

ON J -CONVEXITY AND SOME ERGODIC SUPER-PROPERTIES OF BANACH SPACES

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ABSTRACT. Given two Banach spaces F and X , write F fr X iff for each finite-dimensional subspace F' of F and each number $\epsilon > 0$, there is an isomorphism V of F' into X such that $\|x - \|Vx\| \leq \epsilon$ for each x in the unit ball of F' . Given a property P of Banach spaces, X is called super- P iff F fr X implies F is P . Ergodicity and stability were defined in our articles *On B -convex Banach spaces*, Math. Systems Theory 7 (1974), 294–299, and C. R. Acad. Sci. Paris Ser. A 275 (1972), 993, where it is shown that super-ergodicity and super-stability are equivalent to super-reflexivity introduced by R. C. James [Canad. J. Math. 24 (1972), 896–904]. Q -ergodicity is defined, and it is proved that super- Q -ergodicity is another property equivalent with super-reflexivity. A new proof is given of the theorem that J -spaces are reflexive [Schaffer-Sundaresan, Math. Ann. 184 (1970), 163–168]. It is shown that if a Banach space X is B -convex, then each bounded sequence in X contains a subsequence (y_n) such that the Cesàro averages of $(-1)^i y_i$ converge to zero.

Given two Banach spaces F and X , F is said to be *finitely representable* in X , in symbols F fr X , iff for each finite-dimensional subspace F' of F and each number $\epsilon > 0$, there is an isomorphism V of F' into X such that $\|x - \|Vx\| \leq \epsilon$ for each x in the unit ball of F' . Given a property P of Banach spaces, we say that X is *super- P* iff F fr X implies that F has the property P . Super-reflexive spaces were introduced by James [12], [13]; the result announced in [4] but implicit in the earlier paper [3] is that the following super-properties are equivalent: Super-ergodicity, super-reflexivity, super-Banach-Saks, super-stability. Here we define Q -ergodicity, a notion in appearance weaker than ergodicity, and prove that super- Q -ergodicity is another property equivalent with super-reflexivity. At the same time we give a new proof of James's theorem [10] that $(2, \epsilon)$ -convex spaces are reflexive, and more generally of the recent results of Schaffer-Sundaresan [19], that J -spaces are reflexive. We also show that

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B -convex spaces are *alternate signs Banach-Saks*: Each bounded sequence contains a subsequence (y_n) such that the Cesàro averages of $(-1)^j y_i$ converge to zero.

1. **Preliminaries.** Let X be an arbitrary Banach space with norm $\|\cdot\|$. An *isometry (contraction)* is a linear map $T: X \rightarrow X$ such that $\|Tx\| = \|x\|$ ($\|Tx\| \leq \|x\|$) for each $x \in X$. The Cesàro averages $(1/n)(T^0 + \dots + T^{n-1})$ are denoted by A_n , or $A_n(T)$. The following simple result seems new.

PROPOSITION 1.1. *If T is a contraction on a Banach space X then for each $x \in X$ the limit of $\|A_n x\|$ exists.*

PROOF. Let $x \in X$ and set $\alpha = \liminf \|A_n x\|$. It suffices to show that, for each $\delta > 0$,

$$(1.1) \quad \limsup \|A_n x\| \leq \alpha + \delta.$$

Given a $\delta > 0$, choose a *fixed* integer N such that $\|A_N x\| \leq \alpha + \delta$. If m and n are positive integers, $mN \leq n < (m+1)N$, then as $m \rightarrow \infty$

$$(1.2) \quad \|A_{mN} - A_n\| \rightarrow 0.$$

Therefore it suffices to prove that $\limsup_m \|A_{mN} x\| \leq \alpha + \delta$. $\|T\| \leq 1$ implies that $\|T^{jN} A_N x\| \leq \alpha + \delta$ for each j . Hence for each m

$$(1.3) \quad \|A_{mN}(T)x\| = \|A_m(T^N)A_N(T)x\| \leq \alpha + \delta.$$

This proves that $\lim \|A_n x\|$ exists. It is easy to see that this limit, considered as a function of x , is a seminorm. \square

Note that applying the proposition to the space of bounded operators on X one obtains: for each contraction T , $\lim \|A_n(T)\|$ exists.

We will now define Q -ergodicity. Let S be the space of all sequences $a = (a_i)_{i=1,2,\dots}$ such that $a_i = 0$ but all but finitely many i 's. Assuming T fixed, set, for each $x \in X$ and $a \in S$,

$$(1.4) \quad \begin{aligned} Q(x; a; n_1, n_2, \dots) \\ = \|a_1 A_{n_1} x + a_2 A_{n_2} T^{n_1} x + a_3 A_{n_3} T^{n_1+n_2} x + \dots\|, \end{aligned}$$

$$(1.5) \quad L(x, a) = \limsup_{n \rightarrow \infty} Q(x; a; n_1, n_2, \dots), \quad n = \inf(n_i).$$

The \limsup above becomes *limit* if a is one-dimensional (by Proposition 1.1), or if the norm is "invariant under spreading of the sequence $T^n x$ " (see Proposition 2.2 below).

Let r be an integer ≥ 2 and ϵ a number, $0 \leq \epsilon \leq 1/r$. The space X is called (r, ϵ) -ergodic iff for each isometry T , each $x \in X$, any r elements

a^1, \dots, a^r of S such that $L(x, a^i) \leq 1$, one has

$$(1.6) \quad \min_{1 \leq k \leq r} L(x, a^1 + a^2 + \dots + a^k - a^{k+1} - a^{k+2} - \dots - a^r) \leq r(1 - \epsilon).$$

X is called *Q-ergodic*, or *qualitatively ergodic*, iff it is ergodic for some r and ϵ . We recall that X is called *ergodic* (for isometries) iff $\lim A_n x$ exists for each isometry T and each $x \in X$. We now will show that if X is ergodic, then it is (r, ϵ) -ergodic for each r and ϵ . It is known and easy to see that the ergodic theorem for T implies that, for each x , $\lim_n A_n T^j x$ exists uniformly in j . (Apply, e.g., the decomposition theorem [6, p. 662]; uniform in j converges to the limit is obvious for a T -invariant x , and also for an x of the form $x = y - Ty$.) Let $\bar{x} = \lim A_n x$, $a^j = (a_i^j)$, $\alpha_j = \sum_i a_i^j \|\bar{x}\|$. If T is ergodic then the j th summand in (1.4) converges to $a_j \bar{x}$, hence (1.6) follows from the inequality

$$(1.7) \quad \min_{1 \leq k \leq r} |\alpha_1 + \alpha_2 + \dots + \alpha_k - \alpha_{k+1} - \dots - \alpha_r| \leq (r-1) \sup_j |\alpha_j| \leq r(1 - \epsilon),$$

easy to verify by induction on r .

A Banach space X is called $J(r, \epsilon)$ -convex, where $r \geq 2$, $0 \leq \epsilon < 1$, iff for each r -tuple (x_1, \dots, x_r) of elements of the unit ball U_X of X one has

$$(1.8) \quad \min_{1 \leq k \leq r} \|x_1 + \dots + x_k - x_{k+1} - \dots - x_r\| \leq r(1 - \epsilon).$$

X is called *J-convex* iff it is $J(r, \epsilon)$ -convex for some r and ϵ . It follows from a recent unpublished result of R. C. James [13] that J -convexity is a properly stronger notion than B -convexity introduced in [2]; cf. §3 below. It is easy to see that $J(r, \epsilon)$ -convexity, hence J -convexity, are super-properties; i.e., if X enjoys them, so does every space finitely representable in X . It has been proven by Schaffer-Sunderasan [19], and will be again shown below, that J -convex spaces are reflexive; hence, as already noted in [14], super-reflexive. Since the ergodic theorem holds for reflexive spaces, it follows that J -convex spaces are ergodic. It would be perhaps of interest to give a direct proof of this result; here we only point out that $J(2, \epsilon)$ -convexity easily implies the relation:

$$(1.9) \quad \limsup_{n, p \rightarrow \infty} \|A_n(T)x - A_p(T)x\| \leq 2(1 - \epsilon) \lim \|A_n(T)x\|$$

for each contraction T on X and each $x \in X$: Note that for any fixed positive integers i, N, m one has the identities

$$(1.10) \quad A_{2iN} = A_i(T^N) [\frac{1}{2}(A_N + T^{iN}A_N)],$$

$$(1.11) \quad A_N - A_{mN} = \frac{1}{m} \sum_{i=1}^{m-1} (A_N - T^{iN} A_N).$$

Let $x \in X$; $\lim \|A_n x\| = \alpha$ exists by Proposition 1.1. Select a fixed number δ , $0 < \delta < \epsilon\alpha/(2 - \epsilon)$. Choose a fixed N so large that

$$(1.12) \quad \| \|A_{N+k} x\| - \alpha \| < \delta, \quad k = 0, 1, \dots.$$

Since $\|T\| \leq 1$, either for some integer i

$$(1.13) \quad \|A_N x + T^{iN} A_N x\| \leq 2(1 - \epsilon)(\alpha + \delta),$$

or for all i

$$(1.14) \quad \|A_N x - T^{iN} A_N x\| \leq 2(1 - \epsilon)(\alpha + \delta).$$

In the first case (1.10) implies $\|A_{2iN} x\| \leq (1 - \epsilon)(\alpha + \delta)$, which contradicts (1.12). Therefore (1.14) must hold for all i , and (1.11) implies $\|A_N x - A_{mN} x\| \leq 2(1 - \epsilon)(\alpha + \delta)$ for all m . Since δ may be chosen arbitrarily small, (1.2) now implies (1.9).

2. Ergodic super-properties. A Banach space X with norm $\| \cdot \|$ is given. A bounded sequence (x_n) in X is called *stable* iff there is an element \bar{x} such that

$$(2.1) \quad \lim_n \left\| \frac{1}{n} \sum_{i=1}^n x_{k_i} - \bar{x} \right\| = 0$$

uniformly in the set K of all strictly increasing sequences (k_n) of natural numbers. Actually, the *uniformity* is an easy consequence of convergence for all $(k_n) \in K$. A Banach space is called *stable* iff every bounded sequence contains a stable subsequence; *Banach-Saks* iff every bounded sequence contains a subsequence which converges Cesàro. Professor Paul Erdős has recently informed us that he had shown jointly with Professor M. Magidor that every space which is Banach-Saks is also stable, the proof being based on the combinatorial fact that every analytic set is Ramsey [20].

We now return to the setting of our papers [3], [4], in which we have attempted to connect ergodic properties of X with stability, or the Banach-Saks property. We have at first asked the following question: Does an arbitrary bounded sequence (x_n) in X admit a subsequence (e_n) such that the shift T on (e_n) is defined and power-bounded? (By a shift on (e_n) we understand an operator T satisfying $T e_n = e_{n+1}$ for all n , and acting on the space spanned by the e_n 's.) If the answer to this question had been *positive*, it would follow at once that the ergodic theorem (power-bounded version) for X and its subspaces

implies the Banach-Saks property—therefore the answer is *negative*, since there are reflexive spaces which are not Banach-Saks (Baernstein [1]). This showed the need to change the norm. Denoting the space spanned by (e_n) and a new norm $||$ by F , we could obtain [3] that the shift on (e_n) be an isometry, and yet $||$ be so close to $||$ that the ergodic theorem for T on F implies that (e_n) contains a stable subsequence in X , and F fr X . The implication announced in [4], super-ergodic \Rightarrow super-stable, follows. We recapitulate the construction of (e_n) and F . S is the space of all sequences $a = (a_i)_{i=1,2,\dots}$ with $a_i = 0$ for all but finitely many i . We have

PROPOSITION 2.1 (PROPOSITION 1 OF [3]). *Each bounded sequence (x_n) in X contains a subsequence (e_n) with the following property: For each $a \in S$ there exists a number $L(a)$ such that $\|\sum a_i e_{n_i}\| \rightarrow L(a)$ as the sequences $(n_1), (n_2), \dots$ converge to ∞ so that $n_1 < n_2 < \dots$.*

Now fix (x_n) and let (e_n) be a subsequence of (x_n) satisfying the conditions of the above proposition. Let $\varphi(S)$ be the space of linear combinations $\sum a_i e_i$, $a \in S$. As shown in [3], we may assume without loss of generality that the e_n 's are algebraically independent in X , and that $|\sum a_i e_i|$ defined as equal to $L(a)$ is a *norm* on $\varphi(S)$. We denote by F the completion of $\varphi(S)$ in this norm. We now show that F fr X : If F' is an n -dimensional subspace of F , F' is topologically isomorphic to $l_1^{(n)}$, hence we commit a negligible error assuming that F' is generated by e_1, \dots, e_m for m large. Let the same vectors in X generate a subspace H . Set $S_n = T^n: H \rightarrow X$. Then $\|S_n x\| \rightarrow |x|$ on H implies $M = \sup_n \|S_n\| < \infty$ (uniform boundedness principle), hence $\|T^n x\| \rightarrow |x|$ uniformly on compacts of H ; therefore uniformly on $U_{F'}$. Indeed, if $Y = \{y_i\}$ is a *finite* δ -net in a compact $C \subset H$, then $|\|T^n x\| - |x|| < \delta$ on Y implies that $|\|T^n x\| - |x|| \leq \delta + 2\delta M$ on C . To see this, note that if $\|x - y_i\| < \delta$ then

$$\begin{aligned} |\|T^n x\| - |x|| &\leq |\|T^n x\| - \|T^n y_i\|| + |\|T^n y_i\| - |y_i|| + ||y_i| - |x|| \\ &\leq \delta M + \delta M. \end{aligned}$$

The relation F fr X was already implicitly used in Lemma 6 [3] and in [4]. Parting from F we now propose to introduce a new norm $!!$ on (e_n) , with properties even more pleasing than $||$; the space G generated by (e_n) , $!!$ will still be finitely representable in X . The main virtue of $||$ (not included in isometric character of the shift T) may be described as *invariance under spreading*, or (IS) property: The norm of any finite combination of the e_n 's remains the same when the vectors are shifted, even though their mutual distances (but not positions) may change. This property, formally stated in [3, Lemma 1],

is an immediate consequence of Proposition 2.1. The norm $||$ will inherit from $||$ the (IS) property, but will also be *equal signs additive*, in short of type (ESA): In computing the norm of any finite linear combination of the e_i 's, consecutive terms of equal sign may be combined. Formally, for any vector $x = a_1e_1 + \dots + a_qe_q$, any integers k, p such that $1 \leq k < p \leq q$ and $a_i \geq 0$ for $k \leq i \leq p$, one has

$$(2.2) \quad !x! = \left| \sum_{i=1}^{k-1} a_i e_i + (a_k + \dots + a_p) e_k + \sum_{i=p+1}^q a_i e_i \right|.$$

It is easy to see that it suffices to verify (2.2) for all k and p such that $p - k = 1$.

We now let $A_n(T)$ act on the e_i 's spread so that different averages have disjoint support. More precisely, given a fixed $a = (a_i) \in S$ with $a_i = 0$ for $i > q$, we define

$$(2.3) \quad P(n_1, \dots, n_q; s_1, \dots, s_q) = a_1 A_{n_1} e_{s_1} + \dots + a_q A_{n_q} e_{s_q},$$

$$s_1 > 0, \quad s_2 \geq s_1 + n_1, \dots, \quad s_q \geq s_{q-1} + n_{q-1}.$$

Invariance of $||$ under spreading implies that the F -norm of the first expression in (2.3) does not depend upon the choice of the s_i 's; therefore this norm will be denoted by $Q(n_1, \dots, n_q)$, or $Q(e_1; a; n_1, \dots, n_q)$.

PROPOSITION 2.2. *For each $x = a_1e_1 + \dots + a_qe_q$ in $\varphi(S)$, the limit of $Q(e_1; a; n_1, \dots, n_q)$ as $\inf(n_i)$ converges to infinity exists. This limit, denoted $!x!$, is a seminorm on $\varphi(S)$.*

PROOF. The invariance of $||$ under spreading implies that for any fixed positive integers $N_1, \dots, N_q; m_1, \dots, m_q$

$$(2.4) \quad Q(m_1 N_1, \dots, m_q N_q) \leq Q(N_1, \dots, N_q).$$

The particular case of (2.4) where $m_i = 1$ for $i = 2, \dots, q$ is obtained by taking the Cesàro average of

$$P(a_1 T^{kN_1} A_{N_1} e_{s_1} + a_2 A_{N_2} e_{s_2} + \dots + a_q A_{N_q} e_{s_q})$$

for $k = 0, 1, \dots, m_1 - 1$,

since $s_2 > s_1 + m_1 N_1$, $s_i \geq s_{i-1} + N_i$ for $i = 3, \dots, q$ implies that each term has the norm $= Q(N_1, \dots, N_q)$. An obvious induction argument, again using invariance under spreading of $||$, establishes (2.4). We denote by $\alpha(\beta)$ the limit inferior (limit superior) of $Q(n_1, \dots, n_q)$ as n_i converge independently to infinity. To prove the proposition, it suffices to show that, for each $\delta > 0$, $\beta \leq \alpha + \delta$. Choose N_1, \dots, N_r fixed such that $Q(N_1, \dots, N_q) \leq \alpha + \delta$; (2.4) implies $Q(m_1 N_1, \dots, m_r N_r) \leq \alpha + \delta$ for all m_i . A computation anal-

ogous to (1.2) shows that if $m_i N_i \leq n_i < (m_i + 1)N_i$ for all i , then

$$(2.5) \quad \limsup_{m_i \rightarrow \infty} |Q(m_1 N_1, \dots, m_q N_q) - Q(n_1, \dots, n_q)| = 0.$$

$\beta \leq \alpha + \delta$ follows. Finally, it is easy to see that $!!$ is a seminorm on $\varphi(S)$.

LEMMA 2.1. *The seminorm $!!$ is of type (ESA) on the e_n 's.*

PROOF. We verify (2.2) assuming, as we may, that $p = k + 1$. Since $!!$ is a continuous function of coefficients a_i , we further may suppose that a_k/a_{k+1} is a rational number, and write $a_k = \alpha r$, $a_{k+1} = \alpha s$, where r, s are positive integers. Then for all integers $m > 0$, $t \geq 0$, one has

$$(2.6) \quad a_k A_{mr} e_t + a_{k+1} A_{ms} e_{mr+t} = (a_k + a_{k+1}) A_{m(r+s)} e_t.$$

The relation (2.5) is now applied, with $m_k = m_{k+1} = m$, $N_k = r$, $N_{k+1} = s$ to compute $!x!$, and with $N_k = r + s$ and $m_k = m$ to compute the right-hand side of (2.2) which is thus established. \square

LEMMA 2.2. *If $!e_1 - e_2! = 0$ then (e_n) admits a subsequence stable in X .*

PROOF. $!e_1 - e_2! = 0$ implies that

$$|A_n e_1 - A_r e_{1+n}| \rightarrow 0 \quad \text{and} \quad |A_p e_1 - A_r e_{1+p}| \rightarrow 0 \quad \text{as } n, p, r \rightarrow \infty.$$

Choosing r so that $r/(n + p) \rightarrow \infty$, we have that $|A_r e_{1+n} - A_r e_{1+p}| \rightarrow 0$; therefore by the triangular inequality the sequence $A_n e_1$ is Cauchy in F .

Proposition 3 [3] is now applicable. \square

Since we wish to prove that the space X is stable, we only need to consider the case when $!e_1 - e_2! > 0$; then $!!$ may be easily seen to be a norm on $\varphi(S)$: If $!\sum_{i=1}^q a_i e_i! = 0$, then $!a_1 e_1 + a_3 e_3 + \dots + a_{q+1} e_{q+1}! = 0$ and also $!a_1 e_2 + a_3 e_3 + \dots + a_{q+1} e_{q+1}! = 0$; hence $!a_1(e_1 - e_2)! = 0$ which implies $a_1 = 0$. Similarly one shows that $a_2 = 0, \dots, a_q = 0$. Denote by G the completion of $\varphi(S)$ in this norm.

We show that G is finitely representable in F , hence in X . Let G' be a finite-dimensional subspace of G ; we may assume that G' is generated by e_1, e_2, \dots, e_q . Let $V = V_{n_1, \dots, n_q}$ map each vector $a_1 e_1 + \dots + a_q e_q$ onto

$$a_1 A_{n_1} e_1 + a_2 A_{n_2} T^{n_1} e_1 + \dots + a_q A_{n_q} T^{n_1 + \dots + n_{q-1}} e_1.$$

Then for all $x \in U_{G'}$, by Proposition 2.2 $!|x! - |Vx||$ is small if n_1, \dots, n_q are large. G fr F easily follows (see the proof of F fr X above).

Define a seminorm M on S by

$$(2.7) \quad M(a) = !a_1(e_1 - e_2) + a_2(e_3 - e_4) + a_3(e_5 - e_6) + \dots !.$$

REMARK. The proofs of the following Lemma 2.3 and Proposition 2.3 use only the (IS) property of the norm. Thus they remain valid with $||$ replacing $! !$.

LEMMA 2.3. $M(a) \geq M(b)$ if for each i , $a_i b_i \geq 0$ and $|a_i| \geq |b_i|$. Hence M is orthogonal, i.e., $M(a) \geq M(a^+)$, $M(a) \geq M(a^-)$, where a^+ is a sequence (a_i^+) , a^- the sequence (a_i^-) .

PROOF. The invariance under spreading of $||$ implies that for each j , each n ,

$$(2.8) \quad \begin{aligned} M(a) &= !y + a_j(e_{2j-1} - e_{2j}) + z! = !y + a_j(e_{2j} - e_{2j+1}) + z! \\ &= \dots = !y + a_j(e_{2j-1+n} - e_{2j+n}) + z! \end{aligned}$$

where

$$(2.9) \quad y = \sum_{i=1}^{j-1} a_i(e_{2i-1} - e_{2i}), \quad z = \sum_{i=j+1}^{\infty} a_i(e_{2i+n-1} - e_{2i+n}).$$

(Since $a \in S$, z has only finitely many summands.) Summing the $n + 1$ expressions inside $! !$ in (2.8) and dividing by $n + 1$, one obtains

$$M(a) \geq !y + z! - |a_j|!(e_{2j-1} - e_{2j+n})/(n + 1)!.$$

Let $n \rightarrow \infty$; it follows that $M(a) \geq M(a')$, where $a'_j = 0$, $a'_i = a_i$ for $i \neq j$. The lemma is proved, because M is a convex function of coordinates.

PROPOSITION 2.3. If G does not contain an isomorphic copy of c_0 , then

$$(2.10) \quad \lim_n !e_1 - e_2 + e_3 - e_4 + \dots + e_{2n-1} - e_{2n}! = \infty.$$

PROOF. Set $u_n = e_{2n-1} - e_{2n}$; let G' be the subspace of G generated by the u_i 's. Write $M(\sum a_i u_i) = M(a)$ for $a \in S$; extended to G' , M is a norm coinciding with $! !$. Let $|a| = a^+ + a^-$, $N(a) = M(|a|)$, $N(y) = N(a)$ if $y = \sum a_i u_i$, $a \in S$. $N(a) \leq M(a^+) + M(a^-) \leq 2M(a)$ by Lemma 2.3. Therefore

$$(2.11) \quad \frac{1}{2}N(a) \leq M(a) \leq N(a).$$

Extended to G' , N is a norm equivalent with M . This observation will be useful in §3 below. Now if (2.10) fails, Lemma 2.3 gives a β such that, for all n ,

$$(2.12) \quad !e_1 - e_2 + e_3 - e_4 + \dots + e_{2n-1} - e_{2n}! \leq \beta,$$

and also shows that $M(a^+) \leq \beta \cdot \sup(|a_i|)$, $M(a^-) \leq \beta \sup(|a_i|)$, $M(a) \leq 2\beta \sup(|a_i|)$. Also, $M(a) \geq !a_i(e_{2i-1} - e_{2i})! = |a_i|(e_1 - e_2)$, so that $M(a) \geq$

$(\sup |a_i|)!e_1 - e_2!$. Thus G' is a subspace of G that is isomorphic to c_0 .

PROPOSITION 2.4. *If (2.10) holds, then G is not J -convex.*

PROOF. We show that G is not $J(r, \epsilon)$ -convex by first giving a detailed and "graphic" proof of the case $r = 2$, then a brief proof of the general case.

Set

$$\begin{aligned} v_n &= e_1 && - e_3 && + \cdots + e_{4n-3} && - e_{4n-1} && , \\ w_n &= && + e_2 && - e_4 + \cdots + && e_{4n-2} && - e_{4n}. \end{aligned}$$

We have $!v_n! = !w_n! = !v_n + w_n!/2$, the last equality by (ESA). To prove that G is not $J(2, \epsilon)$ -convex, it will suffice to prove that $!v_n - w_n!/2!v_n!$ converges to 1. This follows from (2.10) because

$$\begin{aligned} !v_n - w_n! &= !e_1 - (e_2 + e_3) + (e_4 + e_5) - \cdots \\ &\quad - (e_{4n-2} + e_{4n-1}) + (e_{4n} + e_{4n+1}) - e_{4n+1}! \\ &\geq -!e_1! + 2!v_n! - !e_{4n+1}!. \end{aligned}$$

We now show, by essentially the same argument, that G is not $J(r, \epsilon)$ -convex, where $r \geq 2$ is arbitrary. Set for $j = 1, 2, \dots, r; n = 1, 2, \dots,$

$$(2.13) \quad v_n^j = e_j - e_{j+r} + e_{j+2r} - \cdots + e_{j+(2n-2)r} - e_{j+(2n-1)r}.$$

Then $!v_n^j! = !v_n^1!$ for each j . In the expression $d_n^k = v_n^1 + v_n^2 + \cdots + v_n^k - v_n^{k+1} - \cdots - v_n^r$ the terms are arranged as follows: First write $S_1 = e_1 + e_2 + \cdots + e_k$. Then $(2n - 1)r$ terms grouped so that r consecutive e_i 's with $-$ sign alternate with r consecutive e_i 's with $+$ sign:

$$\begin{aligned} S_2 &= -(e_{k+1} + e_{k+2} + \cdots + e_{k+r}) + (e_{k+r+1} + \cdots + e_{k+2r}) - \cdots \\ &\quad + (e_{k+2(n-2)r+1} + \cdots + e_{k+2(n-1)r}). \end{aligned}$$

S_3 is composed of the remaining $r - k$ terms of d_n^k . Then $\lim_n !S_i!/r!v_n^1! = 0$ for $i = 1, 3; = 1$ for $i = 2$. Hence $\lim_n !d_n^k!/r!v_n^1! = 1$ for each $k = 1, 2, \dots, r$. The proposition is proved.

Now assume that X is J -convex; then so is G and G cannot contain an isomorphic copy of c_0 (cf. [10] or [8], where this is proved for B -convex spaces). Propositions 2.3 and 2.4 and Lemma 2.2 now imply the following theorem:

THEOREM 2.1. *A J -convex Banach space is stable (hence super-stable).*

THEOREM 2.2. *A super- Q -ergodic Banach space is super-stable.*

PROOF. If X is super- Q -ergodic then G is Q -ergodic, and the proof of Proposition 2.4 yields a contradiction. Lemma 2.2 now implies that (x_n) has a

subsequence stable in X ; since (x_n) is arbitrary, it follows that X is stable. Thus super- Q -ergodicity implies stability; it implies super-stability because the relation “fr” is transitive. \square

Since a Banach-Saks space, and *a fortiori* a stable space, is easily seen to be reflexive (cf. [17]), the argument above provides a new proof that J -convex spaces are reflexive. We finally observe that in the course of the proof of Theorem 2.1 we establish the following: Any sequence (x_n) admits a subsequence (e_n) such that the sequence $(e_{2n-1} - e_{2n})$ is an unconditional basis for the IS norm $|||$, finitely representable in $|||$. (Because an orthogonal norm is unconditional, and, as observed above, the proofs of Proposition 2.3 and Lemma 2.3 are valid for the norm $|||$ as well as $|||$.)

3. Alternate signs Banach-Saks property. A Banach space X is called (r, ϵ) -convex iff for any r elements x_1, \dots, x_r in U_X there is a sequence of signs $\sigma_1, \dots, \sigma_r$ such that $(1/r)(\sigma_1 x_1 + \dots + \sigma_r x_r) \leq 1 - \epsilon$. A Banach space is called B -convex iff it is (r, ϵ) -convex for some integer r and some $\epsilon > 0$.

THEOREM 3.1. *Every bounded sequence (x_n) in a B -convex Banach space admits a subsequence (y_n) such that*

$$(3.1) \quad \frac{1}{n} \sum_{i=1}^n (-1)^{i+1} y_i \rightarrow 0.$$

PROOF. We may assume that (x_n) is not stable, since otherwise (y_n) satisfying (3.1) may be obtained as a union of two stable subsequences of (x_n) . Let F' be the subspace of F generated by $u_1 = e_1 - e_2, u_2 = e_3 - e_4, \dots$. If X is B -convex, then so is F' and therefore, as it is easy to see, there exists a sequence of signs (σ_n) such that

$$(3.2) \quad \liminf_n \left| \frac{1}{n} \sum_{i=1}^n \sigma_i u_i \right| = 0.$$

The proof of (3.2) is only sketched since the argument is known. We may assume $|u_i| \leq 1$ for all i . Let X be (r, ϵ) -convex. First choose signs $\sigma_1^1 = +, \sigma_2^1, \sigma_3^1, \dots$ so that if $y_k = r^{-1} \sum_{i=1+kr}^{r+kr} \sigma_i^1 u_i$, then $|y_k| \leq 1 - \epsilon$ for $k = 0, 1, \dots$. Second choose signs $\sigma_1^2 = +, \sigma_2^2, \sigma_3^2, \dots$ so that if $z_k = r^{-1} \sum_{i=1+kr}^{r+kr} \sigma_i^2 y_i$ then $|z_k| \leq (1 - \epsilon)^2$ for $k = 0, 1, \dots$. Next take Cesàro averages of successive r -tuples of $\sigma_i^3 z_i$, where σ_i^3 are appropriate signs, etc. This procedure yields a sequence of signs σ_i satisfying (3.2).

As already observed, the proofs of Lemma 2.3 and Proposition 2.3, in particular (2.11), use only the (IS) property of the norm, hence remain valid with $|||$ replacing $|||$. Therefore (3.2) remains valid when all the σ_i 's are replaced by the sign $+$. Proposition 1.1 with $-T$ replacing T now implies that

$n^{-1}(u_1 + \dots + u_n)$ converges to zero in F . The proof of Proposition 3 [3] remains valid if (u_n) replaces (e_n) ; hence the sequence (u_n) contains a subsequence stable in X . This proves (3.1). \square

Applying the theorem that every analytic (or only Borel) set is Ramsey (cf. the remarks in the beginning of §2), one may strengthen Theorem 3.1 to read: Every bounded sequence (x_n) in a B -convex space contains a subsequence (z_n) such that (3.1) holds for each subsequence (y_n) of (z_n) .

The alternate signs Banach-Saks property does *not* characterize B -convex spaces since c_0 has it, as has been shown to us by Professor A. Pełczyński.

PROPOSITION 3.1. *Let (x_n) be a sequence of vectors in c_0 , $x_n = (x_n^{(i)})_{i=1}^\infty$, with $\|x_n\| = \sup_i |x_n^{(i)}| \leq 1$ for all n . Then for each $\epsilon > 0$ there exists a subsequence (y_n) of (x_n) such that for all integers m*

$$(3.3) \quad \left\| \sum_{j=1}^m (-1)^{j+1} y_j \right\| = \sup_i \left| \sum_{j=1}^m (-1)^{j+1} y_j^{(i)} \right| \leq 2 + \epsilon.$$

Hence (3.1) holds.

PROOF. Choose $\epsilon > 0$. Since we can pass to subsequences and apply the diagonal procedure, we may and do assume that $\lim_{n \rightarrow \infty} x_n^{(i)} = a_i$ exists for each i and also that $|x_n^{(i)} - a_i| < 2^{-n}\epsilon$ if $|x_k^{(i)}| > 2^{-k}\epsilon$ for some $k < n$. Then, for a subsequence (y_n) ,

$$\left\| \sum_{j=1}^m (-1)^{j+1} y_j \right\| = \sup_i \left| \sum_{j=1}^m (-1)^{j+1} y_j^{(i)} \right| < 2 + \epsilon,$$

since for each i we can replace by a_i each $x_n^{(i)}$ for which there exists $k < n$ such that $|x_k^{(i)}| > 2^{-k}\epsilon$, and obtain

$$\left| \sum_{j=1}^m (-1)^{j+1} y_j^{(i)} \right| < \epsilon \left(\sum_1^\infty 2^{-n} \right) + |x_k^{(i)}| + |a_i| \leq 2 + \epsilon. \quad \square$$

Note that reflexive spaces need not be alternate signs Banach-Saks: The example in [1] is not alternate signs Banach-Saks.

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