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SMOOTH Z_p-ACTIONS ON SPHERES WHICH LEAVE KNOTS POINTWISE FIXED(1)

BY

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ABSTRACT. The paper produces, via handlebody construction, a family of counterexamples to the generalized Smith conjecture; that is, for each pair of integers (n, p) with $n \geqslant 2$ and $p \geqslant 2$ there are infinitely many knots (S^{n+2}, kS^n) which admit smooth semifree Z_p-actions (fixed on the knotted submanifold kS^n and free on the complement $(S^{n+2} - kS^n)$). This produces previously unknown Z_p-actions on (S^4, kS^2) for p even, the one case not covered by the work of C. H. Giffen. The construction is such that all of the knots produced are equivariantly null-cobordant. Another result is that if a knot admits Z_p-actions for all p, then the infinite cyclic cover of the knot complement is acyclic, and thus leads to an unknotting theorem for Z_p-actions.

0. Introduction. Let (M^{n+2}, N^n) denote a smooth codimension 2 manifold pair. The pair (M, N) is said to *admit a Z_p-action* if there exists a diffeomorphism $h: M \longrightarrow M$ such that $h^p = 1_M$, the orbit of each $x \in X = M - N$ is p points, and the orbit of each $x \in N$ is one point. That is, the Z_p-action generated by h is semifree: it is free on the complement X and fixed on the submanifold N. Suppose now that $k: S^n \longrightarrow S^{n+2}$ denotes a smooth embedding. In this setting we have the *generalized Smith conjecture*: If (S^{n+2}, kS^n) admits a Z_p-action $(p \geqslant 2)$ then (S^{n+2}, kS^n) is the unknotted sphere pair.

C. H. Giffen [3] has shown that the conjecture is false for all p when $n \geqslant 3$, and for odd p when $n = 2$. The proof uses the fibration properties of certain twist-spun knots [18] and the generalized Poincaré conjecture for $n \geqslant 3$.

In this paper, we develop a new construction for producing counterexamples to the generalized Smith conjecture which gives infinitely many different counterexamples for each pair of integers (n, p), $n \geqslant 2, p \geqslant 2$. In particular, it gives examples in the outstanding case of Z_p-actions (p even) on S^4. The method

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employs handlebody theory, and replaces use of the generalized Poincaré conjecture by use of the handle cancellation theorem. Counterexamples to the even period 4-dimensional Smith conjecture have also been produced by Gordon [4] and Vinogradov and Kusěl'man [16]. The latter paper, however, seems to have a gap in the proof, which occurs in the application of Smale theory to the low dimensional case (Lemma 2, p. 37). The method of construction in this paper has the advantage of being simpler than the others, and at the same time allows control over and easy calculation of the knot invariants. Moreover, all of the knots it produces are equivariantly null-cobordant, a result not obtainable by the other methods.

In §II we study homology invariants of knots which admit Z_p-actions, using higher-dimensional analogues of theorems of Fox [1]. We prove that if a knot admits Z_p-actions for all p, then the infinite cyclic cover $\widetilde{X}$ of the complement X is acyclic. We use this to prove a special case of the unknotting theorem of Hsiang [6] for S^1-actions.

Since all of the surgery and handlebody theory can be done for the case $n = 1$, we attempt to produce a Z_2-action on S^3. The construction predictably fails, due to linking problems of 1-dimensional attaching spheres in a 3-dimensional manifold. All that is produced is an interesting Z_2-action on a homology 3-sphere leaving a knot pointwise fixed.

I. **The construction.** The main idea in the construction is to produce a knotted ball pair (B^{n+3}, kB^{n+1}) via handle addition to the complement of the unknotted ball pair. At the same time we study the induced handlebody structure for the p-fold branched cyclic covering of the knotted ball pair, which has a natural smooth Z_p-action on it leaving the knotted submanifold pointwise fixed. One then uses the handle cancellation theorem to prove that the p-fold branched cyclic covering is again B^{n+3}, bypassing the Poincaré conjecture.

We will study in detail the construction of a Z_2-action on S^4. The method immediately generalizes to Z_p-actions on S^n. Fix the following notation: Λ denotes the integral group ring of the infinite cyclic group, with t as the multiplicative generator for the infinite cyclic group. If $Y = B^5 - kB^3$ denotes the complement of a knotted ball pair, let $\widetilde{Y}$ denote the infinite cyclic covering space of Y, and Y_p^b denote the p-fold branched cyclic cover of the ball pair (B^5, kB^3). If $(S^4, kS^2) = \partial(B^5, kB^3)$ denotes the boundary sphere pair, we let $X = \partial Y$ denote the n-knot complement, and $\widetilde{X} = \partial\widetilde{Y}$, $X_p^b = \partial Y_p^b$ denote respectively the infinite cyclic cover and the p-fold branched cyclic cover. We let $\approx$ denote diffeomorphism, $\simeq$ denote homotopy of embeddings or homotopy equivalence of spaces, and $\vee$ denote wedge product.

If $\lambda \in \Lambda$, let Λ/λ denote the cyclic Λ-module of order λ. For each integer $\gamma \geqslant 2$, let $\lambda_{2,\gamma}(t) = \gamma + (1-\gamma)t^2$. For each γ, we will produce a knotted ball pair (B^5, kB^3) satisfying:

(i) $Y_2^b \approx B^5$,

(ii) $H_1(\partial\widetilde{Y}, Z) \cong H_1(\widetilde{Y}, Z) \cong \Lambda/\lambda_{2,\gamma}$.

In this case $X_2^b = \partial Y_2^b$ is the desired example of a Z_2-action on S^4 Different choices of γ will produce different examples.

The construction is the same as that in [13], [14], and begins by adding a 1-handle to the complement of the unknotted ball pair $B = B^5 - B^3$. Let $K = B \cup h^1$. Then $K \approx S^1 \times D^4 - B^3$, and we let α denote the homotopy class of $S^1 \times \{*\} \subset \partial K \approx S^1 \times S^3 - S^2$. Now the 2-fold branched cyclic covering of the unknotted ball pair (B^5, B^3) is again the unknotted pair (B^5, B^3). Every handle added to the complement of B^3 in the base space can be covered by adding two handles to the 2-fold (unbranched) covering space. The 2-fold unbranched cyclic covering of K is

$$K_2^u = (B^5 - B^3) \cup h_1^1 \cup h_2^1 \approx (S^1_{\alpha_1} \times D^4 \# S^1_{\alpha_2} \times D^4) - B^3$$

where # denotes connected sum along the boundary. Following [13], we have that $\pi_1(K) \cong \pi_1(\partial K) = \langle\alpha\rangle * \langle\beta\rangle$, the free product of the infinite cyclic groups $\langle\alpha\rangle$ and $\langle\beta\rangle$, generated by the loops α (once around the handle) and β (once around the submanifold). Add a 2-handle to K by the word $g(\gamma) = \alpha^\gamma\beta^2\alpha^{(1-\gamma)}\beta^{-2} \in \pi_1(\partial K)$. Now $Y = K \cup_{g(\gamma)} h^2$ is diffeomorphic to $B^5 - kB^3$ for some knotted embedding k, because $Y \cup B^3 \approx B^5 \cup h^1 \cup_{g(\gamma)} h^2 \approx B^5$. This last diffeomorphism is due to the fact that in the 4-manifold $\partial(B^5 \cup h^1) \approx S^1 \times S^3$, homotopically embedded 1-spheres are concordant (as embeddings) by general position, hence isotopic by "concordance implies isotopy" in codimension 3 [7]. Now in $(B^5 \cup h^1)$, we have $\beta \simeq 1$ so $g(\gamma) \simeq \alpha$; hence by the handle cancellation theorem, h^2 cancels h^1. Similarly, in the 2-fold branched cyclic cover

$$Y_2^b \approx B^5 \cup \left(h_1^1 \underset{g(\gamma)_1}{\cup} h_1^2\right) \cup \left(h_2^1 \underset{g(\gamma)_2}{\cup} h_2^2\right)$$

we have two pairs of cancelling handles. This can be seen by computing the basepoint-free homotopy classes of the two lifts of the attaching sphere $g(\gamma)$, namely $g(\gamma)_1$ and $g(\gamma)_2$. Now $\pi_1(K_2^u) \cong \pi_1(\partial K_2^u) = \langle\widetilde{\beta}\rangle * \langle\alpha_1\rangle * \langle\alpha_2\rangle$ where α_1 and α_2 are the lifts of α, and β^2 lifts to $\widetilde{\beta}$, the loop around the codimension two submanifold in the branched cyclic covering. Since basepoint considerations are irrelevant in computing the homotopy classes of the attaching spheres in ∂K_2^u, by choosing a basepoint on h_1^1, we see that $g(\gamma)_1 \simeq \alpha_1^\gamma\widetilde{\beta}\alpha_1^{(1-\gamma)}\widetilde{\beta}^{-1}$(because β^2 becomes $\widetilde{\beta}$). In the boundary of the branched cyclic cover $K_2^b = K_2^u \cup$

B^3, we have $\widetilde{\beta} \simeq 1$ so $g(\gamma)_1 \simeq \alpha_1$; similarly $g(\gamma)_2 \simeq \alpha_2$. Hence the handles cancel, proving that $Y_2^b \approx B^5$. If $X_2^b = \partial Y_2^b$, then X_2^b is the 4-sphere with a Z_2-action on it leaving a knotted 2-sphere pointwise fixed. The fact that the embedded S^2 is knotted, and that different choices of γ give different knots, will be proved by calculating Δ^1, the 1-dimensional Alexander polynomials. This will be postponed until the next section. For a proof that $H_1(\partial\widetilde{Y}, Z) \cong_\Lambda H_1(\widetilde{Y}; Z) \cong_\Lambda \Lambda/\lambda_{2,\gamma}$ see [13].

The construction of Z_p-actions on S^4 is similar. In the base space we geometrically realize the Alexander polynomials $\lambda_{p,\gamma}(t) = \gamma + (1-\gamma)t^p$. We start as before with $K = (B^5 - B^3) \cup \underset{\alpha}{h^1}$, and its p-fold unbranched cyclic cover is

$$K_p^u = (B^5 - B^3) \cup \underset{\alpha_1^1}{h_1^1} \cup \ldots \cup \underset{\alpha_p^p}{h_p^1}$$

and $\pi_1(K_p^u) = \pi_1(\partial K_p^u) = \langle\widetilde{\beta}\rangle * \langle\alpha_1\rangle * \cdots * \langle\alpha_p\rangle$, where $\{\alpha_i\}_{i=1}^p$ are the lifts of α and β^p lifts to $\widetilde{\beta}$. Attach a 2-handle to K by the word $g(\gamma) = \alpha^\gamma\beta^p\alpha^{(1-\gamma)}\beta^{-p}$. The attaching sphere lifts to p attaching spheres in K_p^u, $g(\gamma)_i \simeq \alpha_i^\gamma\widetilde{\beta}\alpha_i^{(1-\gamma)}\widetilde{\beta}^{-1} (1 \leqslant i \leqslant p)$. Plugging in the branch set as before, in $\partial(K_p^u \cup B^3)$, $\widetilde{\beta} \simeq 1$ so $g(\gamma_i) \simeq \alpha_i$ and we get p cancelling pairs of handles. The analogous construction can clearly be done beginning with any unknotted ball pair (B^{n+3}, B^{n+1}) with $n \geqslant 2$.

Definition. (S^{n+2}, kS^n) is said to be *Z_p-null-cobordant if* $(S^{n+2}, kS^n) = \partial(B^{n+3}, kB^{n+1})$ and (B^{n+3}, kB^{n+1}) admits a Z_p-action.

Modulo the computation of Δ^1, we have proved the following

Theorem 1.1. *For each pair of integers (n, p) with $n \geqslant 2$ and $p \geqslant 2$, there are infinitely many knots (S^{n+2}, kS^n) which admit Z_p-actions; moreover, all of these knots are Z_p-null-cobordant.*

We say that an n-knot (S^{n+2}, kS^n) with complement $X = X^{n+2} - kS^n$ is *q-simple* ($q \geqslant 0$ an integer) if $\pi_i(X) \cong \pi_i(S^1)$ for $i \leqslant q$ [12, p. 114].

Theorem 1.2. *For each triple of integers (n, p, q) with $n \geqslant 2, p \geqslant 2$, and $q \leqslant (n-1)/2$, there are infinitely many q-simple n-knots which admit Z_p-actions. These knots are also Z_p-null-cobordant.*

Proof. Theorem 1.1 produces 0-simple knots, because the surgery changes π_1 of the complement from Z to a knot group. For $q \geqslant 1$, the surgery is performed in dimension $(q+1)$, exactly as in [14, Theorem 3.3] and one produces a ball pair (B^{n+3}, kB^{n+1}) by adding first a $(q+1)$-handle, then a $(q+2)$-handle to the unknotted ball pair (B^{n+3}, B^{n+1}). The resulting knot complement $Y = B^{n+3} - kB^{n+1}$ has the following properties:

(i) $\pi_i(\partial Y) = \pi_i(Y) \cong \pi_i(S^1), i \leqslant q$.

(ii) $\pi_{q+1}(Y) \cong_\Lambda \Lambda/\lambda_{p,\gamma}$.

(iii) $\pi_{q+1}(\partial Y) = \pi_{q+1}(Y)$ if $q < (n-1)/2$, and $\pi_{q+1}(\partial Y)$ is presented as a Λ-module [12] by the 2×2 matrix

$$\begin{pmatrix} \lambda_{p,\gamma}(t) & 0 \\ 0 & \lambda_{p,\gamma}(t^{-1}) \end{pmatrix}$$

if $q = (n-1)/2$.

(iv) $Y_p^b \approx B^{n+3}$.

Note. In fact, the construction of the knots in Theorems 1.1 and 1.2 is arranged so that all the knots produced are Z_p-doubly-null-cobordant [14]. That is, each knot simultaneously bounds two knotted ball pairs such that the union of these two ball pairs along their common knotted boundary is the unknotted sphere pair. Moreover, the standard Z_p-action on the unknotted sphere pair respects the above decomposition: The total spaces are setwise invariant and the submanifolds fixed under the standard Z_p-action.

COROLLARY 1.3. *For each (n, p) with $n \geqslant 2$ and $p \geqslant 2$, there are infinitely many different Z_p-doubly-null-cobordant knots (S^{n+2}, kS^n). If so desired, such examples can be arranged to be q-simple, $q \leqslant (n-1)/2$.*

REMARK. A related Z_p-equivariant decomposition for the unknotted pair (S^4, S^2) into 2 contractible manifold pairs identified along a homology sphere pair will be dealt with in §III.

II. **Homology invariants of cyclic coverings.** Let A be a knot module [5], that is, A is a finitely-generated Λ-module and $(t-1)\colon A \longrightarrow A$ is an isomorphism. For example, if X is an n-knot complement, then $H_q(\widetilde{X}; Z)$ is a knot module $(q \geqslant 1)$. Let $\tau = t^p \in \Lambda$, and $\Lambda_p \subset \Lambda$ the subring of Λ generated by τ, that is, if $J(\tau) \subset J(t)$ is the subgroup of the infinite cyclic multiplicative group $J(t)$ generated by τ, then $\Lambda_p = Z[J(\tau)] \subset Z[J(t)] = \Lambda$. Since A is a finitely-presented Λ-module, and Λ is a finitely-presented Λ_p-module, then A in turn is a finitely-presented Λ_p-module.

Let T be the $p \times p$ matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & \\ 0 & 0 & 1 & 0 & \cdots & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot \\ & & & & & 1 \\ \tau & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

We prove the following lemma due to Fox [1, p. 417]:

LEMMA 2.1. *If A is presented as a Λ-module by the $m \times n$ matrix $M(t) = (m_{ij}(t))$, then A is presented as a Λ_p-module by the $mp \times np$ matrix $M(T) = (m_{ij}(T))$.*

PROOF. Let $\Lambda(\xi)$ denote the free Λ-module generated by the element ξ. As a Λ_p-module, $\Lambda(\xi)$ is free of rank p, generated by $\{t^i\xi\}_{i=0}^{p-1}$. Moreover, in terms of these Λ_p-generators, the Λ-automorphism $\Lambda \xrightarrow[\cong]{t} \Lambda$ is represented by the $p \times p$ matrix T. Hence we have the following commutative diagram of abelian groups with top row an exact sequence of Λ-modules and bottom row an exact sequence of Λ_p-modules:

$$\begin{array}{ccccccc}
\Lambda^m & \xrightarrow{M(t)} & \Lambda^n & \longrightarrow & A & \longrightarrow & 0 \\
\uparrow{\scriptstyle 1} & & \uparrow{\scriptstyle 1} & & \uparrow{\scriptstyle 1} & & \\
(\Lambda_p)^{mp} & \xrightarrow{M(T)} & (\Lambda_p)^{np} & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

Let $\Gamma = \Lambda \otimes_Z Q$ denote the rational group ring of the infinite cyclic group. Then $A \otimes_Z Q \cong_\Gamma \Gamma/\lambda_1 \oplus \ldots \oplus \Gamma/\lambda_k$ where $\lambda_i(t) \in \Lambda$, $\lambda_{i+1} | \lambda_i$ in Λ, and $\lambda_i(1) = \pm 1$. The $\{\lambda_i(t)\}$ are the *Alexander invariants* of A, and $\Delta(t) = \Pi_{i=1}^k \lambda_i(t)$ is the *Alexander polynomial* of A. These are invariants of the Λ-isomorphism type of A.

We have the following theorem due to Fox [1, p. 417]:

THEOREM 2.2. *If $\Delta(t) = \Pi_{i=1}^k \lambda_i(t)$ is the Alexander polynomial of A over Λ, then*

$$\widetilde{\Delta}(\tau) = \prod_{j=0}^{p-1} \Delta(\zeta^j \tau^{1/p}) = \prod_{j=0}^{p-1} \left(\prod_{i=1}^{k} \lambda_i(\zeta^j \tau^{1/p}) \right)$$

is the Alexander polynomial of A over the subring Λ_p, where ζ is a primitive pth root of unity.

Let X denote a knot complement, and X_p^u its unbranched p-fold cyclic covering. Let t denote the meridian curve in X, and τ (the lift of t^p) denote the meridian curve in X_p^u. If $\widetilde{X}$ is the infinite cyclic cover of X with covering translation t, then it is also an infinite cyclic covering of X_p^u with covering translation τ. As homeomorphisms of $\widetilde{X}$, we have $\tau = t^p$. So $H_q(\widetilde{X}; Z)$ is structured both as a Λ-module over X and as a Λ_p-module over X_p^u, with homology invariants related by Theorem 2.2. Hence, if $\Delta_{p,\gamma}(t) = \gamma + (1 - \gamma)t^p$, then $\widetilde{\Delta}_{p,\gamma}(\tau) = [\gamma + (1 - \gamma)\tau]^p$. This completes the proof of all the theorems in §I, because it is now clear that, for fixed p, different choices of γ yield different Alexander polynomials $\widetilde{\Delta}_{p,\gamma}(\tau)$.

Let A be a knot module, and $\mathrm{A}(A)$ denote the group of Z-automorphisms of A. Then $t \in \mathrm{A}(A)$, and $(t-1) \in \mathrm{A}(A)$.

THEOREM 2.3. *If t has pth roots in $\mathrm{A}(A)$ for all $p \geqslant 2$ then $A = 0$.*

PROOF. We first show that A has no Z-torsion. Let $T \subset A$ denote the Λ-submodule of elements of Z-finite order; it is well known that T is a finite group [9]. By restriction, t and $(t-1) \in \mathrm{A}(T)$. But $\mathrm{A}(T)$ is a finite group, so choose $\alpha = \mathrm{order}(\mathrm{A}(T))$. Since t has an αth root y, then $1 = y^\alpha = t$. But then $t - 1 = 0$, so $T = 0$. We now have that A is Z-torsion-free, and will show that A must be finitely-generated free abelian. Let $\widetilde{\Delta}(t) = \Sigma_{i=0}^q a_i t^i$ be the Alexander polynomial of A, with $a_0 \neq 0 \neq a_q$. Fix an integer $p \geqslant 2$. Let $s \in \mathrm{A}(A)$ be such that $s^p = t$. Then the inclusion of infinite cyclic groups

$$J(t) \longrightarrow J(s) = J(t^{1/p})$$

$$t \longmapsto s^p$$

embeds $Z[J(t)] = \Lambda \longrightarrow \Lambda_p \subset \Lambda = Z[J(s)]$. So let $\Delta(t^{1/p}) = \Delta(s) = \Sigma_{j=0}^n \beta_j s^j$, $\beta_0 \neq 0 \neq \beta_n$, be the Alexander polynomial of A over the ring $Z[J(s)]$. Then, by Theorem 2.2, we have $q = n$, and $a_0 = \beta_0^p$, $a_q = \beta_q^p$. This means that the integers a_0 and a_q have pth roots in Z for all $p \geqslant 2$; hence $a_0 = 1 = a_q$. This means [15, Theorem 2.3] that A is f.g. free abelian. Note that the above argument on the Alexander polynomial proves that if A is a knot module, and t has pth roots in $\mathrm{A}(A)$ for infinitely many p, then A is a f.g. abelian group.

Let now t be represented by the $q \times q$ unimodular matrix (a_{ij}) over Z. Let $\mathrm{GL}(q; Z)$ denote the group of $q \times q$ unimodular matrices over Z. The fact that (a_{ij}) has rth roots in $\mathrm{GL}(q; Z)$ for all r will mean that $(a_{ij}) = I_q$. To see this, reduce mod p (p prime)

$$\mathrm{GL}(q; Z) \xrightarrow{\phi_p} \mathrm{GL}(q; Z_p)$$

where $\mathrm{GL}(q; Z_p)$ are the nonsingular $q \times q$ matrices over the field Z_p. Now $\phi_p((a_{ij}))$ has rth roots in $\mathrm{GL}(q; Z_p)$ for all r, but $\mathrm{GL}(q; Z_p)$ is a finite group, so this means

$$(a_{ij}) \equiv I_q \pmod{p}.$$

Since this is true for all primes p, then $(a_{ij}) = I_q$. So $t = 1$ on A; hence $A = 0$.

Suppose now that $(S^{n+2}, \widetilde{k}S^n)$ is an n-knot which admits a Z_p-action generated by h. Let Σ^{n+2} denote the orbit space, and (Σ^{n+2}, kS^n) denote the orbit pair.

LEMMA 2.4. *Σ^{n+2} is a homotopy sphere.*

PROOF. Σ^{n+2} is a smooth manifold, and is simply-connected by a result of Fox [2]. Let $T(\tilde{k}S^n)$ and $T(kS^n)$ denote trivial tubular neighborhoods of the submanifolds. ($T(kS^n)$ is trivial because the action of h on $T(\tilde{k}S^n)$ can be taken to be rotation on the fiber.) Let $X = S^{n+2} - \mathrm{Int}\ T(\tilde{k}S^n)$ and $Y = \Sigma^{n+2} -$ Int $T(kS^n)$ denote the bounded knot complements. Now X is a p-fold covering of Y, and the fact that X is a homology circle will mean that Y also is a homology circle. Let τ denote a meridian curve in X linking the submanifold $\tilde{k}S^n$ once, and t be the meridian curve in Y linking the submanifold kS^n once. Then τ projects to t^p, so if $\tilde{X}$ denotes the infinite cyclic covering space of X with τ generating the infinite cyclic group of covering translations, then the composition of covering projections $\tilde{X} \longrightarrow X \longrightarrow Y$ yields $\tilde{X}$ as the infinite cyclic covering space of Y, with t generating the infinite cyclic group of covering translations. Moreover, as covering translations, $t^p = \tau$.

We have the long exact sequences of homology (integer coefficients) [11], [12]

(i) $\longrightarrow H_i(\tilde{X}) \xrightarrow{\tau - 1} H_i(\tilde{X}) \longrightarrow H_i(X) \longrightarrow,$

(ii) $\longrightarrow H_i(\tilde{X}) \xrightarrow{t - 1} H_i(\tilde{X}) \longrightarrow H_i(Y) \longrightarrow.$

Now $\tau = t^p$ so $(\tau - 1) = (t^p - 1) = (t - 1)(1 + t + \cdots + t^{p-1})$. Since X is a homology S^1, then $H_i(\tilde{X}) \xrightarrow{\tau - 1} H_i(\tilde{X})$ is an isomorphism for all $i \geqslant 1$. Thinking of multiplication of the factors in the factorization of $(\tau - 1)$ as composition of homomorphisms, this means that $(t - 1)$ is surjective. Since multiplication is commutative, this also means that $(t - 1)$ is also injective. So $(t - 1)$ is an isomorphism $(i \geqslant 1)$, and is the trivial homomorphism on $H_0(\tilde{X})$, which means that $H_i(Y; Z) \cong H_i(S^1; Z)$. From the Mayer-Vietoris sequence, one then readily concludes that Σ^{n+2} is a homotopy sphere. This completes the proof of Lemma 2.4. Note that a similar proof will show that if a ball pair (B^{n+3}, kB^{n+1}) admits a Z_p-action generated by h, then B^{n+3}/h is a homotopy ball with boundary a homotopy sphere, hence a real ball if $n \geqslant 5$ by the generalized Poincaré conjecture

What Lemma 2.4 means is that if a knot (S^{n+2}, kS^n) admits a Z_p-action, then (S^{n+2}, kS^n) is the p-fold branched cyclic covering of the quotient pair of homotopy spheres. Hence if τ denotes the covering translation of the infinite cyclic covering $\tilde{X}$, then $\exists$ a homeomorphism $t: \tilde{X} \longrightarrow \tilde{X}$ such that $t^p = \tau$, where t is the covering translation of $\tilde{X}$ over the quotient space. We are now set up to use Theorem 2.3:

THEOREM 2.5. *If a knotted sphere pair (S^{n+2}, kS^n) [or ball pair*

$(B^{n+3}, kB^{n+1})]$ *admits a* Z_p*-action for all* $p \geqslant 2$, *then its infinite cyclic cover is acyclic.*

We have as a corollary a version of the unknotting theorem of Hsiang [6] for S^1-actions (a 1-simple ball pair by definition has a 1-simple boundary sphere pair):

COROLLARY 2.6. *If the* 1*-simple knot* $((S^{n+2}, kS^n)$ $(n \geqslant 3)$ *or* (B^{n+3}, kB^{n+1}) $(n \geqslant 3))$ *admits a* Z_p*-action for all* p *(or an* S^1*-action), then it is unknotted.*

PROOF. Since $\pi_1(X) = Z$, then $\widetilde{X}$ is contractible by Theorem 2.4, and $\pi_i(X) \cong \pi_i(S^1)$ $\forall i$. This is the unknotting criterion [10], [8] for spheres, and for balls because the boundary is unknotted (by the above argument).

Note. The above corollary fails if the requirement "for all p" is relaxed to the requirement "for infinitely many primes p". For example, the 2-twist-spin of any knot admits Z_p-actions for all p odd (see [3, Theorem 3.4] and [4] for the higher-dimensional case). The 2-twist-spin of any q-simple knot is likewise q-simple, so we obtain for $n \geqslant 4$ 1-simple knots (S^{n+2}, kS^n) which admit Z_p-actions for all primes p except $p = 2$.

III. **The classical Smith conjecture.** The classical Smith conjecture (Z_p-actions on S^3) is still unsolved in general. Some partial results are known, for example the result of Waldhausen [17] which says there is no Z_2-action on S^3 leaving a knot pointwise fixed. Since the construction of Z_p-actions in this paper avoids the Poincaré conjecture, it is interesting to attempt an application to the classical situation. It predictably fails, due to the fact that isotopy linking of attaching 1-spheres for the 2-handles in the p-fold cyclic cover occurs, and this prevents us from using the handle cancellation theorem.

We will study an attempt to construct a Z_2-action in detail. The construction is exactly the same as in §I; see [14, p. 249] for a detailed description of handlebody addition in the middle dimension. We get by [14] a doubly-null-cobordant knot (S^3, kS^1) with complement X and with $H_1(\widetilde{X}; Z)$ presented as a Λ-module by the 2×2 matrix

$$\begin{pmatrix} 2 - t^2 & 0 \\ 0 & 2t^2 - 1 \end{pmatrix}$$

by adding a 2-handle to $S^1 \times D^3$ by the attaching curve $g(S^1)$ shown in Figure 1.

The methods of [14, §IV] can be used to calculate the knot resulting from the attaching curve in Figure 1.

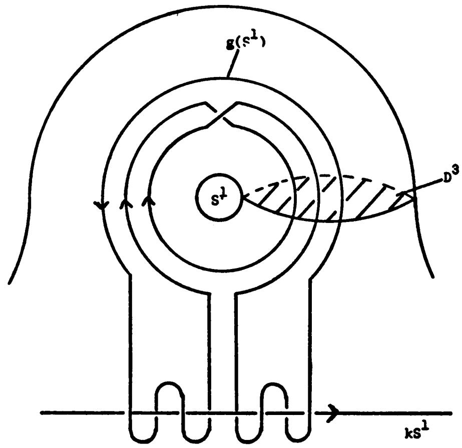

FIGURE 1

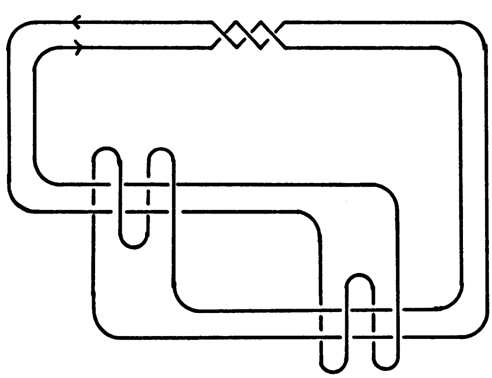

FIGURE 2

The knot (S^3, kS^1) of Figure 2 separates the unknot (S^4, S^2) into two knotted ball pairs (B_1^4, k_1B^2) and (B_2^4, k_2B^2). Let $(Q^3, \tilde{k}S^1)$ be the 2-fold branched cyclic cover of (S^3, kS^1). Because (S^3, kS^1) is doubly-null-cobordant, $(Q^3, \tilde{k}S^1)$ embeds in (S^4, S^2) and separates it into $(M_i^4, \tilde{k}_iB^2)$, $i = 1, 2$, the 2-fold branched cyclic covers of (B_i^4, k_iB^2) respectively. Now M_1^4 is contractible, because the handles cancel up to homotopy; hence Q^3 is a homology 3-sphere. Moreover, $M_1^4 \times I \approx D^5$ because M_1^4 has the same properties as the Mazur manifold: handles which homotopically cancel in M_1^4 will isotopically cancel in $M_1^4 \times I$ because the attaching spheres of the 2-handles now have enough codimension to be isotoped around to cancelling position. I have so far been unable to show $\pi_1(Q^3) \neq 1$.

REFERENCES

1. R. H. Fox, *Free differential calculus. III: Subgroups*, Ann. of Math. (2) **64** (1956), 407–419. MR 20 #2374.

2. ———, *Two theorems about periodic transformations of the 3-sphere*, Michigan Math. J. **14** (1967), 331–334. MR 36 #7136.

3. C. H. Giffen, *The generalized Smith conjecture*, Amer. J. Math. **88** (1966), 187–198. MR 33 #6620.

4. C. M. Gordon, *On the higher-dimensional Smith conjecture*, Proc. London Math. Soc. (3) **29** (1974), 98–110.

5. M. A. Gutierrez, *On knot modules*, Invent. Math. **17** (1972), 329–335.

6. W.-Y. Hsiang, *On the unknottedness of the fixed point set of differentiable circle group actions on spheres—P. A. Smith conjecture*, Bull. Amer. Math. Soc. **70** (1964), 678–680. MR 29 #6491.

7. J. F. P. Hudson, *Concordance, isotopy, and diffeotopy*, Ann. of Math. (2) **91** (1970), 425–448. MR 41 #4549.

8. M. Kato, *Higher-dimensional PL knots and knot manifolds*, J. Math. Soc. Japan **21** (1969), 458–480. MR 40 #2097.

9. M. A. Kervaire, *Les noeuds de dimensions supérieures*, Bull. Soc. Math. France **93** (1965), 225–271. MR 32 #6479.

10. J. Levine, *Unknotting spheres in codimension two*, Topology **4** (1965), 9–16. MR 31 #4045.

11. J. Milnor, *Infinite cyclic coverings*, Conference on the Topology of Manifolds (Michigan State Univ., E. Lansing, Mich., 1967), Prindle, Weber & Schmidt, Boston, Mass., 1968, pp. 115–133. MR 39 #3497.

12. Y. Shinohara and D. W. Sumners, *Homology invariants of cyclic coverings with application to links*, Trans. Amer. Math. Soc. **163** (1972), 101–121.

13. D. W. L. Sumners, *Homotopy torsion in codimension two knots*, Proc. Amer. Math. Soc. **24** (1970), 229–240. MR 40 #6531.

14. ———, *Invertible knot cobordisms*, Comment. Math. Helv. **46** (1971), 240–256. MR 44 #7535.

15. ———, *Polynomial invariants and the integral homology of coverings of knots and links*, Invent. Math. **15** (1972), 78–90. MR 45 #1150.

16. A. M. Vinogradov and M. S. Kušel'man, *The generalized Smith conjecture in dimension four*, Sibirsk. Mat. Ž. **13** (1972), 52–62 = Siberian Math. J. **13** (1972), 36–43.

17. F. Waldhausen, *Über Involutionen der 3-Sphäre*, Topology **8** (1969), 81–91. MR 38 #5209.

18. E. C. Zeeman, *Twisting spun knots*, Trans. Amer. Math. Soc. **115** (1965), 471–495. MR 33 #3290.

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