{-1}z^k)^s\} \).

We eliminate \( w \) using the third relation and obtain relations

(9) \( z = a^{-1}xax^{-1} \),

(10) \( z = a^{-1}z^{-k}xz^kaz^{-k}x^{-1}z^k \), and

(11) \( z^{-k}xz^k = (az^{-k}a^{-1}z^k)^s \) or \( x = (z^kaz^{-k}a^{-1})^s \).

Using (11), we eliminate \( x \) and obtain:

(9') \( z = a^{-1}(z^kaz^{-k}a^{-1})^s(a^kaz^{-k}a^{-1})^{-s} = (a^{-1}z^kaz^{-k}(z^kaz^{-k}a^{-1})^{-s})^s \),

and

\( z = (z^{-k}a^{-1}z^k)^s(az^{-k}a^{-1}z^k)^{-s} \) or

(10') \( z = z^k(z^{-k}a^{-1}z^k)^s z^{-k}z^k(a^{-1}z^{-k}z^k)^{-s}z^{-k} \) or

\( z = (a^{-1}z^kaz^{-k})^s(z^kaz^{-k}a^{-1})^{-s} \).

These calculations show (9') and (10') are the same; hence \( G(T(0, n): -1, s) \) has the presentation

\( \{a, z: z = (a^{-1}z^kaz^{-k})^s(z^kaz^{-k}a^{-1})^{-s}\} \).

We obtain a quotient group by adding the relation \( a^2 = 1 \). The quotient group is \( \{a, z: a^2 = 1, z = (az^kaz^{-k})^s\} \). The second relation is \( z^{2ks+1} = (az^k)^{2s} = az^{2ks}a \) or \( az^{2ks+1} = z^{2ks}a \). Hence the quotient group is

(12) \( \{a, z: a^2 = 1, az^{2ks+1} = z^{2ks}a\} \).

Now the relations in (12) imply \( z^{2ks+1}az = az^{2ks+1}, \) or \( z^{2ks+1}az = z^{2ks}a, \) or \( zaz = a, \) or \( (az)^2 = 1 \). The relations in (12) and this last relation imply \( az^k = z^{-1} \) which implies \( az^{2ks}a = z^{-2ks} \). Using the relations in (12), we change this to \( az^{2ks}a = az^{-2ks-1}a \) or \( z^{4ks+1} = 1 \). Hence the relations in (12) imply \( a^2 = (az)^2 = z^{4ks+1} = 1 \). Now these last relations imply \( az^k = z^{-1} \) which implies \( az^{2ks}a = z^{-2ks}, \) or \( az^{2ks}a = z^{2ks+1}, \) or \( az^{2ks} = z^{2ks+1}a \).

Hence, the group (12) is the same as \( \{a, z: a^2 = (az)^2 = z^{4ks+1} = 1\} \). But this is a presentation of the dihedral group \( D_{4ks+1} \). See [5, p. 6]. Since \( |4ks + 1| \neq 1, G(T(0, n): -1, s) \) is nonabelian because it has the nonabelian quotient \( D_{4ks+1} \).
Case 2. $m$ is odd and $n$ is even. Say $n = 2k$. As in Case 1, for fixed $n$, the knots $T(m, n)$, $m$ odd, are equivalently embedded in $T$. Hence we consider only the case $m = 1$. We find a presentation for $\pi_1(T - T(1, n))$ by considering the link in Figure 6 and adding a relation corresponding to surgery on $K_3$. Hence we have a presentation with generators $a, b, w, x$ and $y$ and relations:

\begin{align*}
(13) \ a^{-1}wa &= b^{-1}yb, \\
(14) \ xax^{-1} &= waw^{-1}, \\
(15) \ y^{-1}by &= wbw^{-1}, \\
(16) \ x &= ay^{-1}, \\
(17) \ w^{-1}xw &= b^{-1}wb, \\
(18) \ b(yw)^{-k} &= 1.
\end{align*}

Now (15) is a consequence of (18). We add a generator $z$ and the relation $z = yw$ and then eliminate $y$. Next we eliminate $b$ using (18) and then $x$ using (16). We are left with generators $a, w, z$ and the following relations:

\begin{align*}
(13') \ a^{-1}wa &= z^{-1-k}w^{-1}z^k, \\
(14') \ azw^{-1}awz^{-1}a^{-1} &= waw^{-1}, \text{ and} \\
(17') \ w^{-1}azw^{-1}a^{-1}w &= z^{-k}wz^k.
\end{align*}

Now (17') is a consequence of (13') and (14'); hence $\pi_1(T - T(1, n))$ has the presentation

\begin{align*}
(19) \ \{a, w, z: a^{-1}wa = z^{-1-k}w^{-1}z^k, waw^{-1} = azw^{-1}awz^{-1}a^{-1}\}.
\end{align*}

Now a meridian of $T(1, n)$ is $c = w$ and a natural longitude, from Figure 6, is $g_1 = ab^{-1}a^{-1}wb^{-1} = az^{-k}a^{-1}wz^{-k}$. Hence a longitude which is null-homologous in $T - T(1, n)$ is $g = w^{4k-1}az^{-k}a^{-1}wz^{-k}$. Therefore, $G(T(1, n): r, s)$ is found by adding to (19) the relation $c^r g^s = 1$ or $w^{r+4ks-s}(az^{-k}a^{-1}wz^{-k})^s = 1$. Now if $G(T(1, n): r, s)$ is abelianized, we obtain the group $\{a, w: aw = wa, w^r = 1\}$; hence we may assume $|r| = 1$. Since the two cases are essentially the same, we assume $r = 1$. Then $G(T(1, n): 1, s)$ has the presentation

![Figure 6](image-url)
We obtain a quotient group by adding the relations $z^k = 1$ and $a^2 = 1$. Then we use the first relation in (20) to eliminate $z$. This yields the group

$\{a, w: a^2 = (aw)^{2k} = w^{4ks+1} = 1\}$. 

This group has the dihedral group $D_{4ks+1}$ as a quotient; hence, as in Case 1, $G(T(1, n): r, s)$ is nonabelian.

![Figure 7](image)

**Figure 7**

Case 3. $m$ is even, $n$ is odd and $n \neq \pm 1$. As in the first two cases, we consider only the case $m = 0$ and we find a presentation for $\pi_1(T - T(0, n))$ by considering the link in Figure 7 and adding a surgery relation. Since a reflection of $T$ takes $T(0, -n)$ onto $T(0, n)$, we assume $n > 0$. Hence we have a presentation with generators $a, b, u, x, y$ and $z$ and relations

(22) $uau^{-1} = y^{-1}ay$,  
(23) $x = a^{-1}ua$,  
(24) $y = azaz^{-1}$,

(25) $b^{-1}xb = z^{-1}yz$,  
(26) $z = b^{-1}ub$,  
(27) $x^{-1}bx = u^{-1}bu$,  and 

(28) $b(xu^{-1})^{-k} = 1$

where $n = 2k + 1$ (hence $k > 0$). Now (27) is a consequence of (28). We add a new generator $t$ with $t = xu^{-1}$ and then eliminate $x$. Then, using (28), we eliminate $b$, and using (24), we eliminate $y$. We are left with the following relations:

(22') $uau^{-1} = az^{-1}aza^{-1}$,

(23') $t = a^{-1}uau^{-1}$,

(25') $t^{1-k}u^{t^k} = z^{-1}aza^{-1}z$,  and 

(26') $z = t^{-k}u^{t^k}$.

Now (25') is a consequence of (22'), (23') and (26'). Also, we may eliminate
z using (26). We are left with generators $a$, $u$, and $t$ and relations

\[(22') \quad uau^{-1}at^{-k}u^{-1}t^kut^{-k}a^{-1}, \quad \text{and} \]

\[(23') \quad t = a^{-1}uau^{-1}. \]

Now, from Figure 7, a meridian of $T(0, n)$ is $c = u$ and a longitude is $g = uabz^{-1}ab^{-1} = uau^{-1}t^k at^{-k}$. Note that $g \sim a^2$, and $a$ is a longitude of the solid torus $T$. Hence, $G(T(0, n): r, s)$ has the presentation

\[
\{a, t, u: uau^{-1} = at^{-k}u^{-1}t^k ut^{-k}a^{-1}, \quad t = a^{-1}uau^{-1},
\]

\[
(29) \quad u^{r+s}(au^{-1}t^k at^{-k})^s = 1. \]

We now obtain some quotient groups. Adding the relation $t^k = 1$ and eliminating $t$ we have

\[(30) \quad \{a, u: auau^{-1} = au^{-1}aua^{-1}, (a^{-1}uau^{-1})^k = 1, \quad u^{r+s}(au^{-1}a)^s = 1\}. \]

Now the first relation is $a^{-1}ua^{-1}u = u^{-1}ua^{-1}$ which is the same as $(au^{-1})^2u = u(ua^{-1})^2$. Using this, the third relation is $u^{r+s} = (a^{-1}uau^{-1})^s$ or $u^{r+s} = [u^{-1}(ua^{-1})^2]^s$ or $u^{r+2s} = (ua^{-1})^2s$. Therefore, adding the relation $(ua^{-1})^2 = 1$ to (30), we obtain the quotient group

\[G_1 = \{a, u: (ua^{-1})^2 = (a^{-1}uau^{-1})^k = u^{r+2s} = 1\}. \]

In terms of $u$ and $d$ with $d = au^{-1}$,

\[G_1 = \{d, u: d^2 = u^{r+2s} = (u^{-1}d^{-1}ud)^k = 1\}. \]

Next we obtain another quotient group of $G(T(0, n): r, s)$ by adding the relation $t^{k+1} = 1$ to (29). After eliminating $t$, we have the group

\[(31) \quad \{a, u: (a^{-1}uau^{-1})^{k+1} = 1, a^{-1}u^{-1}auaua^{-1}u^{-1} = 1,
\]

\[u^{r+s}[u^{-1}auau^{-1}]^s = 1\}. \]

The second relation is $auau = auau$ which is the same as $(ua)^2u = u(ua)^2$. Using this last relation, we may change the third relation to $u^{r-2s}(ua)^2s = 1$. Finally we add the relation $(ua)^2 = 1$ to (31) and obtain the group

\[G_2 = \{a, u: u^{r-2s} = (ua)^2 = (a^{-1}uau^{-1})^{k+1} = 1\}. \]

In terms of $u$ and $d$ with $d = ua$,

\[G_2 = \{d, u: d^2 = u^{r-2s} = (d^{-1}udu^{-1})^{k+1} = 1\}. \]

Now the groups $G_1$ and $G_2$ are both instances of the group

\[(32) \quad \{S, T: S^l = T^2 = (S^{-1}T^{-1}ST)^p = 1\}. \]

From [3], this group is dihedral if $|l| = 2$ and $|p| > 1$, is the direct product
of $A_4$ and a cyclic group if $|l| = 3$ and $|p| = 2$ and is infinite in all other cases with $|l| > 1$ and $|p| > 1$. Hence (32) is nonabelian unless $|l| = 1$ or $|p| = 1$. Therefore, since $k$ is positive, $G_1$ is nonabelian unless $k = 1$ or $|r + 2s| = 1$ and $G_2$ is nonabelian unless $|r - 2s| = 1$. Hence, the groups $G_1$ and $G_2$ may be used to show $G(T(0, n); r, s)$ is nonabelian except for the case $k = 1$ and $|r - 2s| = 1$.

Now for the case $k = 1$, from (29), $G(T(0, n); r, s)$ is presented by

\[(33) \{a, t, u: t^2 = u^{-1}tat^{-1}uta^{-1}, t = a^{-1}uau^{-1}, u^{r+s}(a^{-1}tat^{-1})^s = 1\}.\]

Now the first relation is $u^{-1}tat^{-1} = t^2at^{-1}u^{-1}$. Substituting this into the third relation we obtain $u^r(at^2at^{-1})^s = 1$. (Recall that $u$ and $at^2at^{-1}$ commute since $u$ is a meridian and $at^2at^{-1}$ is derived from a longitude.) Now, by adding the relations $t^3 = (at^{-1})^2 = 1$, we obtain the quotient group

\[(34) \{a, t, u: r^{-1} = u^{-1}tat^{-1}uta^{-1}, t = a^{-1}uau^{-1}, u^r = t^3 = (at^{-1})^2 = 1\}.\]

Now the first relation is $t^{-1}at^{-1}u^{-1} = u^{-1}tat^{-1}$. Substituting $ta^{-1}$ for $at^{-1}$ twice ($ta^{-1} = at^{-1}$ is a consequence of $(at^{-1})^2 = 1$), the first relation in (33) becomes $t^{-1} = ua^{-1}u^{-1}a$, which is the same as the second relation in (33).

Therefore, the first relation in (33) is a consequence of the others. Then, after eliminating $t$, the group (33) becomes

\[(35) \{a, u: u^r = (a^{-1}uau^{-1})^3 = (a^{-1}uau^{-1})a^2 = 1\}.\]

If we add to this the relation $a^2 = 1$, which implies the third relation in (35), we have the group

\[G_3 = \{a, u: u^r = (a^{-1}uau^{-1})^3 = a^2 = 1\}.\]

Now $G_3$ is nonabelian unless $r = \pm 1$. Hence, the groups $G_2$ and $G_3$ may be used to show $G(T(0, 1); r, s)$ is nonabelian unless $|r| = 1$ and $|r - 2s| = 1$.

Solving simultaneously, we find we are left with just one case, $r = s = 1$. Now from (33) and the remarks just after (33), $G(T(0, 1); 1, 1)$ has the presentation

\[(36) \{a, t, u: t^2 = u^{-1}tat^{-1}uta^{-1}, t = a^{-1}uau^{-1}, u = ta^{-1}t^{-2}a^{-1}\}.\]

Eliminating $u$, we find that the remaining two relations are the same; hence (35) is equivalent to

\[\{a, t: t^2a^{-1}t^{-2}a^{-1}t^2 = a^{-1}ta^{-1}\}.\]

In terms of $t$ and $d$, with $d = a^{-1}t$, this is

\[\{d, t: t^2dt^{-3}dt^2 = d^2\}.\]

Adding the relations $t^3 = d^4 = (dt)^4 = 1$ we obtain the quotient
\[ \{d, t: d^4 = t^3 = (dt)^4 = (d^2 t)^2 = 1\}. \]

Now from §1.3 of [4], this is a presentation of the polyhedral group \((4, 4|3, 2)\) which is known to be nonabelian. This concludes the proof of Theorem 3.

**COROLLARY.** If \(K\) is a knot in \(S^3\) which has a doubly twisted knot \(T(m, n)\) as a companion, with \((m, n) \neq (\text{even}, \pm 1)\), then \(K\) has property \(P\).

**Proof.** Let \(T(m, n)\) be the doubly twisted knot in the solid torus \(T\) as shown in Figure 4. \(K\) has \(T(m, n)\) as a companion means there is an embedding \(h\) of \(T\) into \(S^3\) such that \(h(T)\) is knotted and \(h(T(m, n)) = K\). If \(K\) does not have property \(P\) then, for some pair \((r, s)\) of relatively prime integers, \(M^3(K: r, s)\) is simply connected. Hence, by Dehn’s lemma, one closed complementary domain of \(h(Bd T)\) is a homotopy solid torus. But one closed complementary domain is \(\text{Cl}(S^3 - h(T))\) which is a cube with a knotted hole and the other is \(T(T(m, n): r, s)\), neither of which is a homotopy solid torus. Hence \(K\) has property \(P\).

The class of knots considered in the corollary includes all doubled knots, which were shown, except for zero twists, to have property \(P\) by Bing and Martin [2].

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