SPACES OF VECTOR MEASURES

BY

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ABSTRACT. Let $C_{rc} = C_{rc}(X, E)$ denote the space of all continuous functions $f$, from a completely regular Hausdorff space $X$ into a locally convex space $E$, for which $f(X)$ is relatively compact. As it is shown in [8], the uniform dual $C_{rc}'$ of $C_{rc}$ can be identified with a space $M(B, E')$ of $E'$-valued measures defined on the algebra of subsets of $X$ generated by the zero sets. In this paper the subspaces of all $\sigma$-additive and all $\tau$-additive members of $M(B, E')$ are studied. Two locally convex topologies $\beta$ and $\beta_1$ are considered on $C_{rc}$. They yield as dual spaces the spaces of all $\tau$-additive and all $\sigma$-additive members of $M(B, E')$ respectively. In case $E$ is a locally convex lattice, the $\sigma$-additive and $\tau$-additive members of $M(B, E')$ correspond to the $\sigma$-additive and $\tau$-additive members of $C_{rc}$ respectively.

1. Definitions and preliminaries. Let $X$ be a completely Hausdorff space and let $E$ be a real locally convex Hausdorff space. Let $C^b = C^b(X)$ denote the space of all bounded continuous real-valued functions on $X$. We will denote by $C_{rc} = C_{rc}(X, E)$ the space of all continuous functions $f$, from $X$ into $E$, for which $f(X)$ is relatively compact. Clearly $C_{rc}$ consists of those continuous functions $f$, from $X$ into $E$, that have continuous extensions $\tilde{f}$ to all of the Stone-Čech compactification $\beta X$ of $X$. For an $f$ in $C^b$ we will denote also by $\tilde{f}$ its unique continuous extension to all of $\beta X$. The zero sets in $X$ are defined to be the kernels of real continuous functions on $X$. The complement of a zero set is called a cozero set.

Let $\Sigma$ be an algebra of subsets of $X$ and let $m$ be a finitely-additive bounded real set function on $\Sigma$. We say that $m$ is regular with respect to a subfamily $\Sigma_1$ of $\Sigma$ if the following condition is satisfied: For every $F$ in $\Sigma$ and every $\epsilon > 0$ there exists $G$ in $\Sigma_1$ such that $G \subset F$ and $|m(H)| < \epsilon$ for all $H$ in $\Sigma$ which are contained in $F - G$.
Let now \( B = B(X) \) and \( Ba = Ba(X) \) be, respectively, the algebra and \( \sigma \)-algebra of subsets of \( X \) generated by the zero sets. The collection of Borel subsets of \( X \) will be denoted by \( Bo = Bo(X) \). Let \( M(X) \) be the space of all bounded, \( \sigma \)-algebra of subsets of \( X \) generated by the zero sets. The collection of Borel subsets of \( X \) will be denoted by \( Bo = Bo(X) \). Let \( M(X) \) be the space of all bounded, finitely-additive, real-valued, set functions on \( B(X) \) which are regular with respect to the family of zero sets. The space of all bounded, countably-additive, real-valued, regular (with respect to the zero sets) measures on \( Ba \) will be denoted by \( M_\sigma(Ba) \). By \( M_\tau(Bo) \) we will denote the space of all real, regular with respect to the closed sets, Borel measures \( m \) on \( Bo \) such that \( |m|(Z_\alpha) \to 0 \) for each net \( \{Z_\alpha\} \) of zero sets which decreases to the empty set (see Varadarajan [17] or Aleksandrov [1]). Note that an element \( m \) of \( M_\tau(Bo) \) is not necessarily regular with respect to the zero sets. However its restriction to \( Ba \) is an element of \( M_\sigma(Ba) \) by Varadarajan [16, p. 171, Theorem 19]. The subspaces of all \( \sigma \)-additive and \( \tau \)-additive members of \( M(X) \) will be denoted by \( M_\sigma(X) \) and \( M_\tau(X) \) respectively (see Varadarajan [17] for the definition of \( \sigma \)-additive and \( \tau \)-additive measures).

For \( m \) in any one of the spaces \( M(X) \), \( M_\sigma(Ba) \), \( M_\tau(Bo) \), the positive part \( m^+ \), the negative part \( m^- \), and the variation \( |m| \) are understood as, for example, in Aleksandrov [1].

Let now \( \{p: p \in \mathbb{I}\} \) be a family of continuous seminorms on \( E \) generating the topology of \( E \). We choose this family so that it is directed, i.e., given \( p_1, p_2 \) in \( \mathbb{I} \) there exists \( p \in \mathbb{I} \) with \( p \geq p_1, p_2 \). For each \( p \) in \( \mathbb{I} \) we consider the space \( M_p(B, E') \) of all finitely-additive functions \( m: B(X) \to E' \) (\( E' \) is the topological dual of \( E \)) such that the following two conditions are satisfied:

1. For each \( s \in E \), the function \( ms: B \to \mathbb{R}, (ms)(F) = m(F)s, \) is in \( M(X) \).
2. \( \|m\|_p = m_p(X) < \infty \), where for \( F \) in \( B \) we define \( m_p(F) = \sup \{ \sum m(F_i)s_i \} \) the supremum being taken over all finite \( B \)-partitions \( \{F_i\} \) of \( F \) (that is partitions into sets in \( B \)) and all finite collections \( \{s_i\} \) in \( E \) with \( p(s_i) \leq 1 \).

The set function \( m_p \) belongs to \( M(X) \). Indeed it is easy to see that \( m \) is finitely-additive and bounded. For the regularity, consider an \( F \) in \( B \) and let \( \varepsilon > 0 \) be given. By definition there exist a finite \( B \)-partition \( \{F_i\} \) of \( F \) and \( s_i \in E \), with \( p(s_i) \leq 1 \), such that \( \sum m(F_i)s_i > m_p(F) - \varepsilon \). By the regularity of \( ms \), we can choose for each \( i \) a zero set \( Z_i \subset F_i \) such that \( \sum m(Z_i)s_i > m_p(F) - \varepsilon \). The zero set \( Z = \bigcup Z_i \) is contained in \( F \). Moreover we have \( m_p(Z) \geq \sum m(Z_i)s_i > m_p(F) - \varepsilon \). This proves the regularity of \( m_p \). Set \( M(B, E') = \bigcup_{p \in \mathbb{I}} M_p(B, E') \).

Let \( \sigma \) denote the uniform topology on \( C_{rc} \), i.e., the locally convex topology generated by the family of seminorms \( \{ \|f\|_p: p \in \mathbb{I}\} \), where \( \|f\|_p = \sup \{ p(f(x)) : x \in X \} \). In [8] the author defines the integral of a function \( f \) in \( C_{rc} \) with respect to a member of \( M(B, E') \). The integration process employed is a generalization of the process of Aleksandrov to the vector case. It is one of the many integration processes defined by McShane [12]. Every element \( m \) of
$M(B, E')$ generates a linear functional $\phi_m$ on $C_{rc}$ by $\phi_m(f) = \int_X f dm$, $f \in C_{rc}$.

The proof of the following theorem can be found in [8].

**Theorem 1.1.** For each $m \in M(B, E')$, $\phi_m$ is an element of $(C_{rc}, \sigma)' = C_{rc}'$. Moreover, the map $m \rightarrow \phi_m$, from $M(B, E')$ into $C_{rc}$, is linear, one-to-one, and onto.

**Theorem 1.2.** If $m \in M_p(B, E')$, then $\|\phi_m\| = \|m\|_p$, where $\|\phi_m\|_p = \sup \{|\phi_m(f)| : f \in C_{rc}, \|f\|_p \leq 1\}$.

**Proof.** It is clear from the definitions that $\|f dm\| \leq \int p o f dm_p \leq \|f\|_p \|m\|_p$ for all $C_{rc}$ and hence $\|\phi_m\|_p \leq \|m\|_p$. On the other hand, let $\varepsilon > 0$ be given. By the definition of $\|m\|_p$, there exist a finite $B$-partition $\{F_i\}$ of $X$ and $s_i \in E$ with $p(s_i) \leq 1$ such that $\|m\|_p < \Sigma m(F_i) s_i + \varepsilon$. By regularity there are zero sets $Z_i \subset F_i$ such that $\|m\|_p < \Sigma m(Z_i) s_i + \varepsilon$. Again by the regularity of $m s_i$, we can find pairwise disjoint cozero sets $\{U_i\}$, $Z_i \subset U_i$, such that

$$\sum |m s_i|(U_i - Z_i) < \varepsilon.$$ 

For each $i$ choose $h_i$ in $C_b$, $0 < h_i \leq 1$, such that $h_i = 1$ on $Z_i$ and $h_i = 0$ on $X - U_i$. Set $h = \Sigma h_i s_i$. Then $\|h\|_p \leq 1$ and so $\|h dm\| \leq \|\phi_p\|$. But

$$\left|\int h dm\right| \geq \left|\sum \int_{Z_i} s_i dm\right| - \left|\sum \int_{U_i - Z_i} h_i d(m s_i)\right| \geq \sum m(Z_i) s_i - \varepsilon \geq \|m\|_p - 2\varepsilon.$$ 

Since $\varepsilon > 0$ was arbitrary we get that $\|\phi_m\|_p \geq \|m\|_p$ and this completes the proof.

In case $E$ is a locally convex lattice, $(C_{rc}, \sigma)$ becomes also a locally convex lattice under the pointwise ordering (that is, we define $f \geq g$ iff $f(x) \geq g(x)$ for all $x \in X$). We define an order relation $\geq$ on $M(B, E')$ by $m_1 \geq m_2$ iff $m_1(F) \geq m_2(F)$ for all $F$ in $B$. Note that $E'$ is a lattice when ordered by the cone $\{\phi \in E' : \phi(s) \geq 0 \text{ when } s \geq 0\}$. As it is shown in [8], $M(B, E')$ becomes a lattice and the map $m \rightarrow \phi_m$, of Theorem 1.1, is lattice preserving.

2. Extensions of members of $M(B, E')$. Let $p \in I$. We define $M_{a, p}(Ba, E')$ to be the set of all functions $m : Ba \rightarrow E'$ such that the following two conditions are satisfied:

1. For each $s$ in $E$ the function $m s : Ba \rightarrow R$, $(m s)(F) = m(F) s$, is in $M_s(Ba)$.

2. $m_p(X) < \infty$ where, for each $F$ in $Ba$, we define $m_p(F) = \sup \Sigma m(F_i) s_i$, where the supremum is taken over all finite $Ba$-partitions $\{F_i\}$ of $F$ and all finite collections $\{s_i\}$ in $E$ with $p(s_i) \leq 1$.

**Lemma 2.1.** If $m \in M_{a, p}(Ba, E')$, then $m_p \in M_a(Ba)$. 

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Proof. It is easy to see that \( m_p \) is bounded monotone and finitely-additive.

Let \( \{F_n\} \) be a sequence of pairwise Baire sets (i.e., sets in \( \mathcal{B}a \)) and set \( F = \bigcup F_n \).

Since \( m_p \) is monotone and finitely-additive, we have \( m_p(F) \geq m_p(\bigcup F_i) = \sum m_p(F_i) \) for each \( n \). Hence \( m_p(F) \geq \sum m_p(F_i) \). On the other hand, let \( \varepsilon > 0 \) be arbitrary. There exist a \( \mathcal{B}a \)-partition \( G_1, \ldots, G_N \) of \( F \) and \( s_i \in E, p(s_i) \leq 1 \), such that \( \sum m(G_i)s_i > m_p(F) - \varepsilon \). Since \( m(s_i) \) is countably additive we have

\[
\sum_{n=1}^{\infty} \sum_{i=1}^{N} |m(G_i \cap F_n)s_i| \leq \sum_{n=1}^{\infty} m_p(F_n) \leq m_p(F) < \infty.
\]

Hence

\[
m_p(F) - \varepsilon \leq \sum_{i=1}^{N} m(G_i)s_i = \sum_{i=1}^{N} \sum_{n=1}^{\infty} m(G_i \cap F_n)s_i = \sum_{n=1}^{\infty} m(G_i \cap F_n)s_i \leq \sum_{n=1}^{\infty} m_p(F_n) \leq m_p(F).
\]

Since \( \varepsilon > 0 \) was arbitrary we conclude that \( m_p(F) = \sum m_p(F_n) \) and so \( m_p \) is countably-additive. Finally, the proof of the regularity of \( m_p \) is similar to that of the case of a member of \( M_p(B, E') \).

Next we define \( M_{r,p}(B, E') \) to be the set of all \( m: J \rightarrow E' \) having the following two properties:

(a) For each \( s \) in \( E \), \( ms \) belongs to \( M_r(J) \).

(b) \( m_p(X) < \infty \), where for each \( F \) in \( J \) the \( m_p(F) \) is defined by \( m_p(F) = \sup \{ \sum m(F_i) s_i \} \) the supremum being taken over all finite \( \mathcal{B}o \)-partitions of \( F \) and all finite collections \( \{ s_i \} \) in \( F \) with \( p(s_i) \leq 1 \).

Lemma 2.2. If \( m \in M_{r,p}(J, E') \), then \( m_p \in M_r(J) \).

Proof. By using an argument similar to that of 2.1, we show that \( m_p \) is a bounded, countably-additive, regular with respect to the closed sets, \( \mathcal{B}o \)-measure on \( X \). To complete the proof we need to show that \( m_p \) is \( \tau \)-additive. To this end, consider an arbitrary net \( \{ Z_\alpha \} \) of zero sets decreasing to the empty set. For each \( \alpha \) there exists a zero set \( \tilde{Z}_\alpha \in \beta X \) such that \( Z_\alpha = \tilde{Z}_\alpha \cap X \).

Define \( \tilde{m} : \mathcal{B}o(\beta X) \rightarrow E' \) by \( \tilde{m}(F) = m(F \cap X) \). For each \( s \in E \), the function \( \tilde{m}s : \mathcal{B}o(\beta X) \rightarrow R, (\tilde{m}s)(F) = (ms)(F \cap X) \), is a regular \( \mathcal{B}o \)-measure on \( \beta X \) since \( ms \) is \( \tau \)-additive (see Knowles [11]). It follows now easily that \( m \in M_{r,p}(\mathcal{B}o(\beta X), E') \). Moreover \( \tilde{m}_p(F) = m_p(F \cap X) \) for each \( \mathcal{B}o \) set \( F \) in \( \beta X \).

Indeed it is clear that \( \tilde{m}_p(F) \leq m_p(F \cap X) \). On the other hand, if \( \{ G_i \} \) is a finite \( \mathcal{B}o \) partition of \( F \cap X \), then there are pairwise disjoint \( \mathcal{B}o \) sets \( V_i \)
in $\beta X$, which we may choose contained in $F$, such that $G_1 = V_i \cap X$. For $s_i \in E$ with $p(s_i) < 1$, we have $\bar{m}_p(F) \geq |\Sigma \bar{m}(V_i)s_i| = |\Sigma m(G_i)s_i|$. This shows that $\bar{m}_p(F) \geq m_p(F \cap X)$ and so $m_p(F) = m_p(F \cap X)$. Let now $D = \{Z \subset \beta X: Z$ is an intersection of a finite number of $\bar{Z}_\alpha$'s $\}$. Then $D$ is directed downwards to $G = \bigcap \bar{Z}_\alpha$. Hence $\bar{m}_p(G) = \lim_{Z \in D} \bar{m}_p(Z)$. Since $G \cap X = \emptyset$, we have $\bar{m}_p(G) = 0$. Thus given $\epsilon > 0$ there exists $Z = \bar{Z}_{\alpha_1} \cap \cdots \cap \bar{Z}_{\alpha_n}$ in $D$ such that $\bar{m}_p(Z) < \epsilon$. If $\alpha > \alpha_1, \cdots, \alpha_n$, we have $m_p(Z_\alpha) \leq m_p(Z \cap X) = \bar{m}_p(Z) < \epsilon$. This completes the proof.

**Theorem 2.3.** If $m \in M_{a,p}(Ba, E') \setminus \{m \in M_{r,p}(Bo, E')\}$, then $m_p(X) = \sup \{\|f\|_{L^p}: f \in C_{rc}, \|f\|_p \leq 1\}$.

**Proof.** Let $d = \sup \{\|f\|_{L^p}: f \in C_{rc}, \|f\|_p \leq 1\}$. To prove the result in the case of an $m$ in $M_{a,p}(Ba, E')$ one can use the same argument as the one used in the proof of Theorem 1.2. We will prove the result for an $m$ in $M_{r,p}(Bo, E')$. Since $\|f\|_{L^p} < \|f\|_{L^p} m_p(X)$, it follows that $d < m_p(X)$. To prove the reverse inequality, consider an arbitrary $\epsilon > 0$. Define $\bar{m}$ on $Bo(\beta X)$ by $\bar{m}(F) = m(F \cap X)$. As we have seen in the proof of Lemma 2.2, we have $\bar{m} \in M_{r,p}(Bo(\beta X), E')$ and $\bar{m}_p(F) = m_p(F \cap X)$ for each Borel set $F$ in $\beta X$. By the definition of $\bar{m}_p$, there exist a partition $\{F_1, \cdots, F_n\}$ of $\beta X$, $F_i \in Bo(\beta X)$, and $s_i, \cdots, s_n$ in $E$, $p(s_i) < 1$, such that $\sum \bar{m}(F_i)s_i > m_p(\beta X) - \epsilon = m_p(X) - \epsilon$. By regularity there are closed sets $G_i \subset \beta X$, $G_i \subset F_i$, such that $\sum \bar{m}(G_i)s_i > m_p(X) - \epsilon$. Next we choose pairwise disjoint open sets $O_i$ in $\beta X$, $G_i \subset O_i$, such that $|\bar{m}(O_i - G_i)| < \epsilon/n$. For each $i$, $1 \leq i \leq n$, there is an $h_i$ in $C^0(X)$, $0 < h_i \leq 1$, $h_i = 1$ on $G_i$ and $h_i = 0$ in the complement of $O_i$. Set $h = \Sigma h_is_i$. Then $\|h\|_p \leq 1$ and

$$\int_X h \ dm = \int_{\beta X} \hat{h} \ dm = \sum \bar{m}(G_i)s_i + \sum_{O_i \cap \bar{G}_i} \hat{h}_i d(\bar{m}s_i) > m_p(X) - 2\epsilon.$$ 

Thus $d > m_p(X) - 2\epsilon$ and the result follows since $\epsilon > 0$ was arbitrary.

**Lemma 2.4.** Let $m \in M_{r,p}(Bo, E')$ and $\mu = m|_{Ba}$ (= restriction of $m$ to $Ba$). Then (a) $\mu \in M_{a,p}(Ba, E')$, (b) $\mu_p = m_p|_{Ba}$.

**Proof.** Part (a) is clear because the restriction to $Ba$ of an element of $M_r(Bo)$ is in $M_{a}(Ba)$. For (b) we first observe that $\mu_p(X) = m_p(X)$ by 2.3 since $\int f \ dm = \int f \ d\mu$ for all $f$ in $C_{rc}$. It is also clear that $\mu_p(F) \leq m_p(F)$ for all $F$ in $Ba$. Thus (b) follows.

Set $M_o(Ba, E') = \bigcup \{M_{a,p}(Ba, E'): p \in I\}$ and define $M_r(Bo, E')$ analogously.

Let $M_o(B, E')$ be the subspace of $M(B, E')$ consisting of all $m \in M(B, E')$ for which $ms \in M_o(X)$ for all $s$ in $E$. We define $M_r(B, E')$ similarly. We will call
the elements of $M_o(B, E')$ [$M_r(B, E')$] the $\sigma$-additive ($\tau$-additive) members of $M(B, E')$. The next theorem shows that the $\sigma$-additive members of $M(B, E')$ are exactly the ones that have extensions to members of $M_o(Ba, E')$.

**Theorem 2.5.** Let $m \in M(B, E')$. Then $m$ is $\sigma$-additive iff there exists a $\mu$ in $M_o(Ba, E')$ with $m = \mu|_B$. Moreover, if such a $\mu$ exists it is unique.

**Proof.** Clearly $\mu|_B$ is in $M_o(B, E')$ for each $\mu$ in $M_o(Ba, E')$. Moreover if $\lambda$ is another member of $M_o(Ba, E')$ such that $\lambda|_B = \mu|_B$, then $\lambda s|_B = \mu s|_B$ for each $s \in E$. It follows that $\lambda s = \mu s$ by the regularity of $\lambda s$ and $\mu s$. This, being true for all $s$ in $E$, implies that $\mu = \lambda$. Assume next that $m \in M_o(B, E')$. For each $s \in E$, $ms \in M_o(X)$. Hence, for each $s \in E$, there exists a unique extension $\mu_s$ of $ms$ to a member of $M_o(Ba)$ such that $\|ms\| = \|\mu_s\|$ (see Varadarajan [17]). For an $F$ in $Ba$, we define $\mu(F) : E \rightarrow R$, by $\mu(F)s = \mu_s(F)$. Clearly $\mu(F)$ is linear. Moreover, if $m \in M_p(B, E')$, then

$$|\mu(F)s| = |\mu_s(F)| \leq \|\mu_s\| = \|ms\| \leq p(s)\|m\|_p.$$ 

Hence $\mu(F) \in E'$. In this way we define a map $\mu : Ba \rightarrow E'$ such that $\mu s = \mu s \in B_o(Ba)$ for all $s \in E$. To finish the proof it remains to show that $\|\mu\|_p < \infty$. To this end, consider an arbitrary $Ba$-partition $F_1, \ldots, F_n$ of $X$ and let $s_i \in E$ with $p(s_i) \leq 1$. For $e > 0$, there exist zero sets $Z_i, \ldots, Z_n, Z_i \subset F_i$, such that $|\mu s_i|(F_i - Z_i) < e/n$. Thus

$$|\sum \mu_s(F_i)s_i| + e = |\sum \mu s_i s_i| + e \leq p(s)\|m\|_p + e.$$ 

It follows that $\|\mu\|_p \leq \|m\|_p$ and the proof is complete.

We have an analogous theorem for $M_r(Bo, E')$.

**Theorem 2.6.** Let $m \in M(B, E')$. Then $m$ is $\tau$-additive iff there exists a unique $\mu \in M_r(Bo, E')$ such that $m = \mu|_B$.

**Proof.** Clearly $\mu|_B \in M_r(B, E')$ for each $\mu \in M_r(Bo, E')$. Also, if $\mu_1, \mu_2$ are both in $M_r(Bo, E')$ with $\mu_1|_B = \mu_2|_B$, then $\mu_1 s|_B = \mu_2 s|_B$ for each $s$ in $E$. By Kirk [9, Theorem 1.14], we have $\mu_1 s = \mu_2 s$. This, being true for all $s$ in $E$, implies that $\mu_1 = \mu_2$. Assume now that $m \in M_r(B, E')$. Let $C(\beta X, E)$ denote the space of all continuous functions from $\beta X$ into $E$.

Clearly $C(\beta X, E) = \{ \tilde{f} : f \in C_{rc} \}$. Define $\phi$ on $C(\beta X, E)$ by $\phi(\tilde{f}) = \tilde{f} f dm$. Then $\phi$ is continuous with respect to the uniform topology on $C(\beta X, E)$. Hence, by 1.1, there exists $\tilde{m} \in M_p(B(\beta X), E')$ such that $\phi(\tilde{f}) = \tilde{f} \tilde{m}$ for all $f$ in $C_{rc}$. Since each $\tilde{ms}, s \in E$, is $\tau$-additive it has a unique norm-preserving extension to a member $\tilde{\mu}_s$ of $M_r(Bo(\beta X))$ (see Kirk [9]). For each Borel set $F$ in $\beta X$, we define
It is easy to see that \( \mu(F) \in E' \). In this way we get a map \( \tilde{\mu} : Bo(\beta X) \to E' \). We will show that \( \tilde{\mu} \in M_{\tau,p}(Bo(\beta X), E') \). Since \( \tilde{\mu} s = \mu_s(F) \), it only remains to show that \( \|\tilde{\mu}\|_p < \infty \). To this end consider an arbitrary partition \( F_1, \ldots, F_n \) of \( \beta X \) into Borel sets and let \( s_i \in E \) with \( p(s_i) \leq 1 \). There are closed sets \( G_1, \ldots, G_n \) in \( \beta X \), \( G_i \subset F_i \), such that \( |\mu s_i|(F_i - G_i) < \epsilon/n \) (\( \epsilon > 0 \) arbitrary). Since \( G_1, \ldots, G_n \) are pairwise disjoint compact sets and since the cozero sets form a base for the open sets, there are pairwise disjoint cozero sets \( U_1, \ldots, U_n \) in \( \beta X \), \( G_i \subset U_i \), such that \( |\mu s_i|(U_i - G_i) < \epsilon/n \). Thus

\[
\left| \sum \mu(F_i)s_i \right| \leq \left| \sum \mu(U_i)s_i \right| + 2\epsilon = \left| \sum m(U_i)s_i \right| + 2\epsilon \leq m_p(\beta X) + 2\epsilon.
\]

It follows that \( \tilde{\mu}_p(\beta X) \leq m_p(\beta X) \) and so \( \tilde{\mu} \) is in \( M_{\tau,p}(Bo(\beta X), E') \). Next we show that \( \tilde{\mu}_p(F) = 0 \) for each Borel set \( F \) in \( \beta X \) which is disjoint from \( X \). By regularity it suffices to show that \( \mu(F)s = 0 \) for each \( s \in E \) and each closed set \( F \) in \( \beta X \) disjoint from \( X \). So, let \( F \) be such a set and let \( s \in E \). There exists an open set \( 0 \) in \( \beta X \), \( F \subset 0 \), such that \( |\mu s|(0 - F) < \epsilon \) (\( \epsilon > 0 \) arbitrary). There exists a net \( \{f_\alpha\} \) in \( C^b(X) \), \( f_\alpha \downarrow 0, f_\alpha = 1 \) on \( F \) and \( f_\alpha = 0 \) on the complement of \( 0 \), \( 0 < f_\alpha \leq 1 \). Since \( ms \) is \( \tau \)-additive, we have \( \lim f_\alpha d(ms) = 0 \). Hence there exists \( \alpha \) such that \( |f_\alpha d(ms)| < \epsilon \). Thus

\[
\left| \int f_\alpha s d\mu \right| = \left| \int f_\alpha s dm \right| < \epsilon.
\]

Therefore \( |\mu(F)s| \leq 2\epsilon \). Since \( \epsilon > 0 \) was arbitrary, we conclude that \( \mu(F)s = 0 \) which proves the claim: Next we define \( \mu : Bo(X) \to E' \) by \( \mu(F \cap X) = \tilde{\mu}(F) \) for each \( F \in Bo(\beta X) \). If \( F_1, F_2 \) are Borel sets in \( \beta X \) such that \( F_1 \cap X = F_2 \cap X \), then both \( F_1 - F_2 \) and \( F_2 - F_1 \) are disjoint from \( X \) and so \( \mu(F_1) = \mu(F_1 \cap F_2) = \mu(F_2) \). Hence \( \mu \) is well defined. It is easy now to see that \( \mu \in M_{\tau,p}(Bo(\beta X), E') \). Moreover, it is clear that \( ff d\mu = \int f d\tilde{\mu} = \int f d\tilde{m} = \int f dm \) for all \( f \) in \( C_{rc} \). Let \( m_1 = \mu_{|B} \). Then \( m_1 \in M(B, E') \) and \( ff dm = \int f dm_1 \) for each \( f \) in \( C_{rc} \). By Theorem 1.1, \( m = m_1 \) and hence \( \mu \) is an extension of \( m \). The theorem is proved.

The next theorem gives another characterization of the \( \sigma \)-additive and \( \tau \)-additive members of \( M(B, E') \). This characterization will be useful later. Let \( m \in M(B, E') \). Define \( \phi \) on \( C(\beta X, E) \) by \( \phi(f) = \int f dm, f \in C_{rc} \). Then \( \phi \) is continuous with respect to the uniform topology on \( C(\beta X, E) \). Since \( M(\beta X) = M_{\tau}(\beta X) \), there exists \( \tilde{m} \in M_{\tau,p}(Bo(\beta X), E') \) such that \( \phi(f) = \int f \tilde{m} \) for each \( f \) in \( C_{rc} \).
Theorem 2.7. (a) \( m \in M_o(B, E') \) iff \( \widetilde{m}_p(Z) = 0 \) for each zero set \( Z \) in \( \beta X \) which is disjoint from \( X \).

(b) \( m \) is \( \tau \)-additive iff \( \widetilde{m}_p(F) = 0 \) for each closed set \( F \) in \( \beta X \) which is disjoint from \( X \).

Proof. (a) Assume that \( m \) is \( \sigma \)-additive. Let \( s \in E \). For each \( f \in C^b(X) \) we have \( \int f \, d(ms) = \int f \, ds \, dm = \int f \, ds \, dm = \int f \, d(ms) \). Since \( ms \) is \( \sigma \)-additive, we have that \( (ms)(F) = 0 \) for each Baire set \( F \) in \( \beta X \) which is disjoint from \( X \) (see Knowles [11, Theorem 2.1]). Let \( \widetilde{\mu} \) be the restriction of \( \widetilde{m} \) to \( Ba(\beta X) \). Then \( \mu \in M_{\sigma, p}(Ba(\beta X), E') \) and \( \widetilde{m}_p = \widetilde{m}_p \restriction Ba(\beta X) \). By what we proved, \( \widetilde{m}_p(F) = \widetilde{\mu}_p(F) = 0 \) for each Baire set \( F \) in \( \beta X \) which is disjoint from \( X \). Conversely, assume that \( \widetilde{m}_p(Z) = 0 \) for each zero set \( Z \) in \( \beta X \) disjoint from \( X \). By regularity \( \widetilde{\mu}_p(F) = 0 \) for each Baire set \( F \) disjoint from \( X \). Define \( \mu : Ba(X) \rightarrow E' \) by \( \mu(F \cap X) = \widetilde{\mu}_p(F) \) for each Baire set \( F \) in \( \beta X \). This gives us a well-defined element of \( M_{\sigma, p}(Ba(X), E') \). Moreover, if \( m_1 = \mu \restriction B(X) \), then \( \int f \, dm_1 = \int f \, d\mu = \int f \, d\mu = \int f \, dm = \int f \, ds \) for all \( f \) in \( C_{rc} \). Thus \( m_1 = m \) and hence \( m \) is \( \sigma \)-additive by 2.5.

(b) The proof is similar to that of (a).

Theorem 2.8. If we consider on \( C'_{rc} = M(B, E') \) the weak topology \( \sigma(C'_{rc}, C_{rc}) \), then \( M_o(B, E') \) is sequentially closed.

Proof. Let \( \{m_n\} \) be a sequence of elements of \( M_o(B, E') \), \( m \in M(B, E') \), and assume that \( m_n \rightharpoonup m \). Let \( s \in E \). For each \( f \in C^b(X) \) we have

\[
\int f \, d(m_n s) = \int f s \, dm_n \rightharpoonup \int f s \, dm = \int f \, d(ms).
\]

Thus \( m_n s \rightharpoonup ms \) in the \( \sigma(M(X), C^b) \) topology. By Aleksandrov [1], \( ms \) is \( \sigma \)-additive. This, being true for all \( s \in E \), implies that \( m \) is \( \sigma \)-additive.

3. A weighted type topology on \( C_{rc} \). Let \( V \) be a family of bounded continuous real-valued functions on \( X \). Assume that \( V \) has the following two properties:

(1) For each \( x \) in \( X \) there exists \( h \in V \) with \( h(x) \not= 0 \).

(2) Given \( u, v \) in \( V \) and a positive number \( d \), there exists \( w \) in \( V \) with \( |w| \geq du, dv \) (pointwise). We will denote by \( w_V \) the locally convex topology on \( C_{rc} \) generated by the family of seminorms \( \{\| \cdot \|_{p, h} : p \in I, h \in V\} \) where \( \| \cdot \|_{p, h} \) is defined on \( C_{rc} \) by

\[
\| f \|_{p, h} = \sup \{p(h(x)f(x)) : x \in X\} = \|hf\|_p.
\]

It is clear that \( w_V \) has a base at zero consisting of all sets of the form \( \{f \in C_{rc} : \|hf\|_p \leq 1\} \) where \( h \in V \) and \( p \in I \). It is also clear that \( w_V \) is Hausdorff and
that $w_\nu \leq \sigma$. Hence $(C_{rc}, w_\nu)' \subset (C_{rc, \sigma})' = M(B, E')$. We will identify the dual space of $(C_{rc}, w_\nu)$. We begin with an easily established lemma.

**Lemma 3.1.** Let $m \in M(B, E')$ and $h \in C^b$. For each $F$ in $B$ we define $\mu(F)$ on $E$ by $\mu(F)s = \int_F hds(m)$. Then $\mu \in M(B, E')$ and $\int f d\mu = \int fh \, dm$ for each $f$ in $C_{rc}$.

We denote the element $\mu \in M(B, E')$, defined in 3.1, by $hm$. Let

$$V \cdot M(B, E') = \{hm: h \in V, m \in M(B, E')\}.$$ 

We will prove the following.

**Theorem 3.2.** The space $(C_{rc}, w_\nu)'$ is isomorphic to the space $V \cdot M(B, E')$ and the isomorphism $\phi \mapsto m$ is given by the formula, where $\phi(f) = \int f \, dm$ for all $f \in C_{rc}$.

Set $H = (C_{rc}, w_\nu)'$.

**Lemma 3.3.** If $h \in V$ and $m \in M_p(B, E')$, then $hm$ gives an element of the dual space of $(C_{rc}, w_\nu)$.

**Proof.** Set $\mu = hm$. The set $W = \{f: \|fh\|_p < 1\}$ is a $w_\nu$-neighborhood of zero. Moreover, if $f \in W$, then

$$|\int f \, d\mu| = |\int fh \, dm| \leq \|fh\|_p \|m\|_p \leq \|m\|_p.$$ 

This completes the proof.

**Lemma 3.4.** Let $h \in V$ and define $T_h = T: C_{rc} \to C_{rc}, Tf = hf$. Then $T$ is $o(C_{rc}, H) - o(C_{rc}, C_{rc})$ continuous.

Moreover, if $T'$ is the adjoint of $T$ and if $p \in I$, then $T'(B_p^\circ) = W_p^\circ$, where $B_p = \{f \in C_{rc}: \|f\|_p \leq 1\}, W_p = T^{-1}(B_p^\circ), B_p^\circ$ the polar of $B_p$ with respect to the pair $(C_{rc}, C_{rc})$, and $W_p^\circ$ the polar of $W_p$ with respect to the pair $(C_{rc}, H)$.

**Proof.** Let $\{f_\alpha\}$ be a net in $C_{rc}$ converging to zero in the $o(C_{rc}, H)$ topology. Let $m \in M(B, E')$. In view of 3.3 we have $\int f_\alpha d(hm) \to 0$. Thus $\int hf_\alpha \, dm \to 0$ which shows that $Tf_\alpha \to 0$ in the $o(C_{rc}, C_{rc})$ topology. Thus $T$ is $o(C_{rc}, H) - o(C_{rc}, C_{rc})$ continuous. Therefore $T'$ exists and $T'(C_{rc}) \subset H$. Also $T'$ is $o(C_{rc}, C_{rc}) - o(H, C_{rc})$ continuous. The set $B_p$ is clearly $\sigma$-closed. Since $B_p$ is convex and since $\sigma$ and $o(C_{rc}, C_{rc})$ are both compatible with the pair $\langle C_{rc}, C_{rc}\rangle$, $B_p$ is $o(C_{rc}, C_{rc})$ closed. Also $B_p$ is balanced. Thus $B_p = B_p^\circ$ by the bipolar theorem (see Schaefer [14, p. 126]). Let $W = [T'(B_p^\circ)]^\circ$. If $f \in W$ and $m \in B_p^\circ$, then $\|m, Tf\| = \|T'm, f\| \leq 1$. This shows that $Tf \in B_p^\circ = B_p$. Hence $W \subset W_p$. On the other hand, if $f \in W_p$ and $m \in B_p^\circ$, then
Thus $W_p \subseteq W$ and so $W = W_p$. The set $B_p^\circ$ is $\sigma(C_{rc}^\circ, C_{rc})$ compact by the Alaoglu theorem (see Köthe [10, p. 248]). Hence $T'(B_p^\circ)$ is $\sigma(H, C_{rc})$ compact. Also $T'(B_p^\circ)$ is convex and balanced. Therefore, by the bipolar theorem, $T'(B_p^\circ) = [T'(B_p^\circ)]^{oo} = W_p^\circ$. The lemma is proved.

**Lemma 3.5.** If $\phi \in (C_{rc}^\circ, w_V)'$, then there exists $h \in V$, $m \in M(B, E')$ such that $\phi(f) = \text{ff } d(hm)$ for all $f \in C_{rc}^\circ$.

**Proof.** Since $\phi$ is $w_V$-continuous, there exist $h \in V$ and $p \in I$ such that $W_p = \{f: \|hf\|_p \leq 1\} \subseteq \{f: |\phi(f)| \leq 1\}$. Let $T = T_h$ be as in Lemma 3.4. In view of 3.4, we have $T'(B_p^\circ) = W_p^\circ$. Since $\phi \in W_p^\circ$ there exists $m \in B_p^\circ$ such that $\phi = T'm$. Now, for each $f \in C_{rc}^\circ$, we have $\langle f, \phi \rangle = \langle f, T'm \rangle = \langle Tf, m \rangle = \text{ff } d(hm)$. This completes the proof.

Combining 3.3 and 3.5 we get Theorem 3.2.

4. The strict and superstrict topologies on $C_{rc}^\circ$. Buck defined in [4] the strict topology on the space of bounded continuous functions on a locally compact space and he identified the dual in the scalar case. The dual space for the vector case was studied by Wells [18]. Recently Sentilles [15] and Fremlin-Garling-Haydon [5] defined the strict and superstrict topologies on the space of all bounded continuous real-valued functions on a completely regular Hausdorff space. They identified the strict and superstrict dual of $C^b$ with the spaces $M_b(X)$ and $M_\sigma(X)$ respectively. These and other authors completed the result of Hewitt [6] on the representation of linear functionals on spaces of continuous functions. In [3] Bogdanowicz studied the space of continuous linear functionals on the space of continuous mappings from a compact space into a locally convex space. In this section we will introduce on $C_{rc}^\circ$ two locally convex topologies $\beta_1$ and $\beta$ which yield as dual spaces the spaces of all $\sigma$-additive and all $\tau$-additive members of $M(B, E')$ respectively. Our approach will be analogous to that of Sentilles.

Let $\Omega$ ($\Omega_1$) denote the collection of all closed (zero) sets in $\beta X$ which are disjoint from $X$. For $Q$ in $\Omega$, let $B_Q = \{h \in C^b: \hat{h} = 0$ on $Q\}$. Clearly $B_Q$ has all the properties of the family $V$ mentioned in the beginning of §3.

Let $\beta_Q$ be the locally convex topology on $C_{rc}^\circ$ generated by the family of seminorms $f \rightarrow \|hf\|_p$, $h \in B_Q$, $p \in I$. The strict topology $\beta$ on $C_{rc}^\circ$ is defined to be the inductive limit of the topologies $\beta_Q$, $Q \in \Omega$. The superstrict topology $\beta_1$ on $C_{rc}^\circ$ is the inductive limit of the topologies $\beta_Z$, $Z \in \Omega_1$. If $\pi$ is the pointwise convergence topology, one can easily verify the following

**Theorem 4.1.** $\pi \leq \beta \leq \beta_1 \leq \sigma$. 

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Theorem 4.2. \( \beta = \sigma \) iff \( X \) is compact.

Proof. Clearly \( \beta = \sigma \) if \( X \) is compact. On the other hand assume that \( X \) is not compact and that \( \beta = \sigma \).

Let \( x \in \beta X - X, \ Q = \{x\} \). Let \( p \in I, s \in E \) be such that \( p(s) = 2 \). Set \( W = \{f_{C_r} : \|f\|_p \leq 1\} \). Then \( W \) is a \( \sigma \)-neighborhood of zero. By hypothesis \( W \) is also a \( \beta \)-neighborhood of zero. Since \( \beta = \beta_Q \), \( W \) is a \( \beta_Q \)-neighborhood of zero. Thus there exist \( h \in B_Q \) and \( p_1 \) in \( I \) such that \( V = \{f_{C_r} : \|hf\|_{p_1} \leq 1\} \subset W \).

Choose \( \delta > 0 \) such that \( \delta p_1(s) < 1 \), and set \( F = \{y \in \beta X : |\hat{h}(y)| > \delta\} \).

Let \( g \in C^b, 0 \leq g \leq 1, \hat{g}(x) = 1 \) and \( \hat{g} = 0 \) on \( F \). But then the function \( f = gs \) is in \( V \) but not in \( W \). This contradiction completes the proof.

Since \( X \) is pseudocompact iff \( \Omega_1 = \{\emptyset\} \), we have the following theorem for \( \beta_1 \) whose proof is similar to that of Theorem 4.2.

Theorem 4.3. \( \beta_1 = \sigma \) iff \( X \) is pseudocompact.

If \( X \) is locally compact, then \( X \) is open in \( \beta X \). Let \( Q = \beta X - X \). Then \( B_Q \) is the space of all continuous real functions on \( X \) that vanish at infinity. Hence, as one can easily prove, \( \beta = \beta_Q \) coincides with the strict topology as defined by Buck in [4]. We will next identify the dual spaces of \((C_{rc}, \beta)\) and \((C_{rc}, \beta_1)\).

Lemma 4.4. If \( \phi \in (C_{rc}, \beta)' \), then there exists \( m \in M_r(B, E') \) such that \( \phi(f) = \int f dm \) for all \( f \in C_{rc} \).

Proof. Since \( \beta = \sigma \) there exists \( m \in M(B, E') \) such that \( \phi(f) = \int f dm \) for all \( f \) in \( C_{rc} \). Let \( m \in M_{r,p}(Bo(\beta X), E') \) be such that \( \phi(f) = \int f dm \) for all \( f \) in \( C_{rc} \). Let \( Q \in \Omega \). Since \( \phi \) is \( \beta_Q \)-continuous, there exists (by 3.5) \( h \in B_Q \) and \( \mu \in M(B, E') \) such that \( hf d \mu = \phi(f) \) for all \( f \in C_{rc} \). Let \( \mu \in M_r(Bo(\beta X), E') \) be such that \( hf d \mu = \int f d \mu \) for all \( f \) in \( C_{rc} \). Then \( \int f dm = \phi(f) = \int hf d \mu = \int hf d \mu \) for each \( f \in C_{rc} \). It follows that \( \hat{m} = \hat{h}\hat{\mu} \). If \( F \) is a Borel set in \( \beta X \) contained in \( Q \) and if \( s \in E \), then \( \int m(F)s = \int f \hat{h} d(\hat{\mu}s) = 0 \). We conclude that \( m_p(Q) = 0 \). This, being true for all \( Q \in \Omega \), implies that \( m \) is \( r \)-additive by 2.7. This completes the proof.

Lemma 4.5. If \( m \in M_{r,p}(B, E') \), then the map \( \phi_m : C_{rc} \rightarrow R, \phi_m(f) = \int f dm \), is \( \beta \)-continuous.

Proof. It suffices to show that \( \phi_m \) is \( \beta_Q \)-continuous for every \( Q \) in \( \Omega \). So, let \( Q \in \Omega \). Define \( T : C^b \rightarrow R \) by \( T(f) = \int f dm_p \). Since \( m_p \) is \( r \)-additive, \( T \) is \( \beta(C^b) \)-continuous, where \( \beta(C^b) \) is the strict topology on \( C^b \) as defined by Sentilles in [15] (see Sentilles, Theorem 4.3). Hence there exists \( g \) in \( B_Q \) such that 

\[ W = \{f \in C^b : \|gf\| \leq 1\} \subset \{f \in C^b : |T(f)| \leq 1\}. \]
Set $V = \{ f \in C_{rc} : \|gf\|_p \leq 1 \}$ and let $f \in V$.

Define $h : X \to R$, $h(x) = p(f(x))$. Clearly $h \in C^p$. Moreover $|h(x)g(x)| = p(g(x)f(x)) \leq 1$ for all $x \in X$ and hence $h \in W$. It follows that $\|hf dm\| \leq \|h dm_p\| = T(h) \leq 1$. This shows that $\phi_m$ is $\beta$ continuous. The lemma is proved.

Combining Lemmas 4.4 and 4.5 we get

**Theorem 4.6.** The space $M_r(B, E')$ is isomorphic to the space $(C_{rc}, \beta)'$ via the isomorphism $m \mapsto \phi_m$ where $\phi_m(f) = \int f dm$ for all $f$ in $C_{rc}$.

Using similar arguments we prove

**Theorem 4.7.** The space $M_\sigma(B, E')$ is isomorphic to the space $(C_{rc}, \beta_\sigma)$ via the isomorphism $m \mapsto \phi_m$, $\phi_m(f) = \int f dm$.

5. The case of a locally convex lattice $E$. In this section $E$ will be assumed to be a locally convex lattice. By Peressini [13, p. 105] there exists a generating family of continuous seminorms $p$ such that $|x| \leq |y|$ implies $p(x) \leq p(y)$. In view of this, we may assume that every $p \in I$ has the above property. The space $(C_{rc}, \sigma)$ is, under the pointwise ordering, a locally convex lattice. The question we are going to investigate now is the following: Which elements of $C_{rc}$ correspond to members of $M_\sigma(B, E')$ and which to members of $M_r(B, E')$? We will show that these are exactly the $\sigma$-additive and $\tau$-additive members of $C_{rc}$.

**Definition.** For a net $\{f_\alpha\}$ in $C_{rc}$ we say that $\{f_\alpha\}$ decreases to zero, and write $f_\alpha \downarrow 0$, if for each $x \in X$ we have $\lim f_\alpha(x) = 0$ and $0 < f_\alpha(x) < f_\gamma(x)$ if $\alpha \geq \gamma$. We define similarly what we mean by saying that a sequence $\{f_n\}$ in $C_{rc}$ decreases to zero. An element $\phi$ of $C_{rc}$ is called $\sigma$-additive if $\lim \phi(f_\alpha) = 0$ for each sequence $\{f_\alpha\}$ in $C_{rc}$ that decreases to zero. An element $\phi$ of $C_{rc}$ is called $\tau$-additive if $\lim \phi(f_\alpha) = 0$ whenever $f_\alpha \downarrow 0$. We will denote by $L_\sigma(C_{rc})$ and $L_\tau(C_{rc})$ the spaces of all $\sigma$-additive and all $\tau$-additive members of $C_{rc}$ respectively.

**Theorem 5.1.** Let $\phi \in C_{rc}$. Then $\phi$ is $\tau$-additive iff there exists $m \in M_r(B, E')$ such that $\phi(f) = \int f dm$ for all $f \in C_{rc}$.

**Proof.** Let $m \in M_{r,p}(B, E')$ be such that $\phi(f) = \int f dm$ for all $f \in C_{rc}$. Let $\{f_\alpha\}$ be a net in $C_{rc}$ that decreases to zero. For each $\alpha$, let $h_\alpha : X \to R$, $h_\alpha(x) = p(f(x))$. Since $p$ has the property that $p(s) \leq p(t)$ whenever $|s| \leq |t|$, it follows that $h_\alpha \downarrow 0$. Hence $\|f_\alpha dm\| \leq \|h_\alpha dm_p\| \to 0$ since $m_p$ is $\tau$-additive (see Varadarajan [17, p. 174]).

Conversely, assume that $\phi$ is $\tau$-additive. Let $s \geq 0, s \in E$. If $\{f_\alpha\}$ is a net in $C^p(X)$ which decreases to zero, then $f_\alpha s \downarrow 0$. Hence $\int f_\alpha d(ms) = \int f_\alpha s dm = \phi(f_\alpha s) \to 0$. It follows that $ms$ is $\tau$-additive (see Varadarajan [17, p. 174]).
Since every element of \( E \) is the difference of two positive elements, it follows that

\[ m s \in M_s(X) \] for all \( s \in E \) and hence \( m \) is \( \tau \)-additive.

Using an analogous argument we prove the following:

**Theorem 5.2.** Let \( \phi \in C'_{rc} \). Then \( \phi \) is \( \sigma \)-additive iff there exists \( m \in M_o(B, E') \) such that \( \phi(f) = \int f \, dm \) for all \( f \in C_{rc} \).

**Lemma 5.3.** Let \( m \in M_p(B, E') \) and \( |m| = \sup(m, -m) \). Then \( |m| \in M_p(B, E') \) and \( |m|_p = m_p \).

**Proof.** Recall that \( p \) has the property that \( p(s) \leq p(t) \) whenever \( |s| \leq |t| \). As shown in [8], for each \( s \geq 0 \) in \( E \) and each \( F \) in \( B(X) \) we have \( |m|(F)s = \sup \Sigma |m(F_i)s| \) where the supremum is taken over all finite \( B \)-partitions \( \{F_i\} \) of \( F \).

Let now \( F \in B(X) \). If \( F_1, \ldots, F_n \) is a \( B \)-partition of \( F \), and if \( s_i \in E \) with \( p(s_i) \leq 1 \),

\[
|\sum m(F_i)s_i| \leq \sum |m(F_i)||s_i| \leq \sum |m(F_i)||s_i| \leq |m|_p(F)
\]

since \( p(|s_i|) = p(s_i) \leq 1 \). Thus \( m_p(F) \leq |m|_p(F) \). On the other hand, let \( G_1, \ldots, G_n \) be a \( B \)-partition of \( F \) and let \( s_i \in E \) with \( p(s_i) \leq 1 \). We will show that \( \Sigma |m|(G_i)s_i \leq m_p(F) \). Since \( p(|s_i|) \leq 1 \) and since \( \Sigma |m|(G_i)s_i \leq \Sigma |m|(G_i)|s_i| \), we may assume that \( s_i \geq 0 \). Let \( \epsilon > 0 \) be given. For each \( i, 1 \leq i \leq n \), there exists a \( B \)-partition \( F^i_1, \ldots, F^i_{K_i} \) of \( G_i \) such that

\[
\frac{1}{K_i} \sum_{j=1}^{K_i} |m(F^i_j)s_i| > |m|(G_i)s_i - \frac{\epsilon}{n}.
\]

Let \( N = K_1 + \cdots + K_n \). Choose \( t_{ij} \in E, |t_{ij}| \leq s_i \), such that \( |m(F^i_j)t_{ij}| > |m(F^i_j)|s_i - \epsilon/N \). Since \( p(t_{ij}) \leq 1 \) and \( \{F^i_j\} \) is a \( B \)-partition of \( F \), we have

\[
m_p(F) \geq \sum_{i,j} |m(F^i_j)t_{ij}| = \sum_{i=1}^{n} \sum_{j=1}^{K_i} |m(F^i_j)t_{ij}|
\geq \sum_{i=1}^{n} \sum_{j=1}^{K_i} |m(F^i_j)|s_i - \epsilon \geq \sum_{i=1}^{n} |m|(G_i)s_i - 2\epsilon.
\]

Since \( \epsilon > 0 \) was arbitrary, we have \( \Sigma |m|(G_i)s_i \leq m_p(F) \). This proves that \( |m|_p(F) \leq m_p(F) \) and the lemma is proved.

By the above lemma, if \( m \in M_s(B, E') \), then \( |m| \) is also \( \tau \)-additive. From this follows that \( M_{\tau}(B, E') \) is an ideal in \( M(B, E') \). Since the map \( m \rightarrow \phi_m \), of Theorem 1.1, is lattice-preserving and since \( M_{\tau}(B, E') \) corresponds to \( L_{\tau}(C_{rc}) \) in this map, it follows that \( L_{\tau}(C_{rc}) \) is an ideal in the Riesz space \( C'_{rc} \). The same is
true for the space $L_\sigma(C_{rc})$. We have thus the following theorem.

**Theorem 5.4.** Each of the spaces $L_\sigma(C_{rc})$ and $L_\tau(C_{rc})$ is an ideal in the Riesz space $C_{rc}^\prime$.

It is well known (Knowles [11, p. 149]) that any positive linear functional $\phi$ on $C^b$ can be written uniquely as a sum of a positive purely finitely-additive functional on $C^b$, a positive purely $\sigma$-additive and a positive $\tau$-additive functional on $C^b$. It is therefore natural to ask whether this is true in our space $(C_{rc}, \sigma)$.

We will show that the answer to this question is affirmative.

**Definition.** An element $\phi \geq 0$ in $C_{rc}^\prime$ is called purely finitely-additive if the only $\sigma$-additive functional $\phi_1$ in $C_{rc}$ with $0 \leq \phi_1 < \phi$ is the zero functional. Similarly, an element $\phi \geq 0$ of $L_\sigma(C_{rc})$ is purely $\sigma$-additive if $0 < \phi_1 < \phi$ and $\phi_1 \in L_\tau(C_{rc})$ implies that $\phi_1 = 0$.

We are going to prove the following

**Theorem 5.5.** Given $\phi \geq 0$ in $C_{rc}^\prime$ there are $\phi_1, \phi_2, \phi_3$ in $C_{rc}^\prime$, $\phi_1$ purely finitely-additive, $\phi_2$ purely $\sigma$-additive, $\phi_3$ $\tau$-additive, $\phi_1, \phi_2, \phi_3 \geq 0$, such that $\phi = \phi_1 + \phi_2 + \phi_3$. Moreover this decomposition is unique.

To begin with, assume that $\phi, \phi_1, \phi_2 \geq 0$ in $C_{rc}^\prime$, $\phi_1$ $\sigma$-additive, $\phi = \phi_1 + \phi_2$. Let $m, m_1, m_2 \in M_{\tau,p}(B(\beta X), E')$ be such that $\phi(f) = \int f dm$, $\phi_i(f) = \int f dm_i$, $i = 1, 2$. Choose a decreasing sequence $\{U_i\}$ of cozero sets in $\beta X$, such that $m_p(U_i) \to d$. Let $K = \bigcap U_i$. Clearly $m_p(K) = d$. Since $\phi_1$ is $\sigma$-additive we have $(m_1)_p(\beta X - K) = 0$ by 2.7. Hence $m_1(F) = m_1(F \cap K) \leq m(F \cap K)$ for each Borel set $F$ in $\beta X$ since $m_2 \geq 0$ and $m = m_1 + m_2$. Define $m_2$ on $B(\beta X)$ by $m_2(F) = m(F \cap K)$. Then $m_3 \in M_{\tau,p}(B(\beta X), E')$ and $(m_3)_p(F) = m_p(F \cap K)$ for each $F$ in $B(\beta X)$. Since $E$ is locally solid, the positive cone is closed (see Schaefer [14, p. 235]).

Now let $f \geq 0$ in $C_{rc}$. If $y \in \beta X$ and if $\{x_\alpha\}$ is a net in $X$ converging to $y$ in $\beta X$, then $\hat{f}(y) = \lim_{\alpha} \hat{f}(x_\alpha) = \lim f(x_\alpha) \geq 0$. Thus $\hat{f} \geq 0$. It follows that the map $\phi_3 : C_{rc} \to R$, $\phi_3(f) = \int \hat{f} dm_3$ is positive. Moreover $\phi_3 \geq \phi_1$ since $m_3 \geq m_1$.

We next show that $\phi_3$ is $\sigma$-additive. Indeed, let $Z$ be a zero set in $\beta X$ disjoint from $X$. Then $U = \beta X - Z$ is a cozero set containing $X$. Since $X \subset U \cap U_i \downarrow K \cap U$, we have

$$d = \lim m_p(U \cap U_i) = m_p(K \cap U) \leq m_p(K) = d.$$

So $(m_3)_p(\beta X) = m_p(K) = m_p(K \cap U) = (m_3)_p(U)$ and hence $(m_3)_p(Z) = 0$. By 5.2 and 2.7 $\phi_3$ is $\sigma$-additive. Clearly $\phi - \phi_3$ is purely finitely-additive. We have thus proved
Theorem 5.6. If \( \phi \geq 0 \) in \( C'_r(\mathbb{C}) \), then there are unique \( \phi_1, \phi_2 \geq 0 \) in \( C'_r(\mathbb{C}) \), \( \phi_1 \) purely finitely-additive and \( \phi_2 \) \( \sigma \)-additive, such that \( \phi = \phi_1 + \phi_2 \).

Next assume that \( 0 \leq \phi \in L_0(C'_r) \). Suppose that \( \phi = \phi_1 + \phi_2 \), \( \phi_1 \in L_0(C'_r) \), \( \phi_2 \in L_0(C'_r) \), \( \phi_1, \phi_2 \geq 0 \). Let \( m, m_1, m_2 \in M_{r,p}(Bo(\beta X), E') \) be such that \( \phi(f) = \int f d\phi, \phi_i(f) = \int f d\phi_i, i = 1, 2 \), for all \( f \) in \( C'_r \). Let \( d = \inf \{ m_p(O) : O \) open in \( \beta X, X \subset O \} \). Choose a decreasing sequence \( \{ O_n \} \) of open sets in \( \beta X, X \subset O_n \), \( m_p(O_n) \to d \). Since \( \phi_1 \) is \( \tau \)-additive we have that \( (m_1)_p(\beta X \setminus K) = 0 \). Thus \( m_1(F) = m_1(F \cap K) \leq m(F \cap K) \) for all Borel sets \( F \) in \( \beta X \). Define \( m_3 \) on \( Bo(\beta X) \) by \( m_3(F) = m(F \cap K) \). Then \( m_3 \in M_{r,p}(Bo(\beta X), E') \) and \( m_3 \geq m_1 \).

Let \( Q \) be a closed set in \( \beta X \) disjoint from \( X \). Then \( O = \beta X - Q \supset X \) and \( O \) is open. Hence \( d \leq \lim m_p(O \cap O_i) = m_p(O \cap K) \leq m_p(K) = d \). It follows that \( (m_3)_p(\beta X) = (m_3)_p(O) \) and hence \( (m_3)_p(O) = 0 \). This, being true for all closed sets in \( \beta X \) which are disjoint from \( X \), implies that the map \( \phi_3 : C'_r \to R, \phi_3(f) = \int f d\phi_3 \), is \( \tau \)-additive. Also \( \phi_3 \geq 0 \). Moreover, \( \phi - \phi_3 \) is purely \( \sigma \)-additive. We have thus proved

Theorem 5.7. Given \( \phi \geq 0 \) in \( L_0(C'_r) \), there are unique \( \phi_1, \phi_2 \geq 0 \), \( \phi_1 \in L_r(C'_r) \), \( \phi_2 \) purely \( \sigma \)-additive, such that \( \phi = \phi_1 + \phi_2 \).

Combining Theorems 5.6 and 5.7 we get Theorem 5.5.

BIBLIOGRAPHY


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