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ON BOUNDED FUNCTIONS SATISFYING AVERAGING CONDITIONS. I

BY

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ABSTRACT. Let $R(T)$ be the space of real valued L^∞ functions defined on the unit circle C consisting of those functions f for which $\lim_{h\to 0}(1/h)\int_\theta^{\theta+h}f(e^{it})\,dt = f(e^{i\theta})$ for every $e^{i\theta}$ in C. The extreme points of the unit ball of $R(T)$ are found and the extreme points of the unit ball of the space of all bounded harmonic functions in the unit disc which have nontangential limit at each point of the unit circle are characterized. We show that if g is a real valued function in $L^\infty(C)$ and if K is a closed subset of $\{e^{i\theta} \mid \lim_{h\to 0}(1/h)\int_\theta^{\theta+h}g(e^{it})\,dt = g(e^{i\theta})\}$, then there is a function in $R(T)$ whose restriction to K is g. If E is a G_δ subset of C of measure 0 and if F is a closed subset of C disjoint from E, there is a function of norm 1 in $R(T)$ which is 0 on E and 1 on F. Finally, we show that if E and F are as in the preceding result, then there is a function of norm 1 in H^∞(unit disc) the modulus of which has radial limit along every radius, which has radial limit of modulus 1 at each point of F and radial limit 0 at each point of E.

Introduction. Let C denote the unit circle, and let $L^\infty_R(C)$ be the space of all real valued L^∞ functions defined on C.

Define $R(T)$ to be the subspace of $L^\infty_R(C)$ consisting of those functions f for which

$$\lim_{h\to 0}\left(\frac{1}{h}\right)\int_\theta^{\theta+h} f(e^{it})\,dt = f(e^{i\theta})$$

for every $e^{i\theta}$ in C.

It follows from Fatou's theorem and its converse for nonnegative harmonic functions in the unit disc ([6], [3]) that $R(T)$ is isometrically isomorphic to the space of real valued, bounded, harmonic functions in the unit disc which have nontangential limit at each point of the unit circle.

It will be shown that the only extreme points of the unit ball of $R(T)$ are the constant functions 1 and -1. From this it follows immediately via the Krein-Milman theorem that $R(T)$ is not the dual of a Banach space.

If instead of $R(T)$ we consider the space of all complex valued bounded

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harmonic functions in the unit disc which have nontangential limit at each point of C, the extreme points of the unit ball are found to be precisely those functions which have modulus 1 a.e.

A number of theorems concerning the existence of functions in $R(T)$ with special properties are also proved.

Finally, we show that if E is a G_δ subset of C of measure 0 and if F is a closed subset of C disjoint from E, then there is a function of norm 1 in H^∞(unit disc) the modulus of which has radial limit along every radius, which has radial limit of modulus 1 at each point of F and radial limit 0 at each point of E.

Much of the work depends on the fact that if M is a nonempty G_δ subset of measure 0 of the open interval $(0, 1)$, then there exists a real valued function z_M defined on $(0, 1)$ having the following five properties:

(1) $z_M(x) = \infty$ if and only if x is in M.

(2) $1 \leqslant z_M(x) \leqslant \infty$ for all x in $(0, 1)$.

(3) z_M is continuous on M.

(4) If x is not in M, z_M is upper semicontinuous at x, i.e., for every $\epsilon > 0$, there is a $\delta > 0$ such that $z_M(y) > z_M(x) - \epsilon$ whenever $|y - x| < \delta$.

(5) $\lim_{h\to 0}(1/h)\int_x^{x+h} z_M(t)\,dt = z_M(x)$ for all x.

That such a function exists follows from the proof of a theorem due to Zygmunt Zahorski. Because reference will be made to the details of the construction of such a function, both this theorem and an outline of its proof are given.

All of the work was done as part of a Ph.D. thesis guided by Professor Lee Rubel of the University of Illinois.

Background. The notation E' will be used to denote the complement of a set E. $|E|$ will denote the Lebesgue measure of a set E. The letter m when it occurs as a symbol will always denote a positive integer.

A point x in an open interval I is defined to be a point of density of a subset K of I if $\lim_{h\to 0}|K \cap (x, x+h)|/|h| = 1$.

THEOREM 1 [8]. *Let M be a nonempty G_δ of measure 0 contained in $(0, 1)$. Then there exists a real valued, increasing, differentiable function on $(0, 1)$ whose derivative is ∞ exactly on M.*

The following lemma is used in the proof.

LEMMA 1. *Let M_1 be an arbitrary nonempty, measurable subset of $(0, 1)$. Let M_2 be a closed subset of M_1 consisting only of points of density of M_1. Then for every positive number p there exists a closed set M_p with $M_2 \subset M_p \subset M_1$ satisfying*:

(1) *Every point of* M_2 *is a point of density of* M_p *and every point of* M_p *is a point of density of* M_1.

(2) $|M_p| > |M_2| + (1 - 2^{-2-p})(|M_1| - |M_2|)$.

(3) *If* $\theta \in M_2$ *and* $|M_1 \cap (\theta, \theta + t)|/|t| > 1 - \epsilon$ *for* $|t| < 1/m$, *then* $|M_p \cap (\theta, \theta + t)|/|t| > 1 - \epsilon - 2^{-m-p+1}$, *in particular, if* $|M_1' \cap (0, 1)| = 0$, $|M_p \cap (\theta, \theta + t)|/|t| > 1 - 2^{-m-p+1}$ *for* $|t| < 1/m$.

The proof of this lemma will be omitted since it will not be referred to later and since it is precisely the first part of the proof of the Lusin-Menchoff lemma which is stated and proved in Zahorski's paper [8].

PROOF OF THEOREM 1 (OUTLINE). Let $N = M' \cap (0, 1)$. N may be written as a countable union of closed sets, say $N = \bigcup_{k=1}^{\infty} F_k$, F_k closed.

Let P_1 be an arbitrary closed subset of N of measure greater than ½. Set $\Phi_1 = P_1 \cup F_1$, and let p_1 be a positive number greater than 1, for which $|\Phi_1| + (1 - 2^{-2-p_1})(1 - |\Phi_1|) > 3/4$. By Lemma 1, there is a closed set P_2 of measure $> 3/4$ with $\Phi_1 \subset P_2 \subset N$, satisfying (1), (2) and (3). (Here $M_2 = \Phi_1$, $M_1 = N, p = p_1$ and $M_p = P_2$.) Set $\Phi_2 = P_2 \cup F_2$. Then $|\Phi_2| > 3/4$ and $|\Phi_2 \cap (x, x + h)|/|h| > 1 - 2^{-m+1-p_1}$ for $x \in \Phi_1$ and $|h| < 1/m$. Proceeding inductively, let $p_k > p_{k-1} + 1$ satisfy $|\Phi_k| + (1 - 2^{-2-p_k})(1 - |\Phi_k|) > 1 - 2^{-k-1}$. By Lemma 1, there is a closed set P_{k+1} of measure $> 1 - ½^{k+1}$ with $\Phi_k \subset P_{k+1} \subset N$ satisfying (1), (2) and (3). Set $\Phi_{k+1} = P_{k+1} \cup F_{k+1}$.

The sequence $\{\Phi_k\}_{k \geq 1}$ is an increasing sequence of closed sets whose union is N. $|\Phi_k| > 1 - 1/2^k$ for all k and

$$|\Phi_k \cap (x, x + h)|/|h| > 1 - 2^{-m+1-p_{k-1}}$$

for x in Φ_{k-1} and $|h| < 1/m$.

At this point Zahorski proceeds to define closed sets Φ_r, where r is of the form $m/2^n$, $m > 2^n$, m, n positive integers. These sets are so chosen that two properties hold. They are

(a) $\Phi_{r'} \subset \Phi_r$ if $r > r'$,

(b) if $r > r'$, every point of $\Phi_{r'}$ is a point of density of Φ_r.

For our purposes, it will be useful to choose these sets Φ_r in a special way. The essential properties (a) and (b) will hold.

For each positive integer N, let $p_{N+(1/2)} = 2$. Let $\Phi_{N+(1/2)}$ be any closed set with $\Phi_N \subset \Phi_{N+(1/2)} \subset \Phi_{N+1}$ which satisfies (1), (2) and (3) of Lemma 1, with $p = p_{N+(1/2)} = 2$.

Having defined $p_{N+(s/2^n)}$ and $\Phi_{N+(s/2^n)}$ for all N and all $s/2^n$ with $0 < s < 2^n$ and $n \leq k$, let $p_{N+(2r+1)/2^{k+1}} = k + 2$ for each N

and r with $2r+1<2^{k+1}$. Let $\Phi_{N+(2r+1)/2^{k+1}}$ be a closed set constructed as in Lemma 1 with

$$\Phi_{N+(r/2^k)} \subset \Phi_{N+(2r+1)/2^{k+1}} \subset \Phi_{N+(r+1)/2^k} \quad \text{and} \quad p = p_{N+(2r+1)/2^{k+1}} = k+2.$$

If $r = m/2^n$, r in lowest terms, then r may be written uniquely as $r = N + s/2^n$, $s < 2^n$. Let $\Phi_{m/2^n} = \Phi_{N+(s/2^n)}$.

Having defined $\Phi_{m/2^n}$ for all $m/2^n \geqslant 1$, define Φ_λ for λ an arbitrary real number $\geqslant 1$ by $\Phi_\lambda = \bigcap_{m \geqslant \lambda 2^n} \Phi_{m/2^n}$.

Define

$$z_M(\theta) = \begin{cases} \inf\limits_{\lambda} \{\lambda \mid \theta \in \Phi_\lambda\}, & \theta \in N, \\ \infty, & \theta \in M. \end{cases}$$

The function z_M satisfies the five conditions stated previously, and the function $h(\theta) = \int_0^\theta z_M(t)\,dt$ satisfies the conditions of the theorem.

The following corollary will be of use later.

Corollary of the Proof. *For each pair of positive integers N and n,*

$$|\Phi_{N+1/2^n} \cap (\theta, \theta + t)|/|t| > 1 - 2^{-m+1} \quad \textit{for } |t| < 1/m \textit{ and } \theta \in \Phi_N.$$

Proof. By construction, $|\Phi_{N+1} \cap (\theta, \theta + t)|/|t| > 1 - 2^{-m+1-p_N}$ for $\theta \in \Phi_N$ and $|t| < 1/m$. Thus, by (3) of Lemma 1,

$$|\Phi_{N+1/2} \cap (\theta, \theta + t)|/|t| > 1 - 2^{-m+1-p_N} - 2^{-m+1-p_{N+1/2}}$$
$$= 1 - 2^{-m+1-p_N} - 2^{-m+1-2}.$$

Applying (3) of Lemma 1 again yields

$$|\Phi_{N+1/2^2} \cap (\theta, \theta + t)|/|t| > 1 - 2^{-m+1-p_N} - 2^{-m-1} - 2^{-m+1-p_{N+1/2^2}}$$
$$= 1 - 2^{-m+1-p_N} - 2^{-m-1} - 2^{-m-2}.$$

In general

$$|\Phi_{N+1/2^n} \cap (\theta, \theta + t)|/|t| > 1 - 2^{-m+1-p_N} - 2^{-m} \sum_{s=1}^{n} 2^{-s}.$$

Thus, $|\Phi_{N+1/2^n} \cap (\theta, \theta + t)|/|t| > 1 - 2^{-m+1-p_N} - 2^{-m}$. Since $p_N \geqslant 1$ for all N, $|\Phi_{N+1/2^n} \cap (\theta, \theta + t)|/|t| > 1 - 2^{-m+1}$. Q.E.D.

If I is an open subinterval of the real line and if M is a nonempty G_δ of measure 0 contained in I, then a collection of closed sets $\{\Phi_\lambda\}_{\lambda \geqslant 1}$ constructed in the manner of Theorem 1 will be called a Zahorski collection for M on I. If I is a closed interval, such a collection will be called a Zahorski collection for M on I provided each Φ_λ is of the form $\Phi_\lambda = I \cap \Phi_{\lambda'}$ where $\{\Phi_{\lambda'}\}$ is a Zahorski collection for M on some open interval containing I.

A Zahorski function for M on I will be any function z defined on I by

$$z(x) = \begin{cases} \infty, & x \in M, \\ \inf_{\lambda} \{\lambda \mid x \in \Phi_\lambda\}, & x \notin M, \end{cases}$$

where $\{\Phi_\lambda\}_{\lambda \geqslant 1}$ is a Zahorski collection for M. The function z will be referred to as the Zahorski function corresponding to that collection.

If z is a Zahorski function for M on I, then z satisfies the properties (1)–(5) listed in the Introduction, with the interval $(0, 1)$ replaced by I (with appropriate restrictions at the endpoints if I is closed).

We observe that if z is the Zahorski function corresponding to a Zahorski collection $\{\Phi_\lambda\}_{\lambda \geqslant 1}$, then z is identically 1 on Φ_1. Also, if K is a given closed subset of $M' \cap I$, then Φ_1 may be chosen so that $K \subset \Phi_1$.

We turn our attention now to the problem of characterizing the extreme points of the unit ball of $R(T)$.

Extreme points. Let $E \subset [-\pi, \pi]$ be an arbitrary G_δ of measure 0. Let z_E be a Zahorski function for E for which $z_E(\pi) = z_E(-\pi)$, and assume that z_E has been periodically extended to a function on C. Let $u_E(\theta) = 1/z_E(\theta)$. u_E has the following properties:

(1) $u_E(\theta) = 0$ if and only if θ is in E.

(2) $0 \leqslant u_E(\theta) \leqslant 1$ for all θ.

(3) $u_E(\theta)$ is continuous at every θ in E.

(4) If $\theta \notin E$, then for every $\epsilon > 0$, there is a $\delta > 0$ such that $u_E(x) < u_E(\theta)(1/(1-\epsilon))$ when $|x - \theta| < \delta$, i.e., u_E is lower semicontinuous on E'.

(5) $\lim_{h\to 0}(1/h)\int_\theta^{\theta+h}|u_E(\theta) - u_E(t)|\,dt = 0$ for every $\theta \in [-\pi, \pi]$.

Properties (1)–(4) follow directly from the corresponding properties for z_E.

PROOF OF (5). Since u_E tends continuously to 0 at every $\theta \in E$, $\lim_{h\to 0}(1/h)\int_\theta^{\theta+h} u_E(t)\,dt = 0$ for θ in E. Suppose θ is not in E. Let $\{\Phi_\lambda\}_{\lambda\geqslant 1}$ be the Zahorski decomposition corresponding to z_E and let $\epsilon > 0$ be arbitrary. Since every point of Φ_λ is a point of density of $\Phi_{\lambda'}$ whenever $\lambda' > \lambda$, θ is a point of density of $\Phi_{z(\theta)(1+\epsilon)}$ and so also of $\{t \mid z_E(t) \leqslant z_E(\theta)(1+\epsilon)\}$ which contains $\Phi_{z(\theta)(1+\epsilon)}$. In terms of u_E, θ is a point of density of $\{t \mid u_E(t) \geqslant u_E(\theta)(1/(1+\epsilon))\}$. This fact, together with property (4) and the boundedness of u_E, yields the desired conclusion.

A function u_E which is of the form $1/z_E$ where z_E is a Zahorski function for E will be called an inverse Zahorski function for E.

If f is in $L^\infty(C)$, then $S(f)$ will denote

$$\left\{x \in [-\pi, \pi] \mid \lim_{h\to 0}\left(\frac{1}{h}\right)\int_x^{x+h} f(t)\,dt = f(x)\right\}$$

and $L(f)$ will denote

$$\left\{x \in [-\pi, \pi] \mid \lim_{h\to 0}\left(\frac{1}{h}\right)\int_x^{x+h}|f(x)-f(t)|\,dt = 0\right\}.$$

$L(f)$ is a subset of $S(f)$ but the two sets need not be equal as the function

$$f(\theta) = \begin{cases} \sin(1/\theta), & \theta \neq 0, \\ 0, & \theta = 0, \end{cases}$$

shows. $L(f)$ will be referred to as the Lebesgue set of f.

THEOREM 2. *The only extreme points of the unit ball of* $R(T)$ *are the constant functions* 1 *and* -1.

PROOF. It will first be shown that f is an extreme point of the unit ball of $R(T)$ if and only if $|f| = 1$ a.e.

If $|f| = 1$ a.e. and if g is a function in $L_R^\infty(C)$ for which $\|f+g\| \leqslant 1$ and $\|f-g\| \leqslant 1$, then $g = 0$ a.e. Thus f is an extreme point of the unit ball of $L_R^\infty(C)$ and so also of $R(T)$. Suppose $|f| \neq 1$ a.e. Set $g = 1 - |f|$. Let E be a G_δ of measure 0 containing $\{\theta \mid \theta \notin S(g)\} \cup \{-\pi\} \cup \{\pi\}$ and let u_E be an inverse Zahorski function for E on $[-\pi, \pi]$ which has been periodically extended to a function on C. Since u_E tends continuously to 0 on E, θ is in $S(u_E(1-|f|))$ whenever θ is in E. If $\theta \notin E$, θ is in both $L(u_E)$ and in $S(g)$, so that θ is also in $S(u_E g)$. Thus every point of $[-\pi, \pi]$ is in $S(u_E g)$ and $u_E g$ is in $R(T)$. Since $u_E g$ is a function in $R(T)$ which is not identically 0 and for which $u_E(t)g(t) \leqslant 1 - |f(t)|$ for all e^{it} in C, f is not extreme.

The conclusion that the only extreme points are the constant functions 1 and -1 follows from the following lemma:

LEMMA 2. *If f is a real valued, continuous, differentiable function on $(-\pi, \pi)$ whose derivative assumes the values* 0 *or* 1 *a.e., then f' is either identically* 0 *or identically* 1.

Professor C. Weil noticed that this lemma is a consequence of a theorem of Denjoy [2], which asserts that if f' exists everywhere, then for any open set G, $(f')^{-1}(G)$ is either empty or has positive measure.

A direct proof is given here. This method of proof is useful for extending some of the results given here to R^n. These extensions will be given in a later paper.

PROOF OF LEMMA 2. Set $u = f'$ and $K = \{x \mid 0 < u(x) < 1\}$.

The conditions on f imply that f is absolutely continuous [7, p. 207].

Therefore, since u has the Darboux property, it is sufficient to show that K is empty.

Suppose that K is not empty. Let y be an arbitrary number in $(0, 1)$. *If* α is in K and if $h > 0$ is an arbitrary positive number for which $(\alpha - h, \alpha + h) \subset (-\pi, \pi)$, then it follows from the Darboux property for u and the absolute continuity of f that u must assume the value y in $(\alpha - h, \alpha + h)$.

Let α be in K and let $h > 0$ be arbitrary except that $(\alpha - h, \alpha + h) \subset (-\pi, \pi)$. Let α_1 be a point in $(\alpha - h, \alpha + h)$ for which $u(\alpha_1) = 1 - (½)$. Let $0 < T_1 < ½$ be such that $[\alpha_1 - T_1, \alpha_1 + T_1] \subset (\alpha - h, \alpha + h)$ and, for $0 < |t| < T_1$, $|(1/t)\{f(\alpha_1 + t) - f(\alpha_1)\} - \{1 - (½)\}| < ½$ and set $F_1 = [\alpha_1 - T_1, \alpha_1 + T_1]$. Since $\alpha_1 \in K$, there is a point α_2 in $(\alpha_1 - T_1, \alpha_1 + T_1)$ for which $u(\alpha_2) = 1/2^2$. Let $0 < T_2 < 1/2^2$ satisfy $[\alpha_2 - T_2, \alpha_2 + T_2] \subset F_1^\circ$, where $^\circ$ denotes interior, and $|(t^{-1})\{f(\alpha_2 + t) - f(\alpha_2)\} - 1/2^2| < 1/2^2$ for $0 < |t| < T_2$. Set $F_2 = [\alpha_2 - T_2, \alpha_2 + T_2]$.

Continue defining α_k, T_k, F_k inductively as follows: If $k - 1$ is even, let α_k be a point in $(\alpha_{k-1} - T_{k-1}, \alpha_{k-1} + T_{k-1})$ for which $u(\alpha_k) = 1 - (1/2^k)$. Let $0 < T_k < 1/2^k$ satisfy $[\alpha_k - T_k, \alpha_k + T_k] \subset F_{k-1}^\circ$ and

$$|(1/t)\{f(\alpha_k + t) - f(\alpha_k)\} - \{1 - (1/2^k)\}| < 1/2^k \quad \text{for } 0 < |t| < T_k.$$

Let $F_k = [\alpha_k - T_k, \alpha_k + T_k]$. If $k - 1$ is odd, replace $1 - (1/2^k)$ by $1/2^k$, i.e., $\alpha_k \in F_{k-1}^\circ$, $u(\alpha_k) = 1/2^k$ and

$$|(1/t)\{f(\alpha_k + t) - f(\alpha_k)\} - 1/2^k| < 1/2^k \quad \text{for } 0 < |t| < T_k.$$

$\{F_k\}_{k \geq 1}$ is a sequence of nested closed intervals. Let θ be in the intersection of the F_k. Let $\epsilon > 0$ be arbitrary and let $T_\epsilon > 0$ satisfy $|(1/t)\{f(\theta + t) - f(\theta)\} - u(\theta)| < \epsilon$ for $0 < |t| < T_\epsilon$. Let k_ϵ be a positive integer for which $1/2^{k_\epsilon} < \min\{T_\epsilon, \epsilon\}$. Since $\theta \in F_{2k}$ for $k > k_\epsilon$, $|\theta - \alpha_{2k}| < T_{2k}$ and $|1/(\theta - \alpha_{2k})\{f(\theta) - f(\alpha_{2k})\} - 1/2^{2k}| < 1/2^{2k} < \epsilon$. Also since $T_{2k} < 1/2^{k_\epsilon} < T_\epsilon$, $|1/(\alpha_{2k} - \theta)\{f(\alpha_{2k}) - f(\theta)\} - u(\theta)| < \epsilon$. Thus, for $k > k_\epsilon$, $|u(\theta) - 1/2^{2k}| < 2\epsilon$, which implies that $u(\theta) = 0$. On the other hand, the fact that θ is in F_{2k+1} for $k > k_\epsilon$ leads to the conclusion that $u(\theta) = 1$. Thus the assumption that K is not empty leads to a contradiction and K is empty. Q.E.D.

This lemma completes the proof of Theorem 2.

Now consider $L^\infty(C)$, the space of all complex valued L^∞ functions on C, and let $X(T)$ be the subspace consisting of those functions f for which $S(f) = [-\pi, \pi]$.

A proof identical to the one used in the real case can be used to show that f is an extreme point of the unit ball of $X(T)$ if and only if $|f| = 1$ a.e. The functions z^n show that there are many extreme points. Lohwater and Piranian [5] have shown that corresponding to any subset of C of measure 0, which is simultaneously an F_σ and a G_δ, there is an inner function in $H^\infty(D)$ which has radial limit 0 at each point of E and radial limit of modulus 1 at each point of E'. Boehme, Rosenfeld and Weiss have shown [1] that the boundary function of an $H^\infty(D)$ function which has radial limit along every radius is in $X(T)$. Combining these results leads to the conclusion that if E is a subset of C of measure 0, which is simultaneously an F_σ and a G_δ, then there is an extreme point in the unit ball of $X(T)$ which is 0 on E and whose modulus is 1 on E'.

Functions with special properties in $R(T)$ and $H^\infty(D)$. We now state two theorems and a corollary concerning functions in $R(T)$. Since their proofs are all very similar, only the second theorem will be proved here.

If E is a subset of C, the notation E^* will be used to denote $\{\theta \in [-\pi, \pi] \mid e^{i\theta} \in E\}$.

THEOREM 3. *If E is a G_δ of measure 0 contained in C and if F is a closed subset of C disjoint from E, there is a function of norm 1 in $R(T)$ which is 0 at each point of E and 1 at each point of F.*

COROLLARY. *Let $\{w_k\}_{k\geq 1}$ be a convergent sequence of points on C for which $w_i \neq w_j$ unless $i = j$ and $w_i \neq \lim w_k$ for any i. If $\{a_k\}_{k\geq 1}$ is any sequence of 0's and 1's, then there is a function u of norm 1 in $R(T)$ for which $u(w_k) = a_k$ for all k.*

THEOREM 4. *If g is in $L_R^\infty(C)$ and if K is a closed subset of $\{e^{i\theta} \mid \theta \in S(g)\}$, then there is a function in $R(T)$ whose restriction to K is g.*

PROOF. Let E be a G_δ subset of C of measure 0, disjoint from K and containing $\{e^{i\theta} \mid \theta \notin S(g)\}$. Let

$$E^\# = \begin{cases} E & \text{if } e^{i\pi} \in K, \\ E \cup \{e^{i\pi}\} & \text{if } e^{i\pi} \notin K, \end{cases}$$

and let $\{\Phi_\lambda\}_{\lambda\geq 1}$ be a Zahorski collection for $E^{\#*}$ on $[-\pi, \pi]$, where Φ_1 is such that $K^* \subset \Phi_1$. Let u_E be the corresponding inverse Zahorski function extended periodically to a function on C. An argument similar to that used in the proof of Theorem 2 shows that $S(u_E g) = [-\pi, \pi]$. Thus $u_E g$ is in $R(T)$. Since $u_E g/E = 0$ and $u_E g/K = g$, the proof is complete. Q.E.D.

Theorem 4 directly implies that the functions in $R(T)$ are dense in measure in $L_R^\infty(C)$. (A sequence $\{f_n\}$ of measurable functions converges to a function f in measure if, for every $\epsilon > 0$, there is an N such that, for $n \geqslant N$, $|\{x \mid |f(x) - f_n(x)| \geqslant \epsilon\}| < \epsilon$.)

A theorem similar to Theorem 3 holds if the space $R(T)$ is replaced by $\{f \in H^\infty(D) \mid \lim_{r\to 1} |f(re^{i\theta})|$ exists for all $e^{i\theta} \in C\}$. (Here D denotes the unit disc.) Explicitly, this theorem says the following:

THEOREM 5. *If E is a G_δ of measure 0 contained in C and if F is an arbitrary closed subset of E', then there exists a function H in H^∞ of norm 1, the modulus of which has radial limit along every radius, which has radial limit of modulus 1 at each point of F and radial limit 0 at each point of E.*

A lemma will precede the proof of this theorem.

LEMMA 3. *Let E be a nonempty G_δ of measure 0 contained in (0, 1). Let z be a Zahorski function for E with corresponding Zahorski collection $\{\Phi_\lambda\}_{\lambda\geqslant 1}$. Then*

$$\left|\int_0^R (1/t^2)\{J((\theta, \theta + t)) - J((\theta - t, \theta))\}\, dt\right|$$

is finite for every $\theta \in \Phi_1$ and every positive number R less than $\min(1-\theta, \theta)$, *where $J((\theta,\theta+t)) = \int_{(\theta,\theta+t)} z(u)du$ and $J((\theta-t,\theta)) = \int_{(\theta-t,\theta)} z(u)\,du$.*

PROOF. Fix R and fix $\theta \in \Phi_1$. For each measurable subset M of $(0, R)$ define $J(M) = \int_M z(u)\,du$, $M(t, +) = M \cap (\theta, \theta + t)$, $M(t, -) = M \cap (\theta - t, \theta)$, $M_k = \Phi_k \backslash \Phi_{k-1}$. Set

$$(*) = \int_0^R \left(\frac{1}{t^2}\right)\{J((\theta, \theta + t)) - J((\theta - t, \theta))\}\, dt,$$

$$A = \int_0^R \left(\frac{1}{t^2}\right)\left\{\sum_{k=3}^{\infty} J(M_k(t, +)) - J(M_k(t, -))\right\} dt;$$

$$B = \int_0^R \left(\frac{1}{t^2}\right)\left\{J(\Phi_2(t, +)) - J(\Phi_2(t, -))\right\} dt.$$

Since $(*) = A + B$, it is sufficient to show that $|A|$ and $|B|$ are each finite.

Since $z(u)$ is defined $z(u) = \inf_\lambda\{\lambda \mid u \in \Phi_\lambda\}$ for u in E', $z(u) \leqslant k$ if $u \in \Phi_k$. Thus

$$|A| \leqslant \int_0^R \left(\frac{1}{t^2}\right)\left\{\sum_{k=3}^{\infty} k(|M_k(t, +)| + |M_k(t, -)|\right\} dt.$$

For each t in $(0, R)$, let $n(t)$ be the positive integer for which $1/2^{n(t)+1} < t^2 \leqslant 1/2^{n(t)}$. Set $n'(t) = n(t)/2$. Since $\theta \in \Phi_1$, $\theta \in \Phi_{k-1}$ for $k \geqslant 3$, and it follows from the construction of the Φ_k that $|\Phi_{k-1}(t, +)| \geqslant (1 - 2^{-2^{n'(t)} - P_{k-2} + 1})$. Since $|\Phi_k(t, +)| \leqslant t$, $|M_k(\,, +)| \leqslant t 2^{-2^{n'(t)} - P_{k-2} + 1}$.

Similarly, $|M_k(t,-)| \leqslant t2^{-2^{n'(t)}-p_k-2+1}$. Since $p_k \geqslant k$ for all $k \geqslant 1$, $\Sigma_{k=3}^{\infty} k2^{-p_k-2}$ converges and

$$|A| \leqslant \sum_{k=3}^{\infty} k2^{-p_k-2}\int_0^R 4t2^{-2^{n'(t)}}/t^2\,dt.$$

Thus, $|A| \leqslant 4\Sigma_{k=3}^{\infty} k2^{-p_k-2}\int_0^R 2^{-1/\sqrt{2t}}/t\,dt$, which is finite.

Now consider B. Let $M(t) = (1/t^2)\{J(\Phi_2(t,+)) - J(\Phi_2(t,-))\}$, $t \in (0,R)$. Fix t for the time being and let n be the positive integer for which $1/2^{n+1} < t^2 \leqslant 1/2^n$. Let $n' = n/2$. To simplify the notation, write $E(+)$ instead of $E(t,+)$ and $E(-)$ instead of $E(t,-)$ for each subset E of $(0,R)$. For each positive integer $s \geqslant 2$, let $B_s = \Phi_{1+(s/2^n)} \backslash \Phi_{1+(s-1)/2^n}$ and let $B_1 = \Phi_{1+(1/2^n)}$. We have

$$M(t) = \left(\frac{1}{t^2}\right)\left\{J(B_1(+)) - J(B_1(-)) + \sum_{s=2}^{2^n} J(B_s(+)) - J(B_s(-))\right\},$$

$$M(t) \leqslant \left(\frac{1}{t^2}\right)\left\{\left(1+\frac{1}{2^n}\right)|B_1(+)| - |B_1(-)| + \sum_{s=2}^{2^n}\left(1+\frac{s}{2^n}\right)|B_s(+)| - \sum_{s=2}^{2^n}\left(1+\frac{s-1}{2^n}\right)|B_s(-)|\right\}$$

Let $c(s) = |B_s(+)| - |B_s(-)|$. Then,

$$|M(t)| \leqslant \left(\frac{1}{t^2}\right)\left\{\left(\frac{1}{2^n}\right)|B_1(-)| + \left(1+\frac{1}{2^n}\right)|c(1)| + \sum_{s=2}^{2^n}\left(\frac{1}{2^n}\right)|B_s(-)| + \sum_{s=2}^{2^n}\left(1+\frac{s}{2^n}\right)|c(s)|\right\},$$

$$|M(t)| \leqslant \left(\frac{1}{t^2}\right)\left\{\left(\frac{1}{2^n}\right)\left\{|B_1(-)| + \sum_{s=2}^{2^n}|B_s(-)|\right\} + \sum_{s=1}^{2^n}|c(s)| + \left(\frac{1}{2^n}\right)\sum_{s=1}^{2^n} s\,|c(s)|\right\}$$

$$\leqslant \left(\frac{1}{t^2}\right)\left\{\left(\frac{1}{2^n}\right)|\Phi_2(-)| + \sum_{s=1}^{2^n}|c(s)| + \left(\frac{1}{2^n}\right)\sum_{s=1}^{2^n} s\,|c(s)|\right\}$$

$$\leqslant 2t + \left(\frac{1}{t^2}\right)\left\{\sum_{s=1}^{2^n}|c(s)| + \left(\frac{1}{2^n}\right)\sum_{s=1}^{2^n} s\,|c(s)|\right\}.$$

By the corollary of the proof of Theorem 1,

$$|\Phi_{1+1/2^n}(+)| \geqslant t(1-2^{-2^{n'}+1}) \quad \text{and} \quad |\Phi_{1+1/2^n}(-)| \geqslant t(1-2^{-2^{n'}+1}).$$

Thus, $|c(1)| < t^{-2^{n'}+1}$. Similarly, $|c(s)| < t2^{-2^{n'}+1}$ for $s \geqslant 2$, and we have

$$|M(t)| \leqslant 2t + (4/t^2)\{t2^n 2^{-2^{n'}}\} \leqslant 2t + (4/t^3)2^{-1/\sqrt{2t}}.$$

Since the above inequality holds for each fixed t and since $B = \int_0^R M(t)\,dt$, $|B|$ is also finite. Q.E.D.

PROOF OF THEOREM 5. Let

$$E^{\#} = \begin{cases} E & \text{if } e^{i\pi} \in F, \\ E \cup \{e^{i\pi}\} & \text{if } e^{i\pi} \notin F, \end{cases}$$

and let $\{\Phi_\lambda\}_{\lambda \geqslant 1}$ be a Zahorski decomposition for $E^{\#*}$ on $[-\pi, \pi]$, where $F^* \subset \Phi_1$. Let z be the corresponding Zahorski function extended periodically to a function on C.

The function

$$H(w) = e \exp(-1/2\pi) \int_{-\pi}^{\pi} ((e^{it} + w)/(e^{it} - w)) z(t)\,dt, \qquad w \in D,$$

is in $H^\infty(D)$ and its modulus,

$$H(re^{i\theta}) = e \exp(-1/2\pi) \int_{-\pi}^{\pi} \operatorname{Re}\{(e^{it} + w)/(e^{it} - w)\} z(t)\,dt,$$

has radial limit along every radius since $\lim_{h \to 0}(1/h)\int_\theta^{\theta+h} z(t)\,dt = z(\theta)$ for all $e^{i\theta} \in C$. This radial limit is 0 at each point of E, since z tends continuously to ∞ at each point of E.

The function H itself will have radial limit at each $e^{i\theta} \in E'$ for which $\int_{-\pi}^{\pi}(z(\theta + t) - z(\theta - t))/\tan(t/2)\,dt$ converges [4, p. 79].

Set $v(\theta) = \int_{-\pi}^{\theta} z(t)\,dt$. By integration by parts,

$$\begin{aligned} &\int_\epsilon^\pi \{z(\theta + t) - z(\theta - t)\}/\tan(t/2)\,dt \\ &\qquad = \int_\epsilon^\pi \{v(\theta + t) + v(\theta - t) - 2v(\theta)\}/2\sin^2(t/2)\,dt \\ &\qquad\quad - \{(v(\theta + \epsilon) + v(\theta - \epsilon) - 2v(\theta))/\tan(\epsilon/2)\}. \end{aligned}$$

Since v is differentiable at θ, the expression between the $\{,\}$ approaches 0 as $\epsilon \longrightarrow 0$ and $\int_0^\pi (z(\theta + t) - z(\theta - t))/\tan(t/2)\,dt$ will be finite provided $\int_0^\pi (v(\theta + t) + v(\theta - t) - 2v(\theta))/t^2\,dt$ is finite. In terms of z this last integral is $\int_0^\pi J((\theta, \theta + t)) - J((\theta - t, \theta))/t^2\,dt$ (where the notation is the same as in the preceding lemma), which is finite for $\theta \in \Phi_1$ by Lemma 3. Since $F^* \subset \Phi_1$,

H has radial limit at each point of F. This radial limit has modulus 1 since z is identically 1 on Φ_1. Q.E.D.

We conclude with a corollary of Theorem 5.

COROLLARY. *If f is a function in $L_R^\infty(C)$ for which $\log|f|$ is integrable, then for every $\epsilon > 0$ there is a function F_ϵ in $H^\infty(D)$, the modulus of which has radial limit along every radius, which has radial limit of modulus equal to the modulus of f except on a set of measure less than ϵ.*

PROOF. Since $\log|f|$ is integrable, $|\int_{-\pi}^{\pi}(\log|f(\theta+t)| - \log|f(\theta-t)|)t\,dt|$ is finite a.e. Let K be a closed subset of $\{e^{i\theta} \mid \theta \in S(\log|f|)\}$ intersected with $\{e^{i\theta} \mid |\int_{-\pi}^{\pi}(\log|f(\theta+t)| - \log|f(\theta-t)|)/t\,dt| < \infty\}$, with $|K| > 2\pi - \epsilon$. The function

$$G(w) = \exp - (1/2\pi)\int_{-\pi}^{\pi}(-\log|f(t)|)(e^{it} + w)/(e^{it} - w)\,dt, \qquad w \in D,$$

is in H^∞ and has radial limit of modulus equal to the modulus of f at each point of K.

Let E be a G_δ of measure 0 disjoint from K and containing $\{e^{i\theta} \mid |\lim_{r\to 1} G(re^{i\theta})|$ does not exist$\}$. By the previous theorem there is an H in $H^\infty(D)$ whose modulus has radial limit along every radius, which has radial limit 0 at each point of E and radial limit of modulus 1 at each point of K.

The function $F_\epsilon = HG$ satisfies the conditions of the theorem. Q.E.D.

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