ABSTRACT. Let \( X = (x_1, \ldots, x_s) \) be a vector of \( s \) real components and
\[
f_k(X) = \sum_{i=1}^{s} \theta_i x_i^k \quad (k = 2, 3, \ldots; i = 1, \ldots, R)
\]
be \( R \) additive forms, where \( \theta_i \) are arbitrary real numbers. The author obtains some results on the simultaneous approximation of \( \|f_k(X)\| \), where \( \|t\| \) means the distance from \( t \) to the nearest integer.

1. Introduction. In 1948 Heilbronn [5] improved Vinogradov's result [9] and obtained that for any \( \epsilon > 0 \) there exists some positive constant \( C = C(\epsilon) \) which depends on \( \epsilon \) only such that for any real number \( \theta \) and any integer \( N \geq 1 \) there is an integer \( x \) satisfying
\[
1 \leq x \leq N \quad \text{and} \quad \|\theta x^2\| < CN^{-1/2} + \epsilon,
\]
where \( \|t\| \) means the distance from \( t \) to the nearest integer.

In 1967, see [4], Davenport generalized (1.1) and obtained
\[
1 \leq x \leq N \quad \text{and} \quad \|\theta x^k\| < CN^{-1/k} + \epsilon,
\]
where \( K = 2^{k-1} \), \( k = 2, 3, \ldots \) and \( C = C(\epsilon, k) \) is a positive constant depending on \( \epsilon, k \) only. Recently, R. J. Cook [2] extended (1.2) to a finite number of \( \theta \)'s. On the other hand, Cook [3] and the author [8] obtained some results on additive forms. These results are as follows:

**Theorem C** [2]. For any \( \epsilon > 0 \) and any integers \( k > 1, R > 0 \), there exists a positive constant \( C = C(\epsilon, k, R) \) depending on \( \epsilon, k, R \) only such that for any real numbers \( \theta_1, \ldots, \theta_R \) and any integer \( N \geq 1 \) there is an integer \( x \) satisfying
\[
1 \leq x \leq N \quad \text{and} \quad \max_{1 \leq i \leq R} (\|\theta_i x^k\|) < CN^{-1/f(k, R)} + \epsilon,
\]
where \( f(k, R) \) is defined by \( K = 2^{k-1} \) and
\[
f(k, 1) = K, \quad f(k, R) = 2f(k, R - 1) + KR + 1 \quad (R \geq 2).
\]

**Theorem L** [8]. For any integers \( k \geq 2 \) and \( s \geq 1 \), put \( K = 2^{k-1} \) and \( X = (x_1, \ldots, x_s) \), a vector of \( s \) real components. Let \( f(X), g(X) \) be any two additive forms of degree \( k \) in \( s \) variables,
where $\theta_1, \cdots, \theta_s; \phi_1, \cdots, \phi_s$ are arbitrary real numbers. Then for any $\varepsilon > 0$ and any integer $N \geq 1$ there are integers $x_1, \cdots, x_s$ not all zero and some positive constant $C = C(\varepsilon, k, s)$ which depends on $\varepsilon, k, s$ only such that

$$1 \leq \max_{1 \leq i \leq s} |x_i| \leq N,$$

(1.5)

$$\max(\|f(X)\|, \|g(X)\|) < CN^{-1/F(k,s)} + \varepsilon,$$

where

$$F(k, s) =
\begin{cases}
7 & \text{if } s = 1, k = 2, \\
3K + 1/k & \text{if } s = 1, k \geq 3, \\
2K + 1 + K/ks & \text{if } s \geq 2.
\end{cases}$$

(1.6)

In this paper we shall extend Theorem L to a finite number of additive forms with some improvements of Theorem C. We shall prove

**Theorem 1.** For any integers $k \geq 2$ and $s \geq 1$, put $K = 2^{k-1}$ and $X = (x_1, \cdots, x_s)$, a vector of $s$ real components. For any integer $R \geq 3$ let

$$f_i(X) = \sum_{j=1}^s \theta_i \cdot x_j^k \quad (i = 1, \cdots, R)$$

be any $R$ additive forms of degree $k$ in $s$ variables, where $\theta_{ij}$ ($i = 1, \cdots, R; j = 1, \cdots, s$) are arbitrary real numbers. Then for any $\varepsilon > 0$ and any integer $N \geq 1$ there are integers $x_1, \cdots, x_s$ not all zero and some positive constant $C = C(\varepsilon, k, s, R)$ which depends on $\varepsilon, k, s, R$ only such that

$$1 \leq \max_{1 \leq i \leq s} |x_i| \leq N,$$

(1.7)

$$\max_{1 \leq i \leq R} (\|f_i(X)\|) < CN^{-(1/G(k,s,R)) \varepsilon},$$

where $G(k, s, R)$ is defined by

$$G(k, s, R) =
\begin{cases}
7 & \text{if } k = 2, \\
3K + 1/k & \text{if } k \geq 3.
\end{cases}$$

(1.8)

and $g(k, R)$ is defined by

$$g(k, R) =
\begin{cases}
7 & \text{if } k = 2, \\
3K + 1/k & \text{if } k \geq 3.
\end{cases}$$

(1.9)

**Corollary 1.** For any $\varepsilon > 0$ and any integers $k > 1, R > 1$ there exists a positive constant $C = C(\varepsilon, k, R)$ depending on $\varepsilon, k, R$ only such that for any
real numbers \( \theta_1, \cdots, \theta_R \) and any integer \( N \geq 1 \) there is an integer \( x \) satisfying
\[
1 \leq x \leq N \quad \text{and} \quad \max_{1 \leq i \leq R} (\| \theta_i x^k \|) < C N^{-\frac{1}{g(k, R)}} + \varepsilon,
\]
where \( g(k, R) \) is defined by (1.9).

Corollary 1 follows from Theorem 1 \((R = 2)\) and Theorem 1 \((R \geq 3)\).

We shall give the explicit form of \( g(k, R) \) in Lemma 1 \(((2.1), (2.2))\) for the sake of application. For example when \( k \geq 3 \) we have:

<table>
<thead>
<tr>
<th>( R )</th>
<th>( g(k, R) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( 3K + 1/k )</td>
</tr>
<tr>
<td>3</td>
<td>( (6 + 2/k)K + 1 + 2/k )</td>
</tr>
<tr>
<td>4</td>
<td>( (12 + 7/k)K + 3 + 4/k )</td>
</tr>
<tr>
<td>5</td>
<td>( (24 + 18/k)K + 7 + 8/k )</td>
</tr>
</tbody>
</table>

I wish to thank the referee for his very helpful suggestions which brought improvements on the presentation of this paper.

2. Notation and preliminary lemmas. In what follows \( k, s, R \) are the integers given in Theorem 1. We always assume that \( \varepsilon \) is a small enough positive quantity which is not the same \( \varepsilon \) given in Theorem 1 and that \( N \) is a sufficiently large positive integer, say \( N > N_0 = N_0(\varepsilon, k, s, R) \) which is a positive integer depending on \( \varepsilon, k, s, R \) given, such that all the subsequent asymptotic approximations and inequalities in §§3, 4 are satisfied. So it is difficult to define \( \varepsilon \) and \( N_0 \) at the beginning or at any particular point. If \( y > 0 \) we use \( x < y \) to denote \( |x| < Cy \), where \( C \) is some positive constant which can depend on \( \varepsilon, k, s, R \). \([t]\) is the integral part of \( t \). For real \( t \) we write \( e(t) = \exp(2\pi ti) \). We shall use \( B \) to denote some irrelevant numbers which need not be the same from one occurrence to another. For figures \( x, y, (x)_y \) indicates the \( y \)th formula in \( x \).

We need several lemmas.

**Lemma 1.** Let \( k, R \) be positive integers and \( K = 2^{k-1} \). If
\[
g(k, R) = 2g(k, R - 1) + K(R - 1)/k + 1 \quad (R \geq 3),
\]
where
\[
g(k, 2) = \begin{cases} 
7 & \text{if } k = 2, \\
3K + 1/k & \text{if } k \geq 3.
\end{cases}
\]
Lemma 2. Suppose that $\Delta$ satisfies $0 < \Delta < \frac{1}{2}$ and $r$ is a positive integer. Then there exists a real valued function $\psi(x)$, periodic with period 1, which satisfies
\[
\psi(x) = 0 \quad \text{if } \|x\| > \Delta,
\]
(2.3)
\[
\psi(x) = \sum_{u=-\infty}^{\infty} c_u e(ux),
\]
where $c_u$ are real and
\[
c_0 = \Delta, \quad |c_u| < C(r) \min(\Delta, \Delta^{-r} \|u\|^{-r-1}) \quad \text{if } u \neq 0,
\]
(2.4)
where $C(r)$ is some positive constant depending on $r$ only.

Proof. This is a particular case of Lemma 12 in [10, p. 32] with $\beta = -\alpha = \frac{1}{2} \Delta$.

Lemma 3. Let $T = \sum_{x=1}^{N} e(tx^k)$, where $k = 2, 3, \cdots$ and $t$ is any real number. For any $\epsilon > 0$ we have
\[
|T|^K \ll N^{K-1} + N^{K-k+\epsilon} \sum_{j=1}^{L} \min(N, 1/\|j\|),
\]
where $K = 2^{k-1}$ and $L = k!N^{k-1}$.

Proof. See Satz 266 in [7, p. 255].

Lemma 4. For any real $t$ if there are integers $a, q$ with $q > 0$ such that $(a, q) = 1$ and $|t - a/q| < q^{-2}$, then for any positive integers $P$ and $N$,
\[
\sum_{j=P+1}^{P+q} \min(N, 1/\|j\|) \ll N + q \log q.
\]
Proof. Lemma 4 is well known. See, for example, Lemma 3.5 in [6].
SIMULTANEOUS APPROXIMATION

(2.7) \[ \lambda (2g(k, R - 1) + (R - 1)K/sk + 1) < 1 - \epsilon \]

and

(2.8) \[ \alpha = \frac{2}{K} g(k, R - 1) + \frac{(R - 1)(1 - k)}{sk} + \frac{1}{K}, \]

where \( g(k, R) \) is defined by (1.9). Then the following five inequalities hold simultaneously.

(2.9) \[ (\forall s)RK < 1 - \epsilon, \]

(2.10) \[ (\forall s)(sK\alpha + (R - 1)K) < 1 - \epsilon, \]

(2.11) \[ (\forall(k - 1)s)(K(1 + R^2) + sR - s\alpha K) < 1 - \epsilon, \]

(2.12) \[ (\forall(k - 1))(2kg(k, R - 1) - \alpha K + k) < 1 - \epsilon, \]

(2.13) \[ \lambda (2g(k, R - 1) + 1) < 1 - \epsilon. \]

Proof. By (1.9) it is easy to see that \( g(k, R - 1) > KR \) \( (R = 3, 4, \cdots) \). It follows that (2.7) \( \Rightarrow \) (2.13) \( \Rightarrow \) (2.9). On the other hand, substituting (2.8) into (2.10) and (2.12) we see that (2.7) = (2.10) = (2.12). So it remains to show that when \( R > 3 \)

(2.14) \[ 2kg(k, R - 1) + k > K(1 + R^2) + R, \]

i.e. (2.12) \( \Rightarrow \) (2.11). Here we only give the arguments for \( k \geq 3 \). By (2.2) we have

\[ 2kg(k, R - 1) + k = 3K(2R^{-2})(k + 1) + k(2R^{-2} - 1) + 2R^{-2} - 2K. \]

Then (2.14) follows since when \( k \geq 3, R \geq 3 \) we have

\[ 3(2R^{-2})(k + 1) > (1 + R)^2 \quad \text{and} \quad k(2R^{-2} - 1) + 2R^{-2} > R. \]

The proof of Lemma 5 is complete.

Lemma 6. Let \( R, N \) be any positive integers and \( \theta_1, \cdots, \theta_R \) any real numbers. Then there exists an integer \( n \) satisfying

(2.15) \[ 1 \leq n \leq N, \quad \| \theta_i n \| < N^{-1/R} \quad (i = 1, \cdots, R). \]

Proof. See Theorem VI [1, p. 13].

3. Existence of an R-tuple. We come now to the proof of our theorem.

Suppose that for some \( \lambda > 0 \) there are no integral solutions \( X = (x_1, \cdots, x_s) \) of the following inequalities:

(3.1) \[ 1 \leq \max_{1 \leq j \leq s} |x_j| \leq N, \quad \max_{1 \leq i \leq R} (\|f_i(X)\|) \leq N^{-\lambda}, \]
i.e. for each X with integers \(x_1, \cdots, x_s\) satisfying \(1 \leq \max_{1 \leq j \leq s} |x_j| \leq N\), we have some \(i\) \((1 \leq i \leq R)\) such that \(\|f_i(X)\| > N^{-\lambda}\). Putting \(\Delta = N^{-\lambda}\) in Lemma 2, we have

\[
0 = \sum_X \psi(f_1(X)) \cdot \cdots \psi(f_R(X))
\]

(3.2) \[
= \sum_X \left\{ \sum_{m_1 = -\infty}^{\infty} c_{m_1} e(m_1 f_1(X)) \cdots \sum_{m_R = -\infty}^{\infty} c_{m_R} e(m_R f_R(X)) \right\}
\]

\[
= N^s c_0^R + \sum_X \sum_{m} c_{m_1} \cdots c_{m_R} e\left(m_1 \sum_{j=1}^{s} \theta_{ij} x_j^k\right) \cdots e\left(m_R \sum_{j=1}^{s} \theta_{Rj} x_j^k\right),
\]

where \(\Sigma_X\) is taken over integers \(x_j\) with \(1 \leq j \leq s, 1 \leq x_j \leq N\), and \(\Sigma_m\) is taken over \(-\infty < m_1, \cdots, m_R < \infty\) except \(m = (m_1, \cdots, m_R) = (0, \cdots, 0)\). By (2.4) with \(c_0 = N^{-\lambda}\) and (3.2) we have

(3.3) \[
N^{s-R \lambda} \leq \sum_m |c_{m_1} \cdots c_{m_R}| \left|\prod_{j=1}^{s} S(m, j)\right|,
\]

where

(3.4) \[
S(m, j) = \sum_{x_1=1}^{N} e\left(x^k \sum_{i=1}^{R} \theta_{ij} m_i\right) \quad (j = 1, \cdots, s).
\]

Write

(3.5) \[
\sum_m |c_{m_1} \cdots c_{m_R}| \left|\prod_{j=1}^{s} S(m, j)\right| = \left(\sum_1 + \sum_2\right) |c_{m_1} \cdots c_{m_R}| \left|\prod_{j=1}^{s} S(m, j)\right|,
\]

where \(\Sigma_1\) is the summation taken over \(|m_i| < N^{\lambda+\epsilon}\) \((i = 1, \cdots, R)\) and \(m \neq (0, \cdots, 0)\) while \(\Sigma_2\) is taken over all remaining terms in \(\Sigma_m\). We are going to show that \(\Sigma_2 \ll N^{s-R \lambda-\epsilon}\) if we let \(r\) given in Lemma 2 satisfy \(r > R(\lambda + \epsilon)/\epsilon\). So in view of (3.3), (3.5) we may neglect \(\Sigma_2\).

For each \(l = 1, \cdots, R\) by (2.4) we have

(3.6) \[
\sum_{|m_l| < \infty} |c_{m_l}| = \sum_{|m_l| < N^{\lambda+\epsilon}} |c_{m_l}| + \sum_{|m_l| > N^{\lambda+\epsilon}} |c_{m_l}| \ll N^{-\lambda} \sum_{|m_l| < N^{\lambda+\epsilon}} 1 + N^{r \lambda} \sum_{|m_l| > N^{\lambda+\epsilon}} |m_l|^{r-1} \ll N^\epsilon + N^{r \lambda} N^{-r(\lambda + \epsilon)} \ll N^\epsilon.
\]

Since by (3.4), \(|S(m, j)| \leq N\) we see that
\[
\sum_2 \leq \sum_{i=1}^{R} \left\{ \sum_{m_i \geq N^{\lambda + \varepsilon}} |c_{m_1} \cdots c_{m_R}| \prod_{j=1}^{s} |S(m, j)| \right\} \\
\leq N^s \sum_{i=1}^{R} \left\{ \sum_{m_i \geq N^{\lambda + \varepsilon}} |c_{m_i}| \right\} \left\{ \prod_{i \neq i} \sum_{m_i < \infty} |c_{m_i}| \right\}.
\]

By (3.6) we have
\[
\sum_2 \leq N^s \sum_{i=1}^{R} \left\{ \sum_{m_i \geq N^{\lambda + \varepsilon}} |c_{m_i}| \right\} N^{(R-1)e} \\
\leq N^{s+(R-1)e} \sum_{i=1}^{R} N^{-re} \leq N^{s+(R-1)e-re} \leq N^{s-R\lambda - \varepsilon},
\]
if \(r > R(\lambda + \varepsilon)/\varepsilon\).

It follows from (3.3), (3.5) (2.4) that
\[
N^{s-R\lambda} \leq \sum_1 |c_{m_1} \cdots c_{m_R}| \prod_{j=1}^{s} |S(m, j)| \\
\leq \sum_1 N^{-R\lambda} \prod_{j=1}^{s} |S(m, j)|.
\]

That is
\[
(3.7) \quad N^s \leq \prod_{j=1}^{s} |S(m, j)| \leq \sum_1 \sum_1 |S(m, j)|^s.
\]

We see that there exists some \(j_0\) \((1 \leq j_0 \leq s)\) for which we write \(\theta_i = \theta_{j_0}\) and \(S(m) = S(m, j_0)\), such that
\[
(3.8) \quad N^s \leq \sum_1 |S(m)|^s,
\]
where, by (3.4),
\[
(3.9) \quad S(m) = \sum_{x=1}^{N} e \left( x^k \sum_{i=1}^{R} m_i \theta_i \right).
\]

By definition of the notation \(\leq\) there is some positive constant \(B\) such that we may rewrite (3.8) as
\[
(3.8a) \quad BN^s \leq \sum_1 |S(m)|^s.
\]

We are now going to show that by (3.8a) there exists a \(\rho\)
\[
(3.10) \quad 0 \leq \rho \leq \lambda + \varepsilon
\]
(3.11) \[ |S(m)| \geq \left(2^{-R-1}B\right)^{1/s}N^{1-Rp} \quad (\gg N^{1-Rp}), \]

for at least \([N^{R(p-e)}] + 1 \) R-tuples \((m_1, \cdots, m_R)\) in

(3.12) \[ |m_i| < N^{\lambda+e} \quad (i = 1, \cdots, R). \]

Suppose that such a \(p\) does not exist. For some integer \(l\) with \(le > 2R(\lambda + e)\) write

\[ \sum_1 \lambda |S(m)|^s = \sum_3 \lambda |S(m)|^s + \sum_{j=0}^{l-1} T_j, \]

where \(\sum_3\) is taken over \(|m_i| < N^{\lambda+e} \quad (i = 1, \cdots, R), \ m \neq (0, \cdots, 0)\) and \(|S(m)|^s < 2^{-R-1}BN^{s-R(\lambda+e)}|S(m)|^s\) which are taken over \(|m_i| < N^{\lambda+e} \quad (i = 1, \cdots, R), \ m \neq (0, \cdots, 0)\) and

\[ 2^{-R-1}BN^{(s-(i+1)R(\lambda+e)/l)} \leq |S(m)|^s < 2^{-R-1}BN^{(s-iR(\lambda+e)/l)}. \]

According to our assumption on \(p\) there are no such terms \(|S(m)|^s\) satisfying \(2^{-R-1}BN^s \leq |S(m)|^s\) even if \(2^{-R-1}B < 1\). So we neglect this possibility in the above summation \(\sum_{j=0}^{l-1} T_j\). Now by our supposition on \(p\) and \(e/2 > R(\lambda + e)/l\) we have

\[ T_j \leq (2^{-R-1}BN^{(s-jR(\lambda+e)/l)}(N((j+1)R(\lambda+e)/l-e)) \]

\[ = 2^{-R-1}BN^{(s-e+R(\lambda+e)/l)} < 2^{-R-1}BN^{s-e/2} \quad (j = 0, 1, \cdots, l-1). \]

\[ \sum_3 |S(m)|^s < (2^{-R-1}BN^{s-R(\lambda+e)})(2N^{\lambda+e})^R = BN^s/2. \]

Hence

\[ \sum_1 |S(m)|^s = \sum_3 |S(m)|^s + \sum_{j=0}^{l-1} T_j < (BN^s/2) + (2^{-R-1}BN^{s-e/2}) < BN^s \]

if \(N\) is large. This contradicts (3.8a). So a \(p\) satisfying (3.10), (3.11) exists.

In what follows we shall confine our attention to the R-tuples \((m_1, \cdots, m_R)\) satisfying (3.10), (3.11), (3.12). From (2.5), (3.9) we have

(3.13) \[ |S(m)|^K \leq N^{K-1} + N^{K-k+e} \sum_{j=1}^L \min(N, 1/|j|), \]

where \(L = k!N^{k-1}\) and

(3.14) \[ t = \sum_{i=1}^R \theta_im_i. \]

Define
(3.15) \[ Q = N^{k-1+\alpha\lambda K-Kp} \quad (p = \rho/s), \]

where

(3.16) \[ \alpha = \alpha(k, s, R) = \frac{2}{K} g(k, R - 1) + \frac{(R - 1)(1 - k)}{ks} + \frac{1}{K} \]

and \( g(k, R) \) is defined by (1.9). By Dirichlet's theorem for given \( t \) and \( Q \) ((3.14), (3.15)) there are integers \( a, q \) such that

(3.17) \[ (a, q) = 1, \quad 1 \leq q \leq Q, \quad \lvert qt - a \rvert < Q^{-1}. \]

Fix such a \( q \) and divide the sum in the right of (3.13) into blocks of \( q \) terms. It follows from Lemma 4 and (3.13) that

\[ |S(m)|^K \ll N^{K-1} + N^{K-k+\epsilon}(k!N^{k-1}q^{-1} + 1)(N + q \log q). \]

Then by (3.11) we have

(3.18) \[ N^{1-RKp-\epsilon} \ll Nq^{-1} + N^e + N^{1-k+\epsilon}q. \]

Suppose that \( \lambda \) satisfies

(3.19) \[ \lambda \left\{ 2g(k, R - 1) + \frac{(R - 1)K}{sk} + 1 \right\} < 1 - A\epsilon \quad (R \geq 3), \]

where \( g(k, R) \) is defined by (1.9) and \( A \) is a positive constant. The value of \( A \) is so defined such that all following inequalities (3.21), (3.22), (3.26), (4.8), (4.10) will be satisfied.

By (3.15), (3.17), (3.18) we have

(3.20) \[ q^{-1} \gg N^{-RKp-\epsilon} \{ 1 - N^{2e+RKp-1} - N^{RKp+2e-1+\alpha\lambda K-Kp} \}. \]

The last two terms in the curly bracket of (3.20) can be neglected since by (3.10), (3.19) and Lemma 5 ((2.9), (2.10)) we have

(3.21) \[ 2e + RKp - 1 < 0 \]

and

(3.22) \[ RKp + 2e - 1 + \alpha\lambda K - Kp < 0. \]

Then from (3.20) we have

(3.23) \[ (1 \ll) q \ll N^{RKp+e}. \]

By Lemma 6, for given \( \theta_1, \ldots, \theta_R \) and integer \([QN^{-\epsilon}]\) there are integers \( a_1, \ldots, a_R \) and \( b \) such that

(3.24) \[ 1 \leq b \leq [QN^{-\epsilon}], \quad \lvert b\theta_i - a_i \rvert < [QN^{-\epsilon}]^{-1/R} \quad (i = 1, \ldots, R). \]

It follows from (3.12), (3.14), (3.17), (3.23) that
By (3.10), (3.19) and Lemma 5 ((2.11)) we see that

\[(3.26)\]

\[(RKp + \varepsilon) + (\lambda + \varepsilon) - \frac{1}{R}(k - 1 + \alpha\lambda K - Kp - \varepsilon)\]

where \(B\) is some number depending on \(k, s, R\) only. Then from (3.15), (3.25), (3.26) we have \(|q \sum_{i=1}^{R} a_i m_i - ab| \leq N^{-\varepsilon} < 1\), for large \(N\). That is \(q \sum_{i=1}^{R} a_i m_i = ab\). But by (3.17)\(_1\) \(((a, q) = 1)\) we see that \(q\) divides \(b\). Then by (3.15), (3.24) the number of possibilities for \(q\) is \(O(\alpha\lambda^e)\). Since at least \([N^R\rho^{-\varepsilon}] + 1 (> 1)\) \(R\)-tuples \((m_1, \cdots, m_R)\) in (3.12) satisfy (3.11) then \(\geq N^R\rho^{-2\varepsilon}\) (or \(\geq BN^R\rho^{-2\varepsilon}\) for some positive constant \(B\)) of these \(R\)-tuples have the same \(q\). Choose a suitable \(d (d^R \geq 2^{R+1}, \text{ say})\) such that the following pigeonhole argument holds.

Partition the \(R\)-cube, \(|m_i| \leq N^{\lambda + e} (i = 1, \cdots, R)\) by

\[(3.27)\]

\[|m_i| = ldN^{\lambda + e - (2e/R)}\]

where \(i = 1, \cdots, R; l = 0, 1, \cdots\). In all, there are at most

\[\left(\frac{2N^{\lambda + e}}{dN^{\lambda + e - (2e/R)}}\right)^R = (2/d)^R N^{R\rho^{-2\varepsilon}}\]

\(R\)-subcubes in the \(R\)-cube, \(|m_i| \leq N^{\lambda + e} (i = 1, \cdots, R)\). Now by the pigeonhole argument, there is an \(R\)-subcube containing at least two distinct \(R\)-tuples \((m'_1, \cdots, m'_R), (m''_1, \cdots, m''_R)\), say, having the same \(q\). For these two \(R\)-tuples we may suppose that for some integer \(I\) with \(1 \leq I \leq R\) we have \(m'_I > m''_I\). Put

\[(3.28)\]

\[m_i = m'_i - m''_i \quad (i = 1, \cdots, R)\]

In particular, we have

\[(3.29)\]

\[m_I \geq 1\]

Then by (3.27), (3.28) and (3.17)\(_3\) we have

\[(3.30)\]

\[|m_i| \leq N^{\lambda + e(1 + (2/R))^{-\rho}} \quad (i = 1, \cdots, R)\]

since \(R\)-tuples \((m'_1, \cdots, m'_R), (m''_1, \cdots, m''_R)\) have the same \(q\) in (3.17)\(_3\).
4. Completion of the proof. In what follows we shall confine our attention to the new $R$-tuple $(m_1, \cdots, m_R)$ satisfying (3.28), (3.30). We proceed by induction on $R$. As usual, our proof consists of two parts. We first show that Theorem 1 is true for $R (\geq 4)$ if we assume that Theorem 1 is true for $R - 1$. Then we can see that in fact Theorem 1 holds for $R = 3$.

Put
\[
\phi_i = m_i^{k-1} q^k \theta_i,
\]
(4.1)
\[
M = [N^{(2\lambda + 2\epsilon(1+R^{-1})-\rho)g/(1-\epsilon g)}],
\]
(4.2)
where $g = g(k, R - 1), i = 1, \cdots, R; R \geq 3$. Suppose that Theorem 1 is true for $R - 1 (\geq 3)$. Then by Corollary 1, (1.10) is true for $M$ and any $R - 1$ $\phi$'s among $\phi_1, \cdots, \phi_R$. So there is some integer $n$ satisfying
\[
1 \leq n \leq M,
\]
(4.3)
\[
\max_{1 \leq i \leq R ; i \neq I} \| \phi_i n^k \| \ll M^{-1/g(k,R-1)} + \epsilon \ll N^{-2\lambda - 2\epsilon(1+R^{-1}) + \rho}.
\]
Let
\[
x = nqm_I.
\]
(4.4)
It follows from (4.1), (4.3), (3.30) that
\[
\| \theta_i x^k \| \leq \| (m_i^{k-1} q^k \theta_i) n^k m_I \| = \| \phi_i n^k m_I \| \quad (i = 1, \cdots, R; R \geq 4)
\]
and
\[
\max_{1 \leq i \leq R ; i \neq I} \| \theta_i x^k \| \ll N^{-2\lambda - 2\epsilon(1+R^{-1}) + \rho} N^{\lambda + \epsilon(1+(2/R)) - \rho} \ll N^{-\lambda - \epsilon}.
\]
(4.5)
Similarly, we have
\[
\| \theta_I x^k \| = \| \theta_I n^k q^k m_I^k \|
\]
(4.6)
\[
\leq q^{k-1} n^k m_I^{k-1} \left\| q \left( \sum_{i=1}^{R} \theta_i m_i \right) \right\| + \sum_{i=1 ; i \neq I}^{R} |m_i| \| (q^k m_I^{k-1} \theta_i) n^k \|
\]
\[\ll N^{\sigma_1} + N^{-\lambda - \epsilon},
\]
where $\sigma_1$ is defined (so as to make the last part of (4.6) valid) by the first part of (4.7). By (3.23), (4.2), (3.30), (3.15) we see that
\[
\sigma_1 = (k - 1)(RKp + \epsilon) + k(2\lambda + 2\epsilon(1 + R^{-1}) - \rho)g/(1 - \epsilon g)
\]
(4.7)
\[+ (k - 1)(\lambda + \epsilon(1 + (2/R)) - \rho) - (k - 1 + \alpha \lambda K - Kp)
\]
\[= p((k - 1)RK - skg - (k - 1)s + K)
\]
\[+ \lambda(2kg - \alpha K + k) + Be - (k - 1) - \lambda - \epsilon,
\]
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where $B$ depends on $k$, $R$ only. As pointed out in the proof of Lemma 5, $g(k, R - 1) = g > KR$ when $R \geq 4$, whence we see that $(k - 1)RK - skg - (k - 1)s + K < 0$. Then together with (4.7), (3.19), and Lemma 5 ((2.12)), we have

\[(4.8)\quad \sigma_1 < \lambda(2g(k, R - 1) - \alpha K + k) - (k - 1) + Be - \lambda - \epsilon < -\lambda - \epsilon.\]

Hence by (4.5), (4.6), (4.8) we have

\[(4.9)\quad \max_{1 \leq i \leq R} \|\theta_i x^k\| \leq N^{-\lambda - \epsilon}.\]

Next, by (4.2), (3.23), (3.29), (3.30) we see that $1 \leq nq m_i = x \ll N^{\sigma_2}$, where

\[
\sigma_2 = (2\lambda + 2\epsilon(1 + R^{-1}) - \rho)g'(1 - eg) + (RKp + \epsilon) + (\lambda + \epsilon(1 + (2/R)) - \rho) \\
= \rho(RK/s - g(k, R - 1) - 1) + \lambda(2g(k, R - 1) + 1) + Be \\
< \lambda(2g(k, R - 1) + 1) + Be,
\]

for some $B$ depending on $k$, $R$ only. Then by (3.19) and Lemma 5 ((2.13)), we have

\[(4.10)\quad \sigma_2 < \lambda(2g(k, R - 1) + 1) + Be < 1 - \epsilon.\]

Hence

\[(4.11)\quad 1 \leq x \ll N^{1 - \epsilon}.\]

(4.9) and (4.11) show that we have obtained an integer $x$ satisfying

\[(4.12)\quad 1 \leq x \ll N \quad \text{and} \quad \max_{1 \leq i \leq R} (\|\theta_i x^k\|) \leq N^{-\lambda}.\]

This contradicts our supposition (3.1) since if we let $x_{i_0} = x$ (for $i_0$ see statement between (3.7) and (3.8)) and $x_j = 0$ for all $j \neq i_0$ then we have a particular vector $X = (x_1, \cdots, x_j, \cdots, x_s)$ for which

\[f_i(X) = \sum_{j=1}^{s} \theta_{ij} x_j^k = \theta_{i0} x_{i0}^k = \theta_i x^k,
\]

where $i = 1, \cdots, R$; $R \geq 4$. So by (3.19) with a suitable choice of $\epsilon$ Theorem 1 is true for $R \geq 4$, if it is true for $R - 1$.

It remains to see that Theorem 1 is true for $R = 3$. For the case $R = 3$ the proof follows exactly in the same way as that for $R \geq 4$ except that now Corollary 1 ((1.10)) is known for $R = 2$ (a special case of Theorem L). This proves Theorem 1.

REFERENCES


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