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SOME THEOREMS ON (CA) ANALYTIC GROUPS

BY

DAVID ZERLING[1]

ABSTRACT. An analytic group G is called (CA) if the group of inner automorphisms of G is closed in the Lie group of all (bicontinuous) automorphisms of G. We show that each non-(CA) analytic group G can be written as a semidirect product of a (CA) analytic group and a vector group. This decomposition yields a natural dense immersion of G into a (CA) analytic group H, such that each automorphism of G can be extended to an automorphism of H. This immersion and extension property enables us to derive a sufficient condition for the normal part of a semidirect product decomposition of a (CA) analytic group to be (CA).

1. **Introduction.** By an analytic group and an analytic subgroup of a Lie group, we mean a connected Lie group and a connected Lie subgroup, respectively. If G and H are Lie groups and ϕ is a one-to-one (continuous) homomorphism from G into H, ϕ will be called an immersion. ϕ will be called closed or dense, as $\phi(G)$ is closed or dense in H. We have used the term "immersion" rather than "imbedding", which is used in [4], [8], [9], and [10], because of the terminology presently being used in manifold theory. G_0 will denote the identity component group of G.

If G is an analytic group, $A(G)$ will denote the Lie group of all (bicontinuous) automorphisms of G, topologized with the so-called generalized compact-open topology; see Goto [2], and Hochschild [5] and [6]. G will be called (CA) if $I(G)$, the Lie group of all inner automorphisms of G, is closed in $A(G)$. It is well known that G is (CA) if and only if its universal covering group is (CA).

If G is a normal analytic subgroup of an analytic group H, then each element h of H induces an automorphism of G, namely, $g \longmapsto hgh^{-1}$. We will denote this homomorphism from H into $A(G)$ by ρ_G. $I_H(h)$ will denote the inner automorphism of H determined by $h \in H$. More generally, if A is a subset of H, $I_H(A)$ will denote the set of all inner automorphisms of H deter-

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mined by the elements of A. $I_H(H)$ will be written as $I(H)$, and the mapping $h \longmapsto I_H(h)$ of H onto $I(H)$ will be denoted by I_H.

If N is an analytic group and H is an analytic subgroup of $A(N)$, then $N \circledS H$ will denote the semidirect product of N and H. On the other hand, if G is an analytic group containing a closed normal analytic subgroup N and a closed analytic subgroup H, such that $G = NH, N \cap H = \{e\}$, and such that the restriction of ρ_N to H is one-to-one, we will frequently identify G with $N \circledS \rho_N(H)$ and H with $\rho_N(H)$, that is, we may write $G = N \circledS H$.

We now state our main results, which are proved in §2.

MAIN STRUCTURE THEOREM. *Let G ba a non-(CA) analytic group with center Z. Then there exist a (CA) analytic group M, a toral group T in $A(M)$, and a dense vector subgroup V of T, such that*:

(i) *$H = M \circledS T$ is a (CA) analytic group.*

(ii) *G is isomorphic to the dense analytic subgroup $M \circledS V$ of H.*

(iii) *Z is contained in M.*

(iv) *If C denotes the center of H, then $Z_0 = C_0$, and $\pi(C)$ is finite, where π is the natural projection of H onto T. Moreover, if $I(G)$ is homeomorphic to Euclidean space, then $Z = C$.*

(v) *Each automorphism σ of G can be extended to an automorphism $\epsilon(\sigma)$ of H, such that ϵ: $A(G) \longrightarrow A(H)$ is a closed immersion.*

THEOREM 2.4. *Let G be a (CA) analytic group with center Z. Let N and H be a closed normal analytic subgroup and a closed analytic subgroup of G, respectively, such that $G = NH, N \cap H = \{e\}$. Let π denote the natural projection of G onto H.*

(i) *If $\pi(Z)$ is discrete in the relative topology of H, then N is (CA).*

(ii) *If $\pi(Z)$ is closed and $I(N)$ is homeomorphic to Euclidean space, then N is (CA).*

Among other things, these two theorems simplify, generalize, and unify several of the major results in van Est, [8], [9], and [10]. In §3 we will look at several applications of these theorems, as well as an example.

Because of its extreme importance in our work we state the main result in Goto [3]. For a subset S of $\mathrm{GL}(n, \mathbf{R})$, $\bar{S}$ will denote the closure of S in $\mathrm{GL}(n, \mathbf{R})$.

THEOREM (GOTO [3]). *Suppose that G is a nonclosed analytic subgroup of $\mathrm{GL}(n, \mathbf{R})$. Let N be a maximal analytic subgroup of G, which contains the commutator subgroup of G and is closed in $\mathrm{GL}(n, \mathbf{R})$. Then there exists a closed vector subgroup V of G, such that $G = NV$, $N \cap V = \{e\}$, and $\bar{G} = N\bar{V}$,* *where* $\bar{V}$ *is a toral group.* $N \cap \bar{V}$ *is finite, and the space of* $\bar{G}$ *is diffeomorphic to the product space* $N \times \bar{V}$.

The following result of van Est will also be useful.

THEOREM (VAN EST [8]). *If* G *is a dense* (CA) *analytic subgroup of an analytic group* H, *then* $G = H$ *if and only if the center of* G *is closed in* H.

2. Main results.

LEMMA 2.1. *Let* G *be an analytic group. Let* M *and* H *be a closed normal analytic subgroup and a closed abelian analytic subgroup of* G, *respectively, such that* $G = MH, M \cap H = \{e\}$. *If* S *is a subset of* H, *then* $\rho_M(S)$ *is closed in* $A(M)$ *if and only if* $\rho_G(S)$ *is closed in* $A(G)$.

PROOF. Let ψ and ϕ denote the respective restrictions of ρ_M and ρ_G to H. Let $A'(M) = \{\alpha \in A(M)\colon \alpha \circ \psi(h) = \psi(h) \circ \alpha$ for all h in $H\}$. $A'(M)$ is a closed subgroup of $A(M)$. For each α in $A'(M)$ let $E\alpha$ denote the automorphism of G defined by $(E\alpha)(mh) = \alpha(m) \cdot h$. Then $\alpha \longmapsto E\alpha$ is a closed immersion of $A'(M)$ into $A(G)$. Since H is abelian, $\psi(H)$ is contained in $A'(M)$, and $\phi = E \circ \psi$. Therefore, $\psi(S)$ is closed in $A(M) \Longleftrightarrow \psi(S)$ is closed in $A'(M) \Longleftrightarrow E(\psi(S))$ is closed in $A(G) \Longleftrightarrow \phi(S)$ is closed in $A(G)$.

The following lemma is an altered version of a similar result found in Goto [4]. The proof given here is essentially the same as Goto's.

LEMMA 2.2. *Let* M *be an analytic group and let* K *be a compact analytic subgroup of* $A(M)$. *Let* F *be a closed central subgroup of* M, *such that each element of* K *keeps* F *elementwise fixed. Let* $m \in M$ *and suppose that* $\sigma(m) \cdot m^{-1}$ *is in* F *for all* σ *in* K. *Then* $\sigma(m) = m$ *for all* σ *in* K.

PROOF. Suppose that M is simply connected. Then F_0 is a vector group. Let $C\colon K \longrightarrow F$ be defined by $C(\sigma) = \sigma(m) \cdot m^{-1}$. For σ and τ in K we have

$$C(\sigma\tau) = (\sigma\tau)(m) \cdot m^{-1} = (\sigma\tau)(m) \cdot \sigma(m^{-1}) \cdot \sigma(m) \cdot m^{-1}$$

$$= \sigma(\tau(m) \cdot m^{-1}) \cdot \sigma(m) \cdot m^{-1} = \tau(m) \cdot m^{-1} \cdot \sigma(m) \cdot m^{-1} = C(\sigma) \cdot C(\tau).$$

Hence, $C(K)$ is a compact subgroup of F_0 and so $C(K) = \{e\}$. Therefore, $\sigma(m) = m$.

If M is not simply connected, let $\widetilde{M}$ be the universal covering group of M and let $\pi\colon \widetilde{M} \longrightarrow M$ be the natural projection. For each σ in $A(M)$ there is a unique $\widetilde{\sigma}$ in $A(\widetilde{M})$, such that $\pi \circ \widetilde{\sigma} = \sigma \circ \pi$, and $\sigma \longmapsto \widetilde{\sigma}$ is an isomorphism of $A(M)$ onto a closed subgroup of $A(\widetilde{M})$ (cf. Chevalley [1]).

Let $\widetilde{K} = \{\widetilde{\sigma}: \sigma \in K\}$. $\widetilde{K}$ is compact. Let $\widetilde{F} = \pi^{-1}(F)$. $\widetilde{F}$ is a closed central subgroup of $\widetilde{M}$, and $\widetilde{K}$ keeps $\widetilde{F}$ elementwise fixed. Select an $\widetilde{m}$ in $\widetilde{M}$ so that $\pi(\widetilde{m}) = m$. Since $\sigma(m) \cdot m^{-1} \in F$, we see that $\widetilde{\sigma}(\widetilde{m}) \cdot \widetilde{m}^{-1} \in \widetilde{F}$ for all $\widetilde{\sigma}$ in $\widetilde{K}$. We can now apply the simply connected case to conclude that $\widetilde{\sigma}(\widetilde{m}) = \widetilde{m}$ for all $\widetilde{\sigma}$ in $\widetilde{K}$. Therefore, $\sigma(m) = m$ for all σ in K.

THEOREM 2.1. *Let G be a non-(CA) analytic group with center Z. Then there exist an analytic group M, a toral group T in $A(M)$, and a dense vector subgroup V of T, such that*:

(i) *$H = M \circledS T$ is a (CA) analytic group.*

(ii) *G is isomorphic to the dense analytic subgroup $M \circledS V$ of H.*

(iii) *Z is contained in M.*

PROOF. If $d = \dim G$, then $A(G)$ is isomorphic to a closed Lie subgroup of $GL(d, \mathbf{R})$. So we may appeal to the result of Goto [3] we mentioned in §1.

Let N be a maximal analytic subgroup of $I(G)$, which contains the commutator subgroup of $I(G)$ and is closed in $GL(d, \mathbf{R})$. Then there is a closed vector subgroup V' of $I(G)$, such that $I(G) = N \cdot V', N \cap V' = \{e\}$, and $\overline{I(G)} = N \cdot \overline{V}'$, where $\overline{V}'$ is a toral group. Moreover, $N \cap \overline{V}'$ is finite, and the space of $\overline{I(G)}$ is diffeomorphic to the product space $N \times \overline{V}'$. Hence, each one dimensional vector subgroup of V' is not closed in $A(G)$. Let $V' = V'_q \cdot V'_{q-1} \cdots V'_1$ be a direct product decomposition of V' into one dimensional subgroups: $I(G) = NV'_q \cdot V'_{q-1} \cdots V'_1$.

For $I_G: G \longrightarrow I(G)$ let M and H_i, $1 \leqslant i \leqslant q$, denote the identity component groups of the complete inverse images of N and V'_i, respectively. M is closed and normal in G, and each H_i is closed in G. Moreover, $G = M \cdot H_q \cdot H_{q-1} \cdots H_1$, where $M \cap H_i$ is contained in Z for each i. The restriction of I_G to H_i is a homomorphism of H_i onto V'_i having kernel $Z \cap H_i$. Therefore, $Z \cap H_i$ is connected, and so it is contained in M. Also,

$$H_i = (Z \cap H_i) \cdot V_i, \quad Z \cap H_i \cap V_i = Z \cap V_i = \{e\},$$

where V_i is a one dimensional closed vector subgroup of H_i, such that $I_G(V_i) = V'_i$. Therefore,

$$G = M(Z \cap H_q) \cdot V_q \cdots (Z \cap H_1) \cdot V_1 = M \cdot V_q \cdot V_{q-1} \cdots V_1.$$

If $I_G(mv_q \cdots v_1) = e$, then $I_G(m) \cdot I_G(v_q) \cdots I_G(v_1) = e$. Therefore $I_G(m) = I_G(v_q) = \cdots = I_G(v_1) = e$. Since $Z \cap V_i = \{e\}$, we have $v_q = \cdots = v_1 = e$. Hence, Z is contained in M. In the same way we see that each element g in G can be written uniquely in the form $g = mv_qv_{q-1} \cdots v_1, m \in M, v_i \in V_i$. Therefore, G is homeomorphic to $M \times V_q \times V_{q-1} \times \cdots \times V_1$. In particular, $V_qV_{q-1}\cdots V_1$ is closed in G. We will now show that it is actually a closed vector subgroup of G.

Let $M_2 = MV_q \cdot V_{q-1} \cdots V_2$. M_2 is closed and normal in G, and $G = M_2V_1$, $M_2 \cap V_1 = \{e\}$. Let $\psi_1\colon V_1 \longrightarrow A(M_2)$ be given by $\psi_1(v_1)(m_2) = v_1m_2v_1^{-1}$. Since Z is contained in M_2, and since V_1 is abelian, we see that ψ_1 is an immersion. From Lemma 2.1 we see that $\psi_1(V_1)$ is not closed in $A(M_2)$, since $I_G(V_1) = V_1'$ is not closed in $A(G)$. Consider $M_2 \circledS \overline{\psi_1(V_1)}$, where $\overline{\psi_1(V_1)}$ is the closure of $\psi_1(V_1)$ in $A(M_2)$. $\overline{\psi_1(V_1)}$ is a toral group. Let $v_1 \in V_1$ and let $v_j \in V_j$, $2 \leqslant j \leqslant q$. Since $M_2 = M \cdot V_q \cdots V_2$, we see that

$$I_G(\psi_1(v_1)(v_j)) = I_G(v_1v_jv_1^{-1}) = I_G(v_j) \tag{1}$$

because V' is abelian. From (1) we have $(\psi_1(v_1)(v_j))v_j^{-1}$ is in Z. Therefore, $\sigma(v_j)v_j^{-1}$ is in Z for each σ in $\overline{\psi_1(V_1)}$.

Z is a closed central subgroup of M_2 and each element of $\overline{\psi_1(V_1)}$ keeps Z elementwise fixed. Therefore, by Lemma 2.2 we see that $\sigma(v_j) = v_j$ for each σ in $\overline{\psi_1(V_1)}$ and each v_j in V_j; in particular, $v_1v_j = v_jv_1$.

Since $G = MV_{\pi(q)}\cdots V_{\pi(1)}$ for each permutation π on $\{1, 2, 3, \cdots, q\}$, we can show that $v_iv_j = v_jv_i$ for all $v_i \in V_i$, $v_j \in V_j$, $1 \leqslant i, j \leqslant q$. Hence, $V_q \cdot V_{q-1} \cdots V_1$ is a closed vector subgroup of G which is isomorphic to V' under I_G.

Let $V = V_q \cdot V_{q-1} \cdots V_1$. Then $G = M \cdot V$, where M is a closed normal analytic subgroup of G containing Z, V is a closed vector subgroup of G, and $M \cap V = \{e\}$. Let $\psi\colon V \longrightarrow A(M)$ be given by $\psi(v)(m) = vmv^{-1}$. By Lemma 2.1, $\psi(V)$ is not closed in $A(M)$, since $I_G(V) = V'$ is not closed in $A(G)$. In fact, each one parameter subgroup of $\psi(V)$ is not closed in $A(M)$.

Let $T' = \overline{V}'$ and let $T = \overline{\psi(V)}$. Both T' and T are toral groups in $A(G)$ and $A(M)$, respectively. Let the mapping $mv \longmapsto (m, \psi(v))$ of G into $M \circledS T$ be denoted by ψ also, and let $H = M \circledS T$. Then $\psi\colon G \longrightarrow H$ is clearly a dense immersion. Since $I(H) = I_H(M) \cdot I_H(T)$, and since $I_H(T)$ is compact, H will be a (CA) analytic group, if we can show that $I_H(M)$ is closed in $A(H)$.

To this end let $\{I_H(m_n)\}$ converge to σ in $A(H)$, where m_n is in M for all n. Let mv be an element in G. Then $m_nmvm_n^{-1}$ converges in H to $\sigma(mv)$. However,

$$m_nmvm_n^{-1} = (m_nmm_n^{-1}) \cdot (m_nvm_n^{-1}v^{-1}) \cdot v.$$

Let $\overline{m_n} = m_nmm_n^{-1}$ and let $\gamma(m_n) = m_nvm_n^{-1}v^{-1}$. $\gamma(m_n)$ is contained in M, since M contains the commutator subgroup of G. So $\overline{m_n} \cdot \gamma(m_n) \cdot v$ converges in H to $\sigma(mv)$. Therefore, $\overline{m_n}\gamma(m_n)$ converges in H to $\sigma(mv)\cdot v^{-1}$. But, $\overline{m_n}\cdot\gamma(m_n)$ is in M for all n, and M is closed in H. So $\sigma(mv)\cdot v^{-1}$ is in M. Therefore, the restriction of σ to G, $\sigma|_G$, is an automorphism of G.

It is easy to see that $\{I_G(m_n)\}$ converges to $\sigma|_G$ in $A(G)$. Since $N = I_G(M)$ is closed in $A(G)$, we have $\sigma|_G = I_G(\bar{m})$ for some $\bar{m}$ in M. Hence, $\sigma = I_H(\bar{m})$. Therefore, $I_H(M)$ is closed in $A(H)$, and H is (CA). This completes the proof of our theorem.

DEFINITION. Suppose G is a non-(CA) analytic group. Let N, V', T', M, V, T, and H represent the analytic groups constructed in the above proof. Also, let $\psi\colon G \longrightarrow H$ be the dense immersion as constructed above. Then $[G = MV, \psi, H = M \circledS T]$ will be called a canonical triple associated with G, and $[I(G) = NV', \overline{I(G)} = NT']$ will be called the corresponding canonical pair.

LEMMA 2.3. *Let $[G = MV, \psi, H = M \circledS T]$ be a canonical triple associated with the non-(CA) analytic group G, and let $[I(G) = NV', \overline{I(G)} = NT']$ be the corresponding canonical pair. Then $\rho_G(H) = \overline{I(G)}$. Moreover, the restriction of ρ_G to T is an isomorphism of T onto T'.*

PROOF. Since T is compact, $\rho_G(T) = \rho_G(\overline{V}) = \overline{\rho_G(V)} = T'$. Since $\tau(m, v)\tau^{-1} = (\tau(m), v)$ for $m \in M, v \in V, \tau \in T$, we see ρ_G is 1-1 on T. Finally, $\rho_G(H) = \rho_G(M)\cdot\rho_G(T) = \overline{I(G)}$.

THEOREM 2.2. *Let $[G = MV, \psi, H = M \circledS T]$ be a canonical triple associated with the non-(CA) analytic group G, and let $[I(G) = NV', \overline{I(G)} = NT']$ be the corresponding canonical pair. Let Z and C denote the centers of G and H, respectively. Then $Z_0 = C_0$, and $\pi(C)$ is finite, where π is the natural projection of H onto T. Moreover, if $I(G)$ is homeomorphic to Euclidean space, then $Z = C$.*

PROOF. Suppose (m, τ) is in C, $m \in M, \tau \in T$. Then $\rho_G(m) = \rho_G(\tau)^{-1}$, and from Lemma 2.3 we have $\rho_G(\tau) \in T'$. Hence, $\rho_G(\tau)$ is in $T' \cap N$. Therefore, we can define the homomorphism $\sigma\colon C \longrightarrow T' \cap N$ by $\sigma(m, \tau) = \rho_G(\tau)$. Since $T' \cap N$ is finite (see the result of Goto [3] mentioned in our introduction), $\sigma(C_0) = \{e\}$. Since ρ_G is one-to-one on T, $\tau = e$, if $(m, \tau) \in C_0$. Since G is dense in H, $Z_0 = C_0$.

Since ρ_G is one-to-one on T, and since $T' \cap N$ is finite, it is clear that $\pi(C)$ is finite. Moreover, if $I(G)$ is homeomorphic to Euclidean space, then $T' \cap N$ is trivial; thus, $\sigma(C) = \{e\}$. It follows that $Z = C$.

THEOREM 2.3. *Let $[G = MV, \psi, H = M \circledS T]$ be a canonical triple associated with the non-(CA) analytic group G, and let $[I(G) = NV', \overline{I(G)} = NT']$ be the corresponding canonical pair. Then each automorphism σ of G*

can be extended to an automorphism $\epsilon(\sigma)$ *of* H, *such that* $\epsilon\colon A(G) \longrightarrow A(H)$ *is a closed immersion.*

PROOF. We have

$$G = MV \cong M \circledS \rho_M(V) \quad \text{and} \quad H = M \circledS \overline{\rho_M(V)}.$$

Let σ be an automorphism of G. Then

$$G = \sigma(M) \cdot \sigma(V) \cong \sigma(M) \circledS \rho_{\sigma(M)}(\sigma(V)).$$

Since $\rho_G(H) = \overline{I(G)}$ from Lemma 2.3, we can construct the homomorphism $\Psi\colon T \longrightarrow \rho_G(H)$, where

$$\Psi(\tau) = \sigma \circ \rho_G(\tau) \circ \sigma^{-1}. \tag{1}$$

It is clear that Ψ is an isomorphism of T onto a compact subgroup of $\rho_G(H)$.

From (1) we have

$$\Psi(\rho_M(v)) = \rho_G(\rho_{\sigma(M)}(\sigma(v))), \qquad v \in V. \tag{2}$$

Let $\mathrm{Cl}_H(\rho_{\sigma(M)}(\sigma(V)))$ denote the closure of $\rho_{\sigma(M)}(\sigma(V))$ in H. We see from (2) that the restriction of ρ_G to $\mathrm{Cl}_H(\rho_{\sigma(M)}(\sigma(V)))$ is one-to-one. We also see from (1) and (2) that

$$\rho_G(\rho_{\sigma(M)}(\sigma(V))) \subset \Psi(T) \subset \mathrm{Cl}_{A(G)}(\rho_G(\rho_{\sigma(M)}(\sigma(V)))). \tag{3}$$

Since $\Psi(T)$ is compact, we see from (3) that

$$\Psi(T) = \mathrm{Cl}_{A(G)}(\rho_G(\rho_{\sigma(M)}(\sigma(V)))).$$

We now wish to show that $\mathrm{Cl}_H(\rho_{\sigma(M)}(\sigma(V)))$ is a toral group; for once we know this we can conclude that

$$\Psi(T) = \rho_G(\mathrm{Cl}_H(\rho_{\sigma(M)}(\sigma(V)))). \tag{4}$$

To this end we let $A_0(M)$ and $A_0(\sigma(M))$ denote the identity component groups of $A(M)$ and $A(\sigma(M))$, respectively. Let

$$f\colon M \circledS A_0(M) \longrightarrow \sigma(M) \circledS A_0(\sigma(M))$$

be defined by $f(m, \alpha) = (\sigma(m), \alpha')$, where $\alpha'(\sigma(\bar{m})) = \sigma(\alpha(\bar{m}))$, $\bar{m} \in M$. f is an isomorphism onto, and

$$f(H = M \circledS \mathrm{Cl}_{A(M)}(\rho_M(V))) = \sigma(M) \circledS \mathrm{Cl}_{A(\sigma(M))}(\rho_{\sigma(M)}(\sigma(V))),$$

$$f(\mathrm{Cl}_{A(M)}(\rho_M(V))) = \mathrm{Cl}_{A(\sigma(M))}(\rho_{\sigma(M)}(\sigma(V))),$$

$$f(\rho_M(V)) = \rho_{\sigma(M)}(\sigma(V)).$$

Therefore,

$$\mathrm{Cl}_H(\rho_{\sigma(M)}(\sigma(V))) = \mathrm{Cl}_H(f(\rho_M(V))) = f(\mathrm{Cl}_H(\rho_M(V))) = f(\mathrm{Cl}_{A(M)}(\rho_M(V))),$$

where we have identified H and $f(H)$. Since $T = \mathrm{Cl}_{A(M)}(\rho_M(V))$ is a toral group, we see that $\mathrm{Cl}_H(\rho_{\sigma(M)}(\sigma(V)))$ is a toral group. Therefore, (4) is true.

Now let Φ denote the inverse of the isomorphism ρ_G from $\mathrm{Cl}_H(\rho_{\sigma(M)}(\sigma(V)))$ onto $\Psi(T)$. Define $\epsilon(\sigma)\colon H \longrightarrow H$ as follows:

$$\epsilon(\sigma)(m, \tau) = \sigma(m) \cdot (\Phi \circ \Psi)(\tau). \tag{5}$$

First of all, $(\Phi \circ \Psi)(\rho_M(v)) = \Phi(\rho_G(\rho_{\sigma(M)}(\sigma(v))))$, from (2). But $\Phi(\rho_G(\rho_{\sigma(M)}(\sigma(v)))) = \rho_{\sigma(M)}(\sigma(v))$ from the definition of Φ. However, $\rho_{\sigma(M)}(\sigma(v)) = \sigma(\rho_M(v))$ under the identifications $v \longleftrightarrow \rho_M(v)$ and $\sigma(v) \longleftrightarrow \rho_{\sigma(M)}(\sigma(v))$. Therefore, from (5) we see that $\epsilon(\sigma)(g) = \sigma(g)$ for all g in G. Since G is dense in H we also see that $\epsilon(\sigma) \in A(H)$ and $\epsilon\colon A(G) \longrightarrow A(H)$ is a closed immersion. This completes the proof of our theorem.

THEOREM 2.4. *Let G be a (CA) analytic group with center Z. Let N and H be a closed normal analytic subgroup and a closed analytic subgroup of G, respectively, such that $G = NH$, $N \cap H = \{e\}$. Let π denote the natural projection of G onto H.*

(i) *If $\pi(Z)$ is discrete in the relative topology of H, then N is (CA).*

(ii) *If $\pi(Z)$ is closed and $I(N)$ is homeomorphic to Euclidean space, then N is (CA).*

PROOF. Suppose that N is non-(CA). Let N' be a (CA) analytic group containing N as a dense subgroup, where N' is to be constructed according to the method in Theorem 2.1. Let $\epsilon\colon A(N) \longrightarrow A(N')$ be the extension homomorphism constructed in Theorem 2.3. Let $\beta = \epsilon \circ \rho_N$. Then the restriction of β to H is a homomorphism of H into $A(N')$, and we let G' denote the semidirect product of N' and H that is determined by β (cf. Hochschild [5]). Then G is dense in G'.

Let $\{(n_k, h_k)\}$ be a sequence of central elements in G converging in G' to (n', h).

Case (i). In this case we can say that $h_k = h$ for some k on, say, $k = 1$. Therefore, $n_k n_1^{-1} = (n_k, h) \cdot (n_1, h)^{-1}$ is in $Z \cap N$. Since the center of N is closed in N' by Theorem 2.1, and since $\rho_N(H)$ keeps each $n_k n_1^{-1}$ fixed, we see that $n' n_1^{-1}$ is in the center of N and is held fixed by $\rho_N(H)$. Hence, $n' n_1^{-1} \in Z$. Therefore, $(n', h) = \xi \cdot (n_1, h)$, $\xi \in Z$. So the center of G is closed in G'. Since G is (CA), we can appeal to the result of van Est [8] mentioned in our introduction to conclude that $G = G'$. Hence, $N = N'$. Therefore, N is (CA).

Case (ii). In this case, since $\pi(Z)$ is closed in H, there exists an element

$\bar{n}$ in N so that $(\bar{n}, h)$ is in Z. Since $n'\bar{n}^{-1} = (n', h) \cdot (\bar{n}, h)^{-1}$, we see that $n'\bar{n}^{-1} \in$ (center of G') $\cap N'$. Therefore, $n'\bar{n}^{-1}$ is in the center of N'. Since $I(N)$ is homeomorphic to Euclidean space, N and N' have the same center by Theorem 2.2. Thus, $n'\bar{n}^{-1}$ is in the center of N. Therefore, since $n'\bar{n}^{-1}$ is already in the center of G', it follows that $n'\bar{n}^{-1} \in Z$. So $(n', h) = \xi \cdot (\bar{n}, h)$, $\xi \in Z$. As in Case (i) above, we can conclude that N is (CA).

REMARK. In §3 we give an example which shows that the condition in (ii) above that $\pi(Z)$ be closed cannot be deleted, even if $I(N)$ is simply connected and solvable. However, the author does not know at this time if in (i) "discrete" can be replaced by "closed". He suspects that this cannot be done.

THEOREM 2.5. *Let* $[G = MV, \psi, H = M \circledS T]$ *be a canonical triple associated with the non-*(CA) *analytic group* G. *Then* M *is a* (CA) *analytic group.*

PROOF. According to Theorem 2.1, H is (CA), and from Theorem 2.2 we know that the collection of all τ in T, such that (m, τ) is in the center of H, is a finite subgroup of T. Therefore, we see that M is (CA) from Theorem 2.4.

We have now completed the proof of the Main Structure Theorem, which was stated in §1.

3. Applications and an example.

THEOREM 3.1. *An analytic group* G *with center* Z *is non-*(CA) *if and only if* $G \cong M \circledS V$, *where* M *is a* (CA) *analytic group containing* Z, *and* V *is a nonclosed vector subgroup of* $A(M)$.

PROOF. The "only if" part follows from the Main Structure Theorem. On the other hand, if G can be decomposed as above, then G is properly dense in $M \circledS \bar{V}$, such that Z is closed in $M \circledS \bar{V}$. Therefore, G must be non-(CA) by the previously mentioned result of van Est [8].

LEMMA 3.1. *Let* G *be a nilpotent analytic group and let* V *be a vector subgroup of* $A(G)$, *such that* $\bar{V}$ *is a toral group. Then* $G \circledS V$ *is non-*(CA).

PROOF. Since G is nilpotent and, therefore, (CA), $I(G)$ is a closed subgroup of $A(G)$, which is homeomorphic to Euclidean space. Hence, $\bar{V} \cap I(G)$ is trivial. This implies that the center of $G \circledS V$ is contained in G, so that the center of $G \circledS V$ is closed in $G \circledS \bar{V}$. Therefore, $G \circledS V$ is non-(CA) by van Est [8].

The following theorem represents a generalization of a result of Lee and Wu [7, 3.2 Theorem].

THEOREM 3.2. *Let* $[G = MV, \psi, H = M \circledS T]$ *be a canonical triple associated with a solvable, non-*(CA) *analytic group* G. *Let* G_1 *denote the closure of the commutator subgroup of* G. *Then* V *acts effectively on* G_1, *and* $G_1 \circledS V$ *is a non-*(CA) *closed normal analytic subgroup of* G.

PROOF. Let $\widetilde{G}$ be the universal covering group of G and let $\pi: \widetilde{G} \to G$ be the natural projection. Let $\mathcal{G}, \mathcal{M}$, and $\mathcal{V}$ denote the Lie algebras of G, M, and V, respectively. Then $\mathcal{G} = \mathcal{M} + \mathcal{V}$, where $\mathcal{M}$ is an ideal in $\mathcal{G}$, $\mathcal{V}$ is an abelian subalgebra, and $\mathcal{M} \cap \mathcal{V} = \{0\}$. Let $\widetilde{M}$ and $\widetilde{V}$ be the analytic subgroups of $\widetilde{G}$ generated by $\mathcal{M}$ and $\mathcal{V}$, respectively. Since $\widetilde{G}$ is simply connected, we see that $\widetilde{G} = \widetilde{M}\widetilde{V}$, where $\widetilde{M}$ is a closed normal analytic subgroup of $\widetilde{G}$, and $\widetilde{V}$ is a closed vector subgroup of $\widetilde{G}$, such that $\widetilde{M} \cap \widetilde{V} = \{e\}$. Moreover, the center of $\widetilde{G}$ is contained in $\widetilde{M}$ because the center of G is contained in M. Therefore, $\widetilde{G} = \widetilde{M} \circledS \widetilde{V}$, where we have identified $\widetilde{V}$ with $\rho_{\widetilde{M}}(\widetilde{V})$. Since the closure of V in $A(M)$ is a toral group, we see that the closure of $\widetilde{V}$ in $A(\widetilde{M})$ is a toral group.

Let $\widetilde{G}_1$ be the universal covering group of G_1. Using an argument of Lee and Wu [7], we will show that $\widetilde{V}$ acts effectively on $\widetilde{G}_1$. On the contrary, suppose that there is an element $v \in \widetilde{V}$, such that $vxv^{-1} = x$ for all x in $\widetilde{G}_1$. Then $v^n x v^{-n} = x$ for all $n \in \mathbf{Z}$. For $g \in \widetilde{G}$ we let $\gamma_n(g) = v^n g v^{-n} g^{-1} \in G_1$. Let $K(g) = \{\gamma_n(g): n \in \mathbf{Z}\}$. Since $v^k, k \in \mathbf{Z}$, commutes with each $x = \widetilde{G}_1, K(g)$ is an abelian subgroup of $\widetilde{G}_1$. Let X denote the closure of $\{wgw^{-1}g^{-1} \mid w \in \widetilde{V}\}$ in G. Since the closure of $\widetilde{V}$ in $A(\widetilde{G})$ is a toral group, it follows that X is compact. Therefore $\overline{K(g)}$ is compact. Since $\widetilde{G}_1$ is simply connected and solvable, $\overline{K(g)} = \{e\}$. Thus, v is central in $\widetilde{G}$ and so $\pi(v) \in V$ is central in G. Therefore, $v = e$. Hence, $\widetilde{V}$ acts effectively on $\widetilde{G}$.

If $v_1 x v_1^{-1} x^{-1} = e$ for some $v_1 \in V$ and all $x \in G_1$, then $\widetilde{v}\widetilde{x}\widetilde{v}^{-1}\widetilde{x}^{-1}$ is in the kernel of π, where $\widetilde{v} \in \widetilde{V}$ corresponds under π to $v_1 \in V$ and $\widetilde{x}$ is any element in $\widetilde{G}_1$. Since $\{\widetilde{v}\widetilde{x}\widetilde{v}^{-1}\widetilde{x}^{-1}: \widetilde{x} \in \widetilde{G}_1\}$ is connected, and since the kernel of π is discrete, we see that the effective action of $\widetilde{V}$ on $\widetilde{G}_1$ implies $\widetilde{v} = e$. Hence $v_1 = e$, and, therefore, V acts effectively on G_1.

Since the closure of $\rho_G(V)$ in $A(G)$ is a toral group, and since G_1 is characteristic in G, we see that the closure of $\rho_{G_1}(V)$ in $A(G_1)$ is a toral group. Therefore, it follows from Lemma 3.1 that $G_1 \circledS V$ is a non-(CA) closed normal analytic subgroup of G.

In [7] Lee and Wu proved that the dimension of a solvable non-(CA) analytic group is at least five. We now give a direct proof of the following theorem.

THEOREM 3.3. *The dimension of a non-*(CA) *analytic group is at least five.*

PROOF. Let $[G = MV, \psi, H = M \circledS T]$ be a canonical triple associated with a non-(CA) analytic group G. Then V contains a nonclosed one dimensional vector subgroup of $A(M)$. However, $A(M)$ is isomorphic with a closed Lie subgroup of $\mathrm{GL}(m, \mathbf{R})$ for $m = \dim M$, and it is well known that if A is an $m \times m$ complex matrix not all of whose entries are zero, then the one parameter subgroup $\exp \lambda A$, $\lambda \in \mathbf{R}$, will be nonclosed in $\mathrm{GL}(m, \mathbf{C})$ if and only if A is semisimple and its eigenvalues are all purely imaginary or zero, with the ratio of at least two eigenvalues being irrational. Since in our case A is a real matrix, the value of m must be at least four. Then, the dimension of $G = M \circledS V$ must be at least five.

We will now look at some applications of Theorem 2.4. In particular, we give a generalization of van Est's result [10] that the radical of a (CA) analytic group is also (CA).

THEOREM 3.4. *Let G be a (CA) analytic group. Let N be an analytic subgroup of G, such that $N = N_1 \subset N_2 \subset \cdots \subset N_q = G$, where N_i is a closed normal analytic subgroup of N_{i+1} containing the center of N_{i+1}, and N_{i+1}/N_i is a one dimensional vector group. Then N is (CA).*

PROOF. It suffices to show that if N itself is closed and normal in G, such that N contains the center of G and G/N is a one dimensional vector group, then N is (CA). In this case there exists a closed one dimensional vector subgroup V of G, such that $G = NV, N \cap V = \{e\}$. Since N contains the center of G, N is (CA) from Theorem 2.4.

THEOREM 3.5. *Let G be a (CA) analytic group with center Z. Let N and H be a closed normal analytic subgroup and a closed semisimple analytic subgroup of G, respectively, such that $G = NH, N \cap H = \{e\}$. Then N is (CA).*

PROOF. Let π denote the projection of NH onto H. Since $\pi(Z)$ is contained in the center of H, and since the center of H is discrete, we have $\pi(Z)$ is discrete. Therefore, N is (CA) from Theorem 2.4.

COROLLARY. *The radical of a (CA) analytic group G is also (CA).*

PROOF. The proof follows immediately by applying Theorem 3.5 to the Levi decomposition of the universal covering group of G.

EXAMPLE. Let $N = \mathbf{C} \times \mathbf{C}$ and let T denote the two dimensional toral group. Let $H = N \circledS T$, where for real r and s the action of $(e^{2\pi ir}, e^{2\pi is}) \in T$ on $(z_1, z_2) \in N$ is given by

$$(z_1, z_2) \longmapsto (e^{2\pi ir} z_1, e^{2\pi is} z_2).$$

Let h be a fixed irrational number. Let $V = \{(e^{2\pi ir}, e^{2\pi ihr}) \in T\colon\ r \in \mathbf{R}\}$. Then V is a dense one dimensional vector subgroup of T.

Let $G = N \circledS V$. G is clearly non-(CA), since its center is trivial. Since N is the commutator subgroup of G, we see that

$$I(G) = I_G(N) \circledS I_G(V) \cong N \circledS V$$

is a decòmposition of $I(G)$ of the type mentioned in Goto's theorem [3]. Thus $[G = NV, \psi, H = N \circledS T]$ is a canonical triple associated with G. In particular, H is (CA).

Let $L = \{(e^{2\pi ir}, 1) \in T\colon\ r \in \mathbf{R}\}$. Let $\widetilde{H}, \widetilde{G}$, and $\widetilde{L}$ be the universal covering groups of H, G, and L, respectively. Then $\widetilde{H} = \widetilde{G} \cdot \widetilde{L}$, $\widetilde{G} \cap \widetilde{L} = \{e\}$, where $\widetilde{G}$ is a closed normal analytic subgroup of $\widetilde{H}$, and $\widetilde{L}$ is a closed analytic subgroup of $\widetilde{H}$. Let Z denote the center of $\widetilde{H}$, and let π denote the natural projection of $\widetilde{H}$ onto $\widetilde{L}$. Then $\pi(Z)$ is properly dense in $\widetilde{L}$. Therefore the condition that $\pi(Z)$ be closed cannot be deleted in Case (ii) of Theorem 2.4.

BIBLIOGRAPHY

1. C. Chevalley, *Theory of Lie groups.* I, Princeton Math. Series, vol. 8, Princeton Univ. Press, Princeton, N. J., 1946. MR 7, 412.

2. M. Goto, *Semigroup of endomorphisms of a locally compact group,* Trans. Amer. Math. Soc. 87 (1958), 359–371. MR 20 #3229.

3. ———, *Analytic subgroups of* $GL(n, \mathbf{R})$, Tôhoku Math. J. (2) 25 (1973), 197–199.

4. ———, *Absolutely closed Lie groups,* Math. Ann. 204 (1973), 337–341.

5. G. Hochschild, *The structure of Lie groups,* Holden-Day, San Francisco, Calif., 1965. MR 34 #7696.

6. ———, *The automorphism group of a Lie group,* Trans. Amer. Math. Soc. 72 (1952), 209–216. MR 13, 624.

7. D. H. Lee and T. S. Wu, *On* CA *topological groups,* Duke Math. J. 37 (1970), 515–521. MR 42 #417.

8. W. T. van Est, *Dense imbeddings of Lie groups,* Nederl. Akad. Wettensch. Proc. Ser. A 54 = Indag. Math. 13 (1951), 321–328. MR 13, 432.

9. ———, *Dense imbeddings of Lie groups.* II, Nederl. Akad. Wettensch. Proc. Ser. A 55 = Indag. Math. 14 (1952), 255–274. MR 14, 135.

10. ———, *Some theorems on* (CA) *Lie algebras.* I, II, Nederl. Akad. Wettensch. Proc. Ser. A 55 = Indag. Math. 14 (1952), 546–568. MR 14, 720.

DEPARTMENT OF MATHEMATICS AND PHYSICS, PHILADELPHIA COLLEGE OF TEXTILES AND SCIENCE, PHILADELPHIA, PENNSYLVANIA 19144