SPACES OF VECTOR MEASURES

BY

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ABSTRACT. Let $C_{rc} = C_{rc}(X, E)$ denote the space of all continuous functions $f$, from a completely regular Hausdorff space $X$ into a locally convex space $E$, for which $f(X)$ is relatively compact. As it is shown in [8], the uniform dual $C_{rc}'$ of $C_{rc}$ can be identified with a space $M(B, E')$ of $E'$-valued measures defined on the algebra of subsets of $X$ generated by the zero sets. In this paper the subspaces of all $\sigma$-additive and all $\tau$-additive members of $M(B, E')$ are studied. Two locally convex topologies $\beta$ and $\beta_1$ are considered on $C_{rc}$. They yield as dual spaces the spaces of all $\tau$-additive and all $\sigma$-additive members of $M(B, E')$ respectively. In case $E$ is a locally convex lattice, the $\sigma$-additive and $\tau$-additive members of $M(B, E')$ correspond to the $\sigma$-additive and $\tau$-additive members of $C_{rc}$ respectively.

1. Definitions and preliminaries. Let $X$ be a completely Hausdorff space and let $E$ be a real locally convex Hausdorff space. Let $C^b = C^b(X)$ denote the space of all bounded continuous real-valued functions on $X$. We will denote by $C_{rc} = C_{rc}(X, E)$ the space of all continuous functions $f$, from $X$ into $E$, for which $f(X)$ is relatively compact. Clearly $C_{rc}$ consists of those continuous functions $f$, from $X$ into $E$, that have continuous extensions $\hat{f}$ to all of the Stone-Čech compactification $\beta X$ of $X$. For an $f$ in $C^b$ we will denote also by $\hat{f}$ its unique continuous extension to all of $\beta X$. The zero sets in $X$ are defined to be the kernels of real continuous functions on $X$. The complement of a zero set is called a cozero set.

Let $\Sigma$ be an algebra of subsets of $X$ and let $m$ be a finitely-additive bounded real set function on $\Sigma$. We say that $m$ is regular with respect to a subfamily $\Sigma_1$ of $\Sigma$ if the following condition is satisfied: For every $F$ in $\Sigma$ and every $\epsilon > 0$ there exists $G$ in $\Sigma_1$ such that $G \subseteq F$ and $|m(H)| < \epsilon$ for all $H$ in $\Sigma$ which are contained in $F - G$.

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(1) The contents of this paper form part of the work done by the author in his doctoral dissertation under the direction of R. B. Kirk whom the author wishes to thank for his encouragement and advice.
Let now $B = B(X)$ and $Ba = Ba(X)$ be, respectively, the algebra and $\sigma$-algebra of subsets of $X$ generated by the zero sets. The collection of Borel subsets of $X$ will be denoted by $Bo = Bo(X)$. Let $M(X)$ be the space of all bounded, finitely-additive, real-valued, set functions on $B(X)$ which are regular with respect to the family of zero sets. The space of all bounded, countably-additive, real-valued, regular (with respect to the zero sets) measures on $Ba$ will be denoted by $M_\sigma(Ba)$. By $M_\tau(Bo)$ we will denote the space of all real, regular with respect to the closed sets, Borel measures $m$ on $Bo$ such that $|m|(Z_\alpha) \to 0$ for each net $\{Z_\alpha\}$ of zero sets which decreases to the empty set (see Varadarajan [17] or Aleksandrov [1]). Note that an element $m$ of $M_\tau(Bo)$ is not necessarily regular with respect to the zero sets. However its restriction to $Ba$ is an element of $M_\sigma(Ba)$ by Varadarajan [16, p. 171, Theorem 19]. The subspaces of all $\sigma$-additive and $\tau$-additive members of $M(X)$ will be denoted by $M_\sigma(X)$ and $M_\tau(X)$ respectively (see Varadarajan [17] for the definition of $\sigma$-additive and $\tau$-additive measures). For $m$ in any one of the spaces $M(X), M_\sigma(Ba), M_\tau(Bo)$, the positive part $m^+$, the negative part $m^-$, and the variation $|m|$ are understood as, for example, in Aleksandrov [1].

Let now $\{p: p \in I\}$ be a family of continuous seminorms on $E$ generating the topology of $E$. We choose this family so that it is directed, i.e., given $p_1, p_2$ in $I$ there exists $p \in I$ with $p \geq p_1, p_2$. For each $p$ in $I$ we consider the space $M_p(B, E')$ of all finitely-additive functions $m: B(X) \to E'$ ($E'$ is the topological dual of $E$) such that the following two conditions are satisfied:

(1) For each $s \in E$, the function $m_s: B \to R$, $(ms)(F) = m(F)s$, is in $M(X)$.

(2) $\|m\|_p = m_p(X) < \infty$, where for $F$ in $B$ we define $m_p(F) = \sup |\Sigma m(F_i)s_i|$, the supremum being taken over all finite $B$-partitions $\{F_i\}$ of $F$ (that is partitions into sets in $B$) and all finite collections $\{s_i\}$ in $E$ with $p(s_i) \leq 1$.

The set function $m_p$ belongs to $M(X)$. Indeed it is easy to see that $m$ is finitely-additive and bounded. For the regularity, consider an $F$ in $B$ and let $\epsilon > 0$ be given. By definition there exist a finite $B$-partition $\{F_i\}$ of $F$ and $s_i \in E$, with $p(s_i) \leq 1$, such that $\Sigma m(F_i)s_i > m_p(F) - \epsilon$. By the regularity of $m_s$ we can choose for each $i$ a zero set $Z_i \subset F_i$ such that $\Sigma m(Z_i)s_i > m_p(F) - \epsilon$. The zero set $Z = \bigcup Z_i$ is contained in $F$. Moreover we have $m_p(Z) \geq \Sigma m(Z_i)s_i > m_p(F) - \epsilon$. This proves the regularity of $m_p$. Set $M(B, E') = \bigcup_{p \in I} M_p(B, E')$.

Let $\sigma$ denote the uniform topology on $C_{rc}$, i.e., the locally convex topology generated by the family of seminorms $\{\|\cdot\|_p: p \in I\}$, where $\|f\|_p = \sup \{p(f(x)): x \in X\}$. In [8] the author defines the integral of a function $f$ in $C_{rc}$ with respect to a member of $M(B, E')$. The integration process employed is a generalization of the process of Aleksandrov to the vector case. It is one of the many integration processes defined by McShane [12]. Every element $m$ of
M(B, E') generates a linear functional \( \phi_m \) on \( C_{rc} \) by \( \phi_m(f) = \int_X f \, dm \), \( f \in C_{rc} \).

The proof of the following theorem can be found in [8].

**Theorem 1.1.** For each \( m \in M(B, E') \), \( \phi_m \) is an element of \( (C_{rc}, o') \). Moreover, the map \( m \rightarrow \phi_m \), from \( M(B, E') \) into \( C_{rc} \), is linear, one-to-one, and onto.

**Theorem 1.2.** If \( m \in M_p(B, E') \), then \( \| \phi_m \| = \| m \|_p \), where \( \| \phi_m \|_p = \sup \{ \| \phi_m(f) \| : f \in C_{rc}, \| f \|_p \leq 1 \} \).

**Proof.** It is clear from the definitions that \( \| f \, dm \| \leq \int p \circ f \, dm_p \leq \| f \|_p \| m \|_p \) for all \( C_{rc} \) and hence \( \| \phi_m \|_p \leq \| m \|_p \). On the other hand, let \( \epsilon > 0 \) be given. By the definition of \( \| m \|_p \), there exist a finite \( B \)-partition \( \{ F_i \} \) of \( X \) and \( s_i \in E \) with \( p(s_i) \leq 1 \) such that \( \| m \|_p \leq \Sigma(F_i) s_i + \epsilon \). By regularity there are zero sets \( Z_i \subset F_i \) such that \( \| m \|_p \leq \Sigma m(Z_i) s_i + \epsilon \). Again by the regularity of \( m s_i \), we can find pairwise disjoint cozero sets \( \{ U_i \} \), \( Z_i \subset U_i \), such that

\[
\Sigma m s_i(U_i - Z_i) < \epsilon.
\]

For each \( i \) choose \( h_i \in C_b \), \( 0 \leq h_i \leq 1 \), such that \( h_i = 1 \) on \( Z_i \) and \( h_i = 0 \) on \( X - U_i \). Set \( h = \Sigma h_i s_i \). Then \( \| h \|_p \leq 1 \) and so \( \int h \, dm \leq \| \phi \|_p \). But

\[
\left| \int h \, dm \right| = \left| \sum \int_{Z_i} s_i \, dm \right| - \left| \sum \int_{U_i - Z_i} h_i \, dm \right| \geq \Sigma m(Z_i) s_i - \epsilon > \| m \|_p - 2\epsilon.
\]

Since \( \epsilon > 0 \) was arbitrary we get that \( \| \phi_m \|_p \geq \| m \|_p \) and this completes the proof.

In case \( E \) is a locally convex lattice, \( (C_{rc}, o) \) becomes also a locally convex lattice under the pointwise ordering (that is, we define \( f \geq g \) iff \( f(x) \geq g(x) \) for all \( x \in X \)). We define an order relation \( \geq \) on \( M(B, E') \) by \( m_1 \geq m_2 \) iff \( m_1(F) \geq m_2(F) \) for all \( F \) in \( B \). Note that \( E' \) is a lattice when ordered by the cone \( \{ \phi \in E' : \phi(s) \geq 0 \text{ when } s \geq 0 \} \). As it is shown in [8], \( M(B, E') \) becomes a lattice and the map \( m \rightarrow \phi_m \), of Theorem 1.1, is lattice preserving.

2. Extensions of members of \( M(B, E') \). Let \( p \in I \). We define \( M_{o,p}(Ba, E') \) to be the set of all functions \( m: Ba \rightarrow E' \) such that the following two conditions are satisfied:

1. For each \( s \in E \) the function \( m s: Ba \rightarrow R \), \( (ms)(F) = m(F)s \), is in \( M_s(Ba) \).

2. \( m_p(X) < \infty \) where, for each \( F \) in \( Ba \), we define \( m_p(F) = \sup |\Sigma m(F_i) s_i| \) where the supremum is taken over all finite \( Ba \)-partitions \( \{ F_i \} \) of \( F \) and all finite collections \( \{ s_i \} \) in \( E \) with \( p(s_i) \leq 1 \).

**Lemma 2.1.** If \( m \in M_{o,p}(Ba, E') \), then \( m_p \in M_o(Ba) \).
Proof. It is easy to see that $m_p$ is bounded monotone and finitely-additive. Let $\{F_n\}$ be a sequence of pairwise Baire sets (i.e., sets in $Ba$) and set $F = \bigcup F_n$. Since $m_p$ is monotone and finitely-additive, we have $m_p(F) \geq m_p(\bigcup F_i) = \Sigma m_p(F_i)$ for each $n$. Hence $m_p(F) \geq \Sigma m_p(F_i)$. On the other hand, let $\epsilon > 0$ be arbitrary. There exist a $Ba$-partition $G_1, \cdots, G_N$ of $F$ and $s_i \in E$, $p(s_i) \leq 1$, such that $\Sigma m(G_i)s_i > m_p(F) - \epsilon$. Since $m(s_i)$ is countably additive we have $m(G_i)s_i = \Sigma_{n=1}^\infty m(G_i \cap F_n)s_i$. Moreover,

$$\sum_{n=1}^\infty \sum_{i=1}^N |m(G_i \cap F_n)s_i| \leq \sum_{n=1}^\infty m_p(F_n) \leq m_p(F) < \infty.$$ 

Hence

$$m_p(F) - \epsilon \leq \sum_{i=1}^N m(G_i)s_i = \sum_{i=1}^N \sum_{n=1}^\infty m(G_i \cap F_n)s_i$$

$$= \sum_{n=1}^\infty \sum_{i=1}^N m(G_i \cap F_n)s_i \leq \sum m_p(F_n) \leq m_p(F).$$

Since $\epsilon > 0$ was arbitrary we conclude that $m_p(F) = \Sigma m_p(F_n)$ and so $m_p$ is countably-additive. Finally, the proof of the regularity of $m_p$ is similar to that of the case of a member of $M_p(B, E')$.

Next we define $M_{r,p}(B, E')$ to be the set of all $m: Bo \rightarrow E'$ having the following two properties:

(a) For each $s$ in $E$, $m(s)$ belongs to $M_r(Bo)$.

(b) $m_p(X) < \infty$, where for each $F$ in $Bo$ the $m_p(F)$ is defined by $m_p(F) = \sup|\Sigma m(F_i)s_i|$ the supremum being taken over all finite $Bo$-partitions of $F$ and all finite collections $\{s_i\}$ in $E$ with $p(s_i) \leq 1$.

Lemma 2.2. If $m \in M_{r,p}(Bo, E')$, then $m_p \in M_r(Bo)$.

Proof. By using an argument similar to that of 2.1, we show that $m_p$ is a bounded, countably-additive, regular with respect to the closed sets, Borel measure on $X$. To complete the proof we need to show that $m_p$ is $r$-additive. To this end, consider an arbitrary net $\{Z_\alpha\}$ of zero sets decreasing to the empty set. For each $\alpha$ there exists a zero set $Z_\alpha$ in $\beta X$ such that $Z_\alpha = Z_\alpha \cap X$.

Define $\bar{m}: Bo(\beta X) \rightarrow E'$ by $\bar{m}(F) = m(F \cap X)$. For each $s \in E$, the function $\bar{m}s: Bo(\beta X) \rightarrow R$, $(\bar{m}s)(F) = (ms)(F \cap X)$, is a regular Borel measure on $\beta X$ since $ms$ is $r$-additive (see Knowles [11]). It follows now easily that $m \in M_{r,p}(Bo(\beta X), E')$. Moreover $\bar{m}(F) = m_p(F \cap X)$ for each Borel set $F$ in $\beta X$. Indeed it is clear that $\bar{m}(F) \leq m_p(F \cap X)$. On the other hand, if $\{G_i\}$ is a finite $Bo(X)$ partition of $F \cap X$, then there are pairwise disjoint Borel sets $V_i$
in $\beta X$, which we may choose contained in $F$, such that $G_1 = V_i \cap X$. For $s_i \in E$ with $p(s_i) < 1$, we have $\bar{m}_p(F) \geq |\sum \bar{m}(V_i)s_i| = |\sum m(G_i)s_i|$. This shows that $\bar{m}_p(F) \geq m_p(F \cap X)$ and so $m_p(F) = m_p(F \cap X)$. Let now $D = \{Z \subset \beta X: Z$ is an intersection of a finite number of $\hat{Z}_\alpha\}'s\}. Then $D$ is directed downwards to $G = \bigcap \hat{Z}_\alpha$. Hence $\bar{m}_p(G) = \lim_{Z \in D} \bar{m}_p(Z)$. Since $G \cap X = \emptyset$, we have $\bar{m}_p(G) = 0$. Thus given $\varepsilon > 0$ there exists $Z = \hat{Z}_{\alpha_1} \cap \cdots \cap \hat{Z}_{\alpha_n}$ in $D$ such that $\bar{m}_p(Z) < \varepsilon$. If $\alpha > \alpha_1, \cdots, \alpha_n$, we have $m_p(Z_\alpha) \leq m_p(Z \cap X) = \bar{m}_p(Z) < \varepsilon$. This completes the proof.

**Theorem 2.3.** If $m \in M_{\alpha,p}(\beta \alpha, E')$ $[m \in M_{\tau,p}(\beta \alpha, E')]$, then $m_p(X) = \sup \{\|f dm\|: f \in C_{rc}, \|f\|_p \leq 1\}$.

**Proof.** Let $d = \sup \{\|f dm\|: f \in C_{rc}, \|f\|_p \leq 1\}$. To prove the result in the case of an $m$ in $M_{\alpha,p}(\beta \alpha, E')$ one can use the same argument as the one used in the proof of Theorem 1.2. We will prove the result for an $m$ in $M_{\tau,p}(\beta \alpha, E')$. Since $\|f dm\| < \|f\|_p m_p(X)$, it follows that $d < m_p(X)$. To prove the reverse inequality, consider an arbitrary $\varepsilon > 0$. Define $\tilde{m}$ on $Bo(\beta X)$ by $\tilde{m}(F) = m(F \cap X)$. By the definition of $\tilde{m}_p$, there exist a partition $\{F_1, \cdots, F_n\}$ of $\beta X$, $F_i \in Bo(\beta X)$, and $s_i, \cdots, s_n$ in $E$, $p(s_i) < 1$, such that $\sum \tilde{m}(F_i)s_i > m_p(\beta X) - \varepsilon = m_p(X) - \varepsilon$. By regularity there are closed sets $G_i$ in $\beta X$, $G_i \subset F_i$, such that $\sum \tilde{m}(G_i)s_i > m_p(X) - \varepsilon$. Next we choose pairwise disjoint open sets $O_i$ in $\beta X$, $G_i \subset O_i$, such that $|ms_i|(O_i - G_i) < \varepsilon/n$. For each $i$, $1 \leq i \leq n$, there is an $h_i$ in $C^b(X)$, $0 \leq h_i \leq 1$, $h_i = 1$ on $G_i$ and $h_i = 0$ on the complement of $O_i$. Set $h = \sum h_i s_i$. Then $\|h\|_p \leq 1$ and

$$\int_X h dm = \int_{\beta X} h d\tilde{m} = \sum \tilde{m}(G_i)s_i + \sum \int_{O_i - G_i} h_i d(ms_i) > m_p(X) - 2\varepsilon.$$ 

Thus $d > m_p(X) - 2\varepsilon$ and the result follows since $\varepsilon > 0$ was arbitrary.

**Lemma 2.4.** Let $m \in M_{\tau,p}(\beta \alpha, E')$ and $\mu = m|_{\beta \alpha}$ (= restriction of $m$ to $\beta \alpha$). Then (a) $\mu \in M_{\alpha,p}(\beta \alpha, E')$, (b) $\mu_p = m_p|_{\beta \alpha}$.

**Proof.** Part (a) is clear because the restriction to $\beta \alpha$ of an element of $M_{\tau}(\beta \alpha)$ is in $M_{\alpha}(\beta \alpha)$. For (b) we first observe that $\mu_p(X) = m_p(X)$ by 2.3 since $\int f dm = \int f d\mu$ for all $f$ in $C_{rc}$. It is also clear that $\mu_p(F) \leq m_p(F)$ for all $F$ in $\beta \alpha$. Thus (b) follows.

Set $M_{\alpha}(\beta \alpha, E') = \bigcup \{M_{\alpha,p}(\beta \alpha, E'): p \in I\}$ and define $M_{\tau}(\beta \alpha, E')$ analogously.

Let $M_0(\beta \alpha, E')$ be the subspace of $M(\beta \alpha, E')$ consisting of all $m \in M(\beta \alpha, E')$ for which $ms \in M_0(X)$ for all $s$ in $E$. We define $M_{\tau}(\beta \alpha, E')$ similarly. We will call
the elements of $M_\sigma(B, E') \ [M_\tau(B, E')]$ the $\sigma$-additive ($\tau$-additive) members of $M(B, E')$. The next theorem shows that the $\sigma$-additive members of $M(B, E')$ are exactly the ones that have extensions to members of $M_\sigma(Ba, E')$.

**Theorem 2.5.** Let $m \in M(B, E')$. Then $m$ is $\sigma$-additive iff there exists a $\mu$ in $M_\sigma(Ba, E')$ with $m = \mu|_B$. Moreover, if such a $\mu$ exists it is unique.

**Proof.** Clearly $\mu|_B$ is in $M_\sigma(B, E')$ for each $\mu$ in $M_\sigma(Ba, E')$. Moreover if $\lambda$ is another member of $M_\sigma(Ba, E')$ such that $\lambda|_B = \mu|_B$, then $\lambda s|_B = \mu s|_B$ for each $s \in E$. It follows that $\lambda s = \mu s$ by the regularity of $\lambda s$ and $\mu s$. This, being true for all $s$ in $E$, implies that $\mu = \lambda$. Assume next that $m \in M_\sigma(B, E')$. For each $s \in E$, $ms \in M_\sigma(X)$. Hence, for each $s \in E$, there exists a unique extension $\mu_s$ of $ms$ to a member of $M_\sigma(Ba)$ such that $\|ms\| = \|\mu_s\|$ (see Varadarajan [17]).

For an $F$ in $Ba$, we define $\mu(F): E \rightarrow R$, by $\mu(F)s = \mu_s(F)$. Clearly $\mu(F)$ is linear. Moreover, if $m \in M_p(B, E')$, then

$$|\mu(F)s| = |\mu_s(F)| \leq \|\mu_s\| = \|ms\| \leq p(s)\|m\|_p.$$  

Hence $\mu(F) \in E'$. In this way we define a map $\mu: Ba \rightarrow E'$ such that $\mu s = \mu_s \in B_\sigma(Ba)$ for all $s \in E$. To finish the proof it remains to show that $\|\mu\|_p < \infty$. To this end, consider an arbitrary $Ba$-partition $F_1, \cdots, F_n$ of $X$ and let $s_i \in E$ with $p(s_i) \leq 1$. For $\epsilon > 0$, there exist zero sets $Z_1, \cdots, Z_n, Z_i \subset F_i$, such that $|\mu s_i|(F_i - Z_i) < \epsilon/n$. Thus

$$\left|\sum \mu(F_i)s_i\right| < \left|\sum \mu(Z_i)s_i\right| + \epsilon = \left|\sum m(Z_i)s_i\right| + \epsilon \leq \|m\|_p + \epsilon.$$

It follows that $\|\mu\|_p \leq \|m\|_p$ and the proof is complete.

We have an analogous theorem for $M_\tau(Bo, E')$.

**Theorem 2.6.** Let $m \in M(B, E')$. Then $m$ is $\tau$-additive iff there exists a unique $\mu \in M_\tau(Bo, E')$ such that $m = \mu|_B$.

**Proof.** Clearly $\mu|_B \in M_\tau(B, E')$ for each $\mu \in M_\tau(Bo, E')$. Also, if $\mu_1, \mu_2$ are both in $M_\tau(Bo, E')$ with $\mu_1|_B = \mu_2|_B$, then $\mu_1 s|_B = \mu_2 s|_B$ for each $s$ in $E$. By Kirk [9, Theorem 1.14], we have $\mu_1 s = \mu_2 s$. This, being true for all $s$ in $E$, implies that $\mu_1 = \mu_2$. Assume now that $m \in M_\tau(B, E')$. Let $C(\beta X, E)$ denote the space of all continuous functions from $\beta X$ into $E$.

Clearly $C(\beta X, E) = \{ f : f \in C_{rc} \}$. Define $\phi$ on $C(\beta X, E)$ by $\phi(\hat{f}) = \int f dm$. Then $\phi$ is continuous with respect to the uniform topology on $C(\beta X, E)$. Hence, by 1.1, there exists $\tilde{m} \in M_p(B(\beta X), E')$ such that $\phi(\hat{f}) = \int \tilde{m} f$ for all $f$ in $C_{rc}$.

Since each $ms, s \in E$, is $\tau$-additive it has a unique norm-preserving extension to a member $\tilde{\mu}_s$ of $M_\tau(Bo(\beta X))$ (see Kirk [9]). For each Borel set $F$ in $\beta X$, we define
\(\mu(F)\) on \(E\) by \(\mu(F)s = \mu_s(F)\). It is easy to see that \(\mu(F) \in E'\). In this way we get a map \(\mu : Bo(\beta X) \to E'\). We will show that \(\mu \in M_{r,p}(Bo(\beta X), E')\). Since \(\mu_s = \mu_s(F) \in M_{r}(Bo(\beta X))\), it only remains to show that \(\|\mu\|_p < \infty\). To this end consider an arbitrary partition \(F_1, \ldots, F_n\) of \(\beta X\) into Borel sets and let \(s_i \in E\) with \(p(s_i) \leq 1\). There are closed sets \(G_1, \ldots, G_n\) in \(\beta X\), \(G_i \subset F_i\), such that \(|\mu_s|(F_i - G_i) < e/n\) (\(e > 0\) arbitrary). Since \(G_1, \ldots, G_n\) are pairwise disjoint compact sets and since the cozero sets form a base for the open sets, there are pairwise disjoint cozero sets \(U_1, \ldots, U_n\) in \(\beta X\), \(G_i \subset U_i\), such that \(|\mu_s|(U_i - G_i) < e/n\). Thus

\[
\left|\sum_s \mu(F_i)s_i\right| \leq \left|\sum_s \mu(U_i)s_i\right| + 2e = \left|\sum \bar{m}(U_i)s_i\right| + 2e \leq \bar{m}_p(\beta X) + 2e.
\]

It follows that \(\mu_p(\beta X) \leq \bar{m}_p(\beta X)\) and so \(\mu\) is in \(M_{r,p}(Bo(\beta X), E')\). Next we show that \(\mu_p(F) = 0\) for each Borel set \(F\) in \(\beta X\) which is disjoint from \(X\). By regularity it suffices to show that \(\mu(F)s = 0\) for each \(s \in E\) and each closed set \(F\) in \(\beta X\) disjoint from \(X\). So, let \(F\) be such a set and let \(s \in E\). There exists an open set \(0\) in \(\beta X\), \(F \subset 0\), such that \(|\mu_s|\{O - F\} < e\) (\(e > 0\) arbitrary). There exists a net \(\{f_\alpha\}\) in \(C^b(X), f_\alpha \downarrow 0, f_\alpha = 1\) on \(F\) and \(f_\alpha = 0\) on the complement of \(0\), \(0 < f_\alpha \leq 1\). Since \(ms\) is \(r\)-additive, we have \(\lim f_\alpha d(ms) = 0\). Hence there exists \(\alpha\) such that \(|f_\alpha d(ms)| < e\). Thus

\[
\left|\int f_\alpha s d\mu\right| = \left|\int f_\alpha s dm\right| < e.
\]

But

\[
\left|\int f_\alpha s d\mu\right| \geq \left|\int_F s d\mu\right| - \left|\int_{O-F} f_\alpha s d\mu\right| \geq \left|\mu(F)s\right| - \epsilon.
\]

Therefore \(\left|\mu(F)s\right| < 2e\). Since \(e > 0\) was arbitrary, we conclude that \(\mu(F)s = 0\) which proves the claim: Next we define \(\mu : Bo(X) \to E'\) by \(\mu(F \cap X) = \mu(F)\) for each \(F \in Bo(\beta X)\). If \(F_1, F_2\) are Borel sets in \(\beta X\) such that \(F_1 \cap X = F_2 \cap X\), then both \(F_1 - F_2\) and \(F_2 - F_1\) are disjoint from \(X\) and so \(\mu(F_1) = \mu(F_1 \cap F_2) = \mu(F_2)\). Hence \(\mu\) is well defined. It is easy now to see that \(\mu \in M_{r,p}(Bo(X), E')\). Moreover, it is clear that \(\int f df \mu = \int f df \bar{\mu} = \int f d\bar{\mu} = \int f dm\) for all \(f\) in \(C_{rc}\). Let \(m_1 = \mu|_B\). Then \(m_1 \in M(B, E')\) and \(\int f dm = \int f dm_1\) for each \(f\) in \(C_{rc}\). By Theorem 1.1, \(m = m_1\) and hence \(\mu\) is an extension of \(m\). The theorem is proved.

The next theorem gives another characterization of the \(\sigma\)-additive and \(r\)-additive members of \(M(B, E')\). This characterization will be useful later. Let \(m \in M(B, E')\). Define \(\phi\) on \(C(\beta X, E)\) by \(\phi(f) = \int f dm, f \in C_{rc}\). Then \(\phi\) is continuous with respect to the uniform topology on \(C(\beta X, E)\). Since \(M(\beta X) = M_r(\beta X)\), there exists \(\bar{m} \in M_{r,p}(Bo(\beta X), E')\) such that \(\phi(f) = \int f d\bar{m}\) for each \(f\) in \(C_{rc}\).
THEOREM 2.7. (a) \( m \in M_o(B, E') \) iff \( \tilde{m}_p(Z) = 0 \) for each zero set \( Z \) in \( \beta X \) which is disjoint from \( X \).

(b) \( m \) is \( \tau \)-additive iff \( \tilde{m}_p(F) = 0 \) for each closed set \( F \) in \( \beta X \) which is disjoint from \( X \).

PROOF. (a) Assume that \( m \) is \( \sigma \)-additive. Let \( s \in E \). For each \( f \in C^b(X) \) we have \( \int f d(ms) = \int f s \tilde{d}m = \int f \tilde{s} d\tilde{m} = \int f \tilde{d}(ms) \). Since \( ms \) is \( \sigma \)-additive, we have that \( (\tilde{ms})(F) = 0 \) for each Baire set \( F \) in \( \beta X \) which is disjoint from \( X \) (see Knowles [11, Theorem 2.1]). Let \( \mu \) be the restriction of \( m \) to \( Ba(\beta X) \). Then \( \mu \in M_\sigma,Ba(\beta X), E' \) and \( \mu_p = \tilde{m}_p|_{Ba(\beta X)} \). By what we proved, \( \tilde{m}_p(F) = \mu_p(F) = 0 \) for each Baire set \( F \) in \( \beta X \) which is disjoint from \( X \). Conversely, assume that \( \tilde{m}_p(Z) = 0 \) for each zero set \( Z \) in \( \beta X \) disjoint from \( X \). By regularity \( \mu_p(F) = 0 \) for each Baire set \( F \) disjoint from \( X \). Define \( \mu : Ba(X) \rightarrow E' \) by \( \mu(F \cap X) = \mu(F) \) for each Baire set \( F \) in \( \beta X \). This gives us a well-defined element of \( M_\sigma,Ba(\beta X), E' \). Moreover, if \( m_1 = \mu|_{B(\beta X)} \), then \( \int f d m_1 = \int f d\mu = \int \tilde{f} d\tilde{\mu} = \int \tilde{f} d\tilde{m} = \int f d m \) for all \( f \) in \( C_{rc} \). Thus \( m_1 = m \) and hence \( m \) is \( \sigma \)-additive by 2.5.

(b) The proof is similar to that of (a).

THEOREM 2.8. If we consider on \( C_{rc} = M(B, E') \) the weak topology \( \sigma(C_{rc}', C_{rc}) \), then \( M_o(B, E') \) is sequentially closed.

PROOF. Let \( \{m_n\} \) be a sequence of elements of \( M_o(B, E') \), \( m \in M(B, E') \), and assume that \( m_n \rightarrow m \). Let \( s \in E \). For each \( f \) in \( C^b(X) \) we have

\[
\int f d(m_n s) = \int f s d m_n \rightarrow \int f s d m = \int f d(ms).
\]

Thus \( m_n s \rightarrow ms \) in the \( \sigma(M(X), C^b) \) topology. By Aleksandrov [1], \( ms \) is \( \sigma \)-additive. This, being true for all \( s \in E \), implies that \( m \) is \( \sigma \)-additive.

3. A weighted type topology on \( C_{rc} \). Let \( V \) be a family of bounded continuous real-valued functions on \( X \). Assume that \( V \) has the following two properties:

(1) For each \( x \) in \( X \) there exists \( h \in V \) with \( h(x) \neq 0 \).

(2) Given \( u, v \) in \( V \) and a positive number \( d \), there exists \( w \) in \( V \) with \( |w| \geq d u, d v \) (pointwise). We will denote by \( w_V \) the locally convex topology on \( C_{rc} \) generated by the family of seminorms \( \{ || \cdot ||_{p,h} : p \in I, h \in V \} \) where \( || \cdot ||_{p,h} \) is defined on \( C_{rc} \) by

\[
||f||_{p,h} = \sup \{ p(h(x)f(x)) : x \in X \} = ||hf||_p.
\]

It is clear that \( w_V \) has a base at zero consisting of all sets of the form \( \{ f \in C_{rc} : ||hf||_p \leq 1 \} \) where \( h \in V \) and \( p \in I \). It is also clear that \( w_V \) is Hausdorff and
that $w_V \leq \sigma$. Hence $(C_{rc}, w_V)' \subset (C_{rc, \sigma})' = M(B, E')$. We will identify the dual space of $(C_{rc}, w_V)$. We begin with an easily established lemma.

**Remark 3.1.** Let $m \in M(B, E')$ and $h \in C^b$. For each $F$ in $B$ we define $\mu(F)$ on $E$ by $\mu(F) s = \int_F h d(ms)$. Then $\mu \in M(B, E')$ and $\int f d\mu = \int h d m$ for each $f$ in $C_{rc}$.

We denote the element $\mu \in M(B, E')$, defined in 3.1, by $hm$. Let

$$V \cdot M(B, E') = \{hm : h \in V, m \in M(B, E')\}.$$

We will prove the following.

**Theorem 3.2.** The space $(C_{rc}, w_V)'$ is isomorphic to the space $V \cdot M(B, E')$ and the isomorphism $\phi \mapsto m$ is given by the formula, where $\phi(f) = \int f d m$ for all $f \in C_{rc}$.

Set $H = (C_{rc}, w_V)'$.

**Lemma 3.3.** If $h \in V$ and $m \in M_p(B, E')$, then $hm$ gives an element of the dual space of $(C_{rc}, w_V)$.

**Proof.** Set $\mu = hm$. The set $W = \{ f : \| fh \|_p \leq 1 \}$ is a $w_V$-neighborhood of zero. Moreover, if $f \in W$, then

$$\int f d \mu = \int fh d m \leq \| fh \|_p \| m \|_p \leq \| m \|_p.$$

This completes the proof.

**Lemma 3.4.** Let $h \in V$ and define $T_h = T : C_{rc} \to C_{rc}$, $Tf = hf$. Then $T$ is $\sigma(C_{rc}, H) - \sigma(C_{rc}, C'_{rc})$ continuous.

Moreover, if $T'$ is the adjoint of $T$ and if $p \in I$, then $T'(B^o_p) = W^o_p$, where $B_p = \{ f \in C_{rc} : \| f \|_p \leq 1 \}$, $W_p = T^{-1}(B^o_p)$, $B^o_p$ the polar of $B_p$ with respect to the pair $(C_{rc}, C'_{rc})$, and $W^o_p$ the polar of $W_p$ with respect to the pair $(C_{rc}, C'_{rc})$.

**Proof.** Let $\{ f_\alpha \}$ be a net in $C_{rc}$ converging to zero in the $\sigma(C_{rc}, H)$ topology. Let $m \in M(B, E')$. In view of 3.3 we have $\int f_\alpha d(hm) \to 0$. Thus $\int fh_\alpha d m \to 0$ which shows that $Tf_\alpha \to 0$ in the $\sigma(C_{rc}, C'_{rc})$ topology. Thus $T$ is $\sigma(C_{rc}, H) - \sigma(C_{rc}, C'_{rc})$ continuous. Therefore $T'$ exists and $T'(C'_{rc}) \subset H$. Also $T'$ is $\sigma(C_{rc}, C_{rc}) - \sigma(H, C_{rc})$ continuous. The set $B_p$ is clearly $\sigma$-closed. Since $B_p$ is convex and since $\sigma$ and $\sigma(C_{rc}, C'_{rc})$ are both compatible with the pair $(C_{rc}, C'_{rc})$, $B_p$ is $\sigma(C_{rc}, C'_{rc})$ closed. Also $B_p$ is balanced. Thus $B_p = B^o_p$ by the bipolar theorem (see Schaefer [14, p. 126]). Let $W = [T'(B^o_p)]^o$. If $f \in W$ and $m \in B^o_p$, then $\langle m, Tf \rangle = \langle T'm, f \rangle \leq 1$. This shows that $Tf \in B^o_p = B_p$. Hence $W \subset W_p$. On the other hand, if $f \in W_p$ and $m \in B^o_p$, then
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Let $\langle f, T'm \rangle = \langle Tf, m \rangle \leq 1$. Thus $W_p \subset W$ and so $W = W_p$. The set $B_p^\circ$ is $\sigma(C_{rc}, C_{rc})$ compact by the Alaoglu theorem (see Köthe [10, p. 248]). Hence $T'(B_p^\circ)$ is $\sigma(H, C_{rc})$ compact. Also $T'(B_p^\circ)$ is convex and balanced. Therefore, by the bipolar theorem, $T'(B_p^\circ) = [T'(B_p^\circ)]^{\sigma} = W_p^\circ$. The lemma is proved.

**Lemma 3.5.** If $\phi \in (C_{rc}, w_{V'})$, then there exists $h \in V$, $m \in M(B, E')$ such that $\phi(f) = \langle f, hm \rangle$ for all $f \in C_{rc}$.

**Proof.** Since $\phi$ is $w_V$-continuous, there exist $h \in V$ and $p \in I$ such that $W_p = \{f : \|hf\|_p \leq 1\} \subset \{f : |\phi(f)| \leq 1\}$. Let $T = T_h$ be as in Lemma 3.4. In view of 3.4, we have $T'(B_p^\circ) = W_p^\circ$. Since $\phi \in W_p^\circ$ there exists $m \in B_p^\circ$ such that $\phi = T'm$. Now, for each $f \in C_{rc}$, we have $\langle f, \phi \rangle = \langle f, T'm \rangle = \langle Tf, m \rangle = \langle f, d(hm) \rangle$. This completes the proof.

Combining 3.3 and 3.5 we get Theorem 3.2.

4. The strict and superstrict topologies on $C_{rc}$. Buck defined in [4] the strict topology on the space of bounded continuous functions on a locally compact space and he identified the dual in the scalar case. The dual space for the vector case was studied by Wells [18]. Recently Sentilles [15] and Fremlin-Garling-Haydon [5] defined the strict and superstrict topologies on the space of all bounded continuous real-valued functions on a completely regular Hausdorff space. They identified the strict and superstrict dual of $C^0$ with the spaces $M_c(X)$ and $M_o(X)$ respectively. These and other authors completed the result of Hewitt [6] on the representation of linear functionals on spaces of continuous functions. In [3] Bogdanowicz studied the space of continuous linear functionals on the space of continuous mappings from a compact space into a locally convex space. In this section we will introduce on $C_{rc}$ two locally convex topologies $\beta_1$ and $\beta$ which yield as dual spaces the spaces of all $\sigma$-additive and all $\tau$-additive members of $M(B, E')$ respectively. Our approach will be analogous to that of Sentilles.

Let $\Omega$ ($\Omega_1$) denote the collection of all closed (zero) sets in $\beta X$ which are disjoint from $X$. For $Q$ in $\Omega$, let $B_Q = \{h \in C^0 : \widehat{h} = 0$ on $Q\}$. Clearly $B_Q$ has all the properties of the family $V$ mentioned in the beginning of §3.

Let $\beta_Q$ be the locally convex topology on $C_{rc}$ generated by the family of seminorms $f \rightarrow \|hf\|_p$, $h \in B_Q$, $p \in I$. The strict topology $\beta$ on $C_{rc}$ is defined to be the inductive limit of the topologies $\beta_Q$, $Q \in \Omega$. The superstrict topology $\beta_1$ on $C_{rc}$ is the inductive limit of the topologies $\beta_Z$, $Z \in \Omega_1$. If $\pi$ is the point-wise convergence topology, one can easily verify the following

**Theorem 4.1.** $\pi \subseteq \beta \subseteq \beta_1 \subseteq \sigma$. 

Theorem 4.2. \( \beta = \sigma \) iff \( X \) is compact.

Proof. Clearly \( \beta = \sigma \) if \( X \) is compact. On the other hand assume that \( X \) is not compact and that \( \beta = \sigma \).

Let \( x \in \beta X - X \), \( Q = \{x\} \). Let \( p \in I, s \in E \) be such that \( p(s) = 2 \). Set \( W = \{ f \in C_r: \|f\|_p \leq 1 \} \). Then \( W \) is a \( \sigma \)-neighborhood of zero. By hypothesis \( W \) is also a \( \beta \)-neighborhood of zero. Since \( \beta \leq \beta Q \), \( W \) is a \( \beta Q \)-neighborhood of zero. Thus there exist \( h \in B_Q \) and \( p_1 \) in \( I \) such that \( V = \{ f \in C_r: \|hf\|_{p_1} \leq 1 \} \subset W \).

Choose \( \delta > 0 \) such that \( \delta p_1(s) \leq 1 \), and set \( F = \{ y \in \beta X: |\hat{h}(y)| \geq \delta \} \). Let \( g \in C^b, 0 \leq g \leq 1, \hat{g}(x) = 1 \) and \( \hat{g} = 0 \) on \( F \). But then the function \( f = gs \) is in \( V \) but not in \( W \). This contradiction completes the proof.

Since \( X \) is pseudocompact iff \( \Omega = \{\emptyset\} \), we have the following theorem for \( \beta_1 \) whose proof is similar to that of Theorem 4.2.

Theorem 4.3. \( \beta_1 = \sigma \) iff \( X \) is pseudocompact.

If \( X \) is locally compact, then \( X \) is open in \( \beta X \). Let \( Q = \beta X - X \). Then \( B_Q \) is the space of all continuous real functions on \( X \) that vanish at infinity. Hence, as one can easily prove, \( \beta = \beta Q \) coincides with the strict topology as defined by Buck in [4]. We will next identify the dual spaces of \( (C_r, \beta) \) and \( (C_r, \beta_1) \).

Lemma 4.4. If \( \phi \in (C_r, \beta)' \), then there exists \( m \in M_r(B, E') \) such that \( \phi(f) = \int f dm \) for all \( f \in C_r \).

Proof. Since \( \beta \leq \sigma \) there exists \( m \in M(B, E') \) such that \( \phi(f) = \int f dm \) for all \( f \in C_r \). Let \( \tilde{m} \in M_{r,p}(Bo(\beta X), E') \) be such that \( \phi(f) = \int f d\tilde{m} \) for all \( f \in C_r \). Let \( \tilde{Q} \in \Omega \). Since \( \phi \) is \( \beta_Q \)-continuous, there exists \( (3.5) \ h \in B_Q \) and \( \mu \in M(Bo(\beta X), E') \) such that \( \int f d\mu = \phi(f) \) for all \( f \in C_r \). Let \( \tilde{\mu} \in M_r(Bo(\beta X), E') \) be such that \( \int f d\tilde{\mu} = \int \hat{f} d\tilde{\mu} \) for all \( f \in C_r \). Then \( \int \hat{f} d\tilde{m} = \phi(f) = \int hf d\mu = \int \hat{hf} d\tilde{\mu} \) for each \( f \in C_r \). It follows that \( \tilde{m} = \hat{h}\tilde{\mu} \). If \( F \) is a Borel set in \( \beta X \) contained in \( Q \) and if \( s \in E \), then \( \tilde{m}(F)s = \int_F \hat{h} d(\tilde{\mu}s) = 0 \). We conclude that \( \tilde{m}_p(Q) = 0 \). This, being true for all \( Q \in \Omega \), implies that \( m \) is \( r \)-additive by 2.7. This completes the proof.

Lemma 4.5. If \( m \in M_{r,p}(B, E') \), then the map \( \phi_m: C_r \rightarrow R, \phi_m(f) = \int f dm \), is \( \beta \)-continuous.

Proof. It suffices to show that \( \phi_m \) is \( \beta_Q \)-continuous for every \( Q \) in \( \Omega \). So, let \( Q \in \Omega \). Define \( T: C^b \rightarrow R \) by \( T(f) = \int f dm_p \). Since \( m_p \) is \( r \)-additive, \( T \) is \( \beta(C^b) \) continuous, where \( \beta(C^b) \) is the strict topology on \( C^b \) as defined by Sentilles in [15] (see Sentilles, Theorem 4.3). Hence there exists \( g \) in \( B_Q \) such that

\[
W = \{ f \in C^b: \|gf\| \leq 1 \} \subset \{ f \in C^b: |T(f)| \leq 1 \}.
\]
Set \( V = \{ f \in C_{rc} : \|g\|_p \leq 1 \} \) and let \( f \in V \).

Define \( h : X \rightarrow R, h(x) = p(f(x)) \). Clearly \( h \in C^b \). Moreover \(|h(x)g(x)| = p(g(x)f(x)) \leq 1 \) for all \( x \in X \) and hence \( h \in W \). It follows that \( \int f dm \leq \int h dm \rho = T(h) \leq 1 \). This shows that \( \phi_m \) is \( \beta \) continuous. The lemma is proved.

Combining Lemmas 4.4 and 4.5 we get

**Theorem 4.6.** The space \( M_\beta(B, E') \) is isomorphic to the space \( (C_{rc}, \beta)' \) via the isomorphism \( m \mapsto \phi_m \) where \( \phi_m(f) = \int f dm \) for all \( f \in C_{rc} \).

Using similar arguments we prove

**Theorem 4.7.** The space \( M_\alpha(B, E') \) is isomorphic to the space \( (C_{rc}, \beta_1) \) via the isomorphism \( m \mapsto \phi_m, \phi_m(f) = \int f dm \).

**5. The case of a locally convex lattice \( E \).** In this section \( E \) will be assumed to be a locally convex lattice. By Peressini [13, p. 105] there exists a generating family of continuous seminorms \( p \) such that \( |x| \leq |y| \) implies \( p(x) \leq p(y) \). In view of this, we may assume that every \( p \in I \) has the above property. The space \( (C_{rc}, \sigma) \) is, under the pointwise ordering, a locally convex lattice. The question we are going to investigate now is the following: Which elements of \( C_{rc}^e \) correspond to members of \( M_\alpha(B, E') \) and which to members of \( M_\beta(B, E') \)? We will show that these are exactly the \( \sigma \)-additive and \( \tau \)-additive members of \( C_{rc}^e \).

**Definition.** For a net \( \{ f_\alpha \} \) in \( C_{rc} \) we say that \( \{ f_\alpha \} \) decreases to zero, and write \( f_\alpha \downarrow 0 \), if for each \( x \in X \) we have \( \lim f_\alpha(x) = 0 \) and \( 0 \leq f_\alpha(\alpha) \leq f_\beta(\alpha) \) if \( \alpha \geq \beta \). We define similarly what we mean by saying that a sequence \( \{ f_n \} \) in \( C_{rc} \) decreases to zero. An element \( \phi \) of \( C_{rc}^e \) is called \( \sigma \)-additive if \( \lim \phi(f_\alpha) = 0 \) for each sequence \( \{ f_\alpha \} \) in \( C_{rc} \) that decreases to zero. An element \( \phi \) of \( C_{rc}^e \) is called \( \tau \)-additive if \( \lim \phi(f_\alpha) = 0 \) whenever \( f_\alpha \downarrow 0 \). We will denote by \( L_\sigma(C_{rc}) \) and \( L_\tau(C_{rc}) \) the spaces of all \( \sigma \)-additive and all \( \tau \)-additive members of \( C_{rc}^e \) respectively.

**Theorem 5.1.** Let \( \phi \in C_{rc} \). Then \( \phi \) is \( \tau \)-additive iff there exists \( m \in M_\tau(B, E') \) such that \( \phi(f) = \int f dm \) for all \( f \in C_{rc} \).

**Proof.** Let \( \phi \in M_\tau(B, E') \) such that \( \phi(f) = \int f dm \) for all \( f \in C_{rc} \). Let \( \{ f_\alpha \} \) be a net in \( C_{rc} \) that decreases to zero. For each \( \alpha \), let \( h_{\alpha} : X \rightarrow R, h_{\alpha}(x) = p(f(\alpha)) \). Since \( p \) has the property that \( p(s) \leq p(t) \) whenever \( |s| \leq |t| \), it follows that \( h_{\alpha} \downarrow 0 \). Hence \( \int f_\alpha dm \leq \int h_{\alpha} dm \rho \rightarrow 0 \) since \( m_\rho \) is \( \tau \)-additive (see Varadarajan [17, p. 174]).

Conversely, assume that \( \phi \) is \( \tau \)-additive. Let \( s \geq 0, s \in E \). If \( \{ f_\alpha \} \) is a net in \( C^b(X) \) which decreases to zero, then \( f_\alpha s \downarrow 0 \). Hence \( \int f_\alpha d(ms) = \int f_\alpha s dm = \phi(f_\alpha s) \rightarrow 0 \). It follows that \( ms \) is \( \tau \)-additive (see Varadarajan [17, p. 174]).
Since every element of $E$ is the difference of two positive elements, it follows that $ms \in M_s(X)$ for all $s \in E$ and hence $m$ is $\tau$-additive.

Using an analogous argument we prove the following:

**Theorem 5.2.** Let $\phi \in C'_rc$. Then $\phi$ is $\sigma$-additive iff there exists $m \in M_o(B, E')$ such that $\phi(f) = \int f \, dm$ for all $f$ in $C_{rc}$.

**Lemma 5.3.** Let $m \in M_p(B, E')$ and $|m| = \sup(m, -m)$. Then $|m| \in M_p(B, E')$ and $|m|^p = m^p$.

**Proof.** Recall that $p$ has the property that $p(s) \leq p(t)$ whenever $|s| \leq |t|$. As shown in [8], for each $s \geq 0$ in $E$ and each $F$ in $B(X)$ we have $|m|(F)s = \sup \Sigma|m(F_i)|s$ where the supremum is taken over all finite $B$-partitions $\{F_i\}$ of $F$. Let now $F \in B(X)$. If $F_1, \ldots, F_n$ is a $B$-partition of $F$, and if $s_i \in E$ with $p(s_i) \leq 1$,

$$\left| \sum m(F_i)s_i \right| \leq \sum |m(F_i)||s_i| \leq \sum m(F_i)|s_i| \leq |m|^p(F)$$

since $p(|s_i|) = p(s_i) \leq 1$. Thus $m^p(F) \leq |m|^p(F)$. On the other hand, let $G_1, \ldots, G_n$ be a $B$-partition of $F$ and let $s_i \in E$ with $p(s_i) \leq 1$. We will show that $|\Sigma|m(G_i)s_i| \leq m^p(F)$. Since $p(|s_i|) \leq 1$ and since $|\Sigma|m(G_i)s_i| \leq |\Sigma|m(G_i)|s_i|$, we may assume that $s_i \geq 0$. Let $\epsilon > 0$ be given. For each $i$, $1 \leq i \leq n$, there exists a $B$-partition $F^i_1, \ldots, F^i_{K_i}$ of $G_i$ such that

$$\sum_{j=1}^{K_i} |m(F^i_j)||s_i| > |m|(G_i)s_i - \frac{\epsilon}{n}.$$

Let $N = K_1 + \cdots + K_n$. Choose $t_{ij} \in E$, $|t_{ij}| \leq s_i$, such that $|m(F^i_j)t_{ij}| > |m(F^i_j)|s_i - \epsilon/N$. Since $p(t_{ij}) \leq 1$ and $\{F^i_j\}$ is a $B$-partition of $F$, we have

$$m^p(F) \geq \sum_{i,j} |m(F^i_j)t_{ij}| = \sum_{i=1}^{n} \sum_{j=1}^{K_i} |m(F^i_j)t_{ij}|$$

$$\geq \sum_{i=1}^{n} \sum_{j=1}^{K_i} |m(F^i_j)||s_i| - \epsilon \geq \sum_{i=1}^{n} |m|(G_i)s_i - 2\epsilon.$$

Since $\epsilon > 0$ was arbitrary, we have $\Sigma|m|(G_i)s_i \leq m^p(F)$. This proves that $|m|^p(F) \leq m^p(F)$ and the lemma is proved.

By the above lemma, if $m \in M_s(B, E')$, then $|m|$ is also $\tau$-additive. From this follows that $M_\tau(B, E')$ is an ideal in $M(B, E')$. Since the map $m \rightarrow \phi_m$, of Theorem 1.1, is lattice-preserving and since $M_\tau(B, E')$ corresponds to $L_\tau(C_{rc})$ in this map, it follows that $L_\tau(C_{rc})$ is an ideal in the Riesz space $C'_{rc}$. The same is
true for the space $L_\sigma(C_{rc})$. We have thus the following theorem.

**Theorem 5.4.** Each of the spaces $L_\sigma(C_{rc})$ and $L_\tau(C_{rc})$ is an ideal in the Riesz space $C'_{rc}$.

It is well known (Knowles [11, p. 149]) that any positive linear functional $\phi$ on $C^b$ can be written uniquely as a sum of a positive purely finitely-additive functional on $C^b$, a positive purely $\sigma$-additive and a positive $\tau$-additive functional on $C^b$. It is therefore natural to ask whether this is true in our space $(C_{rc}, \sigma)$. We will show that the answer to this question is affirmative.

**Definition.** An element $\phi \geq 0$ in $C'_{rc}$ is called purely finitely-additive if the only $\sigma$-additive functional $\phi_1$ in $C_{rc}$ with $0 \leq \phi_1 \leq \phi$ is the zero functional. Similarly, an element $\phi \geq 0$ of $L_\sigma(C_{rc})$ is purely $\sigma$-additive if $0 \leq \phi_1 \leq \phi$ and $\phi_1 \in L_\tau(C_{rc})$ implies that $\phi_1 = 0$.

We are going to prove the following

**Theorem 5.5.** Given $\phi \geq 0$ in $C'_{rc}$ there are $\phi_1, \phi_2, \phi_3$ in $C'_{rc}$, $\phi_1$ purely finitely-additive, $\phi_2$ purely $\sigma$-additive, $\phi_3$ $\tau$-additive, $\phi_1, \phi_2, \phi_3 \geq 0$, such that $\phi = \phi_1 + \phi_2 + \phi_3$. Moreover this decomposition is unique.

To begin with, assume that $\phi, \phi_1, \phi_2 \geq 0$ in $C'_{rc}$, $\phi_1$ $\sigma$-additive, $\phi = \phi_1 + \phi_2$. Let $m, m_1, m_2 \in M_{r,p}(Bo(\beta X), E')$ be such that $\phi(f) = \int f \, dm, \phi_i(f) = \int f \, dm_i, i = 1, 2$, for all $f \in C_{rc}$.

Set $d = \inf \{m_p(V): V \supset X, V$ a cozero set in $\beta X\}$. Choose a decreasing sequence $\{U_i\}$ of cozero sets in $\beta X$, $X \subset U_i$, such that $m_p(U_i) \to d$. Let $K = \bigcap U_i$. Clearly $m_p(K) = d$. Since $\phi_1$ is $\sigma$-additive we have $(m_1)_p(\beta X - K) = 0$ by 2.7. Hence $m_1(F) = m_1(F \cap K) \leq m(F \cap K)$ for each Borel set $F$ in $\beta X$ since $m_2 \geq 0$ and $m = m_1 + m_2$. Define $m_3$ on $Bo(\beta X)$ by $m_3(F) = m(F \cap K)$. Then $m_3 \in M_{r,p}(Bo(\beta X), E')$ and $(m_3)_p(F) = m_p(F \cap K)$ for each $F$ in $Bo(\beta X)$. Since $E$ is locally solid, the positive cone is closed (see Schaefer [14, p. 235]). Now let $f \geq 0$ in $C_{rc}$. If $y \in \beta X$ and if $\{x_\alpha\}$ is a net in $X$ converging to $y$ in $\beta X$, then $\hat{f}(y) = \lim \hat{f}(x_\alpha) = \lim f(x_\alpha) \geq 0$. Thus $\hat{f} \geq 0$. It follows that the map $\phi_3: C_{rc} \to R, \phi_3(f) = \int \hat{f} \, dm_3$ is positive. Moreover $\phi_3 \geq \phi_1$ since $m_3 \geq m_1$.

We next show that $\phi_3$ is $\sigma$-additive. Indeed, let $Z$ be a zero set in $\beta X$ disjoint from $X$. Then $U = \beta X - Z$ is a cozero set containing $X$. Since $X \subset U \cap U_i \downarrow K \cap U$, we have

$$d \leq \lim m_p(U \cap U_i) = m_p(K \cap U) \leq m_p(K) = d.$$  

So $(m_3)_p(\beta X) = m_p(K) = m_p(K \cap U) = (m_3)_p(U)$ and hence $(m_3)_p(Z) = 0$. By 5.2 and 2.7 $\phi_3$ is $\sigma$-additive. Clearly $\phi - \phi_3$ is purely finitely-additive. We have thus proved
Theorem 5.6. If $\phi \geq 0$ in $C'_r(C_r)$, then there are unique $\phi_1, \phi_2 \geq 0$ in $C'_r(C_r)$, $\phi_1$ purely finitely-additive and $\phi_2$ $\sigma$-additive, such that $\phi = \phi_1 + \phi_2$.

Next assume that $0 \leq \phi \in L_\sigma(C_r)$. Suppose that $\phi = \phi_1 + \phi_2$, $\phi_1 \in L_r(C_r)$, $\phi_2 \in L_\sigma(C_r)$, $\phi_1, \phi_2 \geq 0$. Let $m, m_1, m_2 \in M_{r,p}(Bo(\beta X), E')$ be such that $m(f) = \int f dm$, $\phi_i(f) = \int f dm_i$, $i = 1, 2$, for all $f \in C_r$. Let $d = \inf \{m_p(O) : O \text{ open in } \beta X, X \subset O \}$. Choose a decreasing sequence $\{O_n\}$ of open sets in $\beta X$, $X \subset O_n$, $m_p(O_n) \rightarrow d$. Since $\phi_1$ is $\tau$-additive we have that $(m_1)_p(\beta X - K) = 0$.

Thus $m_1(F) = m_1(F \cap K) \leq m(F \cap K)$ for all Borel sets $F \subset \beta X$. Define $m_3$ on $Bo(\beta X)$ by $m_3(F) = m(F \cap K)$. Then $m_3 \in M_{r,p}(Bo(\beta X), E')$ and $m_3 \geq m_1$. Let $Q$ be a closed set in $\beta X$ disjoint from $X$. Then $O = \beta X - Q \supset X$ and $O$ is open. Hence $d \leq \lim m_p(O \cap O_i) = m_p(O \cap K) \leq m_p(K) = d$. It follows that $(m_3)_p(\beta X) = (m_3)_p(O)$ and hence $(m_3)_p(O) = 0$. This, being true for all closed sets in $\beta X$ which are disjoint from $X$, implies that the map $\phi_3 : C_r \rightarrow R$, $\phi_3(f) = \int f dm_3$, is $\tau$-additive. Also $\phi_3 \geq 0$. Moreover, $\phi - \phi_3$ is purely $\sigma$-additive. We have thus proved

Theorem 5.7. Given $\phi \geq 0$ in $L_\sigma(C_r)$, there are unique $\phi_1, \phi_2 \geq 0$, $\phi_1 \in L_r(C_r)$, $\phi_2$ purely $\sigma$-additive, such that $\phi = \phi_1 + \phi_2$.

Combining Theorems 5.6 and 5.7 we get Theorem 5.5.

BIBLIOGRAPHY


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