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AMALGAMATED PRODUCTS OF SEMIGROUPS: THE EMBEDDING PROBLEM

BY

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ABSTRACT. A necessary and sufficient condition for a semigroup amalgam to be embeddable is given. It is in the form of a countable set of equational implications with existential quantifiers. Furthermore it is shown that no finite set of equational implications can serve as a necessary and sufficient condition. Howie's sufficient condition (see [5]) is derived as a consequence of our main theorem.

1. **Introduction.** Let $\{S_i; i \in I\}$ be a family of semigroups, and suppose that for each $i \in I$ there exists a homomorphism h_i: $U \to S_i$ from a semigroup U into S_i. Consider the set $\Sigma = \{(x, i): i \in I, x \in S_i\}$, sum of the sets S_i, and construct the semigroup S presented by

$$S = \langle \Sigma; (x, i)(x', i) = (xx', i) \text{ for every } i \in I, x \text{ and } x' \in S_i,$$
$$(h_i(u), i) = (h_j(u), j) \text{ for every } i, j \in I, u \in U\rangle.$$

Then S is called the product of the family S_i amalgamated by U and is denoted by $\Pi^*_{i\in I U} S_i$. If φ denotes the canonical homomorphism from the free semigroup on the alphabet Σ onto S, then φ_i defined by $\varphi_i(x) = \varphi((x, i))$ is a homomorphism from S_i into S, and h: $U \to S$ defined by $h(u) = \varphi[(h_i(u), i)]$ is independent of i. Indeed, $\{S; \varphi_i : S_i \to S\}$ is an initial object in the category whose objects are $\{T; f_i: S_i \to T\}$ with $f_i \circ h_i$ independent of i.

Herein, we consider a family $\{S_i; i \in I\}$ of semigroups and mappings h_i: $U \to S_i$ that are injective homomorphisms for all $i \in I$. Under these conditions the system $[S_i; U; h_i]_{i\in I}$ is called a *semigroup amalgam* [2, p. 138], and the set $G = \bigcup_{i\in I} \bar{S}_i$ where $\bar{S}_i = [S_i \backslash h_i(U)] \cup U$, with a partial binary operation defined in such a way that the natural mappings $\bar{S}_i \to S_i$ are isomorphisms, is called the groupoid of the amalgam $[S_i; U; h_i]_{i\in I}$. If there exist a semigroup T and a mapping θ: $G \to T$ such that θ, when restricted to each $\bar{S}_i$, is an injective homomorphism, then the semigroup amalgam is said to be embeddable. The link between embeddable amalgams and amalgamated products is expressed in the next theorem:

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THEOREM 1.1 [2, THEOREM 9.31]. *The following conditions are equivalent:*

(i) *The semigroup amalgam* $[S_i; U; h_i]_{i\in I}$ *is embeddable.*

(ii) *The canonical mapping of the partial groupoid* G *of* $[S_i; U; h_i]_{i\in I}$ *into the product* $\Pi^*_{i\in I U} S_i$ *is an embedding.*

(iii) *The homomorphisms* $\varphi_i: S_i \to \Pi^*_{i\in I U} S_i$ *have the properties*

(a) φ_i *is an injection for every* $i \in I$;

(b) $\varphi_i(S_i) \cap \varphi_j(S_j) = h(U)$ *for* $i, j \in I$, $i \neq j$, *where* $h = \varphi_i \circ h_i$.

So far, only sufficient conditions for a semigroup amalgam to be embeddable have been obtained. They are as follows:

THEOREM 1.2 (HOWIE [5, THEOREM 3.3]). *Let* $[S_i; U; h_i]_{i\in I}$ *be a semigroup amalgam. If, for every* $i \in I$, $h_i(U)$ *is almost unitary* (*see* [2, p. 144]) *in* S_i,*then the amalgam is embeddable.*

THEOREM 1.3 (BOURBAKI [1, PROPOSITION 5, §7, p. I. 81]). *Assume that, for every* $i \in I$, S_i *is a monoid with identity* e_i, *that* U *is a monoid and that* h_i: $U \to S_i$ *are monoid homomorphisms (i.e.* $h_i(e) = e_i$ *for every* $i \in I$). *Assume further that for every* $i \in I$ *there exists a subset* P_i *of* S_i *containing* e_i *such that the mapping* $(u, p) \to h_i(u)p$ *from* $U \times P_i$ *into* S_i *is a bijection. Then every* $x \in \Pi^*_{i\in I U} S_i$ *can be written uniquely as* $x = h(u) \Pi^n_{\alpha=1} \varphi_{i_\alpha}(p_\alpha)$ *with* $i_\alpha \neq i_{\alpha+1}$ $(1 \leqslant \alpha < n)$, $p_\alpha \in S_{i_\alpha}$ *and* $p_\alpha \neq e_{i_\alpha}$ $(1 \leqslant \alpha \leqslant n)$.

In this theorem, the hypothesis that $(u, p) \to h_i(u)p$ is a bijection yields that h_i is an injection. From the conclusion, it is easy to deduce that (iii) of Theorem 1.1 is satisfied, showing that the monoid amalgam $[S_i; U; \varphi_i]$ is embeddable.

The case of $h_i(U)$ being an ideal of S_i is covered by the following

THEOREM 1.4 (GRILLET-PETRICH [3], LJAPIN [6]). *Let* $[S_i; U; h_i]_{i\in I}$ *be a semigroup amalgam. Suppose* $h_i(U)$ *is an ideal of* S_i *for every* $i \in I$. *Then the amalgam is embeddable if and only if for every* $s_i \in S_i$, $s_j \in S_j$, $u, u', u'' \in U$, $s_i h_i(u) = h_i(u')$ *and* $h_j(u)s_j = h_j(u'')$ *imply* $h_j^{-1}(h_j(u')s_j) = h_i^{-1}(s_i h_i(u''))$.

Note that, when expressed as a condition on the groupoid G of the amalgam $[S_i; U; h_i]_{i\in I}$, the condition of Theorem 1.4 is simply $(s_i u)s_j = s_i(us_j)$ for every $s_i \in \bar{S}_i$, $s_j \in \bar{S}_j$, $u \in U$.

Finally we mention an important result concerning the amalgamation of inverse semigroups.

THEOREM 1.5 (HALL [4]). *A semigroup amalgam* $[S_i; U; h_i]_{i\in I}$ *with* S_i *and* U *inverse semigroups is embeddable in an inverse semigroup.*

The main result of this paper (Theorem 3.3) consists of a necessary and sufficient condition for a semigroup amalgam $[S_i; U, h_i]$ to be embeddable. The condition is that any system of equations of a certain nature (balanced system) over the semigroups S_i ($i \in I$) implies another equation, or other equations (two at most) linked to the initial system. Briefly, our condition is in the form of a *countable set* of equational implications with existential quantifiers and with variables taken from card I distinct sets. It is shown further (Theorem 4.4) that *no finite set* of equational implications can serve as a necessary and sufficient condition that a semigroup amalgam be embeddable. The techniques to prove our results are quite comparable to those used by Malcev ([7], [8]) to find necessary and sufficient conditions that a semigroup be embeddable in a group.

Theorem 1.2 and Theorem 1.4 above are derived as corollaries of Theorem 3.3, but to derive Proposition 2.4' (which generalizes Theorem 1.3) or Theorem 1.5 would require a deeper analysis of the structure of balanced systems. This indicates that, at least in its present form, the condition of Theorem 3.3 is not necessarily the shortest way of proving the embeddability of semigroup amalgams in particular situations.

2. Some notation and particular cases of embedding. When considering semigroup amalgams $[S_i; U; h_i]_{i\in I}$, we shall systematically assume that $S_i \cap S_j = \varnothing$ for $i, j \in I$, $i \neq j$. As a consequence, elements of the sum Σ of the sets S_i ($i \in I$) will be denoted s_i or $s_i^{(k)}$, the superscripts being used for the purpose of distinguishing several elements in the same S_i. Similarly we shall write U_i for $h_i(U)$, and elements of U_i are denoted u_i or $u_i^{(k)}$. Utilizing this notation we have $\Sigma = \{s_i\colon s_i \in S_i, i \in I\}$, and denoting by $*$ the multiplication in the free semigroup over Σ, the presentation of $\Pi^*_{i\in I U} S_i$ is written

$$\langle \Sigma; s_i^{(1)} * s_i^{(2)} = s_i^{(1)} s_i^{(2)} \ (i \in I, s_i^{(1)}, s_i^{(2)} \in S_i), u_i = u_j \ (i, j \in I)\rangle.$$

The semigroup $\Pi^*_{i\in I} S_i$ presented by

$$\prod_{i\in I}{}^{*} S_i = \langle \Sigma; s_i^{(1)} * s_i^{(2)} = s_i^{(1)} s_i^{(2)} \text{ for every } i \in I, s_i^{(1)}, s_i^{(2)} \in S_i\rangle$$

is called the *free product* of the family $\{S_i; i \in I\}$. An element $x \in \Pi^*_{i\in I} S_i$ can be written uniquely as $s_{i_1} * s_{i_2} * \cdots * s_{i_m}$ with $i_k \neq i_{k+1}$, for all k, $1 \leqslant k \leqslant m-1$, and the elements s_{i_k} ($1 \leqslant k \leqslant m$) are called the syllables of x. Furthermore,

$$\begin{aligned}(s_{i_1} * \cdots * s_{i_m}^{(1)}) * (s_{j_1}^{(2)} * \cdots * s_{j_n}) &= s_{i_1} * \cdots * s_{i_m}^{(1)} * s_{j_1}^{(2)} * \cdots * s_{j_n} \quad \text{if } i_m \neq j_1, \\ &= s_{i_1} * \cdots * s_{i_{m-1}} * (s_{i_m}^{(1)} s_{j_1}^{(2)}) * s_{j_2} * \cdots * s_{j_n} \quad \text{if } i_m = j_1.\end{aligned}$$

Clearly $\Pi^*_{i\in I U} S_i = \Pi^*_{i\in I} S_i/\rho$ where ρ is the congruence on $\Pi^*_{i\in I} S_i$ generated by all pairs (u_i, u_j), $u_i \in U_i$, $u_j \in U_j$, $i, j, \in I$. A replacement of an occurrence of u_i in a syllable of $x \in \Pi^*_{i\in I} S_i$ by u_j is called a step, and following [5] we distinguish four types of steps (a step is indicated by an arrow):

(1) *S-steps*: $s_{i_1} * \cdots * u_{i_k} * \cdots * s_{i_m} \to (s_{i_1} * \cdots) * u_{j_l} * (\cdots * s_{i_m})$,

(2) *M-steps*: $s_{i_1} * \cdots * s_{i_k} u_{i_k} t_{i_k} * \cdots * s_{i_m} \to s_{i_1} * \cdots * s_{i_k} * u_{j_l} * t_{i_k} * \cdots * s_{i_m}$,

(3) E_r*-steps*: $s_{i_1} * \cdots * s_{i_k} u_{i_k} * \cdots * s_{i_m} \to (s_{i_1} * \cdots * s_{i_k}) * u_{j_l} * (\cdots * s_{i_m})$,

(4) E_l*-steps*: $s_{i_1} * \cdots * u_{i_k} t_{i_k} * \cdots * s_{i_m} \to (s_{i_1} * \cdots) * u_{j_l} * (t_{i_k} * \cdots * s_{i_m})$.

Note that an M-step always introduces a new syllable, while an S-step or an E-step may or may not introduce a new syllable according to the context of u_{i_k}.

We shall use the notation $x \Rightarrow y$ if there is a succession of steps (also called a transition) $x \to x_1 \to \cdots \to x_{n-1} \to y$ linking $x, y \in \Pi^*_{i\in I} S_i$. Thus, we have $x \,\rho\, y$ if and only if there is a transition $\tau: x \Rightarrow y$. Two transitions τ_1 and τ_2 are called equivalent if they both link the same elements of $\Pi^*_{i\in I} S_i$. In terms of transitions, condition (iii) of Theorem 1.1 is obviously equivalent to the following:

(iv) (a) If there is a transition $s_i^{(1)} \Rightarrow s_i^{(2)}$, then $s_i^{(1)} = s_i^{(2)}$.

(b) If there is a transition $s_i \Rightarrow s_j$ $(i \neq j)$ then there exists $u \in U$ such that $s_i = u_i$ and $s_j = u_j$.

We shall refer to (iv) as "condition (iv) of Theorem 1.1".

For further simplification, whenever there is no ambiguity, we suppress the symbol $*$ in denoting products in $\Pi^*_{i\in I} S_i$, thus writing elements in the form $s_{i_1} s_{i_2} \cdots s_{i_m}$.

As an illustration of the concepts introduced, we sketch the proofs of some sufficient conditions, related to the conditions in Theorems 1.2, 1.3, 1.4, for a semigroup amalgam to be embeddable.

Recall that a subsemigroup U of a semigroup S is said to be *left* [resp. *right*] *unitary* if for all $u \in U$, $s \in S$, $us \in U$ implies $s \in U$ [resp. $su \in U$ implies $s \in U$]. U is said to be *unitary* if it is both left and right unitary.

PROPOSITION 2.1 (SEE [5, P. 522]). *If $U_i = h_i(U)$ is unitary in S_i for every $i \in I$, then the semigroup amalgam $[S_i; U; h_i]_{i\in I}$ is embeddable.*

PROOF. The idea is to show that a transition $\tau: x \Rightarrow y$ from x to $y \in \Pi^*_{i\in I} S_i$ is equivalent to a transition $\tau': x \Rightarrow y$ where all possible S-steps occurring in τ' are performed at the beginning. For this purpose one shows that a succession of two steps M-S or E_r-S or E_l-S can be replaced by a transition without S-steps or with S-steps performed at the beginning. This can always be done without any assumption on U_i (see e.g. [5, Lemma 3.10]) except in the following configurations:

$$\text{(a)}\quad \begin{aligned} &\cdots s_i^{(1)}s_j\cdots = \cdots s_i^{(2)}u_i^{(1)}s_j\cdots \xrightarrow{E_r} \cdots s_i^{(2)}u_j^{(1)}s_j\cdots \\ &\cdots s_i^{(2)}u_j^{(1)}s_j\cdots = \cdots s_i^{(2)}u_j^{(2)}\cdots \xrightarrow{S} \cdots s_i^{(2)}u_k^{(2)}\cdots. \end{aligned}$$

Then from $u_j^{(1)}s_j = u_j^{(2)}$ we deduce that $s_j = u_j^{(3)}$, so that (a) can be replaced by

$$\cdots s_i^{(2)}u_j^{(3)}\cdots \xrightarrow{S} \cdots s_i^{(1)}u_i^{(3)}\cdots$$

$$\text{(a}'\text{)}\quad \cdots s_i^{(1)}u_i^{(3)}\cdots = \cdots s_i^{(2)}u_i^{(1)}u_i^{(3)}\cdots = \cdots s_i^{(2)}u_i^{(2)} \cdots \xrightarrow{E_r} \cdots s_i^{(2)}u_k^{(2)}\cdots.$$

There is also a dual configuration E_l-S that can be replaced similarly by S-E_l. Replacing a transition τ: $s_i^{(1)} \Rightarrow s_i^{(2)}$ by an equivalent transition τ' defined as above and taking in account the fact that an E-step or an M-step does not diminish the syllable length, one gets $s_i^{(1)} = s_i^{(2)}$. One shows similarly that $s_i \Rightarrow s_j$ yields $s_i = u_i, s_j = u_j$ for some $u \in U$. A detailed proof is given in [5, Lemma 3.9]. Condition (iv) of Theorem 1.1 is satisfied. Therefore the amalgam is embeddable.

Theorem 1.4, concerning the amalgamation of an ideal, can be proved almost exactly along the same lines as Proposition 2.1.

PROOF OF THEOREM 1.4. The condition of the theorem is clearly necessary. To show sufficiency, consider an E_r-step followed by an S-step in the configuration (a). Since U_i and U_j are ideals of S_i and S_j respectively we have $s_i^{(2)}u_i^{(1)} = u_i^{(3)}$ and $u_j^{(1)}s_j = u_j^{(2)}$. It follows $h_j^{-1}(u_j^{(3)}s_j) = h_i^{-1}(s_i^{(2)}u_i^{(2)})$ and (a) can be replaced by:

$$\cdots s_i^{(1)}s_j\cdots = \cdots u_i^{(3)}s_j\cdots \xrightarrow{S} \cdots u_j^{(3)}s_j\cdots \xrightarrow{S\text{ or }E_r} \cdots s_i^{(2)}u_i^{(2)}\cdots \xrightarrow{E_r} s_i^{(2)}u_k^{(2)}\cdots.$$

The rest of the proof is identical to the proof of Proposition 2.1.

Finally, we proceed to show that the embedding of an amalgam is possible under slightly more general conditions than those of Theorem 1.3. We say that a subsemigroup U of a semigroup S is *strongly left unitary* if

(1) U is left unitary in S ($us \in U$, $u \in U$ imply $s \in U$),

(2) for every $u, v \in U$, $s, t \in S$, $us = vt$ implies that there exist $u', v' \in U$, $p \in S$ such that $s = u'p$, $t = v'p$, $uu' = vv'$.

Note that if, as in Theorem 1.3, there exists a subset P of S such that the mapping $(u, p) \rightarrow up$ from $U \times P$ into S is a bijection, then S is strongly left unitary. We shall prove that if $[S_i; U; h_i]_{i\in I}$ is a monoid amalgam such that $h_i(U)$ is strongly left unitary in S_i for every $i \in I$, then $[S_i; U; h_i]_{i\in I}$ is embeddable. We need the following notion, which will play a key role in the next section.

DEFINITION 2.2. Let $[S_i; U; h_i]_{i\in I}$ be a semigroup amalgam. We say that two consecutive steps (i.e. replacement of occurrences of $u_i = h_i(u)$ by $u_j = h_j(u)$) in $\Pi^*_{i\in I} S_i$ are connected if the following conditions are satisfied:

(a) In case the first step introduces a new syllable, then the second step is an E_r- or E_l- or M-step and affects the syllable just introduced.

(b) In case the first step does not introduce a new syllable but transfers a u_i from a syllable s_i to a syllable s_j, then the second step affects the modified s_j.

A transition τ: $x \Rightarrow y$ is called connected if any two consecutive steps of τ are connected.

LEMMA 2.3. *Let $[S_i; U; h_i]_{i\in I}$ be a semigroup amalgam. Suppose $h_i(U)$ is strongly left unitary in S_i for every $i \in I$. Then connected transitions consisting of two steps of the form E_r-E_l, M-E_l or E_r-S, M-S are equivalent to transitions where the E_l-steps and the S-steps are performed first, or not at all.*

PROOF. *Case E_r-E_l.* If E_r introduces a new syllable we have

$$\cdots s_i \cdots = \cdots s_i'u_i \cdots \xrightarrow{E_r} \cdots s_i'u_j \cdots$$
$$= \cdots s_i'u_j's_j \cdots \xrightarrow{E_l} \cdots s_i'u_i's_j \cdots \text{ or } \cdots s_i'u_k's_j \cdots.$$

From $u_j = u_j's_j$ we deduce that $s_j \in U_j$ and $s_j = u_j''$. Thus we have the equivalent transitions:

$$\cdots s_i \cdots = \cdots s_i'u_i'u_i'' \cdots \xrightarrow{E_r} \cdots s_i'u_i'u_j'' \cdots = \cdots s_i'u_i's_j \cdots \text{ or}$$
$$\cdots s_i \cdots = \cdots s_i'u_i'u_i'' \cdots \xrightarrow{E_r} \cdots s_i'u_i'u_j'' \cdots \xrightarrow{E_r} \cdots s_i'u_k'u_j'' \cdots = \cdots s_i'u_k's_j \cdots.$$

If E_r does not introduce a new syllable we have

$$\cdots s_is_j \cdots = \cdots s_i'u_is_j \cdots \xrightarrow{E_r} \cdots s_i'u_js_j \cdots$$
$$= \cdots s_i'u_j's_j' \cdots \xrightarrow{E_l} \cdots s_i'u_i's_j' \cdots \text{ or } \cdots s_i'u_k's_j' \cdots.$$

But $u_js_j = u_j's_j'$ implies $s_j = v_jp_j$, $s_j' = v_j'p_j$, $u_jv_j = u_j'v_j'$ for some $v_j, v_j' \in U_j, p_j \in S_j$. Thus we obtain equivalent transitions as follows:

$$\cdots s_is_j \cdots = \cdots s_i'u_iv_jp_j \cdots \xrightarrow{E_l} \cdots s_i'u_iv_ip_j \cdots$$
$$= \cdots s_i'u_i'v_i'p_j \cdots \xrightarrow{E_r} \cdots s_i'u_i'v_j'p_j \cdots = \cdots s_i'u_i's_j' \cdots \text{ or}$$
$$\cdots s_i'u_iv_jp_j \cdots \xRightarrow{E_l - S} \cdots s_i'u_kv_kp_j \cdots$$
$$= \cdots s_i'u_k'v_k'p_j \cdots \xrightarrow{E_r} \cdots s_i'u_k'v_j'p_j \cdots = \cdots s_i'u_k's_j' \cdots .$$

The other cases M-E_l, E_r-S, M-S are treated similarly; they use only the left unitary condition or no condition at all for M-S.

As a consequence of Lemma 2.3, it is possible either to suppress or to shift S-steps or E_l-steps forward in a transition, provided they follow an E_r-step or an

M-step. This is proved as in Lemma 3.10 of [5] the most critical cases being when the following M-S configurations occur:

(1) $\cdots s_i u_i u'_i \cdots \xrightarrow{M} s_i u_j u'_i \cdots \xrightarrow{S} \cdots s_i u_j u'_k \cdots,$

(2) $\cdots u'_i u_i t_i \cdots \xrightarrow{M} \cdots u'_i u_j t_i \cdots \xrightarrow{S} \cdots u'_k u_j t_i \cdots.$

They can be replaced respectively by:

(1') $\cdots s_i u_i u'_i \cdots \xrightarrow{E_r} \cdots s_i u_j u'_j \cdots \xrightarrow{E_r} \cdots s_i u_j u'_k \cdots,$

(2') $\cdots u'_i u_i t_i \cdots \xrightarrow{E_l} \cdots u'_k u_k t_i \cdots \xrightarrow{E_r} \cdots u'_k u_j t_i \cdots.$

PROPOSITION 2.4. *Let $[S_i; U; h_i]_{i\in I}$ be a monoid amalgam (i.e. S_i and U are monoids and $h_i(e) = e_i$ for every $i \in I$). Suppose $h_i(U)$ is strongly left unitary in S_i for every $i \in I$. Then the amalgam is embeddable.*

PROOF. Let τ: $x \Rightarrow y$ be a transition from x to y, with $x, y \in \Pi^*_{i\in I} S_i$. Introducing eventually additional steps in τ, where identity elements are shifted from one syllable to another, we may assume that all two consecutive steps in τ form connected transitions. It follows from Lemma 2.3 that τ is equivalent to a transition τ' where all the S-steps and the E_l-steps are performed at the beginning of τ'. We will show that τ' is equivalent to a transition τ'' with the following structure: τ'' begins with a succession of E_l-steps, followed by a succession of S-steps, itself followed by a succession of steps that are not E_l or S-steps. To establish this, it is enough to show that an S-step followed by an E_l-step can be replaced by a sequence of the form E_l-E_l-$\cdots$-E_l-S-S-$\cdots$-S. The cases when an S-step followed by an E_l-step are not immediately interchangeable are

(1) $\cdots s_k u_i s_l \cdots \xrightarrow{S} \cdots s_k u_j s_l \cdots = \cdots s_k u'_j s_j s_l \cdots \xrightarrow{E_l} \cdots s_k u'_m s_j s_l \cdots,$

(2) $\cdots s_k u_i s_l \cdots \xrightarrow{S} \cdots s_k u_k s_l \cdots = \cdots u'_k s'_k s_l \cdots \xrightarrow{E_l} \cdots u'_m s'_k s_l \cdots,$

(3) $\cdots s_k u_i s_l \cdots \xrightarrow{S} \cdots s_k u_l s_l \cdots = \cdots s_k u'_l s'_l \cdots \xrightarrow{E_l} \cdots s_k u'_m s'_l \cdots.$

In case (1), from $u_j = u'_j s_j$ we deduce $s_j = u''_j$ and (1) is equivalent to

(1') $\cdots s_k u_i s_l \cdots = \cdots s_k u'_i u''_i s_l \cdots \xrightarrow{E_l} \cdots s_k u'_m u''_i s_l \cdots \xrightarrow{S} \cdots s_k u'_m u''_j s_l \cdots = \cdots s_k u'_m s_j s_l \cdots.$

In case (2), we use the fact that we have a monoid amalgam; (2) is equivalent to

(2') $\cdots s_k u_i s_l \cdots = \cdots s_k u_i e_i s_l \cdots \xrightarrow{E_l} \cdots s_k u_k e_i s_l \cdots \xrightarrow{E_l} \cdots u'_m s'_k e_i s_l \cdots \xrightarrow{S} \cdots u'_m s'_k e_k s_l \cdots = \cdots u'_m s'_k s_l \cdots.$

In case (3), from $u_l s_l = u'_l s'_l$ we deduce $s_l = v_l p_l$, $s'_l = v'_l p_l$, $u_l v_l = u'_l v'_l$ for some $p_l \in S_l$, and $v_l, v' \in U_l$. It follows that (3) is equivalent to

(3') $\cdots s_k u_i s_l \cdots = \cdots s_k u_i v_l p_l \cdots \xrightarrow{E_l} \cdots s_k u_i v_i p_l \cdots = \cdots s_k u'_i v'_i p_l \cdots \xrightarrow{E_l} \cdots s_k u'_m v'_i p_l \cdots \xrightarrow{S} \cdots s_k u'_m v'_l p_l \cdots = \cdots s_k u'_m s'_l \cdots.$

Suppose now that τ is a transition from $s_i^{(1)} \in S_i$ to $s_j^{(2)} \in S_j$ (eventually $i = j$). Then τ is equivalent to a transition τ'' with the properties indicated above. Since E_r-steps or M-steps do not diminish the syllable length, it follows that when used in τ'' they must be applied to one-syllable words and thus increase the syllable length. The transition τ'' linking two one-syllable words, it follows that τ'' consists in a succession of E_l-steps followed by S-steps. Using the left unitary condition, a simple induction argument shows that a succession of E_l-steps transforms $s_i^{(1)}$ into $u_{i_1}^{(1)}u_{i_2}^{(2)} \cdots u_{i_n}^{(n)}s_i^{(3)}$ where $s_i^{(1)} = u_i^{(1)}u_i^{(2)} \cdots u_i^{(n)}s_i^{(3)}$. In case $j \neq i$, the S-steps in τ'' must affect $s_i^{(3)}$. Thus $s_i^{(3)} = u_i^{(n+1)}$ and $s_j^{(1)} = u_j^{(1)}u_j^{(2)} \cdots u_j^{(n)}u_j^{(n+1)}$, showing that $s_i^{(1)} = u_i$ and $s_j^{(2)} = u_j$. In case $j = i$, the S-steps in τ'' may or not affect $s_i^{(3)}$. If they do, then $s_i^{(1)} = u_i = s_i^{(2)}$ with $u_i = u_i^{(1)}u_i^{(2)} \cdots u_i^{(n)}s_i^{(3)}$ and if they do not $s_i^{(1)} = u_is_i^{(3)} = s_i^{(2)}$ with $u_i = u_i^{(1)}u_i^{(2)} \cdots u_i^{(n)}$. Condition (iv) of Theorem 1.1 is satisfied, proving that the amalgam is embeddable.

3. A necessary and sufficient condition for the embedding of semigroup amalgams. Let $[S_i; U; h_i]_{i \in I}$ be a semigroup amalgam, and let

$$\tau: x = x_0 \to x_1 \to x_2 \to \cdots \to x_n = y$$

be a connected transition (see Definition 2.2) from x to y, with $x = x_0, x_1, \cdots, x_n = y \in \Pi^*_{i \in I} S_i$. Given any two consecutive steps $x_{i-1} \to x_i \to x_{i+1}$ $(0 < i < n)$ in τ, there is a syllable of x_i which admits two factorizations, the first one reflecting the result of the step $x_{i-1} \to x_i$ and the second reflecting the origin of $x_i \to x_{i+1}$. Consider for example, the following connected E_r-E_l steps

$$\cdots s_k u_k^{(1)} s_l^{(1)} \cdots \xrightarrow{E_r} s_k u_l^{(1)} s_l^{(1)} \cdots = \cdots s_k u_l^{(2)} s_l^{(2)} \cdots \xrightarrow{E_l} \cdots s_k u_k^{(2)} s_l^{(2)} \cdots.$$

The syllable with index l obtained after E_r has the factorization $u_l^{(1)}s_l^{(1)}$ ("outcome" of E_r) and the factorization $u_l^{(2)}s_l^{(2)}$ ("input" of E_l).

An equality arising from two connected steps $(u_l^{(1)}s_l^{(1)} = u_l^{(2)}s_l^{(2)})$ of the transition τ is called a *connecting equality* of τ.

DEFINITION 3.1. Let I be a set, and let $s_i^{(k)}, u_i^{(k)}, i \in I, k \in \mathbf{N}$, be sequences of letters forming an alphabet A. A system Σ of equations is a set of (unordered) pairs of elements of the free semigroup over A. If $(w, w') \in \Sigma$ we write $w = w'$. A system Σ is called a *coherent system* of equations, if there exist a semigroup amalgam $[S_i; U; h_i]_{i \in I}$ and a connected transition τ in $\Pi^*_{i \in I} S_i$ such that the equations of Σ are the connecting equalities of τ.

EXAMPLE. Let τ be the connected transition

$$s_2^{(1)}s_1^{(1)}u_3^{(1)} \to s_2^{(1)}s_1^{(1)}u_1^{(1)} = s_2^{(1)}u_1^{(2)}s_1^{(2)} \to s_2^{(1)}u_2^{(2)}s_1^{(2)} = u_2^{(3)}s_1^{(2)}$$
$$\to u_3^{(3)}s_1^{(2)} = s_3^{(1)}u_3^{(4)}s_3^{(2)}s_1^{(2)}.$$

The coherent system corresponding to τ is

$$s_1^{(1)}u_1^{(1)} = u_1^{(2)}s_1^{(2)}, \quad s_2^{(1)}u_2^{(2)} = u_2^{(3)}, \quad u_3^{(3)} = s_3^{(1)}u_3^{(4)}s_3^{(2)}.$$

In order to define a coherent system of equations independently from any reference to a connected transition, note that for any equation E in a coherent system, the left side and the right side of E are words over A of the type $u_i^{(l)}$, $u_i^{(l)}s_i^{(m)}$, $s_i^{(k)}u_i^{(l)}$, $s_i^{(k)}u_i^{(l)}s_i^{(m)}$ with the same index for each side of E and with the exception that E is not $u_i^{(l_1)} = u_i^{(l_2)}$. Thus there are 15 possible types of equations. If the right side of E is $u_i^{(l)}$ we say that the u-variable appears *freely* in E. If the right side of E is $u_i^{(l)}s_i^{(m)}$ [resp. $s_i^{(k)}u_i^{(l)}$] we say that the u-variable is *blocked on the right* [resp. left] in E by $s_i^{(m)}$ [resp. $s_i^{(k)}$]. Finally if the right side of E is $s_i^{(k)}u_i^{(l)}s_i^{(m)}$ we say that the u-variable is *blocked* in E by $s_i^{(k)}$ and $s_i^{(m)}$. This terminology is used in our next inductive construction of a coherent system.

Construction 3.2. Let I be a set, $s_i^{(k)}$, $u_i^{(k)}$ $(i \in I, k \in \mathrm{N})$ sequences of letters forming an alphabet A. We construct a finite system of equations $E_1, E_2, \cdots, E_r$ over A as follows:

(1) For every n $(1 \leqslant n \leqslant r)$ the letters in E_n have all the same index i_n and $i_n \neq i_{n+1}$ for $1 \leqslant n < r$. Furthermore E_n is not $u_{i_n}^{(k)} = u_{i_n}^{(l)}$ for all n, $1 \leqslant n \leqslant r$, $k, l \in \mathrm{N}$.

(2) For every $i \in I$, the letters $u_i^{(k)}$ appearing in all equations E_n such that $i_n = i$ have distinct superscripts, and the letters $s_i^{(k)}$ appearing in a given equation also have distinct superscripts.

(3) E_1 is any one of the 15 possible types of equations indicated above.

(4) Suppose $E_1, E_2, \cdots, E_{n-1}$ $(n > 1)$ have been defined. Then the right side of E_n can be $u_{i_n}^{(l)}$, $u_{i_n}^{(l)}s_{i_n}^{(m)}$, $s_{i_n}^{(k)}u_{i_n}^{(l)}$ or $s_{i_n}^{(k)}u_{i_n}^{(l)}s_{i_n}^{(m)}$, for some $k, l, m \in \mathrm{N}$ except for the case outruled by (1).

The restrictions for the left side of E_n depend first on the nature of the right side of E_{n-1}. They are indicated on the following table:

Right side of E_{n-1}	$u_{i_{n-1}}^{(l)}$	$u_{i_{n-1}}^{(l)}s_{i_{n-1}}^{(m)}$	$s_{i_{n-1}}^{(k)}u_{i_{n-1}}^{(l)}$	$s_{i_{n-1}}^{(k)}u_{i_{n-1}}^{(l)}s_{i_{n-1}}^{(m)}$
Possible left sides of E_n	$u_{i_n}^{(l)}$	$u_{i_n}^{(l)}$	$u_{i_n}^{(l)}$	$u_{i_n}^{(l)}$
	$s_{i_n}^{(k)}u_{i_n}^{(l)}$	$s_{i_n}^{(k)}u_{i_n}^{(l)}$	$u_{i_n}^{(l)}s_{i_n}^{(m)}$	
	$u_{i_n}^{(l)}s_{i_n}^{(m)}$			
	$s_{i_n}^{(k)}u_{i_n}^{(l)}s_{i_n}^{(m)}$			

Further restrictions on the contexts of $u_{i_n}^{(l)}$ are as follows:

(a) If the u-variable appears freely in E_k for every $k < n$, there are no restrictions on the context of $u_{i_n}^{(l)}$.

(b) If, for every $k < n$, the u-variable is blocked on the left [resp. right] by $s_{i_k}^{(k)}$ [resp. $s_{i_k}^{(m)}$] and $s_{i_k}^{(k)}$ [resp. $s_{i_k}^{(m)}$] appears on the left side of $E_{k+1}, \cdots, E_{n-1}$, there is no restriction on the left [resp. right] context of $u_{i_n}^{(l)}$.

(c) If the u-variable does not appear freely in E_k for every $k < n$, let p [resp. q] be the largest integer $< n - 1$ such that the u-variable is blocked on the left by $s_{i_p}^{(k)}$ [resp. on the right by $s_{i_q}^{(m)}$] and such that $s_{i_p}^{(k)}$ [resp. $s_{i_q}^{(m)}$] does not appear on the left side of any of the equations $E_{p+1}, \cdots, E_{n-1}$ [resp. $E_{q+1}, \cdots, E_{n-1}$]. If $i_n \neq i_p$ [resp. $i_n \neq i_q$], then u_{i_n} has no left [resp. right] context. If $i_n = i_p$ [resp. $i_n = i_q$], then the left [resp right] context of u_{i_n} is $s_{i_p}^{(k)}$ [resp. $s_{i_q}^{(m)}$].

Justification. Suppose that the equations of a system Σ are the connecting equalities of a transition τ in $\Pi^*_{i\in I} S_i$ where $[S_i; U; h_i]_{i\in I}$ is a semigroup amalgam. There is no difficulty verifying (1), (2), (3) and the restrictions concerning the type of the right and left side of E_n. To verify (4)(a) assume that the u-variable appears freely in E_k $(k < n)$. Then none of the first n steps of τ introduces an s-variable. Consequently the contexts of $u_{i_n}^{(l)}$ do not depend on the equations $E_1, E_2, \cdots, E_{n-1}$. Similarly if the u-variables are blocked for example on the left, by variables $s_{i_k}^{(k)}$ appearing on the left side of a following equation, the left context of $u_{i_n}^{(l)}$ does not depend on $E_1, E_2, \cdots, E_{n-1}$. Finally s-variables blocking u-variables and not appearing on the left side of a following equation are s-variables introduced by the successive steps of τ and not suppressed by any of them. They are present in the successive elements of $\Pi^*_{i\in I} S_i$ involved in τ, in their order of appearance. Consequently the left context of $u_{i_n}^{(l)}$ (if any) must be $s_{i_p}^{(k)}$ as defined (4)(b).

Conversely, let Σ be a system of equations obeying conditions (1) to (4). For a fixed $i \in I$ let S_i be the semigroup generated by all $s_i^{(k)}, u_i^{(l)}$ where $s_i^{(k)}$ belongs to the set of all s-variables with index i appearing in Σ and $u_i^{(l)}$ belongs to the set of all u-variables such that $u_j^{(l)}$ appears in Σ for some $j \in I$. Suppose that the presentation relations for S_i are all the equations of Σ involving s-variables and u-variables with index i. By condition (2) the subsemigroup of S_i generated by all $u_i^{(l)}$ is free. Thus, if U is the free semigroup on the set $\{u^{(l)}$: there exists $j \in I$ that $u_j^{(l)}$ appears in $\Sigma\}$, then there is a 1-1 homomorphism h_i: $U \to S_i$ for every $i \in I$ and $[S_i; U; h_i]_{i\in I}$ is a semigroup amalgam. We show that there is a transition τ in $\Pi^*_{i\in I} S_i$ admitting the equations $E_1, \cdots, E_r$ of Σ as connecting equalities. We shall say that an s-variable $s_i^{(m)}$ is *at the origin* [resp. the *extremity*] *of* Σ if $s_i^{(m)}$ appears on the left [resp. right] side of an equation E_k $(1 \leqslant k \leqslant r)$ but does not appear on the right [resp. left] side of an equation E_l for $l < k$ [resp. for $l > k$]. Let $x = s_{j_r}^{(l_r)} \cdots s_{j_2}^{(l_2)} s_{j_1}^{(l_1)} u_n^{(1)} s_{k_1}^{(m_1)} s_{k_2}^{(m_2)} \cdots s_{k_t}^{(m_t)}$ be the element of $\Pi^*_{i\in I} S_i$ obtained by the following process: $u_{i_1}^{(1)}$ is the u-variable in the left side of E_1 and $n \neq i_1$; $s_{j_1}^{(l_1)}, s_{j_2}^{(l_2)}, \cdots, s_{j_r}^{(l_r)}$ are s-variables at the origin of Σ that are left contexts of u-variables, written from right to left in their order of appearance; similarly $s_{k_1}^{(m_1)}, s_{k_2}^{(m_2)}, \cdots, s_{k_t}^{(m_t)}$ are right contexts of u-variables at the origin of Σ written from left to right in their order of appearance. Dually define $y \in \Pi^*_{i\in I} S_i$ using as u-variable the variable appearing on the right side of E_r, and using s-variables at the extremity of Σ. Then by conditions (3), (4) it is possible to construct inductively a connected transition $\tau\colon x \Rightarrow y$ with the equations of Σ as connecting equalities. The details involve only notational complications and thus are omitted.

The next definition uses the notion of an s-variable at the origin, or the extremity of a connected system of equations Σ, as defined above in the justification of Construction 3.2.

DEFINITION 3.2. Let Σ be a connected system of equation $E_1, E_2, \cdots, E_r$. Σ is called a *balanced system* if

(1) the left side of E_1 is $u_{i_1}^{(1)}$ and the right side of E_r is $u_{i_r}^{(n)}$,

(2) there is at most one s-variable $s_i^{(l)}$ [resp. $s_j^{(l')}$] at the origin [resp. extremity] of Σ which is a left context, and there is at most one s-variable $s_i^{(m)}$ [resp. $s_j^{(m')}$] at the origin [resp. extremity] of Σ which is a right context.

If $i = j$ we say that Σ is a balanced system of type I, and if $i \neq j$ Σ is called a balanced system of type II. Given a balanced system of type I we call the equation $s_i^{(l)} u_i^{(1)} s_i^{(m)} = s_i^{(l')} u_i^{(n)} s_i^{(m')}$ the *equation locked* to Σ. Note that the s-variables in the locked equation might be present or not. For a balanced system of type II the sentence: "there exists $u_i^{(0)}, u_j^{(0)}$ such that $s_i^{(l)} u_i^{(1)} s_i^{(m)} = u_i^{(0)}$ and $s_i^{(l')} u_i^{(n)} s_i^{(m)} = u_j^{(0)}$" is called the *sentence locked* to Σ.

EXAMPLES. The following are balanced systems with their locked equation or sentence:

TYPE I	TYPE II
$u_2^{(1)} = u_2^{(2)} s_2^{(1)}$	$u_2^{(1)} = s_2^{(1)} u_2^{(2)}$
$s_1^{(1)} u_1^{(2)} = s_1^{(2)} u_1^{(3)} s_1^{(3)}$	$u_3^{(2)} = u_3^{(3)} s_3^{(1)}$
$u_2^{(3)} = s_2^{(2)} u_2^{(4)}$	$s_2^{(1)} u_2^{(3)} = u_2^{(4)} s_2^{(2)}$
$u_1^{(4)} s_1^{(3)} = u_1^{(5)}$	$s_1^{(1)} u_1^{(4)} = u_1^{(5)}$
$s_2^{(2)} u_2^{(5)} s_2^{(1)} = u_2^{(6)}$	$u_2^{(5)} s_2^{(2)} = u_2^{(6)}$

Locked equation: $s_1^{(1)}u_1^{(1)} = s_1^{(2)}u_1^{(6)}$.

Locked sentence: there exists $u_1^{(0)}, u_3^{(0)}$ such that $s_1^{(1)}u_1^{(1)} = u_1^{(0)}, u_3^{(6)}s_3^{(1)} = u_3^{(0)}$.

Our main result is:

THEOREM 3.3. *A semigroup amalgam $[S_i; U; h_i]_{i\in I}$ is embeddable if and only if whenever a balanced system of equations Σ holds in S_i and $U_i = h_i(U)$ $(i \in I)$, then the equation or the sentence locked to Σ also holds in the semigroups S_i $(i \in I)$.*

By condition (iv) of Theorem 1.1 and Definition 3.2 of a balanced system, the condition of Theorem 3.3 is obviously necessary. That it is also sufficient follows directly from Lemma 3.5 below.

Let $[S_i; U; h_i]_{i\in I}$ be a semigroup amalgam and let σ be a transition in $\Pi^*_{i\in I} S_i$ of the form σ: $x = s_{i_1}s_{i_2} \cdots s_{i_k} \Rightarrow y$. We define the notion of descendant of a given syllable s_{i_m} of x as follows. Consider the first step of σ modifying the syllable s_{i_m}. This step might incorporate s_{i_m} into a new syllable of one of the forms $a_{i_m}s_{i_m}b_{i_m}, a_{i_m}s_{i_m}, s_{i_m}b_{i_m}$ or it might create new syllables after a decomposition of s_{i_m} of one the forms $a_{i_m}^{(1)}u_{i_m}, u_{i_m}a_{i_m}^{(2)}, a_{i_m}^{(1)}u_{i_m}a_{i_m}^{(2)}$. The new syllables created by the first modifying step of s_{i_m} are called the direct descendants of s_{i_m}. A *descendant* of s_{i_m} is either s_{i_m} or a syllable t in σ such that there exists a succession of syllables $t_0, t_1, \cdots, t_n$ with $t_0 = s_{i_m}, t_n = t$ and t_{i+1} is a direct descendant of t_i for $i = 0, 1, \cdots, n-1$.

In case the first modifying step of s_{i_m} is due to a factorization of s_{i_m} into $a_{i_m}^{(1)}u_{i_m}$ or $u_{i_m}a_{i_m}^{(2)}$ or $a_{i_m}^{(1)}u_{i_m}a_{i_m}^{(2)}$, then $a_{i_m}^{(1)}$ and $a_{i_m}^{(2)}$ might be modified by subsequent steps of σ, and this modification might be due to factorizations of $a_{i_m}^{(1)}$ or $a_{i_m}^{(2)}$. If all the factorizations due to the successive steps of σ are reported on s_{i_m}, then s_{i_m} takes one of the forms

$$s_{i_m} = u_{i_m} \quad \text{or} \quad s_{i_m} = u_{i_m}^{(1)}a_{i_m}^{(1)}u_{i_m}^{(2)}a_{i_m}^{(2)} \cdots u_{i_m}^{(n_m)}a_{i_m}^{(n_m)}u_{i_m}^{(n_m+1)}$$

where some of the $u_{i_m}^{(k)}$'s might be empty symbols. It follows that

$$(1) \quad x = s_{i_1}s_{i_2} \cdots s_{i_k} = \prod_{m=1}^{k} u_{i_m}^{(1)}a_{i_m}^{(1)}u_{i_m}^{(2)}a_{i_m}^{(2)} \cdots u_{i_m}^{(n_m)}a_{i_m}^{(n_m)}a_{i_m}^{(n_m+1)}.$$

We define the descendants of an element $a_{i_m}^{(j)}$ as the descendants of the syllable containing $a_{i_m}^{(j)}$ when the modification of $a_{i_m}^{(j)}$ occurs (by modification of $a_{i_m}^{(j)}$ we mean the step $x_t \to x_{t+1}$ of σ where $a_{i_m}^{(j)}$ appears in a syllable of x_t but does not appear in x_{t+1}).

On the set $A_x^\sigma = \{a_{i_m}^{(j)} | a_{i_m}^{(j)}$ is in the decomposition (1) of $x\}$ we define a relation γ by $a_{i_m}^{(j)} \gamma a_{i_{m'}}^{(j')}$ if the modifications of $a_{i_m}^{(j)}$ and $a_{i_{m'}}^{(j')}$ occur simultaneously in case both are modified, or if $m = m', j = j'$ if $a_{i_m}^{(j)}$ is not modified. It is easy to verify that γ is an equivalence relation on A_x^σ. Furthermore, $a_{i_m}^{(j)} \gamma a_{i_{m'}}^{(j')}$ implies $m = m'$, because both disappear simultaneously, and thus they are in the same syllable of x_t, there $x_t \to x_{t+1}$ is their modifying step.

Denoting by $\gamma[a_{i_m}^{(j)}]$ the γ-class of $a_{i_m}^{(j)}$, we define a relation $\leqslant$ on the quotient set A_x^σ/γ by

(a) $\gamma[a_{i_m}^{(j)}] \leqslant \gamma[a_{i_{m'}}^{(j')}]$ if the syllable containing $a_{i_m}^{(j)}$ when $a_{i_m}^{(j)}$ is modified, is a descendant of $a_{i_{m'}}^{(j')}$,

(b) $\gamma[a_{i_m}^{(j)}] \leqslant \gamma[a_{i_{m'}}^{(j')}]$ if $a_{i_m}^{(j)}$ is unmodified and $a_{i_{m'}}^{(j')}$ is modified or if $a_{i_m}^{(j)}$ and $a_{i_{m'}}^{(j')}$ are both unmodified and $m = m', j = j'$.

The relation $\leqslant$ is reflexive, antisymmetric (since γ is an equivalence) and transitive (by transitivity of the descendance relation). Thus it is a partial ordering of A_x^σ/γ. Note that if $a_{i_m}^{(j)}$ is unmodified by σ, then $\gamma[a_{i_m}^{(j)}] = \{a_{i_m}^{(j)}\}$ is a minimal element for $\leqslant$.

LEMMA 3.4. *Let $\sigma\colon x \Rightarrow y$ be a transition in $\Pi_{i\in I}^* S_i$. A maximal element for the partial ordering $\leqslant$ defined above on A_x^σ consists of consecutive elements $a_{i_m}^{(j)}$ of the decomposition* (1) *of x, all contained in a single syllable of x.*

PROOF. Suppose $a_{i_m}^{(j)} \gamma a_{i_m}^{(j_1)}$ and $a_{i_n}^{(k)} \gamma a_{i_n}^{(k_1)}$. If these four elements of A_x^σ have the disposition $\cdots a_{i_m}^{(j)} \cdots a_{i_n}^{(k)} \cdots a_{i_m}^{(j_1)} \cdots a_{i_n}^{(k_1)} \cdots$ in the decomposition (1) of x, then the common modification of $a_{i_m}^{(j)}$ and $a_{i_m}^{(j_1)}$ occurs at the same step of σ as the common modification of $a_{i_n}^{(k)}$ and $a_{i_n}^{(k_1)}$. It follows that all four elements are in the same class of γ. If the four elements have the disposition $\cdots a_{i_m}^{(j)} \cdots a_{i_n}^{(k)} \cdots a_{i_n}^{(k_1)} \cdots a_{i_m}^{(j_1)} \cdots$, then the common modification of $a_{i_n}^{(k)}$ and $a_{i_n}^{(k_1)}$ takes place at a step of σ occurring before the step modifying $a_{i_m}^{(j)}$ and $a_{i_m}^{(j_1)}$. But when $a_{i_m}^{(j)}$ and $a_{i_m}^{(j_1)}$ are modified they belong to a syllable of the form $x_{i_m} a_{i_m}^{(j)} y_{i_m} a_{i_m}^{(j_1)} z_{i_m}$ (one or both of x_{i_m}, z_{i_m} are eventually empty symbols), and this syllable is a descendant of $a_{i_n}^{(k)}$. By the definition of $\leqslant$, we have $\gamma[a_{i_m}^{(j)}] \leqslant \gamma[a_{i_n}^{(k)}]$. If $\gamma[a_{i_m}^{(j)}]$ is a maximal element of A_x^σ, we deduce $\gamma[a_{i_m}^{(j)}] = [a_{i_n}^{(k)}]$ and the lemma follows.

LEMMA 3.5. *Let $\sigma\colon x \Rightarrow y$ be a transition in $\Pi_{i\in I}^* S_i$, with $x \in S_i, y \in S_j$ (eventually $i = j$). Then there is a connected transition $\tau\colon x \Rightarrow y$ in $\Pi_{i\in I}^* S_i$.*

PROOF. By induction on the length (i.e. the number of steps) of σ. The result is clear if σ has length 1. We assume it is true for any transition of length strictly less than the length of σ. The factorization (1) of x is $x = u_i$ or

$$(1') \qquad x = u_i^{(1)}a_i^{(1)}u_i^{(2)}a_i^{(2)} \cdots u_i^{(n)}a_i^{(n)}u_i^{(n+1)}.$$

The case $x = u_i$ occurs when the first step of σ is an S-step, i.e. σ has the form $x = u_i \to u_k \Rightarrow y$. The induction hypothesis applied to the transition $u_k \Rightarrow y$ gives the result. Suppose x has the factorization $(1')$ with $n \geqslant 1$. If all the elements $a_i^{(1)}, \cdots, a_i^{(n)}$ are unmodified by σ (i.e. are minimal singleton γ-classes), then $y \in S_i$ and there are transitions in σ such that $u_i^{(k)} \Rightarrow b_i^{(k)}$ for every k, $1 \leqslant k \leqslant n+1$. By the induction hypothesis there are connected transitions $\tau^{(k)}\colon u_i^{(k)} \Rightarrow b_i^{(k)}$, and $\tau^{(1)}, \tau^{(2)}, \cdots, \tau^{(k+1)}$ applied successively give a connected transition $\tau\colon x \Rightarrow y$. If there are elements $a_i^{(1)}, \cdots, a_i^{(n)}$ modified by σ, then by Lemma 3.4 there is a segment of x, say

$$x' = u_i^{(k)}a_i^{(k)}u_i^{(k+1)}a_i^{(k+1)} \cdots u_i^{(k+l)}a_i^{(k+l)}a_i^{(k+l)}a_i^{(k+l)}u_i^{(k+l+1)},$$

such that the elements $a_i^{(k)}, a_i^{(k+1)}, \cdots, a_i^{(k+1)}$ form a maximal γ-class of A_x^σ. Since $\gamma(a_i^{(k)})$ is maximal, all the steps of σ affecting x' until the modifying step common to $a_i^{(k)}, \cdots, a_i^{(k+l)}$ can be performed at the beginning of σ in the order in which they appear in σ. Thus σ has the same length as the transition

$$\begin{aligned} x &= u_i^{(1)}a_i^{(1)} \cdots a_i^{(k-1)}x'a_i^{(k+l+1)} \cdots a_i^{(n)}u_i^{(n+1)} \\ &\overset{\sigma_1}{\Rightarrow} u_i^{(1)}a_i^{(1)} \cdots a_i^{(k-1)}x''a_i^{(k+l+1)} \cdots a_i^{(n)}u_i^{(n+1)} \\ &\overset{\sigma_2}{\Rightarrow} y \end{aligned}$$

where the step following immediately the syllable containing x'' is the modifying step common to $a_i^{(k)}, \cdots, a_i^{(k+l)}$. The induction hypothesis applied to σ_1 and σ_2 gives a connected transition $\tau\colon x \Rightarrow y$. This completes the proof of Lemma 3.5, and thus Theorem 3.3 is established.

As a consequence of Theorem 3.3 we deduce Howie's result [5].

COROLLARY 3.6. *Let $[S_i; U; h_i]_{i\in I}$ be a semigroup amalgam. If $U_i = h_i(U)$ is almost unitary in S_i for every $i \in I$, then the amalgam is embeddable.*

PROOF. Recall that the hypothesis U_i almost unitary in S_i for every $i \in I$ (see [2, p. 144]), means in particular that for every $i \in I$ there is a pair of linked left and right translations λ_i, ρ_i of S_i such that U_i is unitary in $\lambda_i S_i \rho_i$. Denoting $\lambda_i(s_i)\rho_i$ by $\theta^i(s_i)$, we also have $\theta^i(s_i u_i t_i) = \theta^i(s_i)u_i\theta^i(t_i)$, for every $s_i, t_i \in S_i$, $u_i \in U_i$, s_i or t_i being eventually empty symbols (see [2, Lemma 9.36]). In what follows, some of the elements $\theta^i(s_i)$ are elements of U_i. We denote them by $\theta_i^i(s_i)$ and define $\theta_j^i(s_i)$ by $\theta_j^i(s_i) = h_j[h_i^{-1}\theta^i(s_i)]$. Given a balanced system Σ over $[S_i; U; h_i]_{i\in I}$ we prove that the locked equation or the locked sentence holds, as follows. Let i_0 be the index of the locked equation in case Σ is of type I; in this case we will show that all s-variables s_i, $i \neq i_0$, appearing in Σ are such that $\theta^i(s_i) \in U_i$. Similarly let i_0, j_0 be the indices appearing in the locked sentence in case Σ is of the type II; in this case we show that all s-variables s_i, $i \neq i_0$ and $i \neq j_0$, appearing in Σ are such that $\theta^i(s_i) \in U_i$, and that all s-variables in one of any two consecutive equations with indices i_0 and j_0 are such that $\theta^{i_0}(s_{i_0}) \in U_{i_0}$ (or $\theta^{j_0}(s_{j_0}) \in U_{j_0}$). If Σ is of type I, the locked equation is derived from Σ by replacing each step of the connected sequence of Σ by an equality in S_{i_0}. This is possible since each equation of Σ with index $i \neq i_0$ can be replaced by an equation with variables in U_i and thus in U_{i_0}. For example the equation $s_i^{(k)} u_i^{(l)} s_i^{(m)} = s_i^{(n)} u_i^{(l+1)}$ can be replaced by $\theta^i_{i_0}(s_i^{(k)}) u_{i_0}^{(l)} \theta^i_{i_0}(s_i^{(m)}) = \theta^i_{i_0}(s_i^{(n)}) u_{i_0}^{(l+1)}$. A similar argument applies if Σ is a system of type II, but in this case we need the additional property that all the s-variables in the initial or in the final equation of Σ (with respective indices i_0 and j_0) are such that their image under θ^{i_0} or θ^{j_0} are in U_{i_0} or U_{j_0}. In order to prove all the required properties of elements of the form $\theta^i(s_i)$ where s_i is an s-variable in Σ, we transform Σ into a system of equations over the amalgam $[\theta^i(S_i); U; h_i]_{i \in I}$ by applying θ^i to both sides of any equation in θ with index i. It is thus enough to show that the properties indicated for elements $\theta^i(s_i)$ are in fact true for s-variables in a balanced system Σ over an amalgam such that U_i is unitary in S_i for every $i \in I$. This is done by induction on the number of equations in Σ, the idea of the induction being the possibility of moving forward the first S-step in the connected transition defined by Σ (cf. proof of Proposition 2.1). If Σ is not empty, then the connected transition of Σ contains at least one E-step or one E_r-step followed by an S-step. Thus Σ contains two consecutive equations of the form

$$(1)\qquad \begin{array}{ll} (1.a) & \cdots = u_i^{(k)} s_i^{(l)} \\ (1.b) & s_j^{(m)} u_j^{(k)} = u_j^{(k+1)} \end{array}$$

or

$$(2)\qquad \begin{array}{ll} (2.a) & \cdots = s_i^{(l)} u_i^{(k)} \\ (2.b) & u_j^{(k)} s_j^{(m)} = u_j^{(k+1)}. \end{array}$$

In both cases it follows that $s_j^{(m)}$ is an element of U_j which is denoted by $\tau_j(s_j^{(m)})$. Also, we put $\tau_i(s_j^{(m)}) = h_i[h_j^{-1} \tau_j(s_j^{(m)})]$. We construct a balanced system Σ' containing less equations than Σ by deleting the equation $s_j^{(m)} u_j^{(k)} = u_j^{(k+1)}$) (or $u_j^{(k)} s_j^{(m)} = u_j^{(k+1)}$) in the first succession of equations of the form (1) or (2) in Σ and by modifying correspondingly the remaining equations. To indicate the details of these modifications, suppose we suppress equation (1.b) (the case (2.b) is symmetrical). If the index of the equation following immediately (1.b) in Σ is distinct from i (index of (1.a)) the equations of Σ following (1.b) are not modified. Equation (1.a) is replaced by

$$\tau_i(s_j^{(m)}) \cdots = u_i^{(k+1)} s_i^{(l)} \tag{1.a$'$}$$

which is derived from (1.a) by multiplication of both sides on the left by $\tau_i(s_j^{(m)})$. All equations preceding (1.a) are similarly multiplied on the left by $\tau_k(s_j^{(m)})$ which can alternatively be considered as an s-variable or a u-variable, until an equation containing eventually the variable $s_i^{(l)}$ is reached. We replace $s_i^{(m)}$ by $\tau_i(s_i^{(m)})$ in it and continue the multiplication process if necessary in other equations so as to obtain a balanced system Σ'. In case the index of the equation following (1.b) in Σ is the index of (1.a), then this equation has the form $\cdots u_i^{(k+1)} s_i^{(l)} = \cdots$. We replace (1.a) and the equation following (1.b) by a single equation obtained by replacing $u_i^{(k+1)} s_i^{(l)}$ in the latter by its value $\tau_i(s_j^{(m)}) \cdots$ given in (1.a$'$). Other modifications are performed as indicated above, yielding again a system Σ' with less equations than Σ. By the induction hypothesis, all s-variables in Σ' have the required properties. But s-variables in Σ' are either s-variables in Σ or products of s-variables in Σ by an element in one of the U_i's. The unitary condition implies then that all s-variables of Σ in Σ' have the required properties, while variables of Σ, not in Σ', are in one of the U_i's by the reduction process. This completes the proof of Corollary 3.6.

In contrast to the previous result, Theorem 1.4 is an immediate consequence of Theorem 3.3. Indeed, the hypothesis in an ideal amalgamation is such that the sentence locked to a balanced system is directly derivable from the system itself.

It does not seem possible to derive Proposition 2.4, or the embeddability property in the inverse semigroup case, without further analysis of balanced systems in these particular cases. This leads to proofs that are, at the present state of affairs, longer than the original ones.

4. Finite sets of equational implications. In order to define equational implications over a semigroup amalgam we need the following

Definition 4.1. A morphism from the semigroup amalgam $[S_i; U; h_i]_{i\in I}$ to the amalgam $[T_i; V; k_i]_{i\in I}$ is a collection of homomorphisms $[\Phi_i; \Phi]_{i\in I}$ such that the diagrams

$$\begin{tikzcd} S_i \arrow[r, "\Phi_i"] & T_i \\ U \arrow[u, "h_i"] \arrow[r, "\Phi"] & V \arrow[u, "k_i"'] \end{tikzcd}$$

commute for every $i \in I$.

It is easily seen that the existence of a morphism $[\Phi_i, \Phi]_{i\in I}$ is equivalent to the existence of a homomorphism θ: $\Pi^*_{i\in I^U} S_i \to \Pi^*_{i\in I^V} T_i$ such that $\theta[\varphi_i(S_i)] \subseteq \overline{\varphi}_i(T_i)$ and $\theta[h(U)] \subseteq k[V]$ with φ_i [resp. $\overline{\varphi}_i$] the canonical homomorphism $S_i \to \Pi^*_U S_i$ [resp. $T_i \to \Pi^*_V T_i$] and h [resp. k] the canonical homomorphism $U \to \Pi^*_U S_i$ [resp. $V \to \Pi^*_V T_i$] (see §1).

Let Σ_i $(i \in I)$ and Ω be alphabets (i.e. sets of letters) such that there are injections χ_i: $\Omega \to \Sigma_i$ for every $i \in I$. Extend χ_i to an injective homomorphism, also denoted χ_i from Ω^* to Σ_i^* where Ω^* and Σ_i^* are the free semigroups on Ω and Σ_i respectively. Then $[\Sigma_i^*; \Omega^*; \chi_i]_{i\in I}$ is an embeddable semigroup amalgam.

Let $w_i^{(k_p)}, w_i^{(l_p)}, v_i^{(m_q)}, v_i^{(n_q)} \in \Sigma_i^*$ with $p, q = 1, 2, \cdots$.

DEFINITION 4.2. We say that the set of *equational implications*

$$(*) \quad w_i^{(k_p)} = w_i^{(l_p)} (i\in I, p = 1, 2, \cdots) \quad \text{imply} \quad v_i^{(m_q)} = v_i^{(n_q)} (i \in I, q = 1, 2, \cdots)$$

holds in the semigroup amalgam $[S_i; U; h_i]_{i\in I}$ if for any morphism $[\Phi_i; \Phi]_{i\in I}$: $[\Sigma_i^*; \Omega^*; \chi_i]_{i\in I} \to [S_i; U; h_i]_{i\in I}$

$$\Phi_i(w_i^{(k_p)}) = \Phi_i(w_i^{(l_p)}) \ (i \in I, p = 1, 2, \cdots) \quad \text{imply}$$

$$\Phi_i(v_i^{(m_q)}) = \Phi_i(v_i^{(n_q)}) \ (i \in I, q = 1, 2, \cdots).$$

It should be noted that the equational implications $(*)$ contain in general existential quantifiers affecting elements of Ω^* involved in the expressions of $w_i^{(k_p)}, w_i^{(l_p)}, v_i^{(m_q)}, v_i^{(n_q)}$. As an example to write

$$\begin{matrix} u_1^{(1)} s_1^{(1)} = s_1^{(2)} u_1^{(2)} & & s_1^{(1)} u_1^{(2)} = u_1^{(4)} \\ & \text{implies} & \\ u_2^{(2)} = s_2^{(1)} u_2^{(3)} & & s_2^{(1)} = u_2^{(4)} \end{matrix}$$

means: If there exists $u^{(1)}, u^{(2)}, u^{(3)} \in \Omega^*$ such that $\chi_1(u^{(1)})s_1^{(1)} = s_1^{(2)}\chi_1(u^{(2)})$, $\chi_2(u^{(2)}) = s_2^{(1)}\chi_2(u^{(3)})$, then there exists $u^{(4)} \in \Omega^*$ such that $s_1^{(1)}\chi_1(u^{(2)}) = \chi_1(u^{(4)})$, $s_2^{(1)} = \chi_2(u^{(4)})$.

We shall show that no *finite* set of equational implications can serve as a necessary and sufficient condition for a semigroup amalgam to be embeddable. As for the embedding of a semigroup into a group [8], the idea is to show that given any finite set of equational implications, holding in any embeddable semigroup amalgam, it is possible to construct a nonembeddable amalgam in which the implications hold. We construct a nonembeddable amalgam $\mathfrak{A}_n = [S_i; U; h_i]_{i\in I}$ with card $I = 3$ the construction being easily extendible to any finite set I. The semigroup U is the free semigroup on $u^{(1)}, u^{(2)}, \cdots, u^{(4n+4)}$. The semigroup S_i $(i = 1, 2, 3)$ is generated by elements $u_i^{(1)}, u_i^{(2)}, \cdots, u_i^{(4n+4)}$ together with all the elements $s_i^{(k)}$ appearing in the set $\Pi(\mathfrak{A}_n)$ of generating relations indicated in the following board.

S_1	S_2	S_3
	$u_2^{(1)} = s_2^{(1)}u_2^{(2)}$	$u_3^{(2)} = u_3^{(3)}s_3^{(1)}$
$s_1^{(1)}u_1^{(4)} = s_1^{(2)}u_1^{(5)}$	$s_2^{(1)}u_2^{(3)} = u_2^{(4)}s_2^{(2)}$	
	$u_2^{(5)}s_2^{(2)} = s_2^{(3)}u_2^{(6)}$	$u_3^{(6)}s_3^{(1)} = u_3^{(7)}s_3^{(2)}$
$s_1^{(2)}u_1^{(8)} = s_1^{(3)}u_1^{(9)}$	$s_2^{(3)}u_2^{(7)} = u_2^{(8)}s_2^{(4)}$	
-------------	-------------	-------------
$s_1^{(n)}u_1^{(4n)} = s_1^{(n+1)}u_1^{(4n+1)}$	$s_2^{(2n-1)}u_2^{(4n-1)} = u_2^{(4n)}s_2^{(2n)}$	
	$u_2^{(4n+1)}s_2^{(2n)} = s_2^{(2n+1)}u_2^{(4n+2)}$	$u_3^{(4n+2)}s_3^{(n)} = u_3^{(4n+3)}$
	$s_2^{(2n+1)}u_2^{(4n+3)} = u_2^{(4n+4)}$	

The general form of a relation for S_1 is $s_1^{(k)}u_1^{(4k)} = s_1^{(k+1)}u_1^{(4k+1)}$ $(1 \leqslant k \leqslant n)$. For S_3 it is $u_3^{(4k+2)}s_3^{(k)} = u_3^{(4k+3)}s_3^{(k+1)}$ $(1 \leqslant k \leqslant n-1)$, while for S_2, successive relations are

$$s_2^{(k)}u_2^{(2k+1)} = u_2^{(2k+2)}s_2^{(k+1)},$$
$$u_2^{(2k+3)}s_2^{(k+1)} = s_2^{(k+2)}u_2^{(2k+4)}, \qquad 1 \leqslant k \leqslant 2n-1.$$

In each of S_1, S_2, S_3 the word problem is easily solved: Every word can be put into a unique canonical form by lowering, if possible, the superscript of $u_i^{(n)}$. For example, in S_2 replace occurrences of $s_2^{(1)}u_2^{(2)}$ by $u_2^{(1)}$, $u_2^{(2k+2)}s_2^{(2k+2)}$ by $s_2^{(k)}u_2^{(2k+1)}$, etc. Due to the independence of the presentation relations, these replacements can be performed at most once for every occurrence of $u_2^{(n)}$. It follows that $h_i(U)$ is free in S_i and thus that $\mathfrak{A}_n$ is a semigroup amalgam. Furthermore $\mathfrak{A}_n$ is not embeddable, since the set of all presentation relations constitutes a balanced system over $\mathfrak{A}_n$, with locked equation $s_1^{(1)}u_1^{(1)} = s_1^{(n)}u_1^{(4n+4)}$. The locked equation does not hold in S_1, for $s_1^{(1)}u_1^{(1)}$ and $s_1^{(n)}u_1^{(4n+4)}$ are both in canonical forms.

LEMMA 4.3. *Let $\mathfrak{A}_n = [S_i; U; h_i]_{i\in I}$ be the semigroup amalgam where the semigroups S_i have the presentation relations $\Pi(\mathfrak{A}_n)$ indicated above, and let Π be a proper subset of $\Pi(\mathfrak{A}_n)$. For every $i \in I$, denote by S_i^{Π} the semigroup having the same generators as S_i and presentation relations all the relations with index i contained in Π. Then the amalgam $\mathfrak{A}_n^{\Pi} = [S_i^{\Pi}; U; h_i]_{i\in I}$ is embeddable.*

PROOF. We show that condition (iv) of Theorem 1.1 holds in $\mathfrak{A}_n^{\Pi}$. By

Lemma 3.5 it is enough to consider connected transitions τ: $x \Rightarrow y$ with $x \in S_i^{\Pi}$, $y \in S_j^{\Pi}$. If x does not contain any occurrence of $u_i^{(1)}, u_i^{(2)}, u_i^{(4n+3)}$ or $u_i^{(4n+4)}$, then τ is either empty (in case $i = j$) and $x = y$ or τ consists of a single S-step (in case $i \neq j$) performed on $x \in h_i(U)$. A similar result holds in case x contains occurrences of $u_i^{(k)}$ with $k = 1, 2, 4n + 3, 4n + 4$. In this case, possible successive steps of τ are indicated by the factorizations given by the relations in Π. For example, if $x = x_i^{(1)}u_i^{(1)}x_i^{(2)}$ the possibility for τ is

$$x_i^{(1)}u_i^{(1)}x_i^{(2)} \rightarrow x_i^{(1)}u_2^{(1)}x_i^{(2)} = x_i^{(1)}s_2^{(1)}u_2^{(2)}x_i^{(2)}$$
$$\rightarrow x_i^{(1)}s_2^{(1)}u_3^{(2)}x_i^{(2)} = x_i^{(1)}s_2^{(1)}u_3^{(3)}s_3^{(1)}x_i^{(2)} \rightarrow \text{etc.}$$

But such a sequence does not yield an element $y \in S_j^{\Pi}$ in case $i \neq 1$, and also in case $i = 1$ due to the fact that Π is a proper subset of $\Pi(\mathfrak{A}_n)$.

THEOREM 4.4. *Given any finite set of equational implications, each of which holds in every embeddable semigroup amalgam, there is an integer n and a nonembeddable semigroup amalgam* $\mathfrak{A}_n$ *such that the given implications hold in* $\mathfrak{A}_n$.

PROOF. Let the finite set of equational implications be:

$$w_i^{(k_p)} = w_i^{(l_p)} \quad (i \in I, p = 1, 2, \cdots) \quad \text{imply}$$
$$v_i^{(m_q)} = v_i^{(n_q)} \quad (i \in I, q = 1, 2, \cdots).$$

Let n be an integer greater than the total length of all words $w_i^{(k_p)}, w_i^{(l_p)}$ appearing in the equational implications above. Then all the words in these implications can be considered as words in the amalgam $F_n = [\Sigma_i^*, \Omega^*, \chi_i]_{i \in I}$ where $\Omega = \{u^{(1)}, u^{(2)}, \cdots, u^{(4n+4)}\}$ and $\Sigma_i = \{s_i^{(1)}, s_i^{(2)}, \cdots, u_i^{(1)}, u_i^{(2)}, \cdots, u_i^{(4n+4)}\}$. Let $[\Phi_i, \Phi]_{i \in I}$ be a morphism from F_n to $\mathfrak{A}_n$, such that $\Phi_i(w_i^{(k_p)}) = \Phi_i(w_i^{(l_p)})$ for $i \in I, p = 1, 2, \cdots$. Suppose $w_i^{(k_p)} = x_i^{(k_2)} \cdots x_i^{(k_r)}$ and $w_i^{(l_p)} = x_i^{(l_1)}x_i^{(l_2)} \cdots x_i^{(l_s)}$ are the expressions of $w_i^{(k_p)}$ and $w_i^{(l_p)}$ as products of elements of Σ. Then

$$(1) \quad \Phi_i[x_i^{(k_1)}]\Phi_i[x_i^{(k_2)}] \cdots \Phi_i[x_i^{(k_r)}] = \Phi_i[x_i^{(l_1)}]\Phi_i[x_i^{(l_2)}] \cdots \Phi_i[x_i^{(l_s)}].$$

If we assume that each $\Phi_i[x_i^{(k)}]$ and $\Phi_i[x_i^{(l)}]$ is in its canonical form in $\mathfrak{A}_n$, then by the choice of n, to transform the left sides of all the equalities (1) for $i \in I$, $p = 1, 2, \cdots$ into the corresponding right sides we will need a proper subset Π of the relations $\Pi(\mathfrak{A}_n)$. It follows that the morphism $[\Phi_i, \Phi]_{i \in I}$ induces a morphism $[\psi_i; \psi]_{i \in I}$: $F_n \rightarrow \mathfrak{A}_n^{\Pi}$ such that the diagram below, with $[\theta_i, \theta]_{i \in I}$ the canonical morphism: $\mathfrak{A}_n^{\Pi} \rightarrow \mathfrak{A}_n$

$$\begin{tikzcd} F_n \arrow[d, "{[\Phi_i, \Phi]}"'] \arrow[dr, "{[\psi_i, \psi]}"] & \\ \mathfrak{A}_n & \mathfrak{A}_n^{\Pi} \arrow[l, "{[\theta_i, \theta]}"'] \end{tikzcd}$$

commutes. From (1) we deduce $\psi_i(w_i^{(k_p)}) = \psi_i(w_i^{(l_p)})$ for every $i \in I, p = 1, 2, \cdots$. By Lemma 3.3 and the hypothesis on the equational implications we deduce $\psi_i(v_i^{(m_q)}) = \psi_i(v_i^{(n_q)})$ and thus $\Phi_i(v_i^{(m_q)}) = \Phi_i(v_i^{(n_q)})$ for every $i \in I, p = 1, 2, \cdots$, showing that the given implications hold in $\mathfrak{A}_n$.

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