

TOEPLITZ MATRICES GENERATED
 BY THE LAURENT SERIES EXPANSION
 OF AN ARBITRARY RATIONAL FUNCTION

BY

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ABSTRACT. Let $T_n(f) = (a_{i-j})_{i,j=0}^n$ be the finite Toeplitz matrices generated by the Laurent expansion of an arbitrary rational function. An identity is developed for $\det(T_n(f) - \lambda)$ which may be used to prove that the limit set of the eigenvalues of the $T_n(f)$ is a point or consists of a finite number of analytic arcs.

1. Introduction. Let $f(z) = \sum a_m z^m$ be the Laurent expansion of an arbitrary rational function. Define matrices $T_n(f)$ where $T_n(f) = (a_{i-j})$, $i, j = 0, \dots, n$. Such matrices are called Toeplitz matrices and may be generated by functions which are not rational. Denote by σ_n the set of $n + 1$ eigenvalues of $T_n(f)$,

$$\sigma_n = \{\lambda_{n0}, \lambda_{n1}, \dots, \lambda_{nn}\}.$$

Let

$$B = \{\lambda : \lambda = \lim \lambda_m, \lambda_m \in \sigma_{i_m}\}$$

where i_1, i_2, \dots is an increasing sequence of integers. A characterization of this set for complex valued functions was initiated in 1960 and was published for the case: f is a Laurent polynomial, $f(z) = \sum_{-k}^h a_m z^m$, $h, k \geq 1$ [4]. Let

$$D^n(f - \lambda) = \det(T_n(f - \lambda)).$$

Schmidt and Spitzer employed an identity of Harold Widom which up to a constant factor evaluates the $D^n(f - \lambda)$ when f is a Laurent polynomial. We develop an identity for $D^n(f - \lambda)$ for f an arbitrary rational function which, using the techniques of Schmidt and Spitzer, allows one to show that B is a point or consists of a finite number of nondegenerate analytic arcs.

For simplification in the proof of the identity and notational convenience we make certain assumptions about the function f which are essentially nonrestrictive and work directly with the determinants $D^n(f)$. Due to the complexities of notation, we observe the following convention. “(*)” designates a mathematical expression where “*” is the number of the expression, and “(*)” is used to

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represent the object appearing in that expression.

In order that the Laurent series not reduce to a power series in which case the determinants $D^n(f)$ are triangular and the set B reduces to a point we make the following assumptions. Let there be an annulus A with center the origin,

$$(1.1) \quad A = \{z : R_1 < |z| < R_2, 0 \leq R_1 < R_2 < \infty\}.$$

Let $D_k(z)$ and $F_h(z)$ be polynomials of exact degree k and h respectively where the roots of $D_k(z)$ lie in the set $|z| \leq R_1$, and those of $F_h(z)$ lie in the set $|z| \geq R_2$. Let $G_{k+m}(z)$ be a polynomial of exact degree $k + m$ having no common factors with the polynomials $D_k(z)$ and $F_h(z)$.

PROPOSITION 1.1. *Let $k \geq 1$, $m \geq \max(1, h)$, and $G_{k+m}(z)$, $D_k(z)$, and $F_h(z)$ satisfy the conditions given above. If*

$$(1.2) \quad f(z) = G_{k+m}(z)/D_k(z)F_h(z)$$

and

$$(1.3) \quad f(z) = \sum_{-\infty}^{\infty} a_\nu z^\nu$$

is the Laurent series representation of $f(z)$ in the annulus A , then there exist positive and negative powers of z which in the above expansion have nonzero coefficients.

PROOF. Assume that $k, h \geq 1$. If $a_\nu = 0$ for all $\nu \leq -1$, then (1.3) is a power series expansion which converges in the set $|z| < R_2$. This contradicts the existence of at least one pole in the set $|z| \leq R_1 < R_2$. If $a_\nu = 0$ for all $\nu \geq 1$, then (1.3) converges in the set $|z| > R_1$ contradicting the existence of a pole in the set $|z| \geq R_2 > R_1$. In the exceptional case where $h = 0$, then $m \geq 1$. Consequently $f(z)$ defined by (1.2) has a pole of order m at $z = \infty$. So $a_m \neq 0$.

2. Reduction of $D^n(f)$ to a determinant of fixed order m . Let $f(z)$ satisfy the hypothesis of Proposition 1.1. In addition we assume that the roots of $G_{k+m}(z)$ denoted by $r_i, i = 1, \dots, k + m$, are distinct and not equal to zero. We may assume that the coefficient g_0 of the z^{k+m} term of $G_{k+m}(z)$ is equal to 1. For if $f(z)$ is divided by g_0 we have simply divided each a_n in the expansion of f by g_0 . So $D^n(f) = g_0^{n+1} D^n(g_0^{-1}f)$. The proof is long and will be broken up into a series of lemmas. We recommend that the reader turn to Theorem 3.1 for the final result before proceeding with the proof. We introduce the following notation.

$$(2.1) \quad z^{-(k+m)} G_{k+m}(z) = \prod_{i=1}^{k+m} (1 - r_i z^{-1}) = \sum_{i=0}^{k+m} g_{-i} z^{-i},$$

$$(2.2) \quad 1/z^{-(k+m)}G_{k+m}(z) = \sum_{i=0}^{\infty} g_{-i}^* z^{-i},$$

$$(2.3) \quad z^{-k}D_k(z) = \prod_{i=1}^k (1 - \delta_i z^{-1}) = \sum_{j=0}^k d_{-j}^* z^{-j},$$

$$(2.4) \quad 1/z^{-k}D_k(z) = \sum_{j=0}^{\infty} d_{-j} z^{-j},$$

$$(2.5) \quad F_h(z) = \prod_{i=1}^h (1 - \rho_i^{-1} z) = \sum_{i=0}^h f_i^* z^i,$$

$$(2.6) \quad 1/F_h(z) = \sum_{i=0}^{\infty} f_i z^i,$$

$$(2.7) \quad 1/z^{-k}D_k(z) \cdot F_h(z) = \sum_{-\infty}^{\infty} b_n z^n,$$

$$(2.8) \quad z^{-(k+m)}G_{k+m}(z)/z^{-k}D_k(z) = \sum_{i=0}^{\infty} e_{-i} z^{-i},$$

$$(2.9) \quad z^{-k}D_k(z)/z^{-(k+m)}G_{k+m}(z) = \sum_{i=0}^{\infty} e_{-i}^* z^{-i},$$

$$(2.10) \quad f(z) = G_{k+m}(z)/D_k(z) \cdot F_h(z) = \sum_{-\infty}^{\infty} a_n z^n.$$

The first two lemmas are devoted to determining certain information about the coefficients of some of these expansions.

LEMMA 2.1. *If g_{-i}^* is defined by (2.2) and e_{-i}^* by (2.9) then $g_{-i}^* = \sum_{s=1}^{k+m} C_s r_s^i$ holds for $i \geq -(k+m) + 1$, and $e_{-i}^* = \sum_{s=1}^{k+m} C_s D_k(r_s) r_s^{i-k}$ holds for $i \geq k$ where $C_s = r_s^{k+m-1} \prod_{t \neq s} (r_s - r_t)^{-1}$.*

PROOF. By expanding $1/z^{-(k+m)}G_{k+m}(z)$ by partial fractions we obtain the above identity for g_{-i}^* , $i \geq 0$. In addition, the identity is valid for $i = -1, \dots, -(k+m) + 1$ because, for all $i \geq -(k+m) + 1$,

$$(2.11) \quad \sum_{s=1}^{k+m} C_s r_s^i = \sum_{s>t} (r_s - r_t)^{-1} \left\| \begin{array}{cccc} 1 & r_1 & \dots & r_1^{k+m-2} & r_1^{i+k+m-1} \\ 1 & r_2 & \dots & r_2^{k+m-2} & r_2^{i+k+m-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & r_{k+m} & \dots & r_{k+m}^{k+m-2} & r_{k+m}^{i+k+m-1} \end{array} \right\|.$$

The identity for the e_{-i}^* 's follows from the identity for the g_{-i}^* 's and (2.3).

LEMMA 2.2. *If $-i + k < 0$ then $\sum_{j=0}^k d_{-j}^* b_{-i+j} = 0$.*

PROOF. The lemma is clearly true if $D_k(z) = Cz^k$ for then, by (2.6) and (2.7), $b_{-i} = 0$ for $i > 0$. Otherwise expansion of $1/z^{-k}D_k(z)F_h(z)$ by partial fractions shows that

$$(2.12) \quad b_{-i} = \sum_{j=1}^{k'} B_j \hat{\delta}_j^i$$

where $\hat{\delta}_j, j = 1, \dots, k'$, are the distinct nonzero roots of $D_k(z)$, and B_j nonzero constants. Without loss of generality we may assume that all the roots of $D_k(z)$ are nonzero. Proof is by induction.

Let $S_i(k), i = 1, \dots, k$, be the elementary symmetric functions of the roots δ_i of $D_k(z)$ so that from (2.3)

$$\begin{aligned} \prod_{i=1}^k (z - \delta_i) &= d_0 z^k + d_1 z^{k-1} + \dots + d_k \\ &= z^k + S_1(k)z^{k-1} + \dots + S_k(k). \end{aligned}$$

Thus to prove the lemma we need to prove that

$$b_{-i} + S_1(k)b_{-i+1} + S_2(k)b_{-i+2} + \dots + S_k(k)b_{-i+k} = 0.$$

Since

$$\prod_{i=1}^{k-1} (z - \delta_i)(z - \delta_k) = [z^{k-1} + S_1(k-1)z^{k-2} + \dots + S_{k-1}(k-1)](z - \delta_k),$$

it follows that

$$(2.13) \quad \begin{aligned} S_1(k) &= S_1(k-1) - \delta_k, \\ S_i(k) &= S_i(k-1) - \delta_k S_{i-1}(k-1), \quad i = 2, \dots, k-1, \\ S_k(k) &= -\delta_k S_{k-1}(k-1). \end{aligned}$$

By (2.12) it is clear that the lemma will be proved if we can show that

$$(2.14) \quad \delta_j^t + \delta_j^{t-1}S_1(k) + \delta_j^{t-2}S_2(k) + \dots + \delta_j^{t-k}S_k(k) = 0, \\ j = 1, \dots, k, \quad t > k.$$

This is obvious for $k = 1$. We need to prove (2.14) for $k + 1$. From (2.13) it follows that

$$(2.15) \quad \begin{aligned} &\delta_j^t + \delta_j^{t-1}S_1(k+1) + \dots + \delta_j^{t-k-1}S_{k+1}(k+1) \\ &= \delta_j^t + \delta_j^{t-1}S_1(k) + \delta_j^{t-2}S_2(k) + \dots + \delta_j^{t-k}S_k(k) \\ &\quad - \delta_{k+1}(\delta_j^{t-1} + \delta_j^{t-2}S_1(k) + \delta_j^{t-3}S_2(k) + \dots + \delta_j^{t-k-1}S_k(k)). \end{aligned}$$

By induction, (2.15) equals zero for $j = 1, \dots, k$, and (2.15) clearly equals zero for $j = k + 1$.

LEMMA 2.3. $D^n(f)$ is equal to $(-1)^{m(n+1-m)}$ multiplied by the determinant of the product of the following three matrices,

$$(2.16) \quad m \begin{pmatrix} n+1 \\ e_0^* \cdots e_{-n}^* \\ 0 \\ \vdots \\ 0 \cdots e_0^* \cdots e_{-n+m-1}^* \end{pmatrix} \begin{pmatrix} m \\ a_{-n+m-1} \cdots a_{-n} \\ \vdots \\ a_{m-1} \cdots a_0 \end{pmatrix} \begin{pmatrix} m \\ f_0^* \ 0 \cdots 0 \\ f_1^* \\ \vdots \\ \cdots f_1^* \ f_0^* \end{pmatrix}.$$

PROOF. In order to prove the above we multiply $D^n(f)$ on the left and right by determinants each of which is equal to one. In particular, noting by (2.5) and (2.9) that $e_0^* = f_0^* = 1$, we multiply $D^n(f)$ on the left by the upper triangular determinant,

$$(2.17) \quad D^n(D_k(z)/z^{-m}G_{k+m}(z)) = \left\| \begin{matrix} e_0^* e_{-1}^* \cdots e_{-n}^* \\ 0 \\ \vdots \\ 0 \cdots 0 \end{matrix} \begin{matrix} e_{-1}^* \\ \vdots \\ e_{-1}^* \\ e_0^* \end{matrix} \right\| = 1,$$

and on the right by the lower triangular determinant,

$$(2.18) \quad D^n(F_h(z)) = \left\| \begin{matrix} f_0^* & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ f_h^* & & & \vdots \\ 0 & & & \vdots \\ \vdots & & & 0 \\ 0 \cdots 0 & f_h^* & \cdots & f_0^* \end{matrix} \right\| = 1.$$

We make the important remark that the above determinants are asymmetric, and that another reduction of $D^n(f)$ to a determinant of fixed order is possible. We could have multiplied $D^n(f)$ on the left by the upper triangular determinant $D^n(z^{-k}D_k(z))$, and on the right by the lower triangular determinant $D^n(F_h(z)/G_{k+m}(z))$.

Given the relations (2.1)–(2.10) the following may be verified.

$$D^n(f) \cdot D^n(F_h) = \|(a_{i-j})\| \cdot \|(f_{i-j}^*)\|$$

$$= \left\| \begin{matrix} e_{-m} & e_{-m-1} & \cdots & e_{-n} \\ \vdots & & & \vdots \\ e_0 \\ 0 \\ \vdots \\ 0 \end{matrix} \begin{matrix} \vdots \\ \vdots \\ \vdots \\ \vdots \\ e_0 \end{matrix} \right\| \left\| \begin{matrix} * & \cdots & * \\ \vdots & & \vdots \\ \vdots & & \vdots \\ * & \cdots & * \end{matrix} \right\|$$

$n+1-m$ m

where the “*” indicates entries which are left undetermined. Consequently

$$\begin{aligned}
 D^n(f) &= D^n(D_k/z^{-m}G_{k+m}) \cdot [D^n(f) \cdot D^n(F_h)] \\
 &= \|(e_{i-j}^*)\| \cdot [\|(a_{i-j}\| \cdot \|(f_{i-j}^*)\|] \\
 &= \left\| \begin{array}{cccc} e_0^* e_{-1}^* & \cdots & e_{-n}^* & \\ 0 & & \vdots & \\ \vdots & & \vdots & \\ \vdots & & \vdots & \\ 0 & \cdots & 0 & e_0^* \end{array} \right\| \cdot \left\| \begin{array}{ccc|ccc} e_{-m} & \cdots & e_{-n} & * & \cdots & * \\ \vdots & & \vdots & \vdots & & \vdots \\ e_0 & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & e_0 & * & \cdots & * \end{array} \right\| \\
 (2.19) \quad &= \left\| \begin{array}{c|c} 0 & K \\ \hline I & N \end{array} \right\| \begin{array}{l} m \\ n+1-m \end{array}
 \end{aligned}$$

where I is the identity matrix and K equals the product (2.16). By shifting the rows of I in (2.19) into the upper left-hand corner it follows that

$$(2.20) \quad D^n(f) = (-1)^{m(n+1-m)} \|K\|,$$

and the lemma follows.

In the following two lemmas we show that each of the two matrices on the left-hand side of (2.16) may be written as the product of a pair of matrices.

LEMMA 2.4.

$$\begin{aligned}
 &m \left(\begin{array}{cccc} e_0^* & \cdots & e_{-m+1}^* & \cdots e_{-n}^* \\ 0 & & \vdots & \\ \vdots & & \vdots & \\ \vdots & & \vdots & \\ 0 & \cdots & 0 & e_{-n+m-1}^* \end{array} \right) \\
 &\qquad\qquad\qquad n+1 \\
 &= m \left(\begin{array}{cccc} d_0^* & \cdots & d_{-k}^* & 0 \cdots 0 \\ 0 & & \vdots & \\ \vdots & & \vdots & \\ 0 & \cdots & d_0^* & \cdots d_{-k}^* \end{array} \right) \left(\begin{array}{ccc|ccc} g_0^* & \cdots & & g_{-k-m+1}^* & \cdots & g_{-n}^* \\ 0 & & & 0 & & \vdots \\ \vdots & & & \vdots & & \vdots \\ \vdots & & & \vdots & & \vdots \\ 0 & \cdots & 0 & g_0^* & \cdots & g_{-n+k+m-1}^* \end{array} \right) \\
 &\qquad\qquad\qquad k+m \qquad\qquad\qquad n+1 \\
 &= A \cdot B, \text{ say.}
 \end{aligned}$$

PROOF. Verify by referring to (2.2), (2.3), and (2.9).

LEMMA 2.5.

$$\begin{aligned}
 & n+1 \begin{pmatrix} a_{-n+m-1} & \cdots & a_{-n} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \vdots & & \vdots \\ a_{m-1} & \cdots & a_0 \end{pmatrix} \\
 & \qquad \qquad \qquad m \\
 & = n+1 \begin{pmatrix} g_0 & \cdots & g_{-k-m} & 0 & \cdots & 0 \\ 0 & & & & & \vdots \\ \vdots & & & & & \vdots \\ \vdots & & & & & \vdots \\ 0 & \cdots & \cdots & 0 & g_0 & \cdots & g_{-k-m} \end{pmatrix} \begin{pmatrix} b_{-n-1} & \cdots & b_{-n-m} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \vdots & & \vdots \\ b_{k+m-1} & \cdots & b_k \end{pmatrix} \\
 & \qquad \qquad \qquad n+k+m+1 \qquad \qquad \qquad m \\
 & = C \cdot D, \text{ say.}
 \end{aligned}$$

PROOF. Verify by referring to (2.1), (2.7) and (2.10).

It follows from Lemmas 2.4 and 2.5 that if we denote the right-hand matrix of (2.16) by E , then the matrix K which equals (2.16) may be written $K = ABCDE$, where

$$\begin{aligned}
 A &= (d_{i-j}^*), & B &= (g_{i-j}^*), & C &= (g_{i-j}), \\
 D &= (b_{i-j}), & E &= (f_{i-j}^*),
 \end{aligned}$$

as appears in these two lemmas. So by (2.20) it follows that

$$(2.21) \quad D^n(f) = (-1)^{m(n+1-m)} \|A \cdot B \cdot C \cdot D \cdot E\|,$$

and we must simplify the product of these matrices whose orders are increasing with n .

LEMMA 2.6.

$$\begin{aligned}
 (2.22) \quad B \cdot C \cdot D &= k+m \begin{pmatrix} b_{-n-1} & \cdots & \cdots & b_{-n-m} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ b_{-n+k+m-2} & \cdots & \cdots & b_{-n+k-1} \end{pmatrix} \\
 & \qquad \qquad \qquad m \\
 & = \begin{pmatrix} g_{-n-1}^* & \cdots & \cdots & g_{-n-k-m}^* \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ g_{-n+k+m-2}^* & \cdots & \cdots & g_{-n-1}^* \end{pmatrix} \cdot \begin{pmatrix} g_0 & \cdots & \cdots & g_{-k-m+1} \\ 0 & & & \vdots \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 & g_0 \end{pmatrix} \\
 & \qquad \qquad \qquad k+m \qquad \qquad \qquad k+m
 \end{aligned}$$

$$(2.22) \text{ (cont.)} \quad \cdot \begin{pmatrix} b_0 & \cdots & \cdots & \cdots & b_{-m+1} \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ b_{k+m-1} & \cdots & \cdots & \cdots & b_k \end{pmatrix} \cdot$$

m

PROOF. Multiply B and C together and get that

$$B \cdot C = \left(\begin{array}{c|ccc|c} I & 0 & 0 & \cdots & 0 & 0 & \\ \hline & \vdots & \vdots & & \vdots & \vdots & \\ & 0 & 0 & \cdots & 0 & 0 & \\ \hline & & & & & & M \end{array} \right) \begin{matrix} k+m \\ n-k-m+1 \\ k+m \end{matrix}$$

where I is the identity matrix, and

$$M = k+m \begin{pmatrix} g_{-n+k+m-1}^* & \cdots & \cdots & g_{-n}^* \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ g_{-n+2k+2m-2}^* & \cdots & \cdots & g_{-n+k+m-1}^* \end{pmatrix} \begin{pmatrix} g_{-k-m} & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ \vdots & & & 0 \\ g_{-1} & \cdots & \cdots & g_{-k-m} \end{pmatrix}$$

$k+m \qquad k+m$

Using the relations (2.1) and (2.2) that $\sum_{i=0}^{\min(v, k+m)} g_{-i} g_{-v+i}^* = 0$ for $v \geq 1$, we may rewrite the matrix M so that

$$M = (-1) \begin{pmatrix} g_{-n-1}^* & \cdots & \cdots & g_{-n-k-m}^* \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ g_{-n+k+m-2}^* & \cdots & \cdots & g_{-n-1}^* \end{pmatrix} \begin{pmatrix} g_0 & \cdots & \cdots & g_{-k-m+1} \\ 0 & & & \vdots \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 & g_0 \end{pmatrix}$$

Let $B \cdot C = H$. Multiply H by D . We get that

$$H \cdot D = k+m \begin{pmatrix} I & \left| \begin{array}{ccc} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{array} \right| & M \end{pmatrix} \begin{pmatrix} b_{-n-1} & \cdots & \cdots & b_{-n-m} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ b_{k+m-1} & \cdots & \cdots & b_k \end{pmatrix} \begin{matrix} n+k+m+1 \\ m \end{matrix}$$

= (2.22), the desired result.

LEMMA 2.7.

$$\begin{aligned}
 (2.23) \quad A \cdot B \cdot C \cdot D = & (-1) \begin{pmatrix} e_{-n-1}^* & \cdots & e_{-n-k-m}^* \\ \vdots & & \vdots \\ e_{-n+m-2}^* & \cdots & e_{-n-k-1}^* \end{pmatrix} \cdot \begin{pmatrix} g_0 & \cdots & g_{-k-m+1} \\ \vdots & & \vdots \\ 0 & \cdots & g_0 \end{pmatrix} \\
 & \cdot \begin{pmatrix} b_0 & \cdots & b_{-m+1} \\ \vdots & & \vdots \\ b_{k+m-1} & \cdots & b_k \end{pmatrix} \begin{matrix} k+m \\ m \end{matrix}
 \end{aligned}$$

PROOF. We multiply $H \cdot D$ on the left by A where A is the (d_{i-j}^*) matrix of Lemma 2.4. The left-hand component of $H \cdot D$ is the (b_{i-j}) matrix of (2.20). By Lemma 2.2

$$\begin{aligned}
 (d_{i-j}^*)(b_{i-j}) = & m \begin{pmatrix} d_0^* \cdots d_{-k}^* 0 \cdots 0 \\ \vdots & & \vdots \\ 0 \cdots 0 & d_0^* \cdots d_{-k}^* \end{pmatrix} \begin{pmatrix} b_{-n-1} & \cdots & b_{-n-m} \\ \vdots & & \vdots \\ b_{-n+k+m-2} & \cdots & b_{-n+k-1} \end{pmatrix} \\
 = & m \begin{pmatrix} 0 \cdots 0 \\ \vdots & \vdots \\ 0 \cdots 0 \end{pmatrix} \begin{matrix} m \end{matrix}
 \end{aligned}$$

We remark that without this result the identity we obtain for $D^n(f)$ would not be possible. Continuing our multiplication of $H \cdot D$ by A , since by (2.2), (2.3) and (2.9)

$$\begin{aligned}
 & m \begin{pmatrix} d_0^* \cdots d_{-k}^* 0 \cdots 0 \\ \vdots & & \vdots \\ 0 \cdots 0 & d_0^* \cdots d_{-k}^* \end{pmatrix} \begin{pmatrix} g_{-n-1}^* & \cdots & g_{-n-k-m}^* \\ \vdots & & \vdots \\ g_{-n+k+m-2}^* & \cdots & g_{-n-1}^* \end{pmatrix} \\
 = & m \begin{pmatrix} e_{-n-1}^* & \cdots & e_{-n-k-m}^* \\ \vdots & & \vdots \\ e_{-n+m-2}^* & \cdots & e_{-n-k-1}^* \end{pmatrix}, \\
 & \begin{matrix} k+m \end{matrix}
 \end{aligned}$$

it follows that $A \cdot B \cdot C \cdot D = A \cdot H \cdot D = \underline{(2.23)}$.

THEOREM 2.1.

$$(2.24) \quad D^n(f) = (-1)^{m(n+1)} \det \begin{pmatrix} e_{-n-1}^* & \cdots & e_{-n-k-m}^* \\ \vdots & & \vdots \\ e_{-n+m-2}^* & \cdots & e_{-n-k-1}^* \end{pmatrix} \cdot \begin{pmatrix} g_0 & \cdots & g_{-k-m+1} \\ \vdots & & \vdots \\ 0 & \cdots & g_0 \end{pmatrix} \cdot \begin{pmatrix} b_0 & \cdots & b_{-m+1} \\ \vdots & & \vdots \\ b_{k+m-1} & \cdots & b_k \end{pmatrix}.$$

PROOF. Since $\|E\| = 1$, $\|A \cdot B \cdot C \cdot D \cdot E\| = \|A \cdot B \cdot C \cdot D\|$. Since $(-1)^{m(n+1-m)} = (-1)^{m(n+1)-m}$, by (2.21) and (2.23) it follows that $D^n(f) = \underline{(2.24)}$. So $D^n(f)$ equals a determinant of fixed order m .

3. Evaluation of the determinants $D^n(f)$. In the last section we showed that $D^n(f)$ equaled the determinant of the product of three matrices whose orders were independent of n (2.24). In this section we will evaluate this product. We have of course by (2.1) that $G_{k+m}(z) = \prod_{i=1}^{k+m} (z - r_i)$. In general, let

$$(3.1) \quad G_{k+m}^S(z) = \prod_{i=1; i \notin S}^{k+m} (z - r_i)$$

where S is a subset of the integers $(1, 2, \dots, k+m)$. In the event we wish to be explicit about the entries in S we will write $(i_1, i_2, \dots, i_\alpha)$ to indicate that S contains the integers $i_1, i_2, \dots, i_\alpha$. In particular if S consists of the singleton i , we write

$$(3.2) \quad G_{k+m}^{(i)}(z) = (z - r_i)^{-1} G_{k+m}(z) = \prod_{j=1; j \neq i}^{k+m} (z - r_j).$$

In an analogous way we define the coefficient a_m^S by means of

$$(3.3) \quad \sum_{-\infty}^{\infty} a_m^S z^m = G_{k+m}^S(z) / D_k(z) F_h(z) \quad \text{and} \\ \sum_{-\infty}^{\infty} a_m^{(i)} z^m = G_{k+m}^{(i)}(z) / D_k(z) F_h(z).$$

LEMMA 3.1. *The product of the three matrices of (2.24), $(e_{-n-1+i-j}^*) \cdot (g_{i-j})(b_{i-j})$, is equal to the m by m matrix*

$$(3.4) \quad \left[\sum_{i=1}^{k+m} C_i D_k(r_i) r_i^{n+2-k-s} a_{m-t}^{(i)} \right]$$

for $s, t = 1, 2, \dots, m$, and where $C_i = r_i^{k+m-1} \prod_{j \neq i} (r_i - r_j)^{-1}$.

PROOF. By Lemma 2.1, $e^*_{-n} = \sum_{i=1}^{k+m} C_i D_k(r_i) r_i^{n-k}$. By definition

$$z^{-(k+m)} G_{k+m}(z) = \prod_{i=1}^{k+m} (1 - r_i z^{-1}) = \sum_{i=0}^{k+m} g_{-i} z^{-i}.$$

Consequently

$$g_0 = 1, \quad g_{-1} = - \sum_{s=1}^{k+m} r_s, \quad g_{-2} = \sum_{1 \leq s < t}^{k+m} r_s r_t, \quad \text{etc.}$$

If we multiply the top row of the $(e^*_{-n-1+i-j})$ matrix of (2.24) by the columns of the (g_{i-j}) matrix of (2.24), the following may be easily verified.

$$\begin{aligned} e^*_{-n-1} g_0 &= \sum_{i=1}^{k+m} C_i D_k(r_i) r_i^{n+1-k}, \\ e^*_{-n-1} g_{-1} + e^*_{-n-2} g_0 &= - \sum_{i=1}^{k+m} C_i D_k(r_i) r_i^{n+1-k} \cdot \sum_{s=1; s \neq i}^{k+m} r_s, \\ e^*_{-n-1} g_{-2} + e^*_{-n-2} g_{-1} + e^*_{-n-3} g_0 &= \sum_{i=1}^{k+m} C_i D_k(r_i) r_i^{n+1-k} \cdot \sum_{1 \leq s < t; s, t \neq i}^{k+m} r_s r_t, \end{aligned}$$

and similarly for the remaining products. Thus the top row of $(e^*_{-n-1+i-j}) \cdot (g_{i-j})$ is exactly

$$\begin{aligned} &\left(\sum_{i=1}^{k+m} C_i D_k(r_i) r_i^{n+1-k}, - \sum_{i=1}^{k+m} C_i D_k(r_i) r_i^{n+1-k} \cdot \sum_{s=1; s \neq i}^{k+m} r_s, \right. \\ (3.5) \quad &\left. \dots, (-1)^{k+m-1} \sum_{i=1}^{k+m} C_i D_k(r_i) r_i^{n+1-k} \cdot \prod_{s=1; s \neq i}^{k+m} r_s \right). \end{aligned}$$

The second row of $(e^*_{-n-1+i-j}) (g_{i-j})$ will be the same except that the exponent of the r_i factors in each term of the sums will be lowered by one, and similarly for the remaining rows until with the last row each r_i appears in each sum to the power $n - k - m + 2$. Since $G_{k+m}^{(i)}(z)/D_k(z)F_n(z) = G_{k+m}^{(i)}(z) \cdot \sum_{-\infty}^{\infty} b_v z^v = \sum_{-\infty}^{\infty} a_v^{(i)} z^v$, it follows that

$$\begin{aligned} a_{v+m-1}^{(i)} &= b_v - b_{v+1} \sum_{s=1; s \neq i}^{k+m} r_s + b_{v+2} \sum_{1 \leq s < t; s, t \neq i}^{k+m} r_s r_t \\ (3.6) \quad &- \dots (-1)^{k+m-1} b_{v+k+m-1} \prod_{s=1; s \neq i}^{k+m} r_s. \end{aligned}$$

By applying (3.6) to (3.5) it follows that the product of the matrices of (2.24) equals (3.4).

LEMMA 3.2. *The determinant of the m th order matrix of (3.4) is equal to the product of two $(k + m)$ th order determinants,*

(3.7)

$$\left\| \begin{array}{cccc} C_1 D_k(r_1) & C_2 D_k(r_2) & \dots & C_{k+m} D_k(r_{k+m}) \\ C_1 D_k(r_1) r_1^{-1} & C_2 D_k(r_2) r_2^{-1} & \dots & C_{k+m} D_k(r_{k+m}) r_{k+m}^{-1} \\ \vdots & \vdots & & \vdots \\ C_1 D_k(r_1) r_1^{-(k+m)+1} & C_2 D_k(r_2) r_2^{-(k+m)+1} & \dots & C_{k+m} D_k(r_{k+m}) r_{k+m}^{-(k+m)+1} \end{array} \right\|$$

$$\cdot \left\| \begin{array}{cccc} D_k^{-1}(r_1) & \dots & D_k^{-1}(r_1) r_1^{k-1} & a_{m-1}^{(1)} r_1^{n+1} & \dots & a_0^{(1)} r_1^{n+1} \\ D_k^{-1}(r_2) & \dots & D_k^{-1}(r_2) r_2^{k-1} & a_{m-1}^{(2)} r_2^{n+1} & \dots & a_0^{(2)} r_2^{n+1} \\ \vdots & & \vdots & \vdots & & \vdots \\ D_k^{-1}(r_{k+m}) & \dots & D_k^{-1}(r_{k+m}) r_{k+m}^{k-1} & a_{m-1}^{(k+m)} r_{k+m}^{n+1} & \dots & a_0^{(k+m)} r_{k+m}^{n+1} \end{array} \right\|$$

PROOF. By (2.11), $\sum_{i=1}^{k+m} C_i = 1$, $\sum_{i=1}^{k+m} C_i r_i^{-j} = 0$, $j = 1, 2, \dots, k + m - 1$. Consequently (3.7) equals

(3.8)

$$\left\| \begin{array}{cccc|ccc} 1 & * & * & \dots & * & \dots & * \\ 0 & 1 & & \vdots & \vdots & & \vdots \\ \vdots & & \ddots & * & \vdots & & \vdots \\ & & & 1 & * & & \vdots \\ 0 & \dots & 0 & 1 & * & \dots & * \\ \hline 0 & 0 & \dots & 0 & 0 & & \\ \vdots & & & & & & \\ 0 & 0 & \dots & 0 & 0 & & \end{array} \right\|$$

K^*

where K^* is the matrix (3.4), and clearly (3.8) = $\|K^*\|$.

By Theorem 2.1, $D^n(f)$ equals $(-1)^{m(n+1)}$ multiplied by the determinant of (2.24). By Lemma 3.1 and Lemma 3.2,

(3.8) $D^n(f) = (-1)^{m(n+1)} \cdot [(3.7)].$

Because of the representation, except for sign, of $D^n(f)$ by (3.7), we are able to evaluate it. For the left-hand determinant of (3.7) is easy to evaluate, and the right-hand determinant of (3.7) may be expanded in such a way that we will be able to evaluate it.

Clearly the left-hand determinant of (3.7) equals

(3.9) $\prod_{i=1}^{k+m} C_i D_k(r_i) V(r_1^{-1}, r_2^{-1}, \dots, r_{k+m}^{-1})$

where $V(r_1^{-1}, r_2^{-1}, \dots, r_{k+m}^{-1})$ is the Vandermonde determinant based on the numbers r_i^{-1} , $i = 1, 2, \dots, k + m$. It can easily be shown that

$$\begin{aligned} \prod_{i=1}^{k+m} C_i &= \prod_{i=1}^{k+m} r_i^{k+m-1} \cdot \prod_{j \neq i} (r_i - r_j)^{-1} \\ &= (-1)^\sigma \prod_{i=1}^{k+m} r_i^{k+m-1} V^{-2}(r_1, r_2, \dots, r_{k+m}), \end{aligned}$$

and that

$$V(r_1^{-1}, r_2^{-1}, \dots, r_{k+m}^{-1}) = (-1)^\sigma \prod_{i=1}^{k+m} r_i^{-k-m+1} V(r_1, r_2, \dots, r_{k+m})$$

where $\sigma = \frac{1}{2}(k + m - 1)(k + m)$. From this it follows that

$$(3.10) \quad (3.9) = \prod_{i=1}^{k+m} D_k(r_i) V^{-1}(r_1, r_2, \dots, r_{k+m}).$$

Laplacian expansion of the right-hand determinant of (3.7) on the last m columns gives us

$$(3.11) \quad \sum \prod_{j \in \bar{I}} D_k^{-1}(r_j) V(\bar{I}) \prod_{i \in I} r_i^{n+1} V(a_0^{(i)}).$$

The sum is taken over all subsets I of m integers from $(1, 2, \dots, k + m)$, and $\bar{I} = (1, 2, \dots, k + m) - I$, $V(\bar{I})$ is the Vandermonde determinant determined by r_j , $j \in \bar{I}$, and

$$V(a_0^{(i)}) = \left\| \begin{array}{ccc} a_{m-1}^{(i_1)} & a_{m-2}^{(i_1)} & \dots & a_0^{(i_1)} \\ a_{m-1}^{(i_2)} & a_{m-2}^{(i_2)} & \dots & a_0^{(i_2)} \\ \vdots & \vdots & & \vdots \\ a_{m-1}^{(i_m)} & a_{m-2}^{(i_m)} & \dots & a_0^{(i_m)} \end{array} \right\|, \quad (i_1, i_2, \dots, i_m) = I.$$

LEMMA 3.3. $V(a_0^{(i)}) = V(I) \cdot \|(a_{s-t}^I)_{s,t=1}^m\|.$

PROOF. As before, $V(I)$ is the Vandermonde determinant determined by I , and $(a_{s-t}^I)_{s,t=1}^m$ is the m th order Toeplitz matrix generated by the function

$$G_{k+m}^I(z)/D_k(z) \cdot F_h(z) = \sum_{-\infty}^{\infty} a_v^I z^v$$

defined by (3.1) and (3.3). We give a demonstration of this for

$$(3.12) \quad \left\| \begin{array}{ccc} a_{m-1}^{(1)} & a_{m-2}^{(1)} & \dots & a_0^{(1)} \\ a_{m-1}^{(2)} & a_{m-2}^{(2)} & \dots & a_0^{(2)} \\ \vdots & \vdots & & \vdots \\ a_{m-1}^{(m)} & a_{m-2}^{(m)} & \dots & a_0^{(m)} \end{array} \right\|.$$

By the convention (3.3), if S is any subset of the set $(1, 2, \dots, k + m)$ with s elements, then

$$a_{v+m-S}^S = b_v - b_{v+1} \sum_{i \notin S} r_i + \dots + (-1)^{k+m-s} b_{v+k+m-s} \prod_{i \in S} r_i.$$

So we can easily verify that $a_{v+m-1}^{(i)} = a_{v+m-2}^{(i,j)} - r_j a_{v+m-1}^{(i,j)}$ and that

$$(3.13) \quad (3.12) = \begin{vmatrix} a_{m-2}^{(1,i)} - r_i a_{m-1}^{(1,i)} & \dots & a_{-1}^{(1,i)} - r_i a_0^{(1,i)} \\ a_{m-2}^{(1,2)} - r_1 a_{m-1}^{(1,2)} & \dots & a_{-1}^{(1,2)} - r_1 a_0^{(1,2)} \\ \vdots & & \vdots \\ a_{m-2}^{(1,m)} - r_1 a_{m-1}^{(1,m)} & \dots & a_{-1}^{(1,m)} - r_1 a_0^{(1,m)} \end{vmatrix}.$$

Letting i alternately be equal to $2, 3, \dots, m$, and subtracting the first row from each of the others, and rewriting the first row in its original form, we get that the right-hand side of (3.13) equals

$$(3.14) \quad \begin{vmatrix} a_{m-1}^{(1)} & \dots & a_0^{(1)} \\ a_{m-1}^{(1,2)} & \dots & a_0^{(1,2)} \\ \vdots & & \vdots \\ a_{m-1}^{(1,m)} & \dots & a_0^{(1,m)} \end{vmatrix} (r_2 - r_1) (r_3 - r_1) \dots (r_m - r_1).$$

Since $a_v^{(1)} = a_{v-1}^{(1,2)} - r_2 a_v^{(1,2)}$, if we add r_2 times the second row to the first row in (3.14), we obtain the result that (3.12) equals

$$(3.15) \quad \begin{vmatrix} a_{m-2}^{(1,2)} & \dots & a_{-1}^{(1,2)} \\ a_{m-1}^{(1,2)} & \dots & a_0^{(1,2)} \\ \vdots & & \vdots \\ a_{m-1}^{(1,m)} & \dots & a_0^{(1,m)} \end{vmatrix} (r_2 - r_1) \dots (r_m - r_1).$$

Similarly using the second row as a pivot row to reduce the rows below it, we can verify that the determinant in (3.15) equals

$$(3.16) \quad \begin{vmatrix} a_{m-2}^{(1,2)} & \dots & a_{-1}^{(1,2)} \\ a_{m-1}^{(1,2)} & \dots & a_0^{(1,2)} \\ a_{m-1}^{(1,2,3)} & \dots & a_0^{(1,2,3)} \\ \vdots & & \vdots \\ a_{m-1}^{(1,2,m)} & \dots & a_0^{(1,2,m)} \end{vmatrix} (r_3 - r_2) (r_4 - r_2) \dots (r_m - r_2).$$

Adding r_3 times the third row to the second row, and subsequently r_3 times the second row to the first row, we reduce (3.16) to

$$\left\| \begin{array}{cccc} a_{m-3}^{(1,2,3)} & \dots & a_{-2}^{(1,2,3)} \\ a_{m-2}^{(1,2,3)} & \dots & a_{-1}^{(1,2,3)} \\ a_{m-1}^{(1,2,3)} & \dots & a_0^{(1,2,3)} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ a_{m-1}^{(1,2,m)} & \dots & a_0^{(1,2,m)} \end{array} \right\| (r_3 - r_2) \dots (r_m - r_2).$$

Continuing in this manner, successively using the third, fourth, etc. rows as pivot rows and factoring out the appropriate factors, we prove that for $I = (1, 2, \dots, m)$,

$$(3.12) = V(I) \| (a_{s-t}^I)_{s,t=1}^m \|.$$

The general proof follows as above.

Consequently, by (3.8), (3.10), (3.11) and Lemma 3.3,

$$\begin{aligned} D^n(f) &= (-1)^{m(n+1)} \prod_{i=1}^{k+m} D_k(r_i) V^{-1}(r_1, r_2, \dots, r_{k+m}) \\ (3.17) \quad &\cdot \left[\sum_I \prod_{j \in \bar{I}} D_k^{-1}(r_j) V(\bar{I}) \cdot \prod_{i \in I} r_i^{n+1} V(I) \| (a_{s-t}^I)_{s,t=1}^m \| \right] \\ &= (-1)^{m(n+1)} \sum_I \prod_{i \in I; j \in \bar{I}} r_i^{n+1} D_k(r_i) \cdot \| (a_{s-t}^I)_{s,t=1}^m \| (r_i - r_j)^{-1}, \end{aligned}$$

where I runs over all subsets of order m of the set $(1, 2, \dots, k + m)$, and $\bar{I} = (1, 2, \dots, k + m) - I$.

LEMMA 3.4. $\| (a_{s-t}^I)_{s,t=1}^m \| = \| (a_{s-t}^h)_{s,t=1}^h \|.$

PROOF. By (3.3), $\sum_{-\infty}^{\infty} a_v z^v = G_{k+m}^I(z)/D_k(z)F_h(z)$, and $G_{k+m}^I(z) = \prod_{i=1; i \notin I}^{k+m} (z - r_i)$, a monic k th degree polynomial. By (2.3), $D_k(z)$ is also a monic k th degree polynomial. Therefore the power series expansion of

$$(3.18) \quad G_{k+m}^I(z)/D_k(z) = \left(\sum_{-\infty}^{\infty} a_v^I z^v \right) \cdot F_h(z) \text{ around } z = \infty$$

commences with the constant term 1. By (2.5), $F_h(z) = \sum_{i=0}^h f_i^* z^i$ with $f_0^* = 1$. By (3.18) it follows that

$$\begin{aligned}
 & \begin{array}{c} h \\ m-h \end{array} \left\| \begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & \\ 0 & \cdot & & \cdot & \vdots & & \vdots & \\ \vdots & & & \cdot & \vdots & & \vdots & \\ 0 & \cdot & \cdot & 0 & 1 & 0 & \cdot & 0 \end{array} \right\| \left\| \begin{array}{cccccccc} a_0^I & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{-m+1}^I \\ \vdots & & & & & & & \vdots \\ \vdots & & & & & & & \vdots \\ \vdots & & & & & & & \vdots \\ \vdots & & & & & & & \vdots \\ \vdots & & & & & & & \vdots \\ \vdots & & & & & & & \vdots \\ a_{m-1}^I & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_0^I \end{array} \right\| \\
 & = \begin{array}{c} h \\ m-h \end{array} \left\| \begin{array}{ccc|ccc} a_0^I & \cdots & a_{-h+1}^I & * & \cdots & * \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{h-1}^I & \cdots & a_0^I & * & \cdots & * \end{array} \right\| = \|(a_{s-t}^I)_{s,t=1}^h\| \\
 & \left\| \begin{array}{ccc|ccc} 0 & \cdots & 0 & 1 & * & * \\ \cdot & & \cdot & 0 & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & 0 & 0 & \cdot & 0 & 1 \end{array} \right\|
 \end{aligned}$$

By applying the results of this lemma to (3.17), we have derived the following identity for $D^n(f)$.

$$(3.19) \quad D^n(f) = (-1)^{m(n+1)} \sum_I \prod_{i \in I; j \in \bar{I}} r_i^{n+1} D_k(r_i) \cdot \|(a_{s-t}^I)_{s,t=1}^h\| \|(r_i - r_j)^{-1}\|,$$

where I runs over all subsets of order m of the set $(1, 2, \dots, k + m)$, and $\bar{I} = (1, 2, \dots, k + m) - I$. The only factors that are not explicitly evaluated are the h th order determinants $\|(a_{s-t}^I)_{s,t=1}^h\|$. Since the a_s^I 's are generated by the function

$$G_{k+m}^I(z)/D_k(z)F_h(z) = \sum_{-\infty}^{\infty} a_v^I z^v,$$

and by (3.1)

$$G_{k+m}^I(z) = \prod_{j \in \bar{I}} (z - r_j),$$

each a_s^I is a polynomial in the r_j 's, $j \in \bar{I}$. Consequently each determinant $\|(a_{s-t}^I)_{s,t=1}^h\|$ is a polynomial at most of h th degree in each of the r_j 's, $j \in \bar{I}$.

LEMMA 3.5. $\|(a_{s-t}^I)_{s,t=1}^h\| = C \prod_{j \in \bar{I}} F_h(r_j)$ where C is a constant that is independent of the r_j 's, $j \in \bar{I}$.

PROOF. The factor $\prod_{j \in \bar{I}} F_h(r_j)$ is of exact degree h in each of the r_j 's. So by the remarks immediately preceding the lemma we can conclude that C is a constant which is independent of the r_j 's. Direct evaluation of the determinants $\|(a_{s-t}^I)_{s,t=1}^h\|$ seems to be difficult. Consequently we approach the problem in-

directly. We made the remark in the proof of Lemma 2.3 that the reduction of $D^n(f)$ to the m th order determinant (2.24) which led to the identity (3.19) was effected by multiplication of $D^n(f)$ by determinants (2.17) and (2.18) which are asymmetric. We indicated that another reduction is possible.

Define the Laurent series $\Sigma_{-\infty}^{\infty} \hat{a}_v z^v$ by

$$(-1)^{k+m} \prod_{i=1}^{k+m} r_i^{-1} \cdot \sum_{-\infty}^{\infty} a_v z^v = \sum_{-\infty}^{\infty} \hat{a}_v z^v,$$

so that

$$(3.20) \quad (-1)^{k+m} \prod_{i=1}^{k+m} r_i^{-1} \cdot a_v = \hat{a}_v.$$

Since each r_i is assumed to be nonzero, multiplication by $\prod_{i=1}^{k+m} r_i^{-1}$ is possible. Clearly if $\hat{f}(z) = (-1)^{k+m} \prod_{i=1}^{k+m} r_i^{-1} \cdot f(z)$, then $\hat{f}(z) = \Sigma_{-\infty}^{\infty} \hat{a}_v z^v$ in the annulus A . Let $D^n(\hat{f})$ be the $(n + 1)$ st order determinant generated by the \hat{a}_v 's. We may reduce $D^n(\hat{f})$ to a determinant of fixed order exactly as we did in §2 for $D^n(f)$.

Since the numerator of $f(z)$ is $G_{k+m}(z) = \prod_{i=1}^{k+m} (z - r_i)$, the numerator of $\hat{f}(z)$ is

$$(3.21) \quad (-1)^{k+m} r_i^{-1} G_{k+m}(z) = \prod_{i=1}^{k+m} (1 - r_i^{-1} z), \quad \text{and}$$

$$\hat{f}(z) = \prod_{i=1}^{k+m} (1 - r_i^{-1} z) / D_k(z) F_h(z).$$

Let us multiply $D^n(\hat{f})$ on the left by the upper triangular determinant $D^n(z^{-k} D_k(z)) = 1$, and on the right by the lower triangular determinant $D^n(F_h(z) / \prod_{i=1}^{k+m} (1 - r_i^{-1} z)) = 1$. Reduction of the determinant $D^n(\hat{f})$ by the techniques of §2 results in reducing $D^n(\hat{f})$ to a determinant of order k similar to (2.24). Consequent evaluation of this k th order determinant exactly as we have evaluated $D^n(f)$ results in the identity

$$(3.22) \quad D^n(\hat{f}) = (-1)^{k(n+1)} \sum_{\bar{I}} \prod_{j \in \bar{I}; j \in I} r_j^{-n-1-m} F_h(r_j) \cdot \|(\hat{a}_{-k+s-t}^{\bar{I}})_{s,t=1}^k \| (r_j^{-1} - r_i^{-1})^{-1},$$

where \bar{I} runs over all subsets of order k of the set $(1, 2, \dots, k + m)$, and $I = (1, \dots, k + m) - \bar{I}$, and the coefficients $\hat{a}_s^{\bar{I}}$ are generated by the function

$$\prod_{i \in I} (1 - r_i^{-1} z) / D_k(z) F_h(z) = \sum_{-\infty}^{\infty} \hat{a}_v^{\bar{I}} z^v.$$

A comparison of the identity for $D^n(\hat{f})$ (3.22) with the identity for $D^n(f)$ (3.19) results in some apparent discrepancies. In (3.19) appear the factors $\prod_{i \in \bar{I}} r_i^{n+1} D_k(r_i)$, of degree $n + k + 1$ in each of the r_i 's. In (3.22) appear the factors $\prod_{j \in \bar{I}} r_j^{-n-1-m} F_h(r_j)$. Since $F_h(0) \neq 0$, and $m \geq h$, these factors are of degree $n + m + 1$ in each of the r_j^{-1} 's which is what is desired. We note that the coef-

ficient \hat{a}^I_{-k} lies on the main diagonal of the unevaluated determinantal factors appearing in (3.22) rather than \hat{a}^I_0 . This occurs because of differences which arise in simplifying the determinants arising in the alternative approach vis-à-vis the simplification of the determinants appearing in Lemma 3.2.

By (3.20),

$$(3.23) \quad (-1)^{(k+m)(n+1)} \prod_{i=1}^{k+m} r_i^{n+1} D^n(\hat{f}) = D^n(f).$$

Because

$$(-1)^{(k+m)(n+1)} \cdot (-1)^{k(n+1)} = (-1)^{m(n+1)}$$

and

$$\prod_{j \in \bar{I}; i \in I} (r_j^{-1} - r_i^{-1})^{-1} = \prod_{i \in I; j \in \bar{I}} r_i^k r_j^m (r_i - r_j)^{-1},$$

we have, after carrying out the multiplication indicated on the left-hand side of (3.23), that

$$\begin{aligned} & (-1)^{m(n+1)} \sum_I \prod_{i \in I; j \in \bar{I}} r_i^{n+1} \cdot r_i^k \|(\hat{a}^I_{-k+s-t})^k_{s,t=1}\| \cdot F_h(r_j) (r_i - r_j)^{-1} \\ & = (-1)^{m(n+1)} \sum_I \prod_{i \in I; j \in \bar{I}} r_i^{n+1} \cdot D_k(r_i) \|(\alpha^I_{s-t})^h_{s,t=1}\| (r_i - r_j)^{-1}. \end{aligned}$$

Because the r_i may be chosen independently of one another and the above equality holds for all n , we may equate corresponding terms which are defined by the same set of indices I . From this it follows that, for each I ,

$$\begin{aligned} & \prod_{i \in I; j \in \bar{I}} r_i^{n+1} \cdot [r_i^k \|(\hat{a}^I_{-k+s-t})^k_{s,t=1}\|] \cdot F_h(r_j) (r_i - r_j)^{-1} \\ & = \prod_{i \in I; j \in \bar{I}} r_i^{n+1} D_k(r_i) \|(\alpha^I_{s-t})^h_{s,t=1}\| (r_i - r_j)^{-1}. \end{aligned}$$

Consequently

$$(3.24) \quad \prod_{i \in I; j \in \bar{I}} [r_i^k \|(\hat{a}^I_{-k+s-t})^k_{s,t=1}\|] \cdot F_h(r_j) = \prod_{i \in I; j \in \bar{I}} D_k(r_i) \cdot \|(\alpha^I_{s-t})^h_{s,t=1}\|.$$

But, as we remarked immediately preceding this lemma, $\|(\alpha^I_{s-t})^h_{s,t=1}\|$ is a polynomial at most of degree h in each of the r_j 's, $j \in \bar{I}$. Similarly, since the \hat{a}^I_{-k+s-t} are functions of the r_i^{-1} 's, $i \in I$, $\prod_{i \in I} r_i^k \|(\hat{a}^I_{-k+s-t})^k_{s,t=1}\|$ is a polynomial at most of degree k in each r_i , $i \in I$. We may conclude from these observations and (3.24) that

$$\|(\alpha^I_{s-t})^h_{s,t=1}\| = C \prod_{j \in \bar{I}} F_h(r_j)$$

and

$$\prod_{i \in I} r_i^k \|(\hat{a}^I_{-k+s-t})^k_{s,t=1}\| = C \prod_{i \in I} D_k(r_i).$$

This concludes the proof of Lemma 3.5.

We have almost completed the evaluation of $D^n(f)$. Applying the results of Lemma 3.5 to (3.19), the identity for $D^n(f)$ assumes the form

$$(3.26) \quad D^n(f) = (-1)^{m(n+1)} \cdot C \cdot \sum_I \prod_{i \in I; j \in \bar{I}} r_i^{n+1} D_k(r_i) F_h(r_j) (r_i - r_j)^{-1},$$

where I runs over all subsets of order m of the set $(1, 2, \dots, k + m)$ and $\bar{I} = (1, 2, \dots, k + m) - I$. Only the constant term C remains to be determined.

LEMMA 3.6. $C = \prod_{s \in K; t \in H} \rho_t^k / (\rho_t - \delta_s)$ where $K = (1, \dots, k)$ and $H = (1, \dots, h)$.

PROOF. Because the constant C is independent of the roots r_i of the polynomial $G_{k+m}(z)$, we may choose the roots and thereby the polynomial $G_{k+m}(z)$ in such a way that the constant C will be determined. In particular we may assume that $m = h$. Choose the roots $r_i, i = 1, \dots, k$, to lie respectively within an ϵ -neighborhood of the roots $\delta_s, s = 1, \dots, k$, of $D_k(z)$ and the roots $r_i, i = k + 1, \dots, k + h$, to lie within an ϵ -neighborhood of the roots $\rho_t, t = 1, \dots, h$, of $F_h(z)$ but such that $|r_i - r_j| \geq O(\epsilon)$ where ϵ is a small positive number. Let $I_0 = (k + 1, \dots, k + h), \bar{I}_0 = (1, \dots, k)$. We wish to show for all $I \neq I_0$ that if the roots r_i tend to the roots δ_s and ρ_t of $D_k(z)$ and $F_h(z)$ respectively, then

$$(3.27) \quad \prod_{i \in I; j \in \bar{I}} D_k(r_i) F_h(r_j) / (r_i - r_j)$$

tends to zero. This would be obvious except for the possibility that $D_k(z)$ and $F_h(z)$ may have multiple roots and so for some $i, j, r_i - r_j \rightarrow 0$.

For $I \neq I_0, I$ and \bar{I} may each be written as the union of a nonempty disjoint pair of sets as follows. Let $I = I_0^* \cup \bar{I}_0^*$ and $\bar{I} = I_0^{**} \cup \bar{I}_0^{**}$ where $I_0^*, I_0^{**} \subset I_0$, and $\bar{I}_0, \bar{I}_0^{**} \subset \bar{I}_0$. But then

$$(3.27) = \prod_{i \in I_0^*; j \in \bar{I}_0^{**}} \frac{D_k(r_i) F_h(r_j)}{(r_i - r_j)} \cdot \prod_{i \in \bar{I}_0^*; j \in I_0^{**}} \frac{D_k(r_i) F_h(r_j)}{(r_i - r_j)} \\ \cdot \prod_{i \in I_0^*; j \in \bar{I}_0^*} \frac{D_k(r_i) F_h(r_j)}{(r_i - r_j)} \cdot \prod_{i \in I_0^{**}; j \in \bar{I}_0^{**}} \frac{D_k(r_i) F_h(r_j)}{(r_i - r_j)} \\ = \hat{B} \cdot \hat{C} \cdot \hat{D} \cdot \hat{E}, \text{ say.}$$

From the manner in which the roots r_i are chosen,

$$\prod_{i \in I_0^*; j \in \bar{I}_0^{**}} \frac{F_h(r_j)}{(r_i - r_j)}, \quad \prod_{i \in \bar{I}_0^*; j \in I_0^{**}} \frac{D_k(r_i)}{(r_i - r_j)}, \quad \prod_{i \in I_0^*; j \in \bar{I}_0^*} \frac{1}{(r_i - r_j)}$$

$= O(1)$, since the roots δ_s and ρ_t are separated by the annulus A . So $\hat{B} = \hat{C} = \hat{D} = O(1)$. But necessarily $\hat{E} = O(\epsilon)$. Consequently $(3.27) \rightarrow 0$ as the roots r_i tend to the appropriate limits. With the same limits,

$$(3.28) \quad \lim \prod_{i \in I_0; j \in \bar{I}_0} \frac{D_k(r_i) F_h(r_j)}{(r_i - r_j)} = \prod_{s \in K; t \in H} \frac{D_k(\rho_t) F_h(\delta_s)}{(\rho_t - \delta_s)}.$$

So the limit of the right-hand side of (3.26) equals

$$(3.29) \quad (-1)^{h(n+1)} \cdot C \cdot \prod_{s \in K; t \in H} \rho_t^{n+1} D_k(\rho_t) F_h(\delta_s) / (\rho_t - \delta_s).$$

Moreover it is clear that

$$\begin{aligned} f(z) &= \prod_{i=1}^{k+h} (z - r_i) / \left[\prod_{s=1}^k (z - \delta_s) \prod_{t=1}^h (1 - \rho_t^{-1} z) \right] \\ &= \frac{[(-1)^h \prod_{i=1}^k (z - r_i) \prod_{i=k+1}^{k+h} r_i (1 - r_i^{-1} z)]}{[\prod_{s=1}^k (z - \delta_s) \prod_{t=1}^h (1 - \rho_t^{-1} z)]} \end{aligned}$$

tends to $(-1)^h \prod_{t \in H} \rho_t$, a constant.

For each n , $D^n(f)$ varies continuously with the coefficients a_m of the Laurent series representation of f and these coefficients vary continuously with the roots of $G_{k+h}(z)$. Consequently

$$\lim D^n(f) = (-1)^{h(n+1)} \prod_{t \in H} \rho_t^{n+1}.$$

From (3.26) and (3.29)

$$\begin{aligned} (-1)^{h(n+1)} \prod_{t \in H} \rho_t^{n+1} &= (-1)^{h(n+1)} \cdot C \\ &\quad \cdot \prod_{s \in K; t \in H} \rho_t^{n+1} D_k(\rho_t) F_h(\delta_s) / (\rho_t - \delta_s), \end{aligned}$$

and so

$$C = \prod_{s \in K; t \in H} (\rho_t - \delta_s) / D_k(\rho_t) F_h(\delta_s).$$

But $D_k(\rho_t) = \prod_{s \in K} (\rho_t - \delta_s)$ and $F_h(\delta_s) = \prod_{t \in H} (1 - \rho_t^{-1} \delta_s) = \prod_{t \in H} \rho_t^{-1} (\rho_t - \delta_s)$. Consequently

$$C = \prod_{s \in K; t \in H} \frac{\rho_t^k (\rho_t - \delta_s)}{(\rho_t - \delta_s)^2} = \prod_{s \in K; t \in H} \frac{\rho_t^k}{(\rho_t - \delta_s)}.$$

We are now in the position to prove the following theorem.

THEOREM 3.1. *Let $f(z) = G_{k+m}(z) / D_k(z) F_h(z)$ where $G_{k+m}(z)$, $D_k(z)$, and $F_h(z)$ satisfy the conditions of Proposition 1.1. Assume, in addition, that the zeros of $G_{k+m}(z)$ are distinct and not equal to zero, and that the coefficient of the z^{k+m} term of $G_{k+m}(z)$ is equal to 1. Then*

$$(3.30) \quad D^n(f) = (-1)^{m(n+1)} \sum_{I \in \underline{I}, s \in K} \prod_{j \in I, t \in H} r_i^{n+1} \left[\frac{(r_i - \delta_s) (\rho_t - r_j)}{(r_i - r_j) (\rho_t - \delta_s)} \right]$$

where I runs over all subsets of order m of the set $(1, \dots, k+m)$, $\bar{I} = (1, \dots, k+m) - I$, $K = (1, \dots, k)$, and $H = (1, \dots, h)$.

PROOF. By definition of $D_k(z)$ and $F_h(z)$, $D_k(r_i) = \prod_{s \in K} (r_i - \delta_s)$, and

$F_h(r_j) = \prod_{t \in H} \rho_t^{-1}(\rho_t - r_j)$. So by Lemma 3.6 and (3.26),

$$\begin{aligned} D^n(f) &= (-1)^{m(n+1)} \sum_I \prod_{\substack{i \in I, s \in K \\ j \in I, t \in H}} \frac{r_i^{n+1} \rho_t^k (r_i - \delta_s) \rho_t^{-k} (\rho_t - r_j)}{(r_i - r_j) (\rho_t - \delta_s)} \\ &= (-1)^{m(n+1)} \sum_I \prod_{\substack{i \in I, s \in K \\ j \in I, t \in H}} \frac{r_i^{n+1} (r_i - \delta_s) (\rho_t - r_j)}{(r_i - r_j) (\rho_t - \delta_s)}, \end{aligned}$$

which is the conclusion of the theorem.

4. Applications of the identity for $D^n(f)$. Let A be the annulus defined by (1.1). Let $G_s(z), D_k(z), F_h(z)$ be polynomials having no common factors of exact degree s, k , and h respectively. We assume $D_k(z)$ and $F_h(z)$ satisfy the conditions required by Proposition 1.1 and are expressed as in (2.3) and (2.5) respectively. We impose no conditions upon $G_s(z)$. We assume that $k \geq 1$, and if $h = 0$ that $s \geq k + 1$.

It follows that

$$f(z) - \lambda = \frac{G_s(z) - \lambda D_k(z) F_h(z)}{D_k(z) F_h(z)} = \frac{G_{k+m}(\lambda, z)}{D_k(z) F_h(z)}$$

where $G_{k+m}(\lambda, z) = G_s(z) - \lambda D_k(z) F_h(z)$ and $k + m = \max(s, k + h)$.

If $s \leq k + h$ there is one value of λ for which $G_{k+m}(\lambda, z)$ has less than $k + m$ roots, otherwise $G_{k+m}(\lambda, z)$ is of exact degree $k + m$ and satisfies the hypothesis of Proposition 1.1. Thus the Laurent series expansion (1.3) of $f(z) - \lambda$ in A generates matrices $T_n(f - \lambda)$ which are not triangular.

From the theory of algebraic functions [2, pp. 103–104] the set of values of λ for which $G_{k+m}(\lambda, z) = 0$ has multiple roots, has $z = 0$ as a root, or has less than $k + m$ roots is a finite set. For all other λ we denote the roots of $G_{k+m}(\lambda, z) = 0$ by $r_i(\lambda), i = 1, \dots, k + m$, and

$$f(z) - \lambda = c(\lambda) \prod_{i=1}^{k+m} (z - r_i(\lambda)) / D_k(z) F_h(z),$$

$c(\lambda) = a, a - \lambda b$, or $-b$ according to whether $s > k + h, s = k + h$, or $s < k + h$. Note, a is the coefficient of z^s of $G_s(z)$, and b is the coefficient of z^{k+h} of $D_k(z) \cdot F_h(z)$.

By Theorem 3.1 and the first paragraph of §2

$$(4.1) \quad D^n(f - \lambda) = [(-1)^m c(\lambda)]^{n+1} \sum_I \prod_{\substack{i \in I, s \in K \\ j \in I, t \in H}} r_i(\lambda)^{n+1} \frac{(r_i(\lambda) - \delta_s)(\rho_t - r_j(\lambda))}{(r_i(\lambda) - r_j(\lambda))(\rho_t - \delta_s)}.$$

Assume that for fixed λ the roots $r_i(\lambda)$ are indexed by increasing modulus, so that $|r_1(\lambda)| \leq |r_2(\lambda)| \leq \dots \leq |r_{k+m}(\lambda)|$. We define the set C to be the set

$$C = \{\lambda : |r_k(\lambda)| = |r_{k+1}(\lambda)|\}.$$

The analysis provided in [3] and [4] shows that the set C is bounded, contains no isolated points, and consists of a finite union of closed analytic arcs. In addition the arguments of J. L. Ullman [5] may be employed to show that the set C is connected. Since the set C contains no isolated points and the set of λ 's for which the identity (4.1) does not hold is a finite set, the techniques of Schmidt and Spitzer allow us to make the identification of the limit set B of the eigenvalues σ_n with the set C defined above.

A related question is the following. Define a sequence of measures α_n ,

$$\alpha_n(E) = (n + 1)^{-1} \sum_{\lambda_{ni} \in E} 1,$$

where $\lambda_{ni} \in \sigma_n$, and E is an arbitrary set in the λ -plane. Let α be any weak limit of the measures α_n . It will be shown in a later paper that the limit measure α is unique and has at most two atoms. The rational functions f for which α has atoms will be characterized and the weight of the atoms determined.

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