GENERALIZED QUANTIFIERS AND COMPACT LOGIC

SAHARON SHELAH

ABSTRACT. We solve a problem of Friedman by showing the existence of a logic stronger than first-order logic even for countable models, but still satisfying the general compactness theorem, assuming e.g. the existence of a weakly compact cardinal. We also discuss several kinds of generalized quantifiers.

Introduction. We assume the reader is acquainted with Lindström's articles [Li 1] and [Li 2] where he defined "abstract logic" and showed in this framework simple characterizations of first-order logic. For example, it is the only logic satisfying the compactness theorem and the downward Löwenheim-Skolem theorem. Later this was rediscovered by Friedman [Fr 1]; and Barwise [Ba 1] dealt with characterization of infinitary languages.

Keisler asked the following question:

(1) Is there a compact logic (i.e., a logic satisfying the compactness theorem) stronger than first-order logic? It should be mentioned that it is known for many $L(Q_N^\alpha)$ that they satisfy the $\lambda$-compactness theorem for $\lambda < \aleph_\alpha$ (for $\alpha > 0$). $Q_N^\alpha(x) \iff \exists \aleph_\alpha x$'s; the $\lambda$-compactness theorem says that if $T$ is a theory in $L(Q_N^\alpha)$, $|T| \leq \lambda$, and for all finite $t \subseteq T$ there is a model, then $T$ has a model.) For example, this is the case for $\alpha = 1$. See Fuhrken [Fu 1], Keisler [Ke 2] and see [CK] for general information.

At the Cambridge summer conference of 1971 Friedman asked:

(2) Is there a logic satisfying the compactness theorem, or even the $\aleph_0$-compactness theorem, which is stronger than first-order logic even for countable models, i.e., is there a sentence $\psi$ in the logic such that there is no first order sentence $\varphi$ such that for all countable models $M$, $M \models \psi \iff M \models \varphi$?

Notice that the power quantifiers $Q_N^\alpha$ do not satisfy the second part of (2). The quantifier saying "$\varphi(x, y)$ is an ordering with cofinality $\aleph_1$" solves (1) (but obviously not (2)) as proved, in fact in [Sh 2,§4.4] and noticed by me in Cambridge.

The main result of this paper is the presentation in §1 of an example solving both (1) and (2) positively (assuming the existence of a weakly compact cardinal); thus, compactness alone does not characterize first-order logic. In §2 we mention
all kinds of problems about generalized second-order quantifiers, and prove some results.

After the solution Friedman asked:

(3) Is there a compact logic, stronger than first-order logic even for finite models?

**Notation.** $\lambda, \mu, \kappa, \chi$ designate cardinals; $i, j, k, l, \alpha, \beta, \gamma, \delta, \xi$ designate ordinals; and $m, n$ are natural numbers. The power of $A$ is $|A|$. Models are $M, N$, and the universe of $M$ is $|M|$. $a, b, c$ are elements; $\bar{a}, \bar{b}, \bar{c}$ finite sequences of elements; $l(\bar{a})$ is the length of the sequence $\bar{a}$. $x, y, z, v$ will be variables, and $\bar{x}, \bar{y}, \bar{z}, \bar{v}$ sequences of variables.

1. A compact logic different from first-order logic. The following theorem is proven under the assumption of the existence of a weakly compact cardinal (see Silver [Si 1]).

**THEOREM 1.1.** (There is a weakly compact cardinal $\kappa$.) There is a compact logic $L^*$, which is stronger than first-order logic even for countable models.

**Definition 1.1.** $\operatorname{cf}(\{4, <\})$, the cofinality of the ordering $<$ on the set $A$, is the first cardinal $\lambda$ such that there exists $B \subseteq A$, $|B| = \lambda$, $B$ is unbounded from above in $A$. $\operatorname{cf}^*(A, <)$ is $\operatorname{cf}(A, >)$, $>$ the reverse order. When $<$ is understood we just write $\operatorname{cf}(A)$ or $\operatorname{cf}^*(A)$. It is easy to see that the cofinality is a regular cardinal (or 0 or 1).

**Definition 1.2.** $(A_1, A_2)$ is a Dedekind cut of the ordered set $(A, <)$ (or just cut for short) if $A_1 \cup A_2 = A; b_1 \in A_1 \land b_2 \in A_2 \rightarrow b_1 < b_2; b < b_1 \in A_1 \rightarrow b \in A_1$.

**Definition 1.3.** Let $C$ be a class of regular cardinals. We shall define two generalized quantifiers $(Q^c_{\chi}x, y)$ and $(Q^{dc}_{\chi}x, y)$:

(A) $M \vDash (Q^c_{\chi}x, y)\varphi(x, y; \bar{a}) \iff$ the relation $x < y \equiv_\varphi \varphi(x, y; \bar{a})$ linearly orders $A = \{b \in M: M \vDash (\exists x)\varphi(x, b; \bar{a})\}$ and $\operatorname{cf}(A, <) \in C$.

(B) $M \vDash (Q^{dc}_{\chi}x, y)\varphi(x, y; \bar{a}) \iff$ the relation $x < y \equiv_\varphi \varphi(x, y; \bar{a})$ linearly orders $A = \{b \in M: M \vDash (\exists x)\varphi(x, b; \bar{a})\}$ and there is a Dedekind cut $(A_1, A_2)$ of $(A, <)$ such that $\operatorname{cf}(A_1, <), \operatorname{cf}^*(A_2, <) \in C$. Clearly the syntax of $L(Q^c_{\chi}, Q^{dc}_{\chi})$, the logic obtained by adding the two generalized quantifiers to first-order logic, is not dependent on $C$.

**Definition 1.4.** $L^* = L(Q^c_{\{\kappa_0, \kappa\}}, Q^{dc}_{\{\kappa_0, \kappa\}})$ where $\kappa$ is the first weakly compact cardinal. In the following we shall omit writing $\{\kappa_0, \kappa\}$.

**Lemma 1.2.** $L^*$ is stronger than $L$ for countable models.

**Proof.** We must find a sentence $\psi \in L^*$ for which there is no $\psi' \in L$ such that for every countable model $M, M \vDash \psi \iff M \vDash \psi'$.
Let $\psi = [\text{is a linear order}] \land [\text{every element has an immediate follower and an immediate predecessor}] \land \neg (Q^{d,c}x, y)(x < y)$.

Clearly a countable order satisfies $\psi$ iff it is isomorphic to the order of the integers. So clearly there is no sentence of $L$ equivalent to $\psi$ for countable models.

**Theorem 1.3.** $L^*$ is compact.

**Remark.** If we just wanted to prove $\lambda$-compactness for $\lambda < \kappa$, the proof would be somewhat easier.

In order to take care of the possibility that $|L| \geq \kappa$, we encode all the $m$-place relations by one relation with parameters and then we use saturativity. A similar trick was used by Chang [Ch 2] who attributes it to Vaught who attributes it [Va 1] to Chang.

We also use the technique of indiscernibles from Ehrenfeucht-Mostowski [EM]. Helling [He 1] used indiscernibles with weakly compact cardinals.

**Proof of Theorem 1.3.** Let $T$ be a theory in $L^*$ such that every finite subtheory $t \subseteq T$ has a model. We must show that $T$ has a model. Without loss of generality we may make the following assumptions.

**Assumption 1.** There is a singular cardinal $\lambda_0 > |T| + \kappa$ such that every (finite) $t \subseteq T$ has a model of power $\lambda_0$. (There is clearly a singular $\lambda_0 > \kappa + |T|$ such that every $t \subseteq T$ has a model of power $< \lambda_0$. Now let $P$ be a new one-place predicate symbol, and replace every sentence of $T$ by its relativization to $P$ (i.e. replace $(Q^{c}f, y)\varphi(x, y, z)$ by $(Q^{c}f, y)(P(x) \land P(y) \land \varphi(x, y, z))$) and replace $(Q^{d,c}x, y)\varphi(x, y, z)$ by $(Q^{d,c}x, y)(P(x) \land P(y) \land \varphi(x, y, z))$. Let $T'$ be the resulting theory. Clearly every $t \subseteq T'$ has a model of power $\lambda_0$, and $T'$ has a model iff $T$ has a model. Also $|T'| = |T|$.

**Assumption 2.** Every $t \subseteq T$ has a model $M_t$ (of power $\lambda_0$) whose universe set is $\lambda_0 = \{\alpha: \alpha < \lambda_0\}$, $<$ (the order on the ordinals) is a relation of $M_t$, $RC_{M'} = \{\mu: \mu < \lambda_0$ is a regular cardinal$, \omega$ and $\kappa$ are individual constants, and there is a pairing function.

**Assumption 3.** There is $L_a \subseteq L$, $L_a$ countable, and the only symbols in $L - L_a$ are individual constants, and $\omega, \kappa$ are in $L_a$. We can assume that $L$ has no function symbols.

Let $\{R^n_i: i < \alpha_n, n < \omega\}$ be a list of all the predicate symbols in $L$, $R^n_i$ being $n$-place. Define languages $L_0', L_1'$ as follows: $L_0' = \{\omega, \kappa, <\} \cup \{R^n: n < \omega, R^n$ is an $(n + 1)$-place predicate symbol$\}$, $L_1' = L_0' \cup \{c^n_i: i < \alpha_n, n < \omega, c^n_i$ individual constant symbol$\}$. If $\psi \in T$ define $\psi_0$ by replacing every occurrence of $R^n_i(x_1, \ldots, x_n)$ in $\psi$ by $R^n(x_1, \ldots, x_n, c^n_i)$. Let $T_0 = \{\psi_0: \psi \in T\}$, $T_0$ is a theory in $L_0'^*$ and may be taken in place of $T$. 

Let $\psi = [\text{is a linear order}] \land [\text{every element has an immediate follower and an immediate predecessor}] \land \neg (Q^{d,c}x, y)(x < y)$.

Clearly a countable order satisfies $\psi$ iff it is isomorphic to the order of the integers. So clearly there is no sentence of $L$ equivalent to $\psi$ for countable models.
Claim 1.4. For every language $L_b$ containing $<$ there is a language $L_c$ and a theory $T_c = T(L_b)$ in $L_c^*$ such that:

1. $L_b \subseteq L_c$, $|L_b| = |L_c|$.  
2. Every model $M_b$ for $L_b$ has a fixed expansion to a model $M_c$ for $L_c$ which is a model of $T_c$. 
3. Every formula in $L_c^*$ is $T_c$-equivalent to an atomic formula; i.e. for all $\varphi(x) \in L_c^*$ there is a predicate symbol $R_\varphi(x)$ such that $(\forall x)(\varphi(x) \equiv R_\varphi(x)) \in T_c$. 
4. $T_c$ has Skolem functions; i.e., for all $\varphi(y, x) \in L_c^*$ there is a function symbol $F_\varphi \in L_c^*$ such that 
   
   $$(\forall x)[(\exists y)\varphi(y, x) \equiv \varphi(F_\varphi(x), x)] \in T_c.$$ 

5. For every formula $\varphi(x, y, z) \in L_c^*$ there are function symbols $F^i_\varphi \in L_c$ (for $i = 1, \ldots, 5$) such that: if $|M_b| = \lambda_0$ (the universe set of $M_b$), $<^M_b$ is the “natural” order, then for all sequences $\bar{a}$ from $M_b$ if $\varphi(x, y, \bar{a})$ linearly orders $A = \{y \in |M_c|: M_c \models (\exists x)\varphi(x, y, \bar{a})\} \neq \emptyset$ then (in $M_c$):
   
   (i) $F^1_\varphi(\bar{a}) = cf(A, \varphi(x, y, \bar{a})).$
   
   (ii) The sequence $\langle F^2_\varphi(y, \bar{a}): y < F^1_\varphi(\bar{a}) \rangle$ is an increasing unbounded sequence in $A$.
   
   (iii) $A$ has a cut $(A_1, A_2)$ such that $cf^*(A_2, \varphi(x, y, \bar{a})) = \mu$, $cf(A_1, \varphi(x, y, \bar{a})) = \chi$ iff $F^3_\varphi(\mu, \chi, \bar{a}) = 0$ iff $F^3_\varphi(\mu, \chi, \bar{a}) \neq 1$.
   
   (iv) If $F^3_\varphi(\mu, \chi, \bar{a}) = 0$ then $\langle F^4_\varphi(y, \mu, \chi, \bar{a}): y < \chi \rangle$ is an increasing unbounded sequence in $A_1$.
   
   (v) If $F^3_\varphi(\mu, \chi, \bar{a}) = 0$ then $\langle F^5_\varphi(y, \mu, \chi, \bar{a}): y < \mu \rangle$ is a decreasing unbounded sequence in $A_2$ [where $A_1, A_2$ in (iv), (v) are from (iii)].

Proof. If in each stage we were to take $\varphi \in L_b^*$ (instead of $L_c^*$) the proof would be trivial. By repeating this process $\omega$ times we get the desired result.

Notation. Define languages $L_n$ and theories $T_n$ in $L_n^*$ as follows: $L_0 = L_a \cup \{P\}$ where $L_a$ is from Assumption 3 and $P$ is a new unary predicate symbol. If $L_n$ is defined let $L'_n = L_n \cup \{P_n, P^n\}$ where $P_n, P^n$ are new unary predicate symbols. Now $L_{n+1}, T_{n+1}$ will be $L_c$ and $T(L_b)$ from Claim 1.4 where $L'_n$ corresponds to $L_b$. Clearly $L_n$ are countable. Let $L_\omega = \bigcup L_n, T_\omega = \bigcup T_n$.

Definition 1.4. If $M$ is a model, $\Delta$ a set of formulas $\varphi(x)$ (i.e. a formula with a finite sequence of variables, including its free variables) in the language of $M$, $A \subseteq |M|$, then the sequence $\{b_i: i < \alpha\} \subseteq |M|$ is $\Delta$-indiscernible (or a sequence of $\Delta$-indiscernibles) over $A$ if $i \neq j \Rightarrow b_i \neq b_j$ and for all $\varphi(x_0, \ldots, x_{k-1}) \in \Delta, n \leq k$, permutation $\sigma$ of $\{0, \ldots, n-1\}$ and
Claim 1.5. 1. If $A$, $\Delta$, $M$ are as in Definition 1.4, $A$ and $\Delta$ are finite, and $B \subseteq |M|$ is infinite, then there are $b_i \in B$ such that \{b_i: i < \omega\} is $\Delta$-indiscernible over $A$.

2. If $A$, $\Delta$, $M$ are as in Definition 1.4, $\Delta$ is finite, $B \subseteq |M|$, $|A| < \kappa \leq |B|$, then there are $b_i \in B$ such that \{b_i: i < \kappa\} is $\Delta$-indiscernible over $A$ ($\kappa$ is the weakly compact cardinal chosen at the beginning).

Proof. 1. This is a result of the infinite Ramsey theorem. Ehrenfeucht-Mostowski [EM] used this to obtain essentially (1).

2. It is known that $\kappa$ is weakly compact iff $\kappa \rightarrow (\kappa)^\mu_m$ for all $\mu < \kappa$ (see [Si 1]). From here the result is immediate. $\square$

Let \{c_\alpha: \alpha < \alpha_T\} be all the individual constants in $L - L_a$ (see Assumption 3). Let $S = \{(t, n, B): t \subseteq T, n < \omega, B \subseteq \{c_\alpha: \alpha < \alpha_T\}, t$ and $B$ finite\}. Denote elements of $S$ by $s$ or $s_i = (t_i, n_i, B_i)$ and $s_1 \preceq s_2$ will mean $t_1 \subseteq t_2, n_1 \leq n_2, B_1 \subseteq B_2$. Now we define the $\mathcal{L}_n$-model $M(s), s = (t, n, B)$. For $t, B$ fixed, denote $M(s)$ by $M^n$. Define $M^n$ by induction on $n$ such that $M^{n+1}$ expands $M^n, M^n$ is an $\mathcal{L}_n$-model, $P_n(M^{n+1}) \subseteq \omega, P^n(M^{n+1}) \subseteq \kappa, |P^n(M^{n+1})| = \aleph_0, |P^n(M^{n+1})| = \kappa$. For $n = 0$ take $M^0$ to be the expansion $M_0$ by adding the predicate $P(M^0) = B$. Let \{\phi_i(x^i): i < \omega\} be a list of the formulas of $\mathcal{L}_n$, such that the number of variables in $x^i$ is $< i$, and let $\Delta_n = \{\phi_i: i \leq n\} \cap \mathcal{L}_n$. If $M^n$ is defined we define $M^{n+1}$ as follows: Let $A^1 \subseteq P^{n-1}(M^n)$ (or $A^1 \subseteq \{a: a < \kappa\}$ if $n = 0$) be a $\Delta_n$-indiscernible sequence over $B \cup \{a: a < \omega\}$ and let $A^2 \subseteq P^n(M^n)$ (or $A^2 \subseteq \{a: a < \omega\}$ if $n = 0$) be a $\Delta_n$-indiscernible sequence over $B \cup \{a^1, \ldots, a^n\}$, where $a^1, \ldots, a^n$ are the first $n$ elements of $A^1$. (In fact $A^1, A^2$ are sets, but we look on them as sequences by the ordering $<\)$. As for each $\phi(x) \in \Delta_n$ the number of variables in $x$ is $< n, A^2$ is $\Delta_n$-indiscernible over $B \cup A^1$. Expand $M^n$ by interpreting $P^n$ as $A^1$ and $P_n$ as $A^2$, and then expand the result to an $\mathcal{L}_{n+1}$-model by Claim 1.4, so it will be a model of $T_n$ (mentioned in the notation after Claim 1.4). This will be $M^{n+1}$. Let $L_U$ be the language obtained from $L_\omega$ by adding the individual constants \{c_\alpha: \alpha < \alpha_T\} (from $L - L_a$) and new constants $y^i, y_i$ for $i < \kappa$. Now we define a first-order theory $T_U$ in $L_U$. Let $\psi(x, \ldots, x_i; x^1, \ldots, x^m; z_1, \ldots, z_k)$ be a formula in $L_\omega$ and let $j(1) < \cdots < j(m) < \kappa, i(1) < \cdots < i(l) < \kappa$. Then
\[
\psi(y_{i(1)}, \ldots, y_{i(m)}; c_{\alpha(1)}, \ldots, c_{\alpha(k)}) \in T_U
\]

iff there is \( s_1 \in S \) such that, for all \( s \geq s_1, s = (t, n, B) \), and for all \( a_1 < \ldots < a_l \in P_n(M(s)), b_1 < \ldots < b_m \in P^2(M(s)) \), it is the case that

\[
M(s) \models \psi[a_1, \ldots, a_l; b_1, \ldots, b_m; c_{\alpha(1)}, \ldots, c_{\alpha(k)}].
\]

Clearly \( T_U \) is consistent. Let \( M \models T_U \) be \( \kappa^+ \)-saturated (see Morley and Vaught [MV] or e.g. Chang and Keisler [CK]). Let \( N \) be the submodel of \( M \) whose universe set is the closure of \( P^M \) under the functions of \( M \) (and so in particular all the individual constants are in \( N \)). Let \( D \) be a nonprincipal ultrafilter on \( \omega \), and let \( N^* = N^{\omega}/D \). We shall show that \( N^* \models T \), and thus complete the proof of the theorem. We use the fact that \( N^* \) is \( \aleph_1 \)-saturated (see e.g. [CK]).

Because of Claim 1.4(3) it is sufficient to show:

(I) If \( R_1(x, y, z) \) is an atomic formula in \( L^* \) and \( (\forall z)[(Q^c \alpha_x, y) R_1(x, y, z) = R_2(z)] \in T^* \), then for all \( a \in N^* \)

\[
N^* \models (Q^c x, y) R_1(x, y, a) \iff N^* \models R_2(a).
\]

(II) If \( R_1(x, y, z) \) is an atomic formula in \( L^* \) and \( (\forall z)[(Q^d \alpha_x, y) R_1(x, y, z) = R_2(z)] \in T^* \), then for all \( a \in N^* \)

\[
N^* \models (Q^d x, y) R_1(x, y, a) \iff N^* \models R_2(a).
\]

Proof of (I). Clearly the sets \( \{a \in N^* : a < \omega(N^*)\}, \{a \in N^* : a < \kappa(N^*)\} \)
are linearly ordered by \( < \), and both have cofinality \( \kappa \). So by the assumptions and Claim 1.4(5), \( N^* \models R_2(a) \Rightarrow N^* \models (Q^c x, y) R_1(x, y, a) \).

Now assume \( N^* \models \neg R_2(a) \) but \( N^* \models (Q^c x, y) R_1(x, y, a) \). We shall produce a contradiction. Hence \( R_1(x, y, a) \) linearly orders \( A = \{b : N^* \models (\exists x) R_1(x, b, a) \} \neq \emptyset \), and \( A \) has no last element. Since \( N^* \) is \( \aleph_1 \)-saturated, cf \( A > \aleph_0 \) and so by \( N^* \models (Q^c x, y) R_1(x, y, a) \) we have that \( \text{cf} A = \kappa \). By the assumptions and Claim 1.4(5)(ii) we may assume that \( R_1(x, y, a) = x < y \wedge y < a \) (\( a \) is one element in place of the sequence \( a \)), \( N^* \models RC[a] \), and so \( A = \{b : N^* \models b < a\} \). Let \( \{a_i\}_{i < \kappa} \) be an increasing unbounded sequence in \( A \), \( a_\kappa = a \), and suppose that \( a_i = \langle \ldots, a^n_i, \ldots \rangle \in a_i \cap N^* \) (since \( N^* = N^{\omega}/D \)).

Now for all \( \alpha < \beta < \kappa \) define \( f(\alpha, \beta) = \{n < \omega : a^n_\alpha < a^n_\beta < a^n_\kappa, RC[a^n_\kappa] \}, a^n_\kappa \neq \omega, \kappa \} \). Since \( N^* \models (a_\alpha < a_\beta < a_\kappa \wedge RC[a_\kappa] \wedge a_\kappa \neq \omega \wedge a_\kappa \neq \kappa \) we have by Łos' theorem that \( f(\alpha, \beta) \in D \). \( \kappa \), being weakly compact, satisfies \( \kappa \rightarrow (\kappa)^2 \) _\kappa_ and so without loss of generality \( f(\alpha, \beta) = \{0, 1\} \). If, for all \( n \in f(0, 1) \), there exists \( b^n \) such that \( a^n_\alpha < b^n < a^n_\kappa \) for all \( \alpha \in \kappa \), then \( b = \langle \ldots, b^n, \ldots \rangle / D \in N^* \) and \( a_\alpha < b < a \) for all \( \alpha < \kappa \), a contradiction.
So there is \( n \in f(0, 1) \) for which \( \{a^n_\alpha: \alpha < \kappa\} \) is an (increasing) unbounded sequence in \( \{b \in N: b < a^n\} \) and \( N \models RC[a^n] \land a^n \neq \omega \land a^n \neq \kappa \). From now on denote \( a = a^n, a_\alpha = a^n_\alpha \). Let \( a_\alpha = \tau_\alpha(\cdots, y^{(\alpha, m)}, \cdots, \cdots, y_{i(\alpha, l)}, \cdots; \bar{b}_\alpha)_{i<l(\alpha), m<m(\alpha)}, \) where \( \tau_\alpha \) is a term, in \( L_\omega \), \( j(\alpha, m) \) is an increasing sequence in \( m, i(\alpha, l) \) is an increasing sequence in \( l, \) and \( \bar{b}_\alpha \) is a sequence from \( P^N \). Since we may replace \( \{a_\alpha: \alpha < \kappa\} \) by any subset of the same power, we may assume that \( m(\alpha) = m_0, l(\alpha) = l_0, \) and \( \tau_\alpha = \tau \) for all \( \alpha < \kappa \).

Since \( N \models RC[a] \land a > \omega \) and in every \( M(s) \) the interpretation of \( P \) is a finite set, and \( \{b: b < \omega\} \) is a countable set, there is a function symbol \( F \) in \( L_\omega \) such that

\[
F(x^0, \ldots, x^{m_0-1}, x) = \sup \{r(x^0, \ldots, x^{m_0-1}; z_0, \ldots, z_{l_0-1}, v_1, \ldots) < x: z_0, \ldots, < \omega, v_1, \ldots, \in P\}.
\]

Clearly \( \tau(\cdots, y^{(\alpha, m)}, \cdots, \cdots, y_{i(\alpha, l)}, \cdots; \bar{b}_\alpha) < F(\cdots, y^{(\alpha, m)}, \cdots, a) < a, \) and thus without loss of generality \( a_\alpha = F(\cdots, y^{(\alpha, m)}, \cdots, a) \). If \( N \models a < \kappa \) then \( N \) satisfies the sentence "saying:" there is a regular cardinal \( a < \kappa \) such that \( X_\kappa \) is an unbounded subset of \( \{c: c < a\} \), but \( X_b \) is a bounded subset of \( \{c: c < a\} \) for any \( b < \kappa \); where \( X_b = \{F(\cdots, x, \cdots, a) < a: x < b\} \). Hence, for some \( s, M(s) \) satisfies it, contradicting the fact that \( cf \kappa = \kappa \). If \( N \models a > \kappa, \) as we get \( F \) we can get \( F' \) such that \( a_\alpha < F'(a) < a \) for every \( \alpha, \) a contradiction.

**Proof of (ii).** As in the proof of (i) it is clear by Claim 1.4 that \( N^* \models R_2(\bar{a}) \Rightarrow N^* \models (Q^d x, y)R_1(x, y, \bar{a}). \)

Now assuming \( N^* \models (Q^d x, y)R_1(x, y, \bar{a}) \land \neg R_2(\bar{a}) \) we shall arrive at a contradiction. We can restrict ourselves to the case where \( x < y \equiv_{def} R_1(x, y; \bar{a}) \) linearly orders \( A = \{b \in N^*: (\exists x)R_1(x, b, \bar{a})\} \neq \varnothing, A \) has no last element.

Since there are pairing functions we may replace \( \bar{a} \) by \( a \). By hypothesis \( A \) has a Dedekind cut \( (A_1, A_2) \) such that \( cf A_1, cf A_2 \in \{\omega, \kappa\} \).

**Case 1.** \( cf A_1 = cf A_2 = \omega: \) This contradicts the \( \aleph_1 \)-saturation of \( N^* \).

**Case 2.** \( cf A_1 = \omega, cf A_2 = \kappa: \) Let \( \{b_m\}_m < \omega \) be an increasing unbounded sequence in \( A_1 \), let \( \{a_\alpha\}_\alpha < \kappa \) be a decreasing unbounded sequence in \( A_2 \), where \( b_m = (\cdots, b^{n}_m, \cdots)_n < \omega/\kappa, a_\alpha = (\cdots, a^{n}_\alpha, \cdots)_n < \omega/\kappa. \)

For all \( \alpha < \kappa \) define \( f_1(\alpha) = (n < \omega: b^{n}_m < a^{n}_{\alpha}): m < \omega. \) Since the range of \( f_1 \) is a set of power \( \leq 2^{\aleph_0} \) we can assume that \( f_1 \) is constant. Let \( T_m = \{n < \omega: b^{n}_m < a^{n}_{\alpha}\}; \) clearly \( T_m \in D \). Let \( R \) be a new one-place predicate symbol, \( R^n = \{b^n_m: n \in T_m\} \), and \( (N^*, R) = \Pi_{n<\omega}(N, R^n)/\kappa. \) Clearly \( \{b_m: m < \omega\} \subseteq R \cap A \) and \( \langle R \cap A, <^* \rangle \) is an \( \aleph_1 \)-saturated model of the
theory of order, and so it contains an upper bound to the $b_m$'s, and also $b <^* a_\alpha$ for all $b \in R \cap A$, $\alpha < \kappa$. This is a contradiction.

**Case 3.** cf $A_1 = \kappa$, cf* $A_2 = \omega$: The proof is similar to the proof of Case 2.

**Case 4.** cf $A_1 = \text{cf}^* A_2 = \kappa$; Let $\{a_\alpha\}_{\alpha < \kappa}$, $\{b_\alpha\}_{\alpha < \kappa}$ be an increasing (decreasing) unbounded sequence in $A_1$ ($A_2$), where $a_\alpha = (\cdots, a_\alpha^n, \cdots)_{n \in \omega \cup D}$, $b_\alpha = (\cdots, b_\alpha^n, \cdots)_{n \in \omega \cup D}$.

As in (I) we can assume that for all $\alpha < \beta < \kappa$ the following sets are not dependent on the particular $\alpha$ or $\beta$:

$$J_1 = \{n < \omega: a_\alpha^n < a_\beta^n\}, \quad J_2 = \{n < \omega: a_\alpha^n < b_\beta^n\}, \quad J_3 = \{n < \omega: b_\alpha^n < b_\beta^n\}.$$  

Also $J_i \in D_i$ and $J_0 = \{n < \omega: N \vDash \exists R_2[a^n]\} \in D_\varrho$, where $a = (\cdots, a^n, \cdots)$. Thus as in (I), for some $n \in \bigcap J_\varrho R_1(x, y, a^n)$ linearly orders

$$A = \{y \in N: (\exists x)R_1(x, y, a^n)\} \supseteq \{a_\alpha^n, b_\alpha^n: \alpha < \kappa\}$$

and, for no $c \in A$, $a_\alpha^n < c < b_\alpha^n$. So by renaming,

(*) There is $a \in N$, $N \vDash \exists R_2(a^\varrho) \in D$, where $a = (\cdots, a^n, \cdots)$. Thus as in (I), for some $n \in \bigcap J_\varrho R_1(x, y, a^n)$ linearly orders

$$A = \{y \in N: (\exists x)R_1(x, y, a^n)\} \supseteq \{a_\alpha^n, b_\alpha^n: \alpha < \kappa\}$$

and, for no $c \in A$, $a_\alpha^n < c < b_\alpha^n$. So by renaming,

(1) $\tau_\alpha = \tau_0$, $l(\alpha) = l(0)$, $m(\alpha) = m(0)$.

(2) For every formula $\varphi(x^1, x^2, x^3) \in L_\infty$ the truth value of $\varphi(\bar{a}_\alpha, \bar{d}_\alpha, \bar{d})$ is the same for all $\alpha < \beta < \kappa$.

(3) There is $l_1 < l(0)$ such that for every $\alpha < \beta < \kappa$

$$y^j(\alpha, 0) = y^j(\beta, 0) < y^j(\alpha, 1) = y^j(\beta, 1) < \cdots < y^j(\alpha, l_1-1) = y^j(\beta, l_1-1)$$

$$< y^j(\alpha, l_1) < y^j(\alpha, l_1+1) < \cdots < y^j(\alpha, l(0)-1) < y^j(\beta, l_1)$$

$$< \cdots < y^j(\beta, l(0)-1)$$

and $y^k(l) < y^j(\alpha, l_1)$ for any $l$. Denote for $l < l_1$ $y^j(l) = y^j(\alpha, l)$,

$$\bar{y}^* = (y^j(0), \cdots, y^j(l-1), \cdots, y^j(l), \cdots),$$

$$\bar{y}^\alpha = (y^j(\alpha, l_1), \cdots, y^j(\alpha, l(0)-1)).$$
(4) Similar to (3) for the \( y_{l(a,m)} \), we get \( \bar{y}_\alpha \) and \( \bar{y}_\beta \). Thus \( a_\alpha = r_0(\bar{y}_\star, \bar{y}_\alpha, \bar{y}_\beta, \bar{y}_\gamma, \bar{d}_\alpha) \), \( a = r^*(\bar{y}_\star, \bar{y}_\star, \bar{d}) \). By treating the \( b_\alpha \) similarly and making some change in \( \bar{y}_\star, \bar{y}_\alpha, \bar{d}_\alpha \) we may assume

(5) \( b_\alpha = r^0(\bar{y}_\star, \alpha, \bar{y}_\alpha, \bar{y}_\beta, \bar{d}^\alpha) \), and if \( \alpha < \beta \) then every element of \( \alpha_\bar{y} \) comes before every element of \( \beta_\bar{y} \) (in the sequence \( \{y_i^l : i < \kappa \} \)), and after every element of \( \bar{y}_\star \). Similarly for \( \alpha_\bar{y} \). (Of course \( \bar{d}^\alpha \) is a sequence from \( PM; \bar{y}_\star, \alpha_\bar{y} \) from \( \{y_i^l : i < \kappa \} \) and \( \bar{y}_\star, \alpha_\bar{y} \) from \( \{y_i^l : i < \kappa \} \).

(6) As a strengthening of (2), for all \( \varphi(x_1, x_2, x_3) \in L_\alpha \) and all \( \alpha, \beta \) the truth values of \( \varphi(\bar{d}^\alpha, \bar{d}^\beta, \bar{d}) \), \( \varphi(\bar{d}_\alpha, \bar{d}_\beta, \bar{d}) \), and \( \varphi(\bar{d}_\alpha, \bar{d}_\beta, \bar{d}) \) are dependent only on the order between \( \alpha \) and \( \beta \).

**Notation.** \( a_{\alpha, \beta, \gamma} = r_0(\bar{y}_\star, \alpha_\bar{y}, \beta_\bar{y}, \gamma_\bar{y}, \bar{d}) \), \( b_{\alpha, \beta, \gamma} = r^0(\bar{y}_\star, \alpha, \beta_\bar{y}, \gamma_\bar{y}, \bar{d}^\alpha) \).

Notice that by the indiscernibility of the \( y \)'s and (6), \( a_{\alpha, \beta, \gamma}, b_{\alpha, \beta, \gamma} \in A \) and the order between \( a_{\alpha, \beta, \gamma} \) and \( a_{\alpha(1), \beta(1), \gamma(1)} \) depends only on the order between \( \alpha \) and \( \alpha(1) \), the order between \( \beta \) and \( \beta(1) \), and the order between \( \gamma \) and \( \gamma(1) \); and similarly for the \( b_{\alpha, \beta, \gamma} \).

Now for every \( \alpha, \beta, \gamma, \delta < \kappa \) choose \( e, \alpha, \beta, \gamma, \delta < \kappa \). So \( a_\alpha < b_e \Rightarrow a_{\alpha, \alpha, \epsilon} < b_e \Rightarrow a_{\alpha, \beta, \gamma} < b_e \Rightarrow a_{\alpha, \beta, \gamma} < b_\delta \), and hence every \( a_{\alpha, \beta, \gamma} \in A_1 \).

Similarly \( b_{\alpha, \beta, \gamma} \in A_2 \).

If \( a_{0,0,1} < a_{1,1,0} \) then \( \alpha < \alpha(1), \beta > \beta(1) \) imply \( a_{\alpha, \alpha, \beta} < a_{\alpha(1), \alpha(1), \beta(1)} \).

So for all \( \alpha > 0, a_{\alpha, \alpha, \alpha} < a_{\alpha(1), \alpha(1), \alpha(1)}, \) and so \( \{a_{\alpha, \alpha, \alpha} : \alpha < \kappa \} \) is an unbounded subset of \( A_1 \). Similarly, if \( a_{0,0,1} < a_{1,1,0} \) and \( a_{1,2,0} < a_{2,1,0} \) then \( \{a_{0,0,1} : \alpha < \kappa \} \) is unbounded in \( A_1 \), if \( a_{0,0,1} < a_{1,1,0} \) and \( a_{1,2,0} > a_{2,1,0} \) then \( \{a_{0,0,1} : \alpha < \kappa \} \) is unbounded in \( A_1 \), and if \( a_{0,0,1} > a_{1,1,0} \) then \( \{a_{0,0,1} : \alpha < \kappa \} \) is unbounded in \( A_1 \). A parallel claim is true for the \( b \)'s.

So we may change \( r_0 \) and \( r^0 \) such that \( a_{\alpha, \beta, \gamma} \) and \( b_{\alpha, \beta, \gamma} \) will each be dependent only on one index. (If \( a_{\alpha, \beta, \gamma} \) is not dependent on \( \alpha \), then \( \bar{y}_\star \) is empty; if not dependent on \( \beta \), \( \bar{y}_\beta \) is empty, and if not dependent on \( \gamma \), \( \bar{d}_\gamma \) is constant.)

There are, in all, nine possibilities.

We shall now show that there cannot be dependence on \( \gamma \) alone. Assume without loss of generality that \( a_\alpha = r_0(\bar{y}, \bar{d}_\gamma) \) where \( \bar{y} \) is the concatenation of all sequences from \( \{y_i^l : i < \kappa \} \) which are not dependent on \( \gamma \). Consider the following type in the variables \( x_\nu, i < l = l(\bar{d}_\gamma) \): (let \( \bar{x} = (x_1, \ldots, x_l) \):

\[
\{P(x_i) : i < l \} \cup \{ (\exists x) R_1(x, \tau_0(\bar{y}, \bar{x}, i), a) \} \cup \{ \tau_0(\bar{y}, \bar{x}) < b_\alpha : \alpha < \kappa \} \cup \{ a_\alpha < \tau_0(\bar{y}, \bar{x}) : \alpha < \kappa \}\)

This type, containing parameters from \( N \), is finitely satisfiable in \( N \) and thus in \( M \) since \( N \) is an elementary submodel of \( M \). Thus it is satisfiable by \( \bar{c} = (c_1, \ldots, c_l) \) in \( M \), since \( M \) is \( \kappa^+ \)-saturated. But \( c_i \in N \) since \( c_i \in PM \) and thus the type is satisfiable in \( N \). This contradicts the definition of the \( a_\alpha, b_\alpha \).

We are left with four cases. Without loss of generality we shall deal only
with the case \( a_\alpha = \tau_0(\bar{y}^*, \bar{y}^\alpha, \bar{y}^*_\alpha, \bar{d}) \), \( b_\alpha = \tau_0(\bar{y}^*, \bar{y}^*_\alpha, \bar{y}, \bar{d}) \). Without loss of generality all the above sequences are of equal length, and it will be recalled that the sequences of the \( y \)'s here are increasing sequences, \( \bar{y}^* < \bar{y}^\alpha, \bar{y}^*_\alpha < \bar{y} \) (i.e., every element in the left sequence is smaller than every element in the matching right sequence).

For every sentence \( \psi \) which \( N \) satisfies and \( s_1 \in S \) there is \( s \geq s_1 \) such that \( M(s) \) satisfies \( \psi \). Hence there are \( s \in S \), and a sequence \( \bar{d} \in P[M(s)] \) where \( s = (t, n, B) \) such that \( n > 1000l(\bar{y}^*) \) and \( n \) is big enough so that all the formulas we shall need are in \( \Delta_{n-3} \) and (remembering the indiscernibility in the definition of \( P_{n-2}[M(s)] \), \( P_{n-2}[M(s)] \)).

\((**)\) If \( \bar{c}^* < \bar{c}^1 < \bar{c}^2 \) are increasing sequences from \( P_{n-2}[M(s)] \) and \( \bar{c}^*_2 < \bar{c} < \bar{c}^* \) are increasing sequences from \( P_{n-2}[M(s)] \), and \( l(\bar{c}^*) = l(\bar{y}^*) \), \( l(\bar{c}^2) = l(\bar{c}^1) = l(\bar{y}^1), l(\bar{c}^*_2) = l(\bar{y}^*_2), l(\bar{c}) = l(\bar{c}^2) = l(\bar{y}) \) then

(A) \( M(s) \models \neg R_2[\tau(\bar{c}^*, \bar{c}^*_2, \bar{d}')] \), \( R_1(x, y, \tau(\bar{c}^*, \bar{c}^*, \bar{d}')) \) is a linear order \(<^* \) (nonempty) without a last element on a set \( A_s \).

(B) In \( M(s) \) the following holds:

\[
\tau_0(\bar{c}^*, \bar{c}^1, \bar{c}^*_2, \bar{d}') <^* \tau_0(\bar{c}^*, \bar{c}^2, \bar{c}^*_2, \bar{d}') <^* \tau_0(\bar{c}^*, \bar{c}^*_2, \bar{d})
\]

Define \( A^1_s = \{ b \in A_s : \text{there is } \bar{c}^0 > \bar{c}^* \text{ such that } b <^* \tau_0(\bar{c}^*, \bar{c}^0, \bar{c}^*_2, \bar{d}') \} \) and \( A^2_s = \{ b \in A_s : \text{there is } \bar{c} < \bar{c}^*_2 \text{ such that } \tau_0(\bar{c}^*, \bar{c}^*_2, \bar{c}, \bar{d}') <^* b \} \).

Clearly \( A^1_s \cap A^2_s = \emptyset \), cf \( A^1_s = \kappa \), cf \( A^2_s = \omega \), but from \( M(s) \models \neg R_2[\tau(\bar{c}^*, \bar{c}^*_2, \bar{d}')] \) and by the definition of \( R_2 \) it follows that \( M(s) \models \neg (Q^d_c x, y) R_1[x, y, \tau(\bar{c}^*, \bar{c}, \bar{d}')] \).

Thus there is \( b \in A_s, A^1_s < b < A^2_s \). But \( A_s, A^1_s, A^2_s \) are definable by the formulas \( \varphi(x, \bar{c}^*, \bar{c}, \bar{d}'), \varphi^1(x, \bar{c}^*, \bar{c}^*_2, \bar{d}'), \varphi^2(x, \bar{c}^*, \bar{c}^*_2, \bar{d}'), \) where \( \varphi, \varphi^1, \varphi^2 \in L_n \).

Now by 1.4 there is a function symbol \( F \) in \( L_{n+1} \) such that for all \( s_1 \) such that \( n_1 > n \) the following sentence holds in \( M(s_1) \) (abusing our notation the free variables are \( \bar{y}^*, \bar{y}^*, z \))

If \( \neg R_2(\tau(\bar{y}^*, \bar{y}^*, \bar{z})); \) and \( R_1(x, y, \tau(\bar{y}^*, \bar{y}^*, \bar{z})) \) defines a linear order on \( A = \{ v : (\exists x) R_1(x, v, \bar{z}) \} \); \( \bar{y}^*_2 (\bar{y}^*) \) is a sequence of elements \( < \omega (\leq \kappa) \); and for all \( \bar{y}^* < \bar{y}^1 < \bar{y}^2\) such that the elements of \( \bar{y}^1, \bar{y}^2 \) are in \( P^n \), and for all \( \bar{y}^* < \bar{y}_1 < \bar{y}_2 \) such that the elements of \( \bar{y}_1, \bar{y}_2 \), are in \( P^n \), it is true that

\[
\tau_0(\bar{y}^*, \bar{y}^1, \bar{y}^*_2, \bar{z}) <^* \tau_0(\bar{y}^*, \bar{y}^2, \bar{y}^*_2, \bar{z})
\]

\[
<^* \tau_0(\bar{y}^1, \bar{y}^*, \bar{y}^2, \bar{z}) <^* \tau_0(\bar{y}^*, \bar{y}, \bar{y}^*, \bar{z}) \in A
\]

where \( x <^* y \equiv R_1(x, y, \tau(\bar{y}^*, \bar{y}^*, \bar{z})), then F(\bar{y}^*, \bar{y}^*, \bar{z}) \in A \) and for all \( \bar{y}^1, \bar{y}^* \) as above
Thus $M$, and $N$, satisfy the above sentence (because of the suitable indiscernibility of $P^m, P^n$). Thus $F(y^*, y^*, d) \in A$, $a_\alpha < F(y^*, y^*, d) < b_\alpha$, a contradiction. This concludes the proof of Theorem 1.3 and of Theorem 1.1.

2. Discussion. **More on $L^*$**. Some natural problems are:

**Problem 2.1.**
A. In Theorem 1.2, is the condition that $\kappa$ be weakly compact necessary?
B. Give $L^*$ a “nice” axiomatization.

In Theorem 1.2 we prove actually:

**Theorem 2.2.**
A. $L^*$ satisfies the completeness theorem; that is, for every sentence $\psi \in L^*$ we can find (recursively) a recursive set $\Gamma$ of first-order sentences (or even a single sentence) in a richer language such that $\psi$ has a model iff $\Gamma$ has a model.
B. Every $L$-model has $L^*$-elementary extensions of arbitrary large power.

Clearly $L^*$ is interpretable in $L^+_{\kappa, \kappa}$ (the language with conjunction on $\kappa$ formulas and quantification on $\kappa$ variables), and by Hanf [Ha 1] every $L$-model has an $L^+_{\kappa, \kappa}$-elementary submodel of power $\leq |L|^\kappa$. Thus

**Theorem 2.3.**
A. If $|L| \leq \lambda = \lambda^\kappa$, then every $L$-model of power $\geq \lambda^\kappa$ has an $L^*$-elementary submodel of power $\lambda^\kappa$. (If $|L| \leq \kappa$ we can choose $\lambda = 2^\kappa$.)
B. There is a sentence in $L^*$ (having a model) whose models are of power $\geq 2^{\aleph_0}$. There is a consistent theory in $L^*$ of power $\kappa$ whose models are of power $\geq 2^\kappa$. (1)
C. Every consistent theory in $L^*$ of power $\leq \kappa$ has a model of power $\leq \kappa$.

**Proof.** A has already been proved.
B is proved by the sentence “$<$ is a linear order, in which every element has immediate predecessor and successor; $\neg(Q_{dc} x, y)(x < y)$; $P$ is a nonempty convex subset, bounded from above and below, which has no first or last element.” Every model of this sentence is of power $\geq 2^{\aleph_0}$.

Let $T$ be the following theory:

1. “$<$ is a linear order and $\neg(Q_{dc} x, y)x < y$”.
2. “$c_i < c_j$ for all $i < j \in J$”, where $J$ is a dense $\kappa$-saturated order of power $\kappa$.

Clearly $T$ is consistent. Now let $M \models T$ and let $(J_1, J_2)$ be a cut of $J$, $\text{cf} J_1 = \text{cf}^* J_2 = \kappa$. So there is an element $a \in M$, $a_i < a < a_j$, for all $i \in J_1$, $j \in J_2$. Thus $\|M\| \geq 2^\kappa$. This completes the proof of B.

(1) We can improve 2.3B, i.e. there is $\varphi \in L$ which has models only in cardinalities $\geq \kappa$; see 2.24.
C is proved like 1.3, but we do not need the $P$.

Elimination of the assumption of the existence of a weakly compact cardinal.

In place of a weakly compact cardinal we can assume:

(*) There is a proper class of regular cardinals, $C_1$, such that for all $\lambda \in C_1$ there are $\{S_\alpha : \alpha < \lambda^+ \text{, cf} \alpha = \lambda\}$ such that for all $S \subseteq \lambda^+$, $\{\alpha < \lambda^+ : \text{cf} \alpha = \lambda, S \cap \alpha = S_\alpha\}$ is a stationary set of $\lambda^+$.

By Jensen and Kunen [JK, §2, Theorem 1] the class of regular cardinals satisfies (*), if $V = L$.

If (*) holds we can choose $C$ such that $\aleph_0 \in C$ ($\lambda \in C \Rightarrow \lambda^+ \in C$), and $C - \{\aleph_0\}$ satisfies (*).

**Theorem 2.4.** If $C$ and (*) are as above, then $L^* = L(Q_{\lambda}^c, Q_{\lambda}^{dc})$ satisfies the compactness theorem.

**Proof.** The proof is a combination of Keisler [Ke 3, §2] and Chang [Ch 2]. We assume $T$ satisfies the conditions of 1.4, and every finite subtheory has a model. Choose $\lambda \in C$, $\lambda \geq |T|$ (or even $\lambda \geq |T|$). By (*) clearly $\lambda^+ = \lambda$. Now we define an increasing elementary sequence of $\lambda$-saturated models $\{M_\alpha\}_{\alpha < \lambda^+}$, such that for $\alpha < \beta, M_\beta$ is an end extension of $M_\alpha$, and $M = \bigcup M_\alpha$. Also, if $\alpha \in RC^M$ then

$$M \models (Q_{\lambda}^c x, y)(x < y < \alpha) \iff \lambda = \text{cf} \{b \in M : b < \alpha\}$$

and if $(A_1, A_2)$ is a cut of an order in $M$ which is definable (2) (in $M$ by a formula with parameters) such that $\text{cf} A_1 = \lambda^+$ or $\text{cf}^* A_2 = \lambda^+$ then $A_1$ is also definable (in $M$ by a formula with parameters). Clearly $M \models T$. □

Cofinality quantifiers. We shall deal with logics containing just the generalized quantifier $Q_{\leq \alpha}^c$. We write $Q_{\leq \alpha}^c$ in place of $Q_{\lambda}^c$.

**Theorem 2.5.** Let $M$ be an $L$-model of power $\geq \kappa$. Then $M$ has an $L^\kappa$-elementary submodel of power $\kappa$ where $L^\kappa = L(Q_{\leq \alpha}^c, Q_{\leq \alpha}^{df})_{\lambda < \mu, \mu < \kappa}$ if

1. $\lambda < \kappa$, $|L| + \mu < \kappa$,
2. for every $i < n$ there are regular cardinals $\chi_i^l < \cdots < \chi_m^l$ such that if for every $i \chi < \chi_i^l \iff \chi_i^l < \chi_m^l$ then $\chi \in C_i \iff \chi^l \in C_i$; and
3. for all regular $\lambda$ there is a regular $\lambda' < \kappa$ such that $\lambda' \neq \lambda_j$ for all $j$ and $\lambda \in C_i \iff \lambda' \in C_i$.

**Proof.** The proof is by induction on $\lambda = \lVert M \rVert$. As in §1 we can assume that $|M|$ is an ordinal, say $\lambda + 1$, $M$ is the order on the ordinals, $RC^M$ is the set of regular cardinals in $M$, $M$ has Skolem functions, and also cofinality

(2) We assume the order is definable.
Skolem functions (see 1.4(5)). Thus in order that a submodel $N$ of $M$ be an $L^{**}$-elementary submodel; for all $a \in RC^N$ we must have

$$M \vDash (Q_{\alpha_i}^{cf} x, y)(x < y < a) \iff N \vDash (Q_{\alpha_i}^{cf} x, y)(x < y < a),$$

$$M \vDash (Q_{\xi_i}^{cf} x, y)(x < y < a) \iff N \vDash (Q_{\xi_i}^{cf} x, y)(x < y < a).$$

**Case 1.** $\lambda$ is a regular cardinal: Choose regular $\lambda' < \lambda$, $\lambda' \neq \lambda_j$ for all $j$, and $\lambda \in C_i \iff \lambda' \in C_i$. Build an increasing sequence $\{M_{\alpha}\}_{\alpha \in \lambda'}$ of elementary submodels of $M$ such that

1. $M_{\alpha} \subseteq M_{\alpha+1}$, $M_{\delta} = \bigcup_{\alpha < \delta} M_{\alpha}$ for $\delta$ a limit ordinal, $\|M_0\| \geq \kappa$.
2. $|M_{\alpha}|$ is an initial segment of $\lambda$ with the addition of $\lambda$ (which is the last element of $M$). $M_{\lambda'}$ will be the desired model.

**Case 2.** $\lambda$ is singular. Choose regular $\chi < \lambda$ such that $\lambda < \chi' \iff \chi < \chi'$. There is such a $\chi$ since the number of $\chi'$ is finite and they are regular thus $\neq \lambda$, and $\lambda$ is a limit cardinal. Let $M_0$ be an elementary submodel of $M$ of power $\chi' = \chi^+ + \text{cf} \lambda$ which contains $\{\alpha: \alpha \leq \chi\}' \cup \{\lambda\}$. Define by induction on $\alpha \leq \chi^+$ an increasing sequence of elementary submodels of $M$, $\{M_{\alpha}\}_{\alpha \leq \chi^+}$, such that $\|M_{\alpha}\| = \chi'$, $M_{\delta} = \bigcup_{i < \delta} M_i$ for $\delta$ a limit ordinal, and if $a \in RC^M$, $\chi < a$, then there is $a' < a$, $a' \in M_{\alpha+1}$, such that for every $b < a$ if $b \in M_\alpha$, then $b < a'$. Clearly if $a \in RC^M \cap |M_{\chi'}|$ then the cofinality of $\{b \in M_{\chi'}: b < a\}$ is either $\chi^+$ or the cofinality of $\{b \in M: b < a\}$. Thus $M_{\chi'}$ is an $L^{**}$-elementary submodel of $M$.

We may assume now that in the definition of $L^{**}$ the $C_i$ are pairwise disjoint.

**Theorem 2.6.** Assume $\mu < \aleph_0$ in the definition of $L^{**}$ in 2.5.

(A) $L^{**}$ satisfies the completeness theorem and the compactness theorem (and thus the upward Lowenheim-Skolem theorem).

(B) Let $T$ be a theory in $L(Q_{C_i}^{cf}, Q_{\lambda_i}^{cf})$. By substituting $\lambda'_i$ for $\lambda_i$ and $C'_i$ for $C_i$ we get a theory $T'$. $T$ has a model iff $T'$ has a model, on condition that:

1. $\lambda_{i_1} = \lambda_{i_2} \iff \lambda'_{i_1} = \lambda'_{i_2}$.
2. $\lambda_i \in C_i \iff \lambda'_i \in C'_i$.
3. if $C_i = \{\lambda_i: l < l_0\}$ then $C'_i = \{\lambda'_i: l < l_0\}$.

**Remark.** In the completeness theorem we consider a single sentence and the set of quantifiers appearing in it, so there is no need for $\mu < \aleph_0$.

**Sketch of proof.** Let $T$ be a theory in $L^{**}$. Without loss of generality $T$ has Skolem functions, there is a symbol $<$ which is an order on the universe, $RC$ is a unary predicate, there are cofinality Skolem functions (see 1.4(5)), and every formula is equivalent to an atomic formula. By adding cofinality quantifiers
we can assume that $L^{**} = L(Q^{cf}_{C_i})_{i<n}$ where the $C_i$ are disjoint intervals of regular cardinals, $C_n = \{\lambda : \lambda_0 \leq \lambda, \text{ regular}\}; \bigcup_i C_i$ is all the regular cardinals.

By using the previous theorem and the set of sentences from Shelah [Sh 2, §4], we get: if every finite $t \subseteq T$ has a model, then $T \cap L$ has a model $M$ for which if $(\forall x)[R^i(z) \equiv (Q^{cf}_{C_i} x, y)(x < y < z)] \in T$ and $M \models R^i(z) \land RC[z]$, then $cf \{a: a < z\} = \lambda(i)^{(3)}$ From here, by [Sh 2, §4], the theorem is immediate.

Problem 2.7. When in general is $L^{**}$ compact?

Remark. If there is a $C_i$ which is an infinite set of $\lambda_j$'s then $L^{**}$ is not compact. On the other hand, by the previous theorem and ultraproducts, if every finite $t \subseteq T$ has a model, then there is a $T'$, as in (B) of the previous theorem, which has a model.

Problem 2.8. Give a nice axiomatization of $L^{**}$. In one case we have

Theorem 2.9. If $C \neq \emptyset$, and $C$ is not the class of all regular cardinals, then the following system of axioms is complete for $L(Q^{cf}_{C})$:

1. The usual schemes for the first order calculus.

2. The following scheme (in which variables serving as parameters are not explicitly mentioned):

\[(Q^{cf}_{C} x, y)\varphi(x, y) \rightarrow [\varphi(x, y) \text{ is a linear order on } \{y: (\exists x)\varphi\}] \text{ without last element}\]

\[(Q^{cf}_{C} x, y)\varphi(x, y) \land \neg (Q^{cf}_{C} x, y)\psi(x, y) \land [\psi(x, y) \text{ is a linear order on } \{y: (\exists x)\psi\} \text{ without last element}]\]

\[\land (\forall x, y)[\theta(x, y) \rightarrow (\exists x_1)\varphi(x_1, x) \land (\exists y_1)\psi(y_1, y)]\]

\[\land (\forall y)[(\exists y_1)\psi(y_1, y) \rightarrow (\exists x)\theta(x, y)] \rightarrow \neg [(\forall x_0)(\exists y_0)((\exists x)\varphi(x_1 x_0) \rightarrow (\exists y)\psi(y, y_0) \land (\forall x_1, y_1)\varphi(y_0, y_1) \land \theta(x_1, y_1) \rightarrow \varphi(x_0, x_1))]\]

Proof. By the previous theorem it is sufficient to prove that if $T \subseteq L(Q^{cf}_{C})$ is countable, complete, and consistent (by the above axiomatization), then $T$ has a model where we interpret $C$ as $\{N_\emptyset\}$ for example. The proof is like [KM].

A quantifier close to the quantifiers we have discussed is

Definition 2.1. $(Q^{cf}_{C} x, y)[\varphi(x, y), \psi(x, y)]$, which means that the orders defined by $\varphi(x, y)$ and $\psi(x, y)$ on $\{y: (\exists x)\varphi(x, y)\}$ and $\{y: (\exists x)\psi(x, y)\}$, respectively, have the same cofinality.

Conjecture 2.10. The logic $L(Q^{cf}_{C})$ is compact and complete (and even has an axiomatization parallel to that of the last theorem). It is not hard to see that

(3) The $\lambda_i$'s are arbitrary.
Theorem 2.11. (1) There is $\psi \in L(Q^{ec})$ which has a model of power $\aleph_\alpha$ iff $\aleph_\alpha = \alpha$.

(2) If $\|M\| = \kappa$ where $\kappa$ is a Mahlo number of rank $\alpha + 1$, then $M$ has an $L(Q^{ec})$-elementary submodel of power $\lambda$ for some Mahlo number $\lambda < \kappa$ of rank $\alpha$ (actually the set of such $\lambda$'s which corresponds to $M$ is a stationary set). (For information about Mahlo numbers see Lévy [Le 1].)

(3) If $\kappa$ is not a Mahlo number then there is a model of power $\kappa$, with a finite number of relations, which has no $L(Q^{ec})$-elementary submodel of smaller power.

Generalized second-order quantifiers. Henkin [Hn 1] defined first-order generalized quantifiers as follows: The truth value of $(Q\phi(x))$ in a model $M$ is dependent only on the isomorphism type of $(M, \{x: \phi(x)\})$, i.e., on the powers of $\{x: \phi(x)\}$ and $\{x: \neg \phi(x)\}$. This is how the quantifier $(Q^*\phi(x))$ is reached.

Similarly we may define "generalized second-order quantifier" to be such that the truth value of $(Q\phi(P))$ in $M$ is dependent only on the isomorphism type of $(M, \{P: \phi(P)\})$, like [Li 1].

The regular second-order quantifier is too strong from the point of view of model theory, and so there are no nice model theoretic theorems about it. But there could be generalized second-order quantifiers which are weak enough for their model theory to be nice, for example by satisfying Lowenheim-Skolem, compactness or completeness theorems. In fact the cofinality quantifiers we discussed previously are an example.

Definition 2.2. If $<$ is a linear order on $A$ then an initial segment of $A$ is a set $B \subseteq A$ such that $b < a, a \in B \rightarrow b \in B$. An increasing sequence $\{B_\alpha: \alpha < \lambda\}$ of initial segments is unbounded if every initial segment of $A$ is contained in some $B_\alpha$, and it is closed if $B_\delta = \bigcup_{\alpha < \delta} B_\alpha$ for all limit ordinals $\delta$.

If $cf A > \omega$ then the closed and unbounded sequences of initial segments of $A$ generate a (nonprincipal) filter $D(A)$ on the set of all initial segments of $A, H(A)$.

Now we define some generalized second-order quantifiers.

Definition 2.3. Let $C$ be a class of regular cardinals $> \aleph_0$.

$$(Q^{cf}_{\kappa} x, y)[\phi(x, y), \psi(P)] \iff (Q^*_{\kappa} x, y)\phi(x, y)$$

and

if $A = \{y: (\exists x)\phi(x, y)\}$ then $H(A) - \{P: \psi(P), P \in H(A)\} \notin D(A)$; that is, the above set is stationary.

Definition 2.4. Let $\lambda > \aleph_0$ be regular, and let $C \subseteq \lambda$. 


(Q_{\lambda}^{st} C, P, x, y)[\varphi(x, y), \psi(P)] \iff (Q_{\lambda}^{st} P, x, y)[\varphi(x, y), \psi(P)]

There is a sequence \( \{P_i\}_{i<\lambda} \) of initial segments of \( \{y: (\exists x)\varphi(x, y)\} \) which is closed and unbounded, and \( \{i<\lambda: \psi(P_i)\} \cup (\lambda - \mathcal{C}) \subseteq D(\lambda). \)

**Remark.** It is not difficult to see that the above is well defined, for if \( \{P'_i\}_{i<\lambda} \) is another example of such a sequence \( \{i: P_i = P'_i\} \subseteq D(\lambda). \)

In another example we use a filter similar to that of Kueker [Ku 1]:

For a regular power \( \lambda > \aleph_0 \) and set \( A, |A| \geq \lambda \), let \( S_\lambda(A) = \{B: B \subseteq A, |B| < \lambda\} \). \( D_\lambda(A) \) will be the filter on \( S_\lambda(A) \) generated by the families \( S \subseteq S_\lambda(A) \) satisfying

1. for all \( B \in S_\lambda(A) \) there is \( B' \in S \) such that \( B \subseteq B' \), and
2. \( S \) is closed under increasing unions of length \( < \lambda \).

Thus for example if \( M \) is a model \( \|M\| > \lambda \) whose language is of power \( < \lambda \) then \( \{\|N\|: N < M, \|N\| < \lambda\} \subseteq D_\lambda(\|M\|). \)

We can define a suitable quantifier:

**Definition 2.5.** \( (Q_{\lambda}^{st} P, x)[\varphi(x), \psi(P)] \iff S_\lambda(A) - \{P: |P| < \lambda, P \subseteq A \models \psi[P]\} \notin D_\lambda(A) \)

where \( A = \{x: \varphi(x)\} \).

Again it is not hard to check that the definition is valid.

**Problem 2.12.** Investigate the logics with the quantifiers (A) \( Q_{\lambda}^{st} \); (B) \( Q_{\lambda}^{ss} \); (C) \( Q_{\lambda}^{ss} \). In particular in regard to (1) compactness theorems; (2) downward Lowenheim-Skolem theorems; (3) and transfer theorems (from one \( \lambda \) to another).

If necessary use \( V = L \).

We now mention several partial results in this context.

**Theorem 2.13.** (A) If \( \|M\| = \kappa, \kappa \) weakly compact, \( |L(M)| < \kappa, C \) is the class of all regular cardinals \( > \aleph_0 \) then \( M \) has an \( L(Q_{\kappa}^{st}) \)-elementary submodel of smaller power.

(B) \( (V = L.) \) If \( \kappa \) is not weakly compact, then there is a model of power \( \kappa \), whose language is countable, which has no proper \( L(Q_{\kappa}^{st}) \)-elementary submodel. (C as above.)

**Proof.** (A) follows from well-known theorems in set theory.

(B) We shall prove it for regular \( \kappa \); the result for a singular one follows from it.

By Jensen [Je 1] there is a set \( S \) of ordinals \( < \kappa \) of cofinality \( \omega \) such that \( \kappa - S \notin D(\kappa) \) but for all \( \alpha < \kappa \) of cofinality \( > \omega \), \( \alpha - \alpha \cap S \subseteq D(\alpha) \).

Let \( f \) be a two-place function such that for all \( \alpha \) of cofinality \( \omega \) \( \{f(\alpha, n): n < \omega\} \) is an increasing sequence with limit \( \alpha \). We shall choose our model to be \( M = (\kappa, S, f, <, \cdots, n, \cdots) \).

Assume that \( N \) is an \( L(Q_{\kappa}^{st}) \)-elementary submodel of \( M \) of smaller power. Let \( \alpha = \sup\{\beta: \beta \in N\} \), then cf \( \alpha > \omega \) as

\[ Q_{\lambda}^{st} \] means \( Q_{\{\lambda\}}^{st} \).
\[ M \models (Q_{st}^{st}P, x, y) \langle x < y, (\exists z)(\forall v)(P(v) \equiv v < z \land S(z)) \rangle, \]
and there is a closed and unbounded set \( A = \{ a_i : i < \text{cf} \alpha \} \subseteq \alpha \) of type \( \text{cf} \alpha \) which is disjoint with \( S \) because \( \alpha < \kappa, \text{cf} \alpha > \omega \). For every \( a_i \in A \), let \( a'_i = \inf \{ b \in N : b > a_i \} \) and \( A' = \{ a'_i : a_i \in A \} \). Clearly in \( N \) \( a'_\delta = \sup \{ a'_i : i < \delta \} \) for \( \delta \) a limit ordinal. Thus \( A' \) is closed and unbounded in \( N \). If \( a'_i \in S \), \( \text{cf}(a'_i) = \omega \) and so the \( f(a'_i, n) \in N \) converge to \( a'_i \). So \( a_i = a'_i \), contradiction to the disjointness of \( A \) and \( S \). Thus we have
\[ N \models \neg (Q_{st}^{st}P, x, y) \langle x < y, (\exists z)(\forall v)(P(v) \equiv v < z \land S(z)) \rangle, \]
a contradiction.

In regard to the possibility that \( N \) be of power \( \kappa \), by Keisler and Rowbottom [KR] (see [CK]) we can expand \( M \) such that \( M \) will be a Jonsson algebra, and that will be a contradiction. If we restrict ourselves to \( \aleph_1 \) we can get stronger results.

**Theorem 2.14.** (A) \( L(Q_{c'}^{cr}, Q_{st}^{st}, Q_{st}^{st}, A_i)_{i < n} \) is \( \aleph_0 \)-compact and complete. The consistency of a sentence is just dependent on the Boolean algebra generated by \( A_i/D(\aleph_1) \), and not on the particular \( A_i \).

(B) \( L(Q_{st}^{st}, Q_{st}^{st}, A_i)_{i < n} \) is \( \aleph_1 \)-compact.

(C) If \( T \) is a theory in \( L(Q_{c'}^{cr}, Q_{st}^{st}, Q_{st}^{st}, A_i)_{i < n} \) and \( T' \) is the corresponding theory in \( L(Q_{st}^{st}, Q_{st}^{st}, Q_{st}^{st}, A_i)_{i < n} \), and \( \{ B_i \} \) a partition of \( \lambda \), \( \{ A_i \} \) a partition of \( \omega_1, \aleph_1 - A_i \notin D_1(\aleph_1), \lambda - B_i \notin D(\lambda) \) then \( T \) has a model \( \Rightarrow T' \) has a model.

**Proof.** (A) Without loss of generality we shall deal with models of power \( \aleph_1 \) whose universe sets are \( \omega_1 \).

It is not difficult to define a language \( L_1, |L_1| \leq |L| \) such that every \( L \)-model \( M, |M| = \omega_1 \) can be expanded to an \( L_1 \)-model \( M_1 \) such that

1. \( M_1 \) has Skolem functions (dependent only on the formula and not on \( M \)), and every formula (including sentences) is equivalent to an atomic formula,
2. \( < \) is the order on the ordinals, and
3. \( F_{M_1} = A_i \).

Let \( T \) be a theory in the logic from (A) such that every finite \( t \subseteq T \) has a model \( M' \). Let \( T_1 \) be the set of sentences of \( L_1 \) holding in \( M'_1 \) for \( t \) large enough. Define an increasing elementary sequence of countable \( L_1 \)-models:

\( N_0 \) will be any countable model of \( T_1, N_\delta = \bigcup_{\alpha < \delta} N_\alpha \) for \( \delta < \omega_1 \) limit. If \( N_\alpha \) is defined \( N_{\alpha + 1} \) will be an end extension of \( N_\alpha \) (i.e. \( N_{\alpha + 1} \models a < b \in N_\alpha \to a \in N_\alpha \)) such that there is a first element \( a_\alpha \) in \( |N_{\alpha + 1} - N_\alpha| \) and \( a_\alpha \in P_i \iff \alpha \in A_i \). The proof that this is possible is similar to Keisler [Ke 2],

(5) Of course, every model with language \( L \) has an elementary submodel of cardinality \( < |L| + \aleph_1 \) in this logic.
It is not difficult to check that $\bigcup_{\alpha<\omega_1} N_\alpha$ is the required model of $T$.

The proof of the completeness is similar, but $T_1$ must be defined more carefully.

(B) The proof is similar to that of (A); here $N_{\alpha+1}$ will be an expansion (as well as an extension) and instead of the demand that $N_{\alpha+1}$ be an end extension, we only need that for all $\delta \leq \alpha$ limit ordinal the type $\{a_i < x : i < \delta\} \cup \{x < a_\delta\}$ be omitted.\(^{(6)}\)

(C) The proof is similar. □

The class $K_\lambda$. After the proof of the previous theorem it is natural to consider the following class of models which is somewhat parallel to the class of $\kappa$-like models.

**Definition 2.6.** Let $\lambda$ be regular. $M \in K_\lambda$ iff $<\ $ linearly orders $\{x : M \models (\exists y)(x < y \lor y < x)\}$ with cofinality $\lambda$, and there is a continuous increasing unbounded sequence $\{a_i\}_{i<\lambda}$ (i.e. for all $\delta < \lambda$ limit, the type $\{a_i < x < a_\delta : i < \delta\}$ is omitted by $M$).

From the previous theorem follows

**Theorem 2.15.** If $|T| < \aleph_1$ ($T$ a first-order theory) and every finite $t \subseteq T$ has a model in some $K_\lambda, \lambda > \aleph_0$ then $T$ has a model in $K_{\aleph_1}$.

It is easily proven that

**Theorem 2.16.** If $M \in K_\lambda, \mu < \lambda$ regular, $|L(M)| < \lambda$ then $M$ has an elementary submodel in $K_\mu$.

Somewhat less immediate is the following.

**Theorem 2.17.** (A) If for every $n < \omega$ every finite $t \subseteq T$ has a model in some $K_\lambda$ for $\lambda > \aleph_n$, then $T$ has a model in $K_\lambda$ for all $\lambda$.

(B) (Completeness.) The set of sentences true in every model of $K_{\aleph_\omega+1}$ is recursively enumerable.

**Proof.** Without loss of generality assume that $T$ has Skolem functions. For every ordinal $\alpha$ define

$$\Sigma_\alpha = \{\tau(y_{i_1}, \ldots, y_{i_n}) < y_{i_{(k+1)}} \rightarrow \tau(y_{i_1}, \ldots, y_{i_n}) < y_{i_{(k+1)}} : \tau \text{ is a term of } L(T), i_1 < \cdots < i_n < \alpha\}.$$

It is clear that: $T \cup \Sigma_n$ is consistent for all $n \iff T \cup \Sigma_\alpha$ is consistent for all $\alpha \iff$ for all $\lambda$ $T$ has a model in $K_\lambda$; for if $M$ is a model of $T \cup \Sigma_\lambda$.

\(^{(6)}\) We should first assume w.l.o.g. that our language $L$ has a countable sublanguage $L_1$, such that $L - L_1$ consist of individual constants $\{c_i : i < \omega_1\}, P(c_i) \in T$; and every finite $t \subseteq T$ has a model $M^t$, $|M^t| = \omega_1, \mathcal{P}M^t$ is finite, and in $M^t$, every limit ordinal is the universe of a submodel of $M^t$, and (1)-(3) from the proof of (A) holds.
which is the closure of \( \{ y_i : i < \lambda \} \) under Skolem functions, then \( M \in K_\lambda \).

Thus it is sufficient to prove:

\((*)\) For all \( n \) and all finite \( \Sigma' \subseteq \Sigma_n \) and all \( M \in K_{\kappa_n} \) there are \( y_0, \ldots, y_{n-1} \in M \) satisfying \( \Sigma' \).

We shall show by downward induction on \( m < n \) that:

\((**)*\) There are

\((1)\) \( y_{m+1} < \cdots < y_{n-1} \) (when \( m = n - 1 \) this is an empty sequence).
\((2)\) \( a_i^m < a_i^m < y_{m+1} \) for all \( j < i < \kappa_{n-m} \), \( a_i^m = y_{m+1} \) (except when \( n = m \)).
\((3)\) For all \( \delta < \kappa_{n-m} \) limit ordinal there is no \( x \) such that \( a_i^m < x < a_{\delta}^m \) for all \( i < \delta \).
\((4)\) If \( \tau \) occurs in \( \Sigma'_n, b_1, \ldots, b_k \in \{ a_i^m : i < \alpha < \kappa_{n-m} \} \cup \{ y_{m+1}, \ldots, y_{n-1} \} \), then \( M \models \tau(b_1, \ldots, b_k) < y_{m+1} \rightarrow \tau(b_1, \ldots, b_k) < a_{\alpha+1}^m \) (if \( m = n - 1 \) we have instead \( M \models \tau(b_1, \ldots, b_k) < a_{\alpha+1}^m \)).
\((5)\) \( y_{m+1}, \ldots, y_n \) satisfy the corresponding formulas of \( \Sigma'_n \). Now for \( m = n \) choose an increasing unbounded continuous sequence \( \{ b_i \}_{i < \kappa_n} \).

Assume that we have already completed stage \( m + 1 \), and we shall define

for \( m \) (for simplicity let \( m < n \)) there is a closed unbounded set \( S \subseteq \{ \alpha: \alpha < \kappa_{m+1} \} \) such that for \( \alpha \in S, \sigma_i, \ldots, \sigma_l \in \{ a_i^{m+1} : i < \alpha \} \cup \{ y_i : m < i < n \} \), and \( \tau \) which occurs in \( \Sigma_n \) we have \( \tau(\sigma_1, \ldots, \sigma_l) < a_{\alpha+1}^{m+1} \rightarrow \sigma(\sigma_1, \ldots, \sigma_l) < a_{\alpha+1}^{m+1} \). Choose \( \alpha_0 \in S \) such that \( \text{cf}(S \cap \alpha) = \kappa_m \) and define \( y_m = a_{\alpha_0+1}^m \). Let \( \{ \alpha_i: i < \kappa_m \} \) be an increasing unbounded continuous sequence in \( S \cap \alpha \) (it is easy to verify that there is such a sequence), and let \( a_i^m = a_{\alpha_i}^{m+1} \). (If \( m = 0 \) there is no need to choose \( a_i^m \), and thus it was sufficient to assume that \( M \in K_{\kappa_{n-1}+1} \)).

Theorem 2.18. For all \( n < \omega \) there is a sentence \( \psi_n \) having a model in \( K_{\kappa_n} \) but no model in \( K_{\kappa_{n+1}+1} \).

Proof. \( \psi_n \) will more or less characterize \((\omega_n, <)\).

\( \psi_0 \) will say that there is a first element, every element has a successor, and every element (except the first) has a predecessor.

\( \psi_{n+1} \) will say that \( \{ a: a < c_i \} \) satisfies \( \psi_i \) for \( i \leq n \) (\( c_i \) being an individual constant), \( P_0, \ldots, P_n \) is a partition of the limit elements, and if \( a \in P_i \) then \( \langle F_i(a, x): x < c_i \rangle \) is an increasing, continuous, unbounded sequence in \( \{ y: y < a \} \).

Similar theorems may be proved with omitting types as in [Mo 1]. For example if \( T \) is countable and has a model in \( K_{\kappa_{\omega_1}} \) omitting a type \( p \), then for all \( \lambda \) \( T \) has a model in \( K_\lambda \) omitting \( p \).

Problem 2.19. Prove the compactness of \( K_{\kappa_n} \), for \( 1 < n < \omega \).
Remark. If we relax the condition of continuity at \( \delta \) of cofinality \( \omega \) then we can prove this as in [Sh 1]. Since then the class is closed under ultraproducts of \( \aleph_0 \) models. In general it suffices to prove the \( \aleph_0 \)-compactness of \( K_{\aleph_0} \).

General questions. A general problem (which is of course not new) about abstract logic is

Problem 2.20. Find the logical connections between the following properties of the abstract logic \( L^* \):

- (A) \( L^* \) is first-order logic.
- (B) \( \lambda L^* \) satisfies the compactness theorem for theories of power \( \leq \lambda \).
- (C) \( = (B)_{\lambda} L^* \) satisfies the compactness theorem.
- (D) \( L^* \) satisfies the \( \lambda \)-downward Lowenheim-Skolem theorem. (If \( \psi \in L^* \) has a model then \( \psi \) has a model of power \( \leq \lambda \).)
- (E) \( L^* \) satisfies the \( \lambda \)-upward Lowenheim-Skolem theorem. (If \( \psi \) has a model of power \( \geq \lambda \), then \( \psi \) has a model of arbitrarily large power.)
- (F) \( L^* \) satisfies Craig's theorem.
- (G) \( L^* \) satisfies Beth's theorem.
- (H) \( L^* \) satisfies the Feferman-Vaught theorems for
  1. Sum of models.
  2. Product of models.
- (I) \( L^* \) satisfies the completeness theorem (assuming that the set of sentences is recursive in the language).

It is known that (A) implies the others; for \( \mu < \lambda \) (C) \( \rightarrow (B)_{\lambda} \rightarrow (B)_{\mu} \),
(\( (E)_{\mu} \rightarrow (E)_{\lambda} \), (D) \( \mu \rightarrow (D)_{\lambda} \); (F) \( \rightarrow (G), (C) \rightarrow (E)_{\aleph_0}, (H)(3) \rightarrow (H)(2) \rightarrow (H)(1). \)
Lindenström [Li 1], [Li 2] proved (and Friedman [F 1] reproved).

\( (B)_{\aleph_0} \land (D)_{\aleph_0} \rightarrow (A), (E)_{\aleph_0} \land (D)_{\aleph_0} \rightarrow (A), (F) \land (D)_{\aleph_0} \rightarrow (A). \)
The method of proof is by encoding Ehrenfeucht-Fraisse games.

Special questions which look interesting to me are

Problem 2.21. Is there a logic \( L^* \) stronger than first-order logic which is \( \aleph_0 \)-compact and satisfies Craig's theorem? Do sums of models preserve elementary equivalence for \( L^* \)?

Is there an expansion of \( L(Q^{cf}_{\aleph_1}) \) satisfying this? Keisler and Silver showed that \( L(Q^{cf}_{\lambda}) \) does not satisfy Craig's theorem. Friedman [Fr 2] showed that Beth's theorem is also not satisfied. Similarly it is not hard to show that all the logics with the quantifiers \( Q^{cf}, Q^{dc}, Q^{cc}, Q^{st} \) (all or some of them) do not satisfy Craig's theorem, but satisfy (H)(1). \( Q^{as} \) does not satisfy (H)(1).

Problem 2.22. Does \( L(Q^{as}_{\aleph_1}) \) satisfy Craig's theorem, if we restrict ourselves to models of power \( \leq \aleph_1 \)?
Problem 2.23. Find a natural characterization for $L(Q^{st}_{\omega_1})$. (For $L_{\omega_1,\omega}$, $L_{\omega_1,\omega}$, etc. Barwise [Ba 1] found one.)

Lemma 2.24. Let $Q^1$ be the quantifier $Q^{dc}_{\aleph_0}$: there is a sentence $\psi$ in $L(Q^1)$, which has only well-ordered models, and has a model of order type $\alpha$ for every $\alpha \gg 2^{\aleph_0}$. (Thus $L(Q^1)$ is not compact.)

Proof. Let $\psi_1$ say:
1. $P_1, P_2, P_3, P_4$ (one place predicates) are a partition of the universe.
2. $<$ is a total order of the universe, $S$ is the successor function in $P_1$ and $P_2$ (so $P_1$ and $P_2$ are closed under $S$) and each $P_i$ is a convex subset.
3. $F$ is a one place function mapping $P_3$ into $P_2$.
4. $G$ is a two-place function from $P_3$ to $P_1$ and

$$(\forall x \in P_3)(\forall y \in P_3)(\forall z \in P_1)[S(z) \leq G(x, y) \land x < y \equiv (\forall v \in P_2)(\exists x', y' \in P_3)$$

$$(x < x' < y < y' \land \varphi(x', y', v) \land z \leq G(x', y'))]$$

where $\varphi(x, y, z) = P_3(x) \land P_3(y) \land P_2(z) \land x < y \land (\forall v)(x < v < y \rightarrow z < F(v)).$

5. $(\forall z \in P_1)(\exists x, y \in P_3)(x < y \land G(x, y) = z)$.

6. The cofinality of $P_2$ is $\aleph_0$ (just say $F$ is an anti-isomorphism from $(P_2, \leq)$ onto $(P_4, \leq)$, and and

$$(Q^1 xy)(P_2(x) \lor P_4(x)) \land (P_2(y) \lor P_4(y)) \land x < y$$

$$(Q^1 xy)(P_2(x) \land P_2(y) \land x < y)$$

7. $(Q^1 xy)(P_3(x) \land P_3(y) \land x < y)$.

Suppose $M \models \psi_1$ and $c_n$ is a strictly decreasing sequence in $P_1^M$; let $d_n (n < \omega)$ be an increasing unbounded sequence in $P_2^M$, and define inductively $x_n, y_n \in P_3^M$, $x_n < x_{n+1} < y_{n+1} < y_n$, and $G(x_n, y_n) \geq c_n$, and $\varphi(x_{n+1}, y_{n+1}, d_n)$. For $n = 0$ use 5, for $n + 1$ use 4. So by $\varphi$'s definition for no $z, x_n < z < y_n$ for every $n$ (as then $F(z)$ cannot be defined); contradicting 7). So in every model of $\psi_1, P_1$ is well-ordered. Now we define by induction on $\alpha$ orders $I_\alpha$ and functions $F_\alpha: I_\alpha \rightarrow \omega$ as follows:

$I_0$ is $\aleph_1$-saturated order of cardinality $2^{\aleph_0}$; $F_0$ is constantly zero.

$I_{\alpha+1} = \{ (i, a): i \in \alpha + 1, a \in I_\alpha \}$ ordered lexicographically.

$F_{\alpha+1}(i, a) = F(a) + i$ for $i < \omega$, and zero otherwise.

$I_\delta = \{ (\alpha, a): \alpha \leq \delta + 1, a \in I_\alpha \}$ ordered lexicographically.

$F_\delta(a) = F_\alpha(a)$ for $\alpha < \delta$, and zero otherwise.

Now we can easily define $M^\alpha \models \psi_1, P_1^{M^\alpha} = 1 + \alpha, P_2^{M^\alpha} = \omega, P_3^{M^\alpha} = I_\alpha, F^{M^\alpha} \supset F_\alpha, P_4^{M^\alpha} = \omega^*$. The change to $\psi$ is now only technical.

Added in proof. 1. Schmerl, in a preprint “On $\kappa$-like structures which embed stationary and closed unbounded subsets” proved interesting results on problems closely related to $(Q^{st}_{\omega_1})$.
2. The author proved that a variant of Feferman-Vaught theorem and Beth theorem implies Craig theorem. This and other results will appear.

3. Why do we use $O_{\{N_0, \kappa\}}^c$, $O_{\{N_0, \kappa\}}^{de}$, and not just $O_{\{N_0\}}^c$, $O_{\{N_0\}}^{de}$ in Definition 1.4? (Note that $O_{\{N_0\}}^c$ is added just for convenience.)

REFERENCES


[Fr 1] H. Friedman, Why first order logic, Mimeographed Notes, Stanford University, 1970.


INSTITUTE OF MATHEMATICS, THE HEBREW UNIVERSITY, JERUSALEM, ISRAEL