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ON THE 2-REALIZABILITY OF 2-TYPES

BY

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This paper is respectfully dedicated to the memory of Professor Andrew Moursund

ABSTRACT. A 2-type is a triple (π, π_2, k), where π is a group, π_2 a π-module and $k \in H^3(\pi, \pi_2)$. The following question is studied: When is a 2-type (π, π_2, k) realizable by 2-dimensional CW-complex X such that the 2-type $(\pi_1 X, \pi_2 X, k(X))$ is equivalent to (π, π_2, k)? A long list of necessary conditions is given (2.2). One necessary and sufficient condition (3.1) is proved, provided π has the property that stably free, finitely generated π-modules are free. "Stable" 2-realizability is characterized (4.1) in terms of the Wall invariant of [15]. Finally, techniques of [5] are used to extend C. T. C. Wall's Theorem F of [15] to a space X which is dominated by a finite CW-complex of dimension 2, provided $\pi_1 X$ is finite cyclic. Under these conditions X has the homotopy type of a finite 2-complex if and only if the Wall invariant vanishes.

1. **Introduction.** In [9], S. Mac Lane and J. H. C. Whitehead introduced the notion of the 2-*type of a connected* CW-*complex* X. This is the triple $T(X) = (\pi_1 X, \pi_2 X, k(X))$ consisting of the fundamental group of X, the $\pi_1 X$-module $\pi_2 X$ and the obstruction invariant

$$k[X] \in H^3(\pi_1 X, \pi_2 X)$$

of [8]. An abstract 2-type is a triple (π, π_2, k) consisting of a group π, a π-module π_2, and an element $k \in H^3(\pi, \pi_2)$. Two 2-types $T = (\pi, \pi_2, k)$, $T' = (\pi, \pi_2', k')$ with the same fundamental group π are *equivalent* $(T \cong T')$ if there are isomorphisms

$$f\colon \pi \longrightarrow \pi, \qquad f'\colon \pi_2 \longrightarrow \pi_2'$$

where $f'(xa) = f(x) \cdot f'(a)$ $(x \in \pi,\ a \in \pi_2)$ and $f'_*(k) = f^*(k')$ *in*

$$f'_*\colon H^3(\pi, \pi_2) \longrightarrow H^3(\pi, (\pi_2')_f) \longleftarrow H^3(\pi, \pi_2')\colon f^*.$$

Let $\mathrm{A}(\pi)$ *be the set of equivalence classes of* 2-*types* (π, π_2, k) *with the same group* π; $[T]$ is the equivalence class of 2-types containing T.

We say that connected CW-*complexes* X, Y *have the same* (*topological*) 2-*type if and only if there exist maps* $f\colon X^{(3)} \longrightarrow Y^{(3)}$, $g\colon Y^{(3)} \longrightarrow X^{(3)}$ *such*

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that $gf|_{X^{(2)}} \simeq i\colon X^{(2)} \longrightarrow X^{(3)}$, $fg|_{Y^{(2)}} \simeq i\colon Y^{(2)} \longrightarrow Y^{(3)}$. Theorem 1 of [9] shows that X, Y have the same topological 2-type $\Longleftrightarrow T(X) \cong T(Y)$. We will call f, g above 2-equivalences; g will be called the 2-inverse of f, etc.

It is also known from [9, Theorem 2] that every abstract 2-type $T = (\pi, \pi_2, k)$ can be realized by a connected 3-complex X such that $T \cong T(X)$. *Let* $\mathrm{FA}(\pi)$ *be the subset of* $\mathrm{A}(\pi)$ *consisting of those 2-types which can be realized by a finite connected 3-complex. For any* $T \in (\mathrm{F})\mathrm{A}(\pi)$, *let* $(\mathrm{F})\mathrm{X}^3(T)$ *be the set of (finite)* CW-*complexes of dimension* $\leqslant 3$ *such that* $T(X) \cong T$. It follows from Theorem 1 of [9] that any 2-complexes X, Y have the same homotopy type $\Longleftrightarrow [T(X)] = [T(Y)] \in \mathrm{A}(\pi_1 X)$. We say that an element $T = [(\pi, \pi_2, k)] \in (F)\mathrm{A}(\pi)$ is (finitely) 2-realizable if there is a (finite) connected 2-dimensional CW-complex X such that $T(X) \in T$. Let

$$(F)\mathrm{R}(\pi) = \{T \in \mathrm{A}(\pi) \mid T \text{ is (finitely) 2-realizable}\}.$$

Thus $(F)\mathrm{R}(\pi)$ is the set of homotopy types of (finite) connected 2-complexes with fundamental group π.

In this paper we will study the following problem: *For any* $T \in (F)\mathrm{A}(\pi)$ *give necessary and sufficient conditions that* T *be a member of* $(F)\mathrm{R}(\pi)$.

For example, if $\pi = Z_n$, the cyclic group of order n generated by x, then $T \in \mathrm{FR}(Z_n) \Longleftrightarrow \pi_2 = (x-1)Z[Z_n] \oplus (Z[Z_n])^m$ and $k \in H^3(Z_n, \pi_2) \cong Z_n$ is a generator. See [5, I] and [4]. If $\pi = F^n$, the free group of rank n, then a result of H. Bass [1] and C. T. C. Wall [15, I] shows that $T \in \mathrm{FR}(F^n)$ $\Longleftrightarrow \pi_2 \cong (Z[F^n])^m$ and $k = 0$.

We will give a long list of necessary conditions that T be 2-realizable (Theorem 2.2) and one necessary and sufficient condition, provided π is suitably restricted (Theorem 3.1). In general, we are able to give sufficient conditions only to the "stable" 2-realizability of a 2-type T (Theorem 4.1). Finally, in §5, we study *chain* 2-realizability.

The problem of 2-realizability is clearly connected to the difficulty C. T. C. Wall experienced in deciding whether or not a CW-complex X dominated by a finite 2-complex had the homotopy type of a finite 2-complex. For if X is dominated by a finite 2-complex and the obstruction in the projective class group $\widetilde{K}^0(\pi_1 X)$ vanishes, then X has the homotopy type of a *finite 3-complex* Y [15, I, Theorem F]. Let $\widetilde{Y}$ denote the universal cover of Y, $p\colon \widetilde{Y} \longrightarrow Y$ the covering map.

THEOREM 1.1. *A connected 3-complex* Y *has the homotopy type of a (finite) connected 2-complex* $\Longleftrightarrow T(Y) \in (F)\mathrm{R}(\pi, Y)$ *and* $H_3(\widetilde{Y}) = 0$.

PROOF. The necessity is obvious. If $T(Y)$ is (finitely) 2-realizable, then there is a (finite) 2-complex Z and a 2-equivalence $f\colon Z \longrightarrow Y$. By part N(h) of Theorem 2.2 there is an isomorphism

$$I = \begin{pmatrix} g_\# \\ hp_\#^{-1} \end{pmatrix}$$

such that the following commutes:

$$\begin{tikzcd} \pi_3(Z) \arrow[r, "f_\#"] \arrow[dr, "\binom{\mathrm{id}}{0}"'] & \pi_3(Y) \arrow[d, "I"', "\approx"] \\ & \pi_3(Z) \oplus H_3(\widetilde{Y}) \end{tikzcd}$$

where h is the Hurewicz homomorphism and g is a 2-inverse to f.

Since $H_3(\widetilde{Y}) = 0, f_\#\colon \pi_3(Z) \longrightarrow \pi_3(Y)$ is an isomorphism $\Rightarrow f$ is a 3-equivalence $\Rightarrow f$ is a homotopy equivalence by Whitehead's theorem [16, I].

COROLLARY 1.2. *X is dominated by a finite 2-complex, the obstruction in $\widetilde{K}^0(\pi_1 X)$ is zero, and $T(X) \in \mathrm{FR}(\pi_1 X) \Longleftrightarrow X$ has the homotopy type of a finite 2-complex.*

In §5, Corollary 5.3, we extend C. T. C. Wall's Theorem F [15, I] to the following: *Let X be a connected CW-complex dominated by a finite 2-complex and let $\pi_1 X \cong Z_n$. Then X has the homotopy type of a finite 2-complex $\Longleftrightarrow \mathrm{Wa}_2[X] = 0$.*

Here $\mathrm{Wa}_2[X]$ = class of the π-module $C_2(\widetilde{X})/B_2(\widetilde{X})$ in the projective class group $\widetilde{K}^0(\pi_1 X)$, where $\widetilde{X}$ is the universal cover of X, $C(\widetilde{X})$, the cellular chain complex of $\widetilde{X}$, and $B_2(\widetilde{X}) = \mathrm{im}\{\partial_3\colon C_3(\widetilde{X}) \longrightarrow C_2(\widetilde{X})\}$. X satisfies D_2 [15, I, p. 61] $\Rightarrow H_2(\widetilde{X}, \widetilde{X}^{(1)}) \cong C_2(\widetilde{X})/B_2(\widetilde{X})$ is projective.

2. **Necessary conditions that $T \in \mathrm{FR}(\pi)$.** For any $T \in \mathrm{A}(\pi)$ we define the *homotopy modules of T, $\pi_i(T)$,* as

$$\pi_i(T) = \mathrm{im}\{\pi_i(X^{(2)}) \longrightarrow \pi_i(X)\}, \qquad i = 1, 2, \cdots,$$

where X is any connected CW-complex of dimension $\leqslant 3$ having the 2-type $T(X) \in T$. This definition makes sense because if X, Y are any CW-complexes of dimension $\leqslant 3$ having 2-type T, then there exist 2-inverses

$$f\colon X \rightleftarrows Y\colon g.$$

An easy argument on the homotopy ladder of the pairs $(X, X^{(2)}), (Y, Y^{(2)})$ shows that $f_\#|_{\pi_i(T)}\colon \pi_i(T) \longrightarrow \mathrm{im}\{\pi_i(Y^{(2)}) \longrightarrow \pi_i(Y)\}$ is an isomorphism.

Note. If $T \in \mathrm{R}(\pi)$, $\pi_*(T) \cong \pi_*(Y)$ for any 2-complex Y such that $T(Y) \in T$.

LEMMA 2.1. *For any connected 3-complex X, the following is exact*:

$$0 \longrightarrow \pi_i(T(X)) \xrightarrow{\varphi_i} \pi_i X \xrightarrow{\psi_i} L_i(X) \longrightarrow 0$$

where $L_i(X) = \ker\{\partial\colon \pi_i(X, X^{(2)}) \longrightarrow \pi_{i-1}(X^{(2)})\}$. $L_3(X) \cong H_3(\widetilde{X})$, *where* $\widetilde{X}$ *is the universal cover of* X. *Furthermore, under this isomorphism,* $\psi_3 = h \circ p_{\#}^{-1}$, *where* h *is the Hurewicz homomorphism and* $p\colon \widetilde{X} \longrightarrow X$ *is the covering projection.*

PROOF. The only interesting portion is $i = 3$. We will show that $L_3(X) \cong H_3(\widetilde{X})$ and that $\psi_3 = h \circ p_{\#}^{-1}$. Consider the following commutative diagram:

$$\begin{tikzcd}
\pi_3(X^{(2)}) \arrow[r] & \pi_3(X) \arrow[r, "i_\#"] & \pi_3(X, X^{(2)}) \arrow[r, "\partial"] & \pi_2(X^{(2)}) \arrow[r] & \pi_2(X) \\
\pi_3(\widetilde{X}^{(2)}) \arrow[r] \arrow[u, "\approx"'] \arrow[d] & \pi_3(\widetilde{X}) \arrow[r, "\tilde{i}_\#"] \arrow[u, "\approx", "p_\#"'] \arrow[d, "h"'] & \pi_3(\widetilde{X}, \widetilde{X}^{(2)}) \arrow[r, "\partial'"] \arrow[u, "\bar{p}_\#", "\approx"'] \arrow[d, "\bar{h}", "\approx"'] & \pi_2(\widetilde{X}^{(2)}) \arrow[r] \arrow[u, "\bar{\bar{p}}_\#", "\approx"'] \arrow[d, "\bar{\bar{h}}", "\approx"'] & \pi_2(\widetilde{X}) \arrow[u, "\approx"'] \arrow[d] \\
H_3(\widetilde{X}^{(2)}) \arrow[r] \arrow[d, equal] & H^3(\widetilde{X}) \arrow[r, "i_*"] & H_3(\widetilde{X}, \widetilde{X}^{(2)}) \arrow[r, "\partial''"] \arrow[d, equal] \arrow[dr, "\bar{\partial}"] & H_2(\widetilde{X}^{(2)}) \arrow[r] \arrow[d, "j_*"] & H_2(\widetilde{X}) \\
0 & & C_3(\widetilde{X}) & H_2(\widetilde{X}^{(2)}, \widetilde{X}^{(1)}) \arrow[d, equal] & \\
 & & & C_2(\widetilde{X}) &
\end{tikzcd}$$

In the top ladder all vertical arrows are isomorphisms; in the bottom ladder $\bar{h}$, $\bar{\bar{h}}$ are isomorphisms by the Hurewicz theorem.

$$\begin{aligned}
H_3(\widetilde{X}) &= \ker\{\bar{\partial}\colon C_3(\widetilde{X}) \longrightarrow C_2(\widetilde{X})\} \\
&= \ker\{\partial''\colon H_3(\widetilde{X}, \widetilde{X}^{(2)}) \longrightarrow H_2(\widetilde{X}^{(2)})\} \\
&\cong \ker\{\partial\colon \pi_3(X, X^{(2)}) \longrightarrow \pi_2(X^{(2)})\} \quad \text{via } \bar{h} \circ \bar{p}_{\#}^{-1} \\
&= L_3(X).
\end{aligned}$$

It follows from the commutativity of the diagram that $\psi_3 = h \circ p_{\#}^{-1}$, if we identify $L_3(X)$ with $H_3(\widetilde{X})$. □

We denote the projective class group of the integral group ring $Z[\pi]$ by $\widetilde{K}^{\circ}(\pi)$; $[P]$ means the class in $\widetilde{K}^{\circ}(\pi)$ represented by the finitely generated projective π-module P. A finitely generated projective π-module P is *stably free* if and only if $[P] = 0$ in $\widetilde{K}^{\circ}(\pi)$.

THEOREM 2.2. *Let* π *be any finitely presentable group and* $(\pi, \pi_2, k) \in T \in \mathbf{FR}(\pi)$. *Then the following are true.*

N(a): π_2 *is a submodule of a free finitely generated* π*-module (hence* π_2 *is a free abelian group). If* $Z[\pi]$ *is Noetherian, then* π_2 *is a finitely generated* π*-module.*

N(b): *If* π *is a finite group of order* $|\pi|$, *Z-rank* $\pi_2 \equiv -1$ *(modulo* $|\pi|$*) and* k *is a generator of the finite cyclic group* $H^3(\pi, \pi_2) \cong Z_{|\pi|}$.

N(c): *If* π *is a group which has some element of finite order, then* π_2 *cannot be a projective* π*-module.*

N(d): *If* $[(\pi, 0, 0)] \in \mathbf{FR}(\pi)$, *then* $[\pi_2] = 0$ *in* $\widetilde{K}^0(\pi)$. *Furthermore,* $H^3(\pi, \pi_2) = 0 \Rightarrow$ *there is only a single 2-type for each pair* (π, π_2).

N(e): *If* $(\pi, \pi_2', k') \in T^1 \in \mathbf{FR}(\pi)$ *is any other finitely 2-realizable 2-type, then* $\pi_2 \oplus F \cong \pi_2' \oplus F'$ *for certain free finitely generated* π*-modules* F, F'.

N(f): *For any* $X \in \mathbf{FX}^3(T)$, *the Wall invariant* $\mathrm{Wa}_2[X] = [C_2(\widetilde{X})/B_2(\widetilde{X})] = 0 \in \widetilde{K}^0(\pi)$.

N(g): *For any* $X \in \mathbf{FX}^3(T)$, $[H_3(\widetilde{X})] = 0$.

N(h): *For any* $X \in \mathbf{FX}^3(T)$,

$$\pi_3(T) \oplus H_3(\widetilde{X}) \xrightarrow[\cong]{\binom{i}{s}} \pi_3(X)$$

where i *is the inclusion,* $s: H_3(\widetilde{X}) \longrightarrow \pi_3(X)$ *is any* π*-module homomorphism such that* $\psi_3 \circ s = 1$. *Furthermore, if* Y *is any 2-complex in* $\mathbf{FX}^3(T)$ *and* $f: X \rightleftarrows Y: g$ *are 2-inverses, then the following diagram commutes*:

$$\begin{tikzcd}
\pi_3(Y) \arrow[r, "f_\#"] \arrow[dr, "\binom{\mathrm{id}}{0}"'] & \pi_3(X) \arrow[d, "\cong"', "\binom{g\#}{hp\#\,1}"] \\
& \pi_3(Y) \oplus H_3(\widetilde{X})
\end{tikzcd}$$

N(i): *If* π *is any infinite group such that* $Z[\pi]$ *is weakly injective as a* π*-module, then* $H^3(\pi, \pi_2) \cong Z$ *and* k *is a generator provided* $\pi_2 \neq 0$. *In this case each pair* (π, π_2) *determines a single equivalence class in* $\mathrm{A}(\pi)$.

Note 1. In N(i), no such group can have finite cohomological dimension. (See [18].)

Note 2. For completeness, let me add two more necessary conditions that a 2-realizable two-type must satisfy. Their proofs will appear elsewhere.

DEFINITION. A π-module M has the *cancellation property* (CP) $\Longleftrightarrow$

any isomorphism $M \oplus (Z\pi)^i \cong M' \oplus (Z\pi)^j$ $(i \geq j)$ implies that $M' \cong M \oplus (Z\pi)^{i-j}$.

N(j) (R. Swan): $\pi_2 \oplus Z\pi$ has CP. (See [19].)

N(k) ([18]): π_2 satisfies $H_i(\pi; \pi_2) \cong H_{i+3}(\pi)$ $(i > 0)$. If π is finite, then $H^{i+3}(\pi; \pi_2) \cong H^i(\pi)$ $(i > 0)$.

PROOF. For the duration of this proof let Y be a finite 2-complex in $\mathbf{FX}^3(T)$.

(a) $\pi_2(Y)$ is a submodule of $C_2(\widetilde{Y})$, the second cellular chain module of the universal cover $\widetilde{Y}$, which is a free π-module with rank the number of 2-cells in Y.

It is known that $Z[\pi]$ is Noetherian if π is a finite extension of a polycyclic group [6]. This is true if π is finite or finitely generated abelian. An example of a finite 2-complex K with $\pi_2(K)$ a nonfinitely generated $Z[\pi]$-module is given by J. Stallings in [13].

(b) If π is finite, then

$$\chi(\widetilde{Y}) = |\pi| \cdot \chi(Y) \Rightarrow Z\text{-rank } \pi_2 + 1 = |\pi|(\text{rank } H_2(Y) + 1)$$

$$\Rightarrow Z\text{-rank } \pi_2 = |\pi|(\text{rank } H_2(Y) + 1) - 1.$$

That k is a generator in $H^3(\pi, \pi_2) \cong Z_{|\pi|}$ follows from the same argument as W. Cockcroft and R. Swan in [4] and uses the fact that for π finite, $Z[\pi]$ is weakly injective [3, p. 199].

(c) If π_2 were projective, then the chain complex $0 \longrightarrow \pi_2 \longrightarrow C_2(\widetilde{Y}) \longrightarrow C_1(\widetilde{Y}) \longrightarrow C_0(\widetilde{Y}) \longrightarrow Z \longrightarrow 0$ gives a projective resolution of the trivial π-module Z of length 3, which is impossible according to a classical theorem of P. A. Smith [12, p. 287], provided π has an element of finite order.

(d) Let $Z \in \mathbf{FX}^3([(\pi, 0, 0)])$ be a 2-complex. Then by a theorem of J. H. C. Whitehead [17] there exist integers m, n such that

$$Z \vee \left(\bigvee^m S^2\right) \simeq_s Y \vee \left(\bigvee^n S^2\right) \Rightarrow \pi_2\left(Z \vee \left(\bigvee^m S^2\right)\right) \cong (Z[\pi])^m$$

$$\cong \pi_2(Y) \oplus (Z[\pi])^n \Rightarrow \pi_2(Y)$$

is stably free. Furthermore, $C(\widetilde{Z})$ is a projective resolution of Z of length $2 \Rightarrow H^3(\pi, \pi_2) = 0$.

(e) follows from the above theorem of J. H. C. Whitehead.

(f) $X \in \mathbf{FX}^3(T)$. Again the theorem of J. H. C. Whitehead implies

$$X \vee \left(\bigvee^m S^3\right) \simeq_s Y \vee \left(\bigvee^n S^3\right) \tag{2.3}$$

for some integers m, n. We attach n 4-cells to both sides of (2.3) to kill the $\bigvee^n S^3$, obtaining

$$W = \left(X \vee \left(\bigvee^m S^3\right)\right) \cup e_1^4 \cup \cdots \cup e_n^4 \simeq Y.$$

Thus W satisfies D2 of [15, I, p. 67]. The proof of Lemma 2.1 of [15, I] implies $\mathrm{Wa}_2[W] = 0$. But $C_2(\widetilde{W}) = C_2(\widetilde{X})$, $B_2(\widetilde{W}) = B_2(\widetilde{X}) \Rightarrow \mathrm{Wa}_2[X] = 0$ as well.

$$\begin{aligned}
\text{(g) } C_2(\widetilde{X})/B_2(\widetilde{X}) \text{ is stably free } &\Rightarrow C_2(\widetilde{X}) \cong B_2(\widetilde{X}) \oplus C_2(\widetilde{X})/B_2(\widetilde{X}) \\
&\Rightarrow B_2(\widetilde{X}) \text{ stably free} \\
&\Rightarrow C_3(\widetilde{X}) \cong B_2(\widetilde{X}) \oplus H_3(\widetilde{X}) \\
&\Rightarrow H_3(\widetilde{X}) \text{ stably free.}
\end{aligned}$$

(h) If $X \in \mathrm{FX}^3(T)$, then there are 2-inverses $f: Y \rightleftarrows X : g$ such that $gf \cong 1: Y \longrightarrow Y$, $fg|_{X^{(2)}} \cong i: X^{(2)} \longrightarrow X$. By (2.1)

$$0 \longrightarrow \pi_3(T) \longrightarrow \pi_3(X) \xrightarrow{h \circ p_{\#}^{-1}} H_3(\widetilde{X}) \longrightarrow 0$$

is exact; by (g) $H_3(\widetilde{X})$ is projective. Thus

$$\pi_3(T) \oplus H_3(X) \xrightarrow[\cong]{\binom{i}{s}} \pi_3(X)$$

where s is any π-splitting such that $(h \circ p_{\#}^{-1})s = 1$. It is not difficult to see that the above maps f, g induce

$$\begin{aligned}
\pi_3(X) &\cong \operatorname{im} f_{\#} \oplus \ker g_{\#} \\
&\cong \pi_3(Y) \oplus \ker g_{\#} \\
&\cong \pi_3(T) \oplus H(\widetilde{X}).
\end{aligned}$$

(i) An easy computation similar to [4] gives $H^3(\pi, \pi_2) \cong Z$ and k a generator, provided π is infinite and $Z[\pi]$ is weakly injective as a π-module. $(\pi, \pi_2, k) \cong (\pi, \pi_2, -k)$ via id: $\pi \longrightarrow \pi$ and $\lambda: \pi_2 \longrightarrow \pi_2$, where $\lambda(x) = -x$ for each $x \in \pi_2$.

3. **Fundamental groups possessing** $(SF \Rightarrow F)$. We say that a group π *has* $(SF \Rightarrow F)$ provided any finitely generated projective π-module P such that $[P] = 0 \in \widetilde{K}^0(\pi)$ is free. It is known by a theorem of H. Jacobinski (see [7], [11, Theorem 19.8] or [14, p. 178]) that if π is a finite group which has no quotient group isomorphic to a generalized quaternion group or any one of three exceptional groups (the binary tetrahedral, octahedral, or icosahedral groups) then π has $(SF \Rightarrow F)$. Thus any finite group which is abelian, simple, or of odd order has $(SF \Rightarrow F)$. Also if π is free of finite rank, then π has $(SF \Rightarrow F)$ [1].

THEOREM 3.1. *Let* π *have* $(SF \Rightarrow F)$. $T \in \mathrm{FR}(\pi) \Longleftrightarrow$ *for any* $X \in \mathrm{FX}^3(T)$, X *has the homotopy type of a finite* 2*-complex wedged with a finite number of* 3*-spheres.*

PROOF. If $X \in \mathbf{FX}^3(T) \Rightarrow X \simeq Y \vee (\bigvee^k S^3)$, where Y is a finite 2-complex, then clearly $T \in \mathbf{FR}(\pi)$. If $T \in \mathbf{FR}(\pi)$, then there exists a finite 2-complex $Y \in \mathbf{FX}^3(T)$. Then there are 2-inverses $f : Y \rightleftarrows X : g$ inducing a 2-equivalence. By parts (g) and (h) of Theorem 2.2

$$\pi_3(X) \xrightarrow[\binom{g_\#}{hp_\#^{-1}}]{\cong} \pi_3(Y) \oplus H_3(\widetilde{X})$$

and $[H_3(\widetilde{X})] = 0 \Rightarrow H_3(\widetilde{X})$ is a free π-module of finite rank k on generators $\{\alpha_1, \cdots, \alpha_k\}$. Then $Y \vee (\bigvee^k S_i^3) \xrightarrow{\bar{f}} X$ is a homotopy equivalence, where $\bar{f}|_Y = f$ and the homotopy class of $\bar{f}|_{S_i^3} : S_i^3 \longrightarrow X$ is $\alpha_i \in \pi_3(X)$. $\square$

Two corollaries follow easily from 3.1.

COROLLARY 3.2 (CANCELLATION). *Let X, Y be finite 3-complexes such that $X \vee (\bigvee^n S_i^3) \simeq Y \vee (\bigvee^m S_i^3)$ where $m \geq n$. Assume that $\pi_1 X$ has $(SF \Rightarrow F)$ and that $T(X) \in \mathbf{FR}(\pi_1 X)$. Then $X \simeq Y \vee (\bigvee^{m-n} S_i^3)$.* $\square$

Let $\mathbf{HFX}^3(T)$ be the set of homotopy classes of connected, finite 3-complexes with 2-types in T and $[*]$ be the homotopy class of $*$.

COROLLARY 3.3 (HOMOTOPY CLASSIFICATION). *Let $T \in \mathbf{FR}(\pi)$ and π have $(SF \Rightarrow F)$. If W is any finite 2-complex having 2-type T, then* $\mathbf{HFX}^3(T) = \{[W \vee (\bigvee^n S^3)] \mid n = 0, 1, 2, \cdots\}$. $\square$

These same theorems are true for $(n+1)$-dimensional finite connected CW-complexes whose n-types $T(n)$ are n-realizable $(n = 1, 2, 3, \cdots)$, provided π_1 has $(SF \Rightarrow F)$. The case $n = 1$ is due to H. Bass and C. T. C. Wall [15, I, Theorem 3.3].

4. **Stably 2-realizable 2-types.** We say that a 2-type

$$T = [(\pi, \pi_2, k)] \in \mathbf{FA}(\pi)$$

is *stably 2-realizable* if there is a free π-module F^n of rank n such that the 2-type $T \oplus F^n = [(\pi, \pi_2 \oplus F^n, \bar{k})]$ is finitely 2-realizable. Here $\bar{k}$ is the image of k under the homomorphism

$$H^3(\pi, \pi_2) \xrightarrow{i_*} H^3(\pi, \pi_2 \oplus F^n) \cong H^3(\pi, \pi_2) \oplus H^3(\pi, F^n)$$

induced by the inclusion $\pi_2 \longrightarrow \pi_2 \oplus F^n$.

Note. If $Z[\pi]$ is a *weakly injective*, then i_* is an *isomorphism.*

THEOREM 4.1. *$T \in \mathbf{FA}(\pi)$ is stably 2-realizable $\Longleftrightarrow$ for any $X \in \mathbf{FX}^3(T)$ the Wall invariant $\mathrm{Wa}_2[X] = 0$ in $\widetilde{K}^0(\pi)$.*

PROOF. Suppose there is an integer n and a finite 2-complex Y such that $T(Y) \in T \oplus F^n$. Let $X \in \mathrm{FX}^3(T)$; then $Z = X \vee (\bigvee^n S^2) \in \mathrm{FX}^3(T \oplus F^n)$. By the theorem of Whitehead [17] there are integers s, t such that

$$Y \vee (\bigvee^s S^3) \simeq Z \vee (\bigvee^t S^3) = X \vee (\bigvee^n S^2) \vee (\bigvee^t S^3).$$

We adjoin s 4-cells to both sides of the above equation to kill $\bigvee^s S^3$. This gives $Y \simeq (X \vee (\bigvee^n S^2) \vee (\bigvee^t S^3)) \cup e_1^4 \cup \cdots \cup e_s^4 = W$. W is a finite 4-complex satisfying D2 $\Rightarrow \mathrm{Wa}_2[W] = 0$ [13, I, Theorem F]. But

$$\begin{aligned} &C_2(\widetilde{W}) = C_2(\widetilde{X}) \oplus F^n, \\ &B_2(\widetilde{W}) = B_2(\widetilde{X}) \Rightarrow \mathrm{Wa}_2[W] = \mathrm{Wa}_2[X] \oplus [F^n] = 0 \\ &\qquad\qquad\qquad \Rightarrow \mathrm{Wa}_2[X] = 0. \end{aligned}$$

Let $X \in FX^3(T)$ be such that $\mathrm{Wa}_2[X] = 0$. Assume that the zero skeleton $X^{(0)}$ of X is a single point. Choose an integer n such that $C_2(\widetilde{X})/B_2(\widetilde{X}) \oplus F^n$ is free. Then

$$\begin{array}{ccccccccc} & & & \begin{pmatrix} \partial_2 & 0 \\ 0 & 1' \end{pmatrix} & (\bar{x}_1 - 1, \bar{x}_2 - 1, \cdots, \bar{x}_m - 1, \underbrace{0, \cdots, 0}_{n}) & & & & \\ & & & \| & \| & & & & \\ 0 \longrightarrow \pi_2(X) \longrightarrow & C_2(\widetilde{X})/B_2(\widetilde{X}) \oplus F^n & & \xrightarrow{\widetilde{\partial}_2} & C_1(\widetilde{X}) \oplus F^n & \xrightarrow{\widetilde{\partial}_1 = \partial_1 + 0} & Z[\pi] & \xrightarrow{\epsilon} Z \longrightarrow 0 \\ & \| & & & \| & & \| & & \\ & \langle y_1, \cdots, y_k \rangle & & & \langle x_1, \cdots, x_m, z_1, \cdots, z_n \rangle & & \widetilde{C}_0 & & \\ & \| & & & \| & & & & \\ & \widetilde{C}_2 & & & \widetilde{C}_1 & & & & \end{array}$$

is an exact chain complex such that $\widetilde{C}_0, \widetilde{C}_1$, and $\widetilde{C}_2$ are free π-modules with the indicated bases and the set $\{\bar{x}_1, \cdots, \bar{x}_n\}$ forms a set of generators for the group $\pi = \pi_1 X$ (see [15, II, p. 136]). The element $k(X) \in H^3(\pi_1 X, \pi_2 X)$ is the cohomology class of the π-homomorphism k in the diagram as follows

$$\begin{tikzcd} & B_3 \arrow[r, "\bar{\partial}_3"] \arrow[d, "k"] & B_2 \arrow[r, "\bar{\partial}_2"] \arrow[d, "\alpha_2"] & B_1 \arrow[dr, "\bar{\partial}_1"] \arrow[d, "\alpha_1"] & & \\ 0 \arrow[r] & \pi_2(X) \arrow[r] & \widetilde{C}_2 \arrow[r, "\widetilde{\partial}_2"'] & \widetilde{C}_1 \arrow[r, "\widetilde{\partial}_1"'] & Z[\pi] \arrow[r, "\epsilon"] & Z \arrow[r] & 0 \end{tikzcd}$$

where $B_3 \longrightarrow B_2 \longrightarrow B_1 \longrightarrow Z\pi \longrightarrow Z \longrightarrow 0$ is a portion of the bar construction, α_1, α_2 are chain maps, and $k = \alpha_2 \circ \bar{\partial}_3$.

We may assume that this set of generators $\{\bar{x}_1, \cdots, \bar{x}_m\}$ *is the set of generators for some standard (preassigned) presentation* $P = \{a_1, \cdots, a_m : r_1, \cdots, r_l\}$ *of* π. Let

$$\begin{array}{l} 1 \longrightarrow R \longrightarrow F(a_1, \cdots, a_m) \xrightarrow{\varphi} \pi \longrightarrow 1 \\ \qquad \| \\ \quad \langle r_1, \cdots, r_l \rangle \end{array}$$

be the exact sequence of groups where $\varphi(a_i) = \bar{x}_i$ $(i = 1, \cdots, m)$, $F(a_1, \cdots, a_m)$ is the free group of rank m freely generated by $\{a_1, \cdots, a_m\}$, and R is the smallest normal subgroup generated by $\{r_1, \cdots, r_l\}$. This is done by letting Y be the 2-complex determined by the presentation P; i.e., $Y = (\bigvee^m S_i^1) \cup_{r_1} e_1^2 \cup_{r_2} e_2^2 \cup_{r_3} \cdots \cup_{r_l} e_l^2$, and choosing a map $f\colon Y \longrightarrow X$ inducing an isomorphism on $\pi = \pi_1 Y$. Then we use Lemma 1.1 of [15, I] to add 2-cells and 3-cells to Y to create a homotopy equivalence.

In order to realize $(\pi, \pi_2 \oplus F^s, \bar{k})$ for some $s \geqslant 0$, we use the homomorphism ρ described in [5, II]. Consider the expanded presentation

$$\begin{array}{l} 1 \longrightarrow \langle r_1, \cdots, r_l, b_1, \cdots, b_n \rangle \longrightarrow F(a_1, \cdots, a_m, b_1, \cdots, b_n) \xrightarrow{\varphi} \pi \longrightarrow 1 \\ \qquad \| \\ \qquad R' \end{array}$$

given by $P' = \{a_1, \cdots, a_m, b_1, \cdots, b_n : r_1, \cdots, r_l, b_1, \cdots, b_n\}$. There is a *surjective* group homomorphism

$$\rho\colon R' \longrightarrow \ker \tilde{\partial}_1 = \ker \partial_1 \oplus F^n$$

(see 4.2) which has kernel $[R', R']$, the commutator subgroup of R'. See [5, II] and [16, II]. Briefly, ρ is defined as follows: define the free *crossed homomorphism*

$$\bar{\rho}\colon F(a_1, \cdots, a_m, b_1, \cdots, b_n) \longrightarrow \tilde{C}_1(x_1, \cdots, x_m, z_1, \cdots, z_n)$$

by

(a) $\bar{\rho}(a_i) = x_i$ $(i = 1, \cdots, m)$, $\bar{\rho}(b_j) = z_j$ $(j = 1, \cdots, n)$,

(b) $\bar{\rho}(a_i^{-1}) = -\bar{x}_i^{-1} x_i$, $\bar{\rho}(b_j^{-1}) = -z_j$.

(c) If $W_1, W_2 \in F$, then $\bar{\rho}(W_1 W_2) = \bar{\rho}(W_1) + \varphi(W_1) \cdot \bar{\rho}(W_2)$.

We define $\rho \equiv \bar{\rho}|_{R'}$. By (c), ρ is a homomorphism. For each $\tilde{\partial}_2 y_i \in \ker \tilde{\partial}_1$, we choose $\bar{r}_i \in R'$ such that $\rho(\bar{r}_i) = \tilde{\partial}_2 y_i$ $(i = 1, \cdots, k)$.

Express each $\rho(r_i)$ $(i = 1, \cdots, l)$ as a π-linear combination of $\{\rho(\bar{r}_j) | j = 1, \cdots, k\}$ ($\{\partial_2 y_i | i = 1, \cdots, k\}$ generates $\ker \tilde{\partial}_1$)

$$\rho(r_i) = \sum_{j=1}^{k} \alpha_{ij} \rho(\bar{r}_j) \qquad (\alpha_{ij} \in Z[\pi], j = 1, \cdots, k).$$

Using the definition of ρ, it is easy to see that one can choose words $W_i(\{\bar{r}_j\})$ $(i = 1, \cdots, l)$ in conjugates of the $\{\bar{r}_j\}$ so that

$$\rho(W_i(\{\bar{r}_j\})) = \sum_j \alpha_{ij} \rho(\bar{r}_j) = \rho(r_i) \qquad (i = 1, \cdots, l).$$

Therefore, there exists $\kappa_i \in \ker \rho$ $(i = 1, \cdots, l)$ such that

$$W_i \kappa_i = r_i \qquad (i = 1, \cdots, l).$$

Thus $Q = \{a_1, \cdots, a_m, b_1, \cdots, b_n : \bar{r}_1, \cdots, \bar{r}_k, \kappa_1, \cdots, \kappa_l\}$ is a presentation of π and the 2-complex X_Q modeled on the presentation Q has 2-type $(\pi, \pi_2 \oplus F^l, \bar{k})$.

We note from the proof that if $l_\pi = \min\{\#$ of relators in $P | P$ is a finite presentation of $\pi\}$ then $T \oplus F^{l_\pi} = [(\pi, \pi_2 \oplus F^{l_\pi}, \bar{k})]$ is finitely 2-realizable $\Longleftrightarrow$ for each $X \in \mathrm{FX}^3(T)$, $\mathrm{Wa}_2[X] = 0$. This remark gives rise to a rather amusing corollary.

COROLLARY 4.2. *Let F^n be the free group of rank n. F^n has $(SF \Rightarrow F) \Longleftrightarrow$ the tree $\mathrm{HT}(F^n)$ of homotopy types of connected, finite 2-complexes with fundamental group F^n is a single stalk generated by $\bigvee_{i=1}^n S_i^1$.*

PROOF. $(\Rightarrow)$ This is the standard proof given in [15, I].

$(\Leftarrow)$ Suppose M is a stably free, finitely generated F^n-module. There exists an integer $k \geq 0$ such that $M \oplus (Z[F^n])^k$ is free. Thus the two-type $(F^n, M, 0)$ is stably 2-realizable $\Rightarrow$ (since $l_{F^n} = 0$) (F^n, M) is 2-realizable $\Rightarrow M$ is free. $\square$

5. **Chain 2-realizable 2-types.** We say that $T = [(\pi, \pi_2, k)]$ is *finitely chain 2-realizable* $\Longleftrightarrow$ there is an *exact* chain complex of π-modules

$$\begin{tikzcd} C\colon 0 \arrow[r] & \pi_2 \arrow[r] & C_2 \arrow[r] & C_1 \arrow[r, "\partial_1"] & C_0 \arrow[r, "\epsilon"] \arrow[d, equal] & Z \arrow[r] & 0 \\ & & & & Z[\pi] & & \end{tikzcd}$$

where C_1, C_2 are free finitely generated π-modules, ϵ is the augmentation, and C_1 has a basis $\{x_1, \cdots, x_n\}$ such that $\partial_1 x_1 = g_i - 1$, where $g_i \in \pi$ $(i = 1, \cdots, n)$. k is the cohomology class of the π-module homomorphism k' in the diagram

$$\begin{tikzcd} & B_3 \arrow[r, "\bar{\partial}_3"] \arrow[d, "k'"] & B_2 \arrow[r, "\bar{\partial}_2"] \arrow[d, "\alpha_2"] & B_1 \arrow[dr, "\bar{\partial}_1"] \arrow[d, "\alpha_1"] & & & \\ 0 \arrow[r] & \pi_2 \arrow[r] & C_2 \arrow[r, "\partial_2"] & C_1 \arrow[r, "\partial_1"'] & Z[\pi] \arrow[r, "\epsilon"] & Z \arrow[r] & 0 \end{tikzcd}$$

where the top rung is part of the bar construction, α_1, α_2 are chain maps, and $k' = \alpha_2 \circ \partial_3$. Let $\mathrm{FCR}(\pi) = \{T \in \mathrm{FA}(\pi) | T$ is finitely chain 2-realizable$\}$. Always $\mathrm{FR}(\pi) \subset \mathrm{FCR}(\pi)$. $\mathrm{FR}(\pi) = \mathrm{FCR}(\pi)$ would imply the truth of a conjecture of C. T. C. Wall for the group π [15, II, p. 131].

THEOREM 5.1. *Let* Z_n *be the finite cyclic group of order* n. *Let* $x \in Z_n$ *be a generator. The following are equivalent*:

(a) $T \in \mathrm{FR}(Z_n)$,

(b) *for every* $X \in \mathrm{FX}^3(T)$, $X \simeq$ (*2-complex*) $\vee (\bigvee^k S^3)$,

(c) $T \in \mathrm{FCR}(Z_n)$,

(d) $\pi_2 = (x-1)Z[Z_n] \oplus (Z[Z_n])^m$ *and* k *is a generator of* $H^3(Z_n, \pi_2) \cong Z_n$,

(e) *for each* $X \in \mathrm{FX}^3(T)$, $\mathrm{Wa}_2[X] = 0$.

PROOF. (a) $\Longleftrightarrow$ (b) is (3.1) for $\pi = Z_n$. (a) $\Longleftrightarrow$ (d) is given in [4] and [5, I]. (a) $\Rightarrow$ (e) is (4.1). (e) $\Rightarrow$ (c) is clear. We show (c) $\Rightarrow$ (a). Let T be represented by the chain complex

$$\begin{array}{c} (x^{h_1}-1, x^{h_2}-1, \cdots, x^{h_l}-1) \\ \| \end{array}$$

$$0 \longrightarrow \pi_2 \longrightarrow C_2(y_1, \cdots, y_m) \xrightarrow{\partial_2} C_1(x_1, \cdots, x_l) \xrightarrow{\partial_1} Z[Z_n] \xrightarrow{\epsilon} Z \longrightarrow 0$$

where x generates Z_n, $\{h_1, \cdots, h_l\}$ are integers such that there are integers $\{\alpha_1, \cdots, \alpha_l\}$ where $\Sigma\, \alpha_i h_i \equiv 1 \pmod n$ (i.e., $\{x^{h_1}, \cdots, x^{h_l}\}$ generate Z_n). We claim that there is a basis $\{x'_1, \cdots, x'_l\}$ for C_1 such that the matrix for ∂_1 with respect to the new basis is

$$(\bar{x}-1, \underbrace{0, \cdots, 0}_{l-1}) \tag{5.2}$$

where $\bar{x}$ is a generator (possibly distinct from x) of Z_n. This can be done by extending the generators $\{x^{h_1}, \cdots, x^{h_l}\}$ to a finite presentation for Z_n:

$$(a_1, \cdots, a_l : \{[a_i, a_j] \mid 1 \leqslant i < j \leqslant l\},$$
$$\{(a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_l^{\alpha_l})^{h_i} a_i^{-1} \mid i = 1, \cdots, l\}, (a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_l^{\alpha_l})^n).$$

Apply the Nielson transformations (as in [5, I] or [10, p. 140]) to this presentation to alter it to a pre-abelian form

$$(b_1, \cdots, b_l \colon \{[b_i, b_j] \mid 1 \leqslant i < j \leqslant l\}, b_1^n W_1, b_2 W_2, \cdots, b_l W_l, W_{l+1})$$

where each word W_i $(i = 1, \cdots, l+1)$ has total exponent zero with respect to each b_j $(j = 1, \cdots, l)$. The elementary Neilson transformations used to transform $\{a_1, \cdots, a_l\} \longrightarrow \{b_1, \cdots, b_l\}$ as generators of the free group of rank l give a prescription for changing the basis $\{x_1, \cdots, x_l\}$ of C_1 to a basis $\{\bar{x}_1, \cdots, \bar{x}_l\}$ with the "right" matrix (5.2) for ∂_1. Specifically, each Nielson operation $a_i \longrightarrow a_i a_j^\epsilon$ $(\epsilon = \pm 1, i \neq j)$ corresponds to the elementary basis change

$$x_i \longrightarrow \begin{cases} x_i + (x^{h_i})x_j & \text{if } \epsilon = 1, \\ x_i - x^{(h_i - h_j)}x_j & \text{if } \epsilon = -1, \end{cases}$$

in C_1. Note that $\partial_1(x_i + ex^{h_i}x_j) = x^{h_i + \epsilon h_j} - 1$.

T is now represented by a chain complex of the form

$$\begin{array}{c} (\bar{x} - 1, 0, \cdots, 0) \\ \| \\ 0 \longrightarrow \pi_2 \longrightarrow C_2(y_1, \cdots, y_n) \xrightarrow{\partial_2} C_1(\bar{x}_1, \cdots, \bar{x}_l) \xrightarrow{\partial_1} Z[Z_n] \xrightarrow{\epsilon} Z \longrightarrow 0. \end{array}$$

The arguments of [5, I] show that we may change the basis of C_2 (again denoted by $\{y_1, \cdots, y_n\}$) so that the matrix of ∂_2 is given by

$$\begin{pmatrix} N & 0 \cdots 0 & 0 \cdots 0 \\ 0 & & \\ \vdots & \boxed{I_{l-1}} & \mathbf{0} \\ 0 & & \end{pmatrix}$$

where $N = \Sigma_{i=0}^{n-1} \bar{x}^i$. The chain complex above with these new bases is clearly realizable by a presentation [5, II] and hence by a 2-complex. □

THEOREM 5.2. *Let π be a finitely presentable group. The following statements are equivalent*:

(a) $T = [(\pi, \pi_2, k)] \in \mathrm{FCR}(\pi)$,

(b) T *is stably* 2*-realizable*,

(c) *for each* $X \in \mathrm{FX}^3(T)$, $\mathrm{Wa}_2[X] = 0$,

(d) *there is an* $X \in \mathrm{FX}^3(T)$ *such that* $\mathrm{Wa}_2[X] = 0$.

PROOF. (b) $\Longleftrightarrow$ (c) is (4.1); (c) $\Rightarrow$ (d) is clear; (d) $\Rightarrow$ (a) is given by (4.2). W will show that (a) $\Rightarrow$ (b); i.e., $T \in \mathrm{FCR}(\pi) \Rightarrow T \oplus F^n \in \mathrm{FR}(\pi)$ for some nonnegative integer n. We assume that

$$\begin{array}{c} (\bar{x}_1 - 1, \cdots, \bar{x}_n - 1) \\ \| \\ 0 \longrightarrow \pi_2 \longrightarrow C_2(y_1, \cdots, y_m) \xrightarrow{\partial_2} C_1(x_1, \cdots, x_n) \xrightarrow{\partial_1} Z[\pi] \longrightarrow Z \xrightarrow{\epsilon} 0 \end{array}$$

has 2-type T. By [15, II, p. 136], we know that $\{\bar{x}_i \in \pi | \, i = 1, \cdots, n\}$ generate π. Define a homomorphism φ from the free group F of rank n with generators $\{a_1, \cdots, a_n\}$ to π by $\varphi(a_i) = \bar{x}_i$ $(i = 1, \cdots, n)$. φ is surjective since $\{\bar{x}_i\}$ generate π. Denote $\ker \varphi$ by R. Since π is finitely presentable there exist words $\{r_1, \cdots, r_s\}$ such that R is the smallest normal subgroup containing $\{r_1, \cdots, r_s\}$ (see [20, pp. 73–74]). Then an argument similar to that used in Theorem 4.1 shows that $T \oplus F^s$ is 2-realizable. $\square$

COROLLARY 5.3. *Let X be a connected CW-complex with fundamental group $\pi_1 X \cong Z_n$ such that X is dominated by a finite 2-complex. X has the homotopy type of a finite 2-complex* $\Longleftrightarrow$ $\mathrm{Wa}_2[X] = 0$.

PROOF. ($\Rightarrow$) X has homotopy type of a finite 2-complex $\Rightarrow T(X) \in \mathrm{FR}(Z_n) \Rightarrow^{5.1(e)} \mathrm{Wa}_2[X^{(3)}] = \mathrm{Wa}_2[X] = 0$ since $X^{(3)} \in \mathrm{FX}^3(T(X))$.

($\Leftarrow$) X dominated by a finite 2-complex and $\mathrm{Wa}_2[X] = 0 \Rightarrow X \cong$ finite 3-complex Y [15, I, Theorem F]. $T(X) \cong T(Y)$ and $\mathrm{Wa}_2[X] = 0 \Rightarrow^{5.2} T(Y)$ is stably 2-realizable $\Rightarrow^{5.1} T(Y)$ finitely 2-realizable. Y dominated by a finite 2-complex $\Rightarrow H_3(\widetilde{Y}) = 0 \Rightarrow^{1.1} Y \simeq$ finite 2-complex. $\square$

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