ABSTRACT. Let $T_n(f) = (a_{i,j})_{i,j=0}^n$ be the finite Toeplitz matrices generated by the Laurent expansion of an arbitrary rational function. An identity is developed for \( \det(T_n(f) - \lambda) \) which may be used to prove that the limit set of the eigenvalues of the $T_n(f)$ is a point or consists of a finite number of analytic arcs.

1. Introduction. Let $f(z) = \sum m z^m$ be the Laurent expansion of an arbitrary rational function. Define matrices $T_n(f)$ where $T_n(f) = (a_{i-j})$, $i, j = 0, \cdots, n$. Such matrices are called Toeplitz matrices and may be generated by functions which are not rational. Denote by $\sigma_n$ the set of $n + 1$ eigenvalues of $T_n(f)$,

$$\sigma_n = \{\lambda_0, \lambda_1, \cdots, \lambda_n\}.$$

Let

$$B = \{\lambda : \lambda = \lim \lambda_m, \lambda_m \in \sigma_{i_m}\}$$

where $i_1, i_2, \cdots$ is an increasing sequence of integers. A characterization of this set for complex valued functions was initiated in 1960 and was published for the case: $f$ is a Laurent polynomial, $f(z) = \sum_{-k}^{h} a_m z^m$, $h, k \geq 1$ [4]. Let

$$D^n(f - \lambda) = \det(T_n(f - \lambda)).$$

Schmidt and Spitzer employed an identity of Harold Widom which up to a constant factor evaluates the $D^n(f - \lambda)$ when $f$ is a Laurent polynomial. We develop an identity for $D^n(f - \lambda)$ for $f$ an arbitrary rational function which, using the techniques of Schmidt and Spitzer, allows one to show that $B$ is a point or consists of a finite number of nondegenerate analytic arcs.

For simplification in the proof of the identity and notational convenience we make certain assumptions about the function $f$ which are essentially nonrestrictive and work directly with the determinants $D^n(f)$. Due to the complexities of notation, we observe the following convention. "(*)" designates a mathematical expression where "*" is the number of the expression, and "(\#)" is used to

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represent the object appearing in that expression.

In order that the Laurent series not reduce to a power series in which case the determinants $D^n(f)$ are triangular and the set $B$ reduces to a point we make the following assumptions. Let there be an annulus $A$ with center the origin,

\[(1.1) \quad A = \{z : R_1 < |z| < R_2, \ 0 < R_1 < R_2 < \infty \}.
\]

Let $D_k(z)$ and $F_h(z)$ be polynomials of exact degree $k$ and $h$ respectively where the roots of $D_k(z)$ lie in the set $|z| < R_1$, and those of $F_h(z)$ lie in the set $|z| \geq R_2$. Let $G_{k+m}(z)$ be a polynomial of exact degree $k + m$ having no common factors with the polynomials $D_k(z)$ and $F_h(z)$.

**Proposition 1.1.** Let $k \geq 1$, $m \geq \max(1, h)$, and $G_{k+m}(z)$, $D_k(z)$, and $F_h(z)$ satisfy the conditions given above. If

\[(1.2) \quad f(z) = \frac{G_{k+m}(z)}{D_k(z)}F_h(z)\]

and

\[(1.3) \quad f(z) = \sum_{-\infty}^{\infty} a_v z^v\]

is the Laurent series representation of $f(z)$ in the annulus $A$, then there exist positive and negative powers of $z$ which in the above expansion have nonzero coefficients.

**Proof.** Assume that $k, h \geq 1$. If $a_v = 0$ for all $v \leq -1$, then (1.3) is a power series expansion which converges in the set $|z| < R_2$. This contradicts the existence of at least one pole in the set $|z| < R_1 < R_2$. If $a_v = 0$ for all $v \geq 1$, then (1.3) converges in the set $|z| > R_1$ contradicting the existence of a pole in the set $|z| \geq R_2 > R_1$. In the exceptional case where $h = 0$, then $m \geq 1$. Consequently $f(z)$ defined by (1.2) has a pole of order $m$ at $z = \infty$. So $a_m \neq 0$.

2. Reduction of $D^n(f)$ to a determinant of fixed order $m$. Let $f(z)$ satisfy the hypothesis of Proposition 1.1. In addition we assume that the roots of $G_{k+m}(z)$ denoted by $r_i$, $i = 1, \cdots, k + m$, are distinct and not equal to zero. We may assume that the coefficient $g_0$ of the $z^{k+m}$ term of $G_{k+m}(z)$ is equal to 1. For if $f(z)$ is divided by $g_0$ we have simply divided each $a_n$ in the expansion of $f$ by $g_0$. So $D^n(f) = g_0^{n+1}D^n(g_0^{-1}f)$. The proof is long and will be broken up into a series of lemmas. We recommend that the reader turn to Theorem 3.1 for the final result before proceeding with the proof. We introduce the following notation.

\[(2.1) \quad z^{-(k+m)}G_{k+m}(z) = \prod_{d=1}^{k+m} (1 - r_i z^{-1}) = \sum_{i=0}^{k+m} g_{-i} z^{-i},\]
\[1/z^{-(k+m)}G_{k+m}(z) = \sum_{i=0}^{\infty} g^*_iz^{-i},\]

\[z^{-k}D_k(z) = \prod_{i=1}^{k} (1 - \delta_i z^{-1}) = \sum_{j=0}^{k} d^*_jz^{-j},\]

\[1/z^{-k}D_k(z) = \sum_{j=0}^{\infty} d^*_jz^{-j},\]

\[F_h(z) = \prod_{i=1}^{h} (1 - \rho_i^{-1}z) = \sum_{i=0}^{h} f^*_iz^i,\]

\[1/F_h(z) = \sum_{i=0}^{\infty} f_iz^i,\]

\[1/z^{-k}D_k(z) \cdot F_h(z) = \sum_{n=-\infty}^{\infty} b_nz^n,\]

\[z^{-(k+m)}G_{k+m}(z)/z^{-k}D_k(z) = \sum_{i=0}^{\infty} e^*_iz^{-i},\]

\[z^{-k}D_k(z)/z^{-(k+m)}G_{k+m}(z) = \sum_{i=0}^{\infty} e^*_iz^{-i},\]

\[f(z) = G_{k+m}(z)/D_k(z) \cdot F_h(z) = \sum_{n=-\infty}^{\infty} a_nz^n.\]

The first two lemmas are devoted to determining certain information about the coefficients of some of these expansions.

**Lemma 2.1.** If \(g^*_i\) is defined by (2.2) and \(e^*_i\) by (2.9) then \(g^*_i = \sum_{s=1}^{k+m} C_s r_s^i\) holds for \(i \geq -(k + m) + 1\), and \(e^*_i = \sum_{s=1}^{k+m} C_s D_k(r_s) r_s^{i-k}\) holds for \(i \geq k\) where \(C_s = r_s^{k+m-1} \Pi_{t \neq s} (r_s - r_t)^{-1}\).

**Proof.** By expanding \(i/z^{-(k+m)}G_{k+m}(z)\) by partial fractions we obtain the above identity for \(g^*_i\), \(i \geq 0\). In addition, the identity is valid for \(i = -1, \cdots, -(k + m) + 1\) because, for all \(i \geq -(k + m) + 1\),

\[\sum_{s=1}^{k+m} C_s r_s^i = \sum_{s > t} (r_s - r_t)^{-1}\]

The identity for the \(e^*_i\)'s follows from the identity for the \(g^*_i\)'s and (2.3).

**Lemma 2.2.** If \(-i + k < 0\) then \(\sum_{j=0}^{k} d^*_j b_{-i+j} = 0\).
Proof. The lemma is clearly true if \( D_k(z) = Cz^k \) for then, by (2.6) and (2.7), \( b_{-i} = 0 \) for \( i > 0 \). Otherwise expansion of \( 1/z^{-k}D_k(z)F_h(z) \) by partial fractions shows that

\[
(2.12) \quad b_{-i} = \sum_{j=1}^{k'} B_j \delta_j^i
\]

where \( \delta_j, j = 1, \cdots, k' \), are the distinct nonzero roots of \( D_k(z) \), and \( B_j \) non-zero constants. Without loss of generality we may assume that all the roots of \( D_k(z) \) are nonzero. Proof is by induction.

Let \( S_i(k), i = 1, \cdots, k \), be the elementary symmetric functions of the roots \( \delta_i \) of \( D_k(z) \) so that from (2.3)

\[
\prod_{i=1}^{k} (z - \delta_i) = d_0 z^k + d_1 z^{k-1} + \cdots + d_k = z^k + S_1(k) z^{k-1} + \cdots + S_k(k).
\]

Thus to prove the lemma we need to prove that

\[
b_{-i} + S_1(k) b_{-i+1} + S_2(k) b_{-i+2} + \cdots + S_k(k) b_{-i+k} = 0.
\]

Since

\[
\prod_{i=1}^{k-1} (z - \delta_i)(z - \delta_k) = [z^{k-1} + S_1(k-1) z^{k-2} + \cdots + S_{k-1}(k-1)] (z - \delta_k),
\]

it follows that

\[
(2.13) \quad S_1(k) = S_1(k - 1) - \delta_k, \\
S_i(k) = S_i(k - 1) - \delta_k S_{i-1}(k - 1), \quad i = 2, \cdots, k - 1, \\
S_k(k) = -\delta_k S_{k-1}(k - 1).
\]

By (2.12) it is clear that the lemma will be proved if we can show that

\[
(2.14) \quad \delta_j + \delta_j^{-1} S_1(k) + \delta_j^{-2} S_2(k) + \cdots + \delta_j^{-k} S_k(k) = 0, \\
\quad j = 1, \cdots, k, \ t > k.
\]

This is obvious for \( k = 1 \). We need to prove (2.14) for \( k + 1 \). From (2.13) it follows that

\[
(2.15) \quad \delta_j + \delta_j^{-1} S_1(k) + \delta_j^{-2} S_2(k) + \cdots + \delta_j^{-k} S_k(k) = \delta_k + \delta_k^{-1} S_1(k) + \delta_k^{-2} S_2(k) + \cdots + \delta_k^{-k} S_k(k).
\]

By induction, (2.15) equals zero for \( j = 1, \cdots, k \), and (2.15) clearly equals zero for \( j = k + 1 \).

Lemma 2.3. \( D^n(f) \) is equal to \((-1)^m(n+1-m)\) multiplied by the determinant of the product of the following three matrices,
PROOF. In order to prove the above we multiply \(D^n(f)\) on the left and right by determinants each of which is equal to one. In particular, noting by \((2.5)\) and \((2.9)\) that \(e_0^* = f_0^* = 1\), we multiply \(D^n(f)\) on the left by the upper triangular determinant,

\[
\begin{bmatrix}
e_0^* & \cdots & e_{-n}^* \\
0 & \ddots & \vdots \\
0 & \cdots & e_{-m+n-1}^*
\end{bmatrix}
\]

and on the right by the lower triangular determinant,

\[
\begin{bmatrix}
f_0^* & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
f_h^* & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & f_0^* \\
\end{bmatrix}
\]

We make the important remark that the above determinants are asymmetric, and that another reduction of \(D^n(f)\) to a determinant of fixed order is possible. We could have multiplied \(D^n(f)\) on the left by the upper triangular determinant \(D^n(z^{-k}D_k(z))\), and on the right by the lower triangular determinant \(D^n(F_h(z)/G_{k+m}(z))\).

Given the relations \((2.1)\)–\((2.10)\) the following may be verified.

\[
D^n(f) \cdot D^n(F_h(z)) = \|a_{i-j}\| \cdot \|(f_{i-j}^*)\|
\]

where the "\*" indicates entries which are left undetermined. Consequently
\[ D^n(f) = D^n(D_k z^{-m} G_{k+m}) \cdot [D^n(f) \cdot D^n(F_h)] \]
\[ = \|e_{i-j}\| \cdot [\|a_{i-j}\| \cdot \|f_{i-j}\|] \]
\[ = \begin{vmatrix} e_0^* e_{-1} \cdots e_{-n}^* \\ 0 \cdots e_0^* \end{vmatrix} \begin{vmatrix} e_{-m} \cdots e_{-n}^* \\ \vdots \end{vmatrix} \begin{vmatrix} e_0 \cdots \end{vmatrix} \]
\[ = \begin{vmatrix} 0 \\ I \end{vmatrix} \begin{vmatrix} K \\ N \end{vmatrix} = m \]
\[ \begin{vmatrix} 0 \\ I \end{vmatrix} \begin{vmatrix} K \\ N \end{vmatrix} = m \]

where \( I \) is the identity matrix and \( K \) equals the product (2.16). By shifting the rows of \( I \) in (2.19) into the upper left-hand corner it follows that

\[ D^n(f) = (-1)^m (n+1-m) \|K\|, \]

and the lemma follows.

In the following two lemmas we show that each of the two matrices on the left-hand side of (2.16) may be written as the product of a pair of matrices.

**Lemma 2.4.**

\[ m \begin{pmatrix} e_0^* \cdots e_{-n}^* \\ 0 \cdots e_0^* \end{pmatrix} = A - B, \text{ say.} \]
\[ m \begin{pmatrix} e_0^* \cdots e_{-n}^* \\ 0 \cdots e_0^* \end{pmatrix} = A - B, \text{ say.} \]

**Proof.** Verify by referring to (2.2), (2.3), and (2.9).
Lemma 2.5.

\[
\begin{pmatrix}
  a_{-n+m-1} & \cdots & a_{-n} \\
  \vdots & & \vdots \\
  a_{m-1} & \cdots & a_0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
  \cdots & g_{-k-m} & \cdots & 0 \\
  0 & & & \vdots \\
  \vdots & & & 0 \\
  0 & \cdots & 0 & g_0 & \cdots & g_{-k-m} \\
\end{pmatrix}
\begin{pmatrix}
  b_{-n-1} & \cdots & b_{-n-m} \\
  \vdots & & \vdots \\
  \vdots & & \vdots \\
  b_{k+m-1} & \cdots & b_k
\end{pmatrix}
\]

\[
= n + 1 \\
\]

\[
\begin{pmatrix}
  \cdots & g_0 & \cdots & g_{k-m} & \cdots & 0 \\
  0 & & & \vdots & & \vdots \\
  \vdots & & & 0 & & \vdots \\
  0 & \cdots & 0 & g_0 & \cdots & g_{k-m} \\
\end{pmatrix}
\begin{pmatrix}
  b_{-n-1} & \cdots & b_{-n-m} \\
  \vdots & & \vdots \\
  \vdots & & \vdots \\
  b_{k+m-1} & \cdots & b_k
\end{pmatrix}
\]

\[
= C \cdot D,
\]

say.

Proof. Verify by referring to (2.1), (2.7) and (2.10).

It follows from Lemmas 2.4 and 2.5 that if we denote the right-hand matrix of (2.16) by \( E \), then the matrix \( K \) which equals (2.16) may be written \( K = A B C D E \), where

\[
A = (a^*_{i-1}), \quad B = (g^*_{i-1}), \quad C = (g_{i-1}),
\]

\[
D = (b^*_{i-1}), \quad E = (f^*_{i-1}),
\]

as appears in these two lemmas. So by (2.20) it follows that

\[
D^n(f) = (-1)^{m(n+1-m)} \|A \cdot B \cdot C \cdot D \cdot E\|,
\]

and we must simplify the product of these matrices whose orders are increasing with \( n \).

Lemma 2.6.

\[
B \cdot C \cdot D = k + m
\]

\[
\begin{pmatrix}
  b_{-n-1} & \cdots & b_{-n-m} \\
  \vdots & & \vdots \\
  b_{n+k+m-2} & \cdots & b_{n+k-1}
\end{pmatrix}
\]

(2.22)

\[
- \begin{pmatrix}
  g_{-n-1}^* & \cdots & g_{-n-k-m}^* \\
  \vdots & & \vdots \\
  \vdots & & \vdots \\
  g_{n+k+m-2}^* & \cdots & g_{n-1}^*
\end{pmatrix}
\begin{pmatrix}
  g_0 & \cdots & g_{k-m+1} \\
  \vdots & & \vdots \\
  \vdots & & \vdots \\
  0 & \cdots & 0 & g_0
\end{pmatrix}
\]

\[
k + m
\]

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PROOF. Multiply $B$ and $C$ together and get that

\[
B \cdot C = \begin{pmatrix} I & \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} & M \end{pmatrix}^{k + m}
\]

where $I$ is the identity matrix, and

\[
M = k + m \begin{pmatrix} g_{-n+k+m-1}^* & \cdots & g_{-n}^* \\ \vdots & \ddots & \vdots \\ g_{-n+2k+2m-2}^* & \cdots & g_{-n+k+m-1}^* \end{pmatrix}^{k + m} \begin{pmatrix} g_{-k-m}^* & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & g_{-k-m}^* \end{pmatrix}^{k + m}
\]

Using the relations (2.1) and (2.2) that $\Sigma_{i=0}^{\min(u,k+m)} g_{-i} g_{-u+i}^* = 0$ for $u \geq 1$, we may rewrite the matrix $M$ so that

\[
M = \begin{pmatrix} g_{-n-1}^* & \cdots & \cdots & g_{-n-k-m}^* \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ g_{-n+k+m-2}^* & \cdots & \cdots & g_{-n-1}^* \end{pmatrix} \begin{pmatrix} g_{0} & \cdots & \cdots & g_{-k-m+1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & g_{0} \end{pmatrix}
\]

Let $B \cdot C = H$. Multiply $H$ by $D$. We get that

\[
H \cdot D = k + m \begin{pmatrix} I & \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} & M \end{pmatrix}^{k + m} \begin{pmatrix} b_{-n-1} & \cdots & b_{-n-m} \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ b_{k+m-1} & \cdots & b_{k} \end{pmatrix}^{n + k + m + 1}
\]

= (2.22), the desired result.
\textbf{Lemma 2.7.}

\[ A \cdot B \cdot C \cdot D = (-1) \begin{pmatrix} \varepsilon_{n-1}^* & \cdots & \varepsilon_{n-k-m}^* \\ \vdots & \ddots & \vdots \\ \varepsilon_{n+m-2}^* & \cdots & \varepsilon_{n-k-1}^* \end{pmatrix} \begin{pmatrix} g_0 & \cdots & g_{k-m+1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & g_0 \end{pmatrix} \]

\[ = m \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}. \]

\textbf{Proof.} We multiply $H \cdot D$ on the left by $A$ where $A$ is the $(d^*_{i-j})$ matrix of Lemma 2.4. The left-hand component of $H \cdot D$ is the $(b^*_{i-j})$ matrix of (2.20). By Lemma 2.2

\[ (d^*_{i-j})(b^*_{i-j}) = m \begin{pmatrix} d_0^* & \cdots & d_{-k}^* \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} b_{-n-1} & \cdots & b_{-n-m} \\ \vdots & \ddots & \vdots \\ b_{-n+k+m-2} & \cdots & b_{-n+k-1} \end{pmatrix} \]

\[ = m \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}. \]

We remark that without this result the identity we obtain for $D^T(f)$ would not be possible. Continuing our multiplication of $H \cdot D$ by $A$, since by (2.2), (2.3) and (2.9)

\[ m \begin{pmatrix} d_0^* & \cdots & d_{-k}^* \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} g_{-n-1}^* & \cdots & g_{-n-k-m} \\ \vdots & \ddots & \vdots \\ g_{-n+k+m-2}^* & \cdots & g_{-n-1}^* \end{pmatrix} \]

\[ = m \begin{pmatrix} \varepsilon_{-n-1}^* & \cdots & \varepsilon_{-n-k-m}^* \\ \vdots & \ddots & \vdots \\ \varepsilon_{-n+m-2}^* & \cdots & \varepsilon_{-n-k-1}^* \end{pmatrix}. \]
it follows that \( A \cdot B \cdot C \cdot D = A \cdot H \cdot D = (2.23) \).

**Theorem 2.1.**

\[
D^n(f) = (-1)^m(n+1) \det \begin{pmatrix}
  e^*_{-n-1} & \cdots & e^*_{-n-k-m} \\
  \vdots & \ddots & \vdots \\
  e^*_{-n+m-2} & \cdots & e^*_{-n-k-1}
\end{pmatrix}
\]

(2.24)

\[
\cdot \begin{pmatrix}
  g_0 & \cdots & g_{-k-m+1} \\
  \vdots & \ddots & \vdots \\
  0 & \cdots & g_0
\end{pmatrix} \cdot \begin{pmatrix}
  b_0 & \cdots & b_{-m+1} \\
  \vdots & \ddots & \vdots \\
  b_{k+m-1} & \cdots & b_k
\end{pmatrix}
\]

**Proof.** Since \( \|E\| = 1 \), \( \|A \cdot B \cdot C \cdot D \cdot E\| = \|A \cdot B \cdot C \cdot D\| \). Since 
\((-1)^m(n+1) = (-1)^m(n+1)-m\), by (2.21) and (2.23) it follows that \( D^n(f) = (2.24) \). So \( D^n(f) \) equals a determinant of fixed order \( m \).

3. Evaluation of the determinants \( D^n(f) \). In the last section we showed that \( D^n(f) \) equaled the determinant of the product of three matrices whose orders were independent of \( n \) (2.24). In this section we will evaluate this product. We have of course by (2.1) that \( G_{k+m}(z) = \prod_{i=1}^{k+m} (z - r_i) \). In general, let

\[
G_{k+m}^S(z) = \prod_{i=1; i \in S}^{k+m} (z - r_i)
\]

where \( S \) is a subset of the integers \((1, 2, \cdots, k + m)\). In the event we wish to be explicit about the entries in \( S \) we will write \((i_1, i_2, \cdots, i_a)\) to indicate that \( S \) contains the integers \( i_1, i_2, \cdots, i_a \). In particular if \( S \) consists of the singleton \( i \), we write

\[
G_{k+m}^{(i)}(z) = (z - r_i)^{-1} G_{k+m}(z) = \prod_{j=1; j \not= i}^{k+m} (z - r_j).
\]

In an analogous way we define the coefficient \( a_{m}^{S} \) by means of

\[
\sum_{m=-\infty}^{\infty} a_{m}^{S} z^{m} = G_{k+m}^{S}(z)/D_{k}(z) F_{h}(z) \quad \text{and}
\]

(3.3)

\[
\sum_{m=-\infty}^{\infty} a_{m}^{(i)} z^{m} = G_{k+m}^{(i)}(z)/D_{k}(z) F_{h}(z).
\]

**Lemma 3.1.** The product of the three matrices of (2.24), \((e^*_{-n-1}+l-j)\)
\((g_{-l-j})(b_{l-j})\), is equal to the \( m \) by \( m \) matrix

\[
\begin{pmatrix}
  \sum_{i=1}^{k+m} C_i D_k(r_i) r_i^{n+2-k-s_{m-t}} a_{m-t}^{(i)}
\end{pmatrix}
\]

(3.4)

for \( s, t = 1, 2, \cdots, m \), and where \( C_i = r_i^{k+m-1} \prod_{i \not= i}^{m} (r_i - r_j)^{-1} \).
PROOF. By Lemma 2.1, 
\[ e^*_{n-i} = \sum_{i=1}^{k+m} C_i D_k(r_i) r_{i}^{n+k}. \]
By definition
\[ z^{-(k+m)} G_{k+m}(z) = \prod_{i=1}^{k+m} (1 - r_i z^{-1}) = \sum_{i=0}^{k+m} g_{-i} z^{-i}. \]
Consequently
\[ g_{0} = 1, \quad g_{-1} = - \sum_{s=1}^{k+m} r_s, \quad g_{-2} = \sum_{1 \leq s < t} r_s r_t, \quad \text{etc.} \]
If we multiply the top row of the \( (e^*_{n-1+i-j}) \) matrix of (2.24) by the columns of the \( (g_{i-j}) \) matrix of (2.24), the following may be easily verified.

\[ e^*_{n-1} g_0 = \sum_{i=1}^{k+m} C_i D_k(r_i) r_i^{n+1-k}, \]

\[ e^*_{n-2} g_0 + e^*_{n-2} g_0 = - \sum_{i=1}^{k+m} C_i D_k(r_i) r_i^{n+1-k} \cdot \sum_{s=1; s \neq i}^{k+m} r_s, \]

\[ e^*_{n-1} g_0 + e^*_{n-2} g_0 + e^*_{n-3} g_0 = \sum_{i=1}^{k+m} C_i D_k(r_i) r_i^{n+1-k} \cdot \sum_{1 \leq s < t; s \neq t} r_s r_t, \]

and similarly for the remaining products. Thus the top row of \( (e^*_{n-1+i-j}) \)
\( (g_{i-j}) \) is exactly

\[ \left( \sum_{i=1}^{k+m} C_i D_k(r_i) r_i^{n+1-k} \right) \cdot \left( \sum_{s=1; s \neq i}^{k+m} r_s \right). \]

(3.5)

The second row of \( (e^*_{n-1+i-j}) (g_{i-j}) \) will be the same except that the exponent of the \( r_i \) factors in each term of the sums will be lowered by one, and similarly
for the remaining rows until with the last row each \( r_i \) appears in each sum to the power \( n - k - m + 2 \). Since
\[ G_{k+m}^{(f)}(z)/D_k(z) F_{n}^{(f)}(z) = G_{k+m}^{(f)}(z) \cdot \sum_{-\infty}^{\infty} b_v z^v = \sum_{-\infty}^{\infty} a_v^{(f)} z^v, \]
it follows that

\[ a_v^{(f)} = b_v - b_v+1 \sum_{s=1; s \neq i}^{k+m} r_s + b_v+2 \sum_{1 \leq s < t; s \neq t} r_s r_t \]

(3.6)

By applying (3.6) to (3.5) it follows that the product of the matrices of (2.24) equals (3.4).

**Lemma 3.2.** The determinant of the \( m \)th order matrix of (3.4) is equal
to the product of two \( (k + m) \)th order determinants.
PROOF. By (2.11), $\Sigma_{i=1}^{k+m} C_i = 1$, $\Sigma_{i=1}^{k+m} C_i r_i^{-1} = 0$, $j = 1, 2, \ldots, k + m - 1$. Consequently (3.7) equals

$$\begin{vmatrix}
C_1 D_k(r_1) & C_2 D_k(r_2) & \cdots & C_{k+m} D_k(r_{k+m}) \\
C_1 D_k(r_1) r_1^{-1} & C_2 D_k(r_2) r_2^{-1} & \cdots & C_{k+m} D_k(r_{k+m}) r_{k+m}^{-1} \\
\vdots & \vdots & \ddots & \vdots \\
C_1 D_k(r_1) r_1^{-(k+m)+1} & C_2 D_k(r_2) r_2^{-(k+m)+1} & \cdots & C_{k+m} D_k(r_{k+m}) r_{k+m}^{-(k+m)+1}
\end{vmatrix}$$

$$\begin{vmatrix}
D_k^{-1}(r_1) & \cdots & D_k^{-1}(r_1) r_1^{k-1} & a_{m-1}^{(1)} r_{k+1}^{-1} & \cdots & a_0^{(1)} r_{k+m+1}^{-1} \\
D_k^{-1}(r_2) & \cdots & D_k^{-1}(r_2) r_2^{k-1} & a_{m-1}^{(2)} r_{k+1}^{-1} & \cdots & a_0^{(2)} r_{k+m+1}^{-1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
D_k^{-1}(r_{k+m}) & \cdots & D_k^{-1}(r_{k+m}) r_{k+m}^{k-1} & a_{m-1}^{(k+m)} r_{k+1}^{-1} & \cdots & a_0^{(k+m)} r_{k+m+1}^{-1}
\end{vmatrix}$$

where $K^*$ is the matrix (3.4), and clearly (3.8) = $\|K^*\|$.

By Theorem 2.1, $D^n(f)$ equals $(-1)^{m(n+1)}$ multiplied by the determinant of (2.24). By Lemma 3.1 and Lemma 3.2,

$$D^n(f) = (-1)^{m(n+1)} \cdot [(3.7)]$$

Because of the representation, except for sign, of $D^n(f)$ by (3.7), we are able to evaluate it. For the left-hand determinant of (3.7) is easy to evaluate, and the right-hand determinant of (3.7) may be expanded in such a way that we will be able to evaluate it.

Clearly the left-hand determinant of (3.7) equals

$$\prod_{i=1}^{k+m} C_i D_k(r_i) V(r_1^{-1}, r_2^{-1}, \ldots, r_{k+m}^{-1})$$
where \( V(r_1^{-1}, r_2^{-1}, \ldots, r_{k+m}^{-1}) \) is the Vandermonde determinant based on the numbers \( r_i^{-1}, \ i = 1, 2, \ldots, k + m \). It can easily be shown that
\[
\prod_{i=1}^{k+m} C_i = \prod_{i=1}^{k+m} r_i^{k+m-1} \prod_{i \neq j} (r_i - r_j)^{-1}
\]
\[
= (-1)^\sigma \prod_{i=1}^{k+m} r_i^{k+m-1} V^{-2}(r_1, r_2, \ldots, r_{k+m}),
\]
and that
\[
V(r_1^{-1}, r_2^{-1}, \ldots, r_{k+m}^{-1}) = (-1)^\sigma \prod_{i=1}^{k+m} r_i^{-k-m+1} V(r_1, r_2, \ldots, r_{k+m})
\]
where \( \sigma = \frac{1}{2} (k + m - 1) (k + m) \). From this it follows that
\[
(3.10) \quad (3.9) = \prod_{i=1}^{k+m} D_k(r_i) V^{-1}(r_1, r_2, \ldots, r_{k+m}).
\]

Laplacian expansion of the right-hand determinant of (3.7) on the last \( m \) columns gives us
\[
(3.11) \quad \sum_{I \in \tilde{I}} \prod_{j \in \tilde{I}} D_k^{-1}(r_j) V(I) \prod_{i \in \tilde{I}} r_i^{-1} V(a_0^{(I)}).
\]
The sum is taken over all subsets \( I \) of \( m \) integers from \( (1, 2, \ldots, k + m) \), and
\( I = (1, 2, \ldots, k + m) \setminus I \), \( V(I) \) is the Vandermonde determinant determined by \( r_j, \ j \in \tilde{I} \), and
\[
V(a_0^{(I)}) = 
\begin{vmatrix}
    a_0^{(i_1)} & a_0^{(i_1)} & \cdots & a_0^{(i_1)} \\
    a_0^{(i_2)} & a_0^{(i_2)} & \cdots & a_0^{(i_2)} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_0^{(i_m)} & a_0^{(i_m)} & \cdots & a_0^{(i_m)} \\
\end{vmatrix}, \quad (i_1, i_2, \ldots, i_m) = I.
\]

**Lemma 3.3.** \( V(a_0^{(I)}) = V(I) \cdot \| a_{s-t}^{m} \|_{s, t=1} \).  

**Proof.** As before, \( V(I) \) is the Vandermonde determinant determined by \( I \), and \( a_{s-t}^{m} \) is the \( m \)th order Toeplitz matrix generated by the function
\[
G_k^{I}(z)/D_k(z) \cdot F_k(z) = \sum_{-\infty}^{\infty} a_0^{I} z^\sigma
\]
defined by (3.1) and (3.3). We give a demonstration of this for
\[
(3.12) \quad \begin{vmatrix}
    a_0^{(1)} & a_0^{(1)} & \cdots & a_0^{(1)} \\
    a_0^{(2)} & a_0^{(2)} & \cdots & a_0^{(2)} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_0^{(m)} & a_0^{(m)} & \cdots & a_0^{(m)} \\
\end{vmatrix}.
\]
By the convention (3.3), if $S$ is any subset of the set $(1, 2, \cdots, k + m)$ with $s$ elements, then

$$a_{v+m-s}^S = b_v - b_{v+1} \sum_{i \in S} r_i + \cdots + (-1)^{k+m-s} b_{v+k+m-s} \prod_{i \in S} r_i.$$ 

So we can easily verify that $a_{v+m-1}^{(i)} = a_{v+m-2}^{(i)} - r_i a_{v+m-1}^{(i)}$ and that

$$a_{m-1}^{(1,2)} - r_1 a_{m-2}^{(1,2)} \cdots \cdot a_{1}^{(1,2)} - r_1 a_0^{(1,2)}$$

$$\begin{pmatrix}
a_{m-2}^{(1,1)} - r_1 a_{m-1}^{(1,1)} \cdots \cdot a_{-1}^{(1,1)} - r_1 a_0^{(1,1)} \\
\vdots \quad \vdots \\
a_{m-2}^{(1,m)} - r_1 a_{m-1}^{(1,m)} \cdots \cdot a_{-1}^{(1,m)} - r_1 a_0^{(1,m)}
\end{pmatrix}.$$ 

(3.13) \quad (3.12) =

Letting $i$ alternately be equal to 2, 3, \ldots, $m$, and subtracting the first row from each of the others, and rewriting the first row in its original form, we get that the right-hand side of (3.13) equals

$$\begin{pmatrix}
a_{m-1}^{(1)} \cdots a_0^{(1)} \\
a_{m-1}^{(1,2)} \cdots a_0^{(1,2)} \\
\vdots \quad \vdots \\
a_{m-1}^{(1,m)} \cdots a_0^{(1,m)}
\end{pmatrix}$$

$$\begin{pmatrix}
(r_2 - r_1) (r_3 - r_1) \cdots (r_m - r_1)
\end{pmatrix}.$$ 

(3.14)

Since $a_0^{(1)} = a_{v-1}^{(1,2)} - r_2 a_0^{(1,2)}$, if we add $r_2$ times the second row to the first row in (3.14), we obtain the result that (3.12) equals

$$\begin{pmatrix}
a_{m-2}^{(1,2)} \cdots a_0^{(1,2)} \\
a_{m-2}^{(1,2)} \cdots a_0^{(1,2)} \\
\vdots \quad \vdots \\
a_{m-1}^{(1,m)} \cdots a_0^{(1,m)}
\end{pmatrix}$$

$$\begin{pmatrix}
(r_2 - r_1) \cdots (r_m - r_1)
\end{pmatrix}.$$ 

(3.15)

Similarly using the second row as a pivot row to reduce the rows below it, we can verify that the determinant in (3.15) equals

$$\begin{pmatrix}
a_{m-2}^{(1,2,3)} \cdots a_0^{(1,2,3)} \\
a_{m-1}^{(1,2,3)} \cdots a_0^{(1,2,3)} \\
\vdots \quad \vdots \\
a_{m-1}^{(1,2,m)} \cdots a_0^{(1,2,m)}
\end{pmatrix}$$

$$\begin{pmatrix}
(r_3 - r_2) (r_4 - r_2) \cdots (r_m - r_2)
\end{pmatrix}.$$ 

(3.16)
Adding $r_3$ times the third row to the second row, and subsequently $r_3$ times the second row to the first row, we reduce (3.16) to

\[
\begin{pmatrix}
  a_{m-3}^{(1,2,3)} & \cdots & a_{-3}^{(1,2,3)} \\
  a_{m-2}^{(1,2,3)} & \cdots & a_{-2}^{(1,2,3)} \\
  a_{m-1}^{(1,2,3)} & \cdots & a_{-1}^{(1,2,3)} \\
  \vdots & \ddots & \vdots \\
  \vdots & & \vdots \\
  a_{m-1}^{(1,2,m)} & \cdots & a_{0}^{(1,2,m)} \\
\end{pmatrix}
(r_3 - r_2) \cdots (r_m - r_2).
\]

Continuing in this manner, successively using the third, fourth, etc. rows as pivot rows and factoring out the appropriate factors, we prove that for $I = (1, 2, \cdots, m)$,

\[ (3.12) = V(I) \| (a_{s-t}^I)_{s,t=1}^m \| . \]

The general proof follows as above.

Consequently, by (3.8), (3.10), (3.11) and Lemma 3.3,

\[ D^n(f) = (-1)^m(n+1) \prod_{i=1}^{k+m} D_k(r_i)V^{-1}(r_1, r_2, \cdots, r_{k+m}) \]

\[ \left( \sum_{I} \prod_{i \in I} D_{-1}^k(r_i)V(I) \prod_{i \notin I} r_i^{n+1} V(I)\| (a_{s-t}^I)_{s,t=1}^m \| \right) \]

\[ = (-1)^m(n+1) \sum_{I} \prod_{i \in I, j \notin I} r_i^{n+1} D_k(r_i) \| (a_{s-t}^I)_{s,t=1}^m \| (r_i - r_j)^{-1}, \]

where $I$ runs over all subsets of order $m$ of the set $(1, 2, \cdots, k + m)$, and $\bar{I} = (1, 2, \cdots, k + m) - I$.

**Lemma 3.4.** $\| (a_{s-t}^I)_{s,t=1}^m \| = \| (a_{s-t}^I)_{s,t=1}^h \|$.

**Proof.** By (3.3), $\Sigma_{-\infty}^\infty a_v z^v = G^I_{k+m}(z)/D_k(z)F_h(z)$, and $G^I_{k+m}(z) = \prod_{i=1; i \notin I} (z - r_i)$, a monic $k$th degree polynomial. By (2.3), $D_k(z)$ is also a monic $k$th degree polynomial. Therefore the power series expansion of

\[ G^I_{k+m}(z)/D_k(z) = \left( \sum_{-\infty}^\infty a_v^I z^v \right) \cdot F_h(z) \text{ around } z = \infty \]

commences with the constant term 1. By (2.5), $F_h(z) = \Sigma_{i=0}^h f_i^* z^i$ with $f_0^* = 1$. By (3.18) it follows that
By applying the results of this lemma to (3.17), we have derived the following identity for $D^n(f)$.

(3.19)  \[ D^n(f) = (-1)^{m(n+1)} \sum_{\mathcal{I}} \prod_{i \in \mathcal{I}, j \in \bar{\mathcal{I}}} r_i^{n+1} D_k(r_i) \cdot \| (a_{s-t})^h_{s,t=1} \| (r_i - r_j)^{-1}, \]

where $\mathcal{I}$ runs over all subsets of order $m$ of the set $(1, 2, \cdots, k + m)$, and $\bar{\mathcal{I}} = (1, 2, \cdots, k + m) - \mathcal{I}$. The only factors that are not explicitly evaluated are the $h$th order determinants $\| (a_{s-t})^h_{s,t=1} \|$. Since the $a_s^I$'s are generated by the function

\[ G_{k+m}(z)/D_k(z)F_h(z) = \sum_{-\infty}^\infty a_s^I z^u, \]

and by (3.1)

\[ G_{k+m}(z) = \prod_{j \in \bar{\mathcal{I}}} (z - r_j), \]

each $a_s^I$ is a polynomial in the $r_j$'s, $j \in \bar{\mathcal{I}}$. Consequently each determinant $\| (a_{s-t})^h_{s,t=1} \|$ is a polynomial at most of $h$th degree in each of the $r_j$'s, $j \in \bar{\mathcal{I}}$.

**Lemma 3.5.** $\| (a_{s-t})^h_{s,t=1} \| = C \Pi_{j \in \bar{\mathcal{I}}} F_h(r_j)$ where $C$ is a constant that is independent of the $r_j$'s, $j \in \bar{\mathcal{I}}$.

**Proof.** The factor $\Pi_{j \in \bar{\mathcal{I}}} F_h(r_j)$ is of exact degree $h$ in each of the $r_j$'s. So by the remarks immediately preceding the lemma we can conclude that $C$ is a constant which is independent of the $r_j$'s. Direct evaluation of the determinants $\| (a_{s-t})^h_{s,t=1} \|$ seems to be difficult. Consequently we approach the problem in-
directly. We made the remark in the proof of Lemma 2.3 that the reduction of
$D^n(f)$ to the $m$th order determinant (2.24) which led to the identity (3.19) was
effected by multiplication of $D^n(f)$ by determinants (2.17) and (2.18) which are
asymmetric. We indicated that another reduction is possible.

Define the Laurent series $\sum_{-\infty}^{\infty} \hat{a}_v z^v$ by
\[
(-1)^{k+m} \prod_{i=1}^{k+m} r_i^{-1} \cdot \sum_{-\infty}^{\infty} a_v z^v = \sum_{-\infty}^{\infty} \hat{a}_v z^v,
\]
so that
\[
(-1)^{k+m} \prod_{i=1}^{k+m} r_i^{-1} \cdot a_v = \hat{a}_v.
\]
(3.20)

Since each $r_i$ is assumed to be nonzero, multiplication by $\prod_{i=1}^{k+m} r_i^{-1}$ is possible.
Clearly if $\hat{f}(z) = (-1)^{k+m} \prod_{i=1}^{k+m} r_i^{-1} \cdot f(z)$, then $\hat{f}(z) = \sum_{-\infty}^{\infty} \hat{a}_v z^v$ in the annulus $A$. Let $D^n(\hat{f})$ be the $(n + 1)$st order determinant generated by the $\hat{a}_v$'s. We may
reduce $D^n(\hat{f})$ to a determinant of fixed order exactly as we did in §2 for $D^n(f)$.

Since the numerator of $f(z)$ is $G_{k+m}(z) = \prod_{i=1}^{k+m} (z - r_i)$, the numerator of
$\hat{f}(z)$ is
\[
(-1)^{k+m} r_i^{-1} G_{k+m}(z) = \prod_{i=1}^{k+m} (1 - r_i^{-1} z), \quad \text{and}
\]
(3.21)

Let us multiply $D^n(\hat{f})$ on the left by the upper triangular determinant
$D^n(z^{-k} D_k(z)) = 1$, and on the right by the lower triangular determinant
$D^n(F_h(z)/\prod_{i=1}^{k+m} (1 - r_i^{-1} z)) = 1$. Reduction of the determinant $D^n(\hat{f})$ by the
techniques of §2 results in reducing $D^n(\hat{f})$ to a determinant of order $k$ similar to
(2.24). Consequent evaluation of this $k$th order determinant exactly as we have
evaluated $D^n(f)$ results in the identity
\[
D^n(\hat{f}) = (-1)^{k(n+1)} \sum_{I} \prod_{j \in I; i \notin I} r_j^{-n-1-m} F_h(r_j) \cdot \prod_{s \in I} \prod_{r \in I} (1 - r_i^{-1} z)/D_k(z)F_h(z).
\]
(3.22)

A comparison of the identity for $D^n(\hat{f})$ (3.22) with the identity for $D^n(f)$ (3.19)
results in some apparent discrepancies. In (3.19) appear the factors $\prod_{i=1}^{n+1} D_k(r_i)$,
of degree $n + k + 1$ in each of the $r_i$'s. In (3.22) appear the factors
$\prod_{j \in I} r_j^{-n-1-m} F_h(r_j)$. Since $F_h(0) \neq 0$, and $m \geq h$, these factors are of degree
$n + m + 1$ in each of the $r_j^{-1}$'s which is what is desired. We note that the coef-
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ficient $T_{-k}$ lies on the main diagonal of the unevaluated determinantal factors appearing in (3.22) rather than $T^h_0$. This occurs because of differences which arise in simplifying the determinants arising in the alternative approach vis-à-vis the simplification of the determinants appearing in Lemma 3.2.

By (3.20),

(3.23) $(-1)^{k+m}(n+1) \prod_{i=1}^{k+m} r_i^{n+1} D^n(f) = D^n(f)$.

Because

$(-1)^{k+m}(n+1) \cdot (-1)^k(n+1) = (-1)^m(n+1)$

and

$\prod_{j \in I, i \in I} (r_j^{-1} - r_i^{-1})^{-1} = \prod_{i \in I, j \in I} r_i^k p_j^m (r_j - r_i)^{-1}$,

we have, after carrying out the multiplication indicated on the left-hand side of (3.23), that

$(-1)^m(n+1) \sum_{I} \prod_{i \in I, j \in I} r_i^{n+1} \cdot \prod_{i \in I, j \in I} r_i^k p_j^m (a_{k+s-t})^h_{s,t=1} \cdot F_h(r_j) (r_j - r_i)^{-1}$

$= (-1)^m(n+1) \sum_{I} \prod_{i \in I, j \in I} r_i^{n+1} \cdot D_k(r_j) ||(a_{s-t})^h_{s,t=1} \cdot (r_j - r_i)^{-1}$.

Because the $r_i$ may be chosen independently of one another and the above equality holds for all $n$, we may equate corresponding terms which are defined by the same set of indices $I$. From this it follows that, for each $I$,

$\prod_{i \in I, j \in I} r_i^{n+1} \cdot \prod_{i \in I, j \in I} r_i^k p_j^m (a_{k+s-t})^h_{s,t=1} \cdot F_h(r_j) (r_j - r_i)^{-1}$

$= \prod_{i \in I, j \in I} r_i^{n+1} D_k(r_j) ||(a_{s-t})^h_{s,t=1} \cdot (r_j - r_i)^{-1}$.

Consequently

(3.24) $\prod_{i \in I, j \in I} r_i^k (a_{k+s-t})^h_{s,t=1} \cdot F_h(r_j) = \prod_{i \in I, j \in I} D_k(r_j) \cdot ||(a_{s-t})^h_{s,t=1}$.

But, as we remarked immediately preceding this lemma, $||(a_{s-t})^h||$ is a polynomial at most of degree $h$ in each of the $r_j$'s, $j \in I$. Similarly, since the $T_{-k}$'s are functions of the $r_j$'s, $i \in I$, $\prod_{i \in I} r_i^k ||(a_{k+s-t})^h_{s,t=1}||$ is a polynomial at most of degree $k$ in each $r_i$, $i \in I$. We may conclude from these observations and (3.24) that

$||(a_{s-t})^h_{s,t=1}|| = C \prod_{j \in I} F_h(r_j)$

and

$\prod_{i \in I} r_i^k ||(a_{k+s-t})^h_{s,t=1}|| = C \prod_{i \in I} D_k(r_i)$.

This concludes the proof of Lemma 3.5.

We have almost completed the evaluation of $D^n(f)$. Applying the results of Lemma 3.5 to (3.19), the identity for $D^n(f)$ assumes the form
(3.26) \( D^n(f) = (-1)^m(n+1) \cdot C \cdot \sum_{I} \prod_{i \in I, j \in \bar{I}} r_i^{n+1} D_k(r_i) F_h(r_j) (r_i - r_j)^{-1} \),

where \( I \) runs over all subsets of order \( m \) of the set \((1, 2, \ldots, k + m)\) and \( \bar{I} = (1, 2, \ldots, k + m) - I \). Only the constant term \( C \) remains to be determined.

**Lemma 3.6.** \( C = \Pi_{s \in K} \; t \in H \; b_t^k/ (\rho_t - \delta_s) \) where \( K = (1, \ldots, k) \) and \( H = (1, \ldots, h) \).

**Proof.** Because the constant \( C \) is independent of the roots \( r_i \) of the polynomial \( G_{k+m}(z) \), we may choose the roots and thereby the polynomial \( G_{k+m}(z) \) in such a way that the constant \( C \) will be determined. In particular we may assume that \( m = h \). Choose the roots \( r_i, \; i = 1, \ldots, k \), to lie respectively within an \( \varepsilon \)-neighborhood of the roots \( \delta_s, \; s = 1, \ldots, k \), of \( D_k(z) \) and the roots \( r_t, \; t = k + 1, \ldots, k + h \), to lie within an \( \varepsilon \)-neighborhood of the roots \( \rho_t, \; t = 1, \ldots, h \), of \( F_h(z) \) but such that \( |r_i - r_j| \geq O(\varepsilon) \) where \( \varepsilon \) is a small positive number. Let \( I_0 = (k + 1, \ldots, k + h) \), \( \bar{I}_0 = (1, \ldots, k) \). We wish to show for all \( I \neq I_0 \) that if the roots \( r_i \) tend to the roots \( \delta_s \) and \( \rho_t \) of \( D_k(z) \) and \( F_h(z) \) respectively, then

\[
(3.27) \quad \prod_{i \in I, j \in \bar{I}} D_k(r_i) F_h(r_j) / (r_i - r_j)
\]

tends to zero. This would be obvious except for the possibility that \( D_k(z) \) and \( F_h(z) \) may have multiple roots and so for some \( i, j, r_i - r_j \to 0 \).

For \( I \neq I_0 \), \( I \) and \( \bar{I} \) may each be written as the union of a nonempty disjoint pair of sets as follows. Let \( I = I_0^* \cup \bar{I}_0^* \) and \( \bar{I} = I_0^{**} \cup \bar{I}_0^{**} \) where \( I_0^*, \bar{I}_0^* \subset I_0, \) and \( I_0^{**}, \bar{I}_0^{**} \subset \bar{I}_0 \). But then

\[
(3.27) = \prod_{i \in I_0^*, j \in I_0^{**}} D_k(r_i) F_h(r_j) / (r_i - r_j) \cdot \prod_{i \in \bar{I}_0^{**}, j \in \bar{I}_0^{**}} D_k(r_i) F_h(r_j) / (r_i - r_j)
\]

\[
= \hat{\beta} \cdot \hat{C} \cdot \hat{D} \cdot \hat{E}, \quad \text{say.}
\]

From the manner in which the roots \( r_i \) are chosen,

\[
\prod_{i \in I_0^*, j \in I_0^{**}} F_h(r_j) / (r_i - r_j) \cdot \prod_{i \in \bar{I}_0^{**}, j \in \bar{I}_0^{**}} D_k(r_i) / (r_i - r_j) \cdot \prod_{i \in I_0^*, j \in I_0^{**}} 1 / (r_i - r_j)
\]

= \( O(1) \), since the roots \( \delta_s \) and \( \rho_t \) are separated by the annulus \( A \). So \( \hat{\beta} = \hat{C} = \hat{D} = O(1) \). But necessarily \( \hat{E} = O(\varepsilon) \). Consequently \( (3.27) \to 0 \) as the roots \( r_i \) tend to the appropriate limits. With the same limits,

\[
(3.28) \quad \lim_{I \neq I_0} \prod_{i \in I_0, j \in \bar{I}_0} D_k(r_i) F_h(r_j) / (r_i - r_j) = \prod_{s \in K} \prod_{t \in H} D_k(\rho_t) F_h(\delta_s) / (\rho_t - \delta_s).
\]
So the limit of the right-hand side of (3.26) equals

\[(3.29) \quad (-1)^{h(n+1)} \cdot C \cdot \prod_{s \in K; t \in H} \rho_{t}^{n+1}D_{k}(\rho_{t})F_{h}(\delta_{s})/(\rho_{t} - \delta_{s}).\]

Moreover it is clear that

\[
f(z) = \prod_{i=1}^{k+h} (z - r_{i})\left[\prod_{s=1}^{k} (z - \delta_{s}) \prod_{t=1}^{h} (1 - \rho_{t}^{-1}z)\right]
\]

\[
= \frac{[(-1)^{h} \prod_{i=1}^{k} (z - r_{i}) \prod_{t=k+1}^{k+h} r_{t}(1 - r_{t}^{-1}z)]}{[\prod_{s=1}^{k} (z - \delta_{s}) \prod_{t=1}^{h} (1 - \rho_{t}^{-1}z)]}
\]

tends to \((-1)^{h} \Pi_{t \in H} \rho_{t}\), a constant.

For each \(n\), \(D^{n}(f)\) varies continuously with the coefficients \(a_{m}\) of the Laurent series representation of \(f\) and these coefficients vary continuously with the roots of \(G_{k+m}(z)\). Consequently

\[
\lim_{n \to \infty} D^{n}(f) = (-1)^{h(n+1)} \prod_{t \in H} \rho_{t}^{n+1}.
\]

From (3.26) and (3.29)

\[
(-1)^{h(n+1)} \prod_{t \in H} \rho_{t}^{n+1} = (-1)^{h(n+1)} \cdot C \cdot \prod_{s \in K; t \in H} \rho_{t}^{n+1}D_{k}(\rho_{t})F_{h}(\delta_{s})/(\rho_{t} - \delta_{s}),
\]

and so

\[
C = \prod_{s \in K; t \in H} \rho_{t}^{k}(\rho_{t} - \delta_{s})/D_{k}(\rho_{t})F_{h}(\delta_{s}).
\]

But \(D_{k}(\rho_{t}) = \Pi_{s \in K} (\rho_{t} - \delta_{s})\) and \(F_{h}(\delta_{s}) = \Pi_{t \in H} (1 - \rho_{t}^{-1}\delta_{s}) = \Pi_{t \in H} \rho_{t}^{-1}(\rho_{t} - \delta_{s})\).

Consequently

\[
C = \prod_{s \in K; t \in H} \rho_{t}^{k}/(\rho_{t} - \delta_{s})^{2} = \prod_{s \in K; t \in H} \rho_{t}^{k}/(\rho_{t} - \delta_{s}).
\]

We are now in the position to prove the following theorem.

**Theorem 3.1.** Let \(f(z) = G_{k+m}(z)/D_{k}(z)F_{h}(z)\) where \(G_{k+m}(z), D_{k}(z),\) and \(F_{h}(z)\) satisfy the conditions of Proposition 1.1. Assume, in addition, that the zeros of \(G_{k+m}(z)\) are distinct and not equal to zero, and that the coefficient of the \(z^{k+m}\) term of \(G_{k+m}(z)\) is equal to 1. Then

\[(3.30) \quad D^{n}(f) = (-1)^{m(n+1)} \sum_{I \subseteq \overline{I}, s \in K} \prod_{j \in I, t \in H} r_{t}^{n+1}\left[\frac{(r_{i} - \delta_{s})(\rho_{t} - r_{j})}{(r_{i} - r_{j})(\rho_{t} - \delta_{s})}\right]
\]

where \(I\) runs over all subsets of order \(m\) of the set \((1, \cdots, k + m)\) with \(\overline{I} = (1, \cdots, k + m) - I, K = (1, \cdots, k)\), and \(H = (1, \cdots, h)\).

**Proof.** By definition of \(D_{k}(z)\) and \(F_{h}(z)\), \(D_{k}(\rho_{t}) = \Pi_{s \in K} (\rho_{t} - \delta_{s})\), and
\[ F_h(r_j) = \prod_{i \in H} \rho_i^{-1}(\rho_i - r_j). \]
So by Lemma 3.6 and (3.26),
\[
D^n(f) = (-1)^m(n+1) \sum_{I} \prod_{i \in I, j \in K} \frac{\rho_i^{n+1}(\rho_i - \delta_s) \rho_i^{-k}(\rho_i - r_j)}{(\rho_i - r_j)(\rho_i - \delta_s)} \]
\[ = (-1)^m(n+1) \sum_{I} \prod_{i \in I, j \in H} \frac{r_i^{l+1}(\rho_i - \delta_s) (\rho_i - r_j)}{(\rho_i - r_j)(\rho_i - \delta_s)}, \]
which is the conclusion of the theorem.

4. Applications of the identity for \(D^n(f)\). Let \(A\) be the annulus defined by (1.1). Let \(G_s(z), D_k(z), F_h(z)\) be polynomials having no common factors of exact degree \(s, k,\) and \(h\) respectively. We assume \(D_k(z)\) and \(F_h(z)\) satisfy the conditions required by Proposition 1.1 and are expressed as in (2.3) and (2.5) respectively. We impose no conditions upon \(G_s(z)\). We assume that \(k \geq 1\), and if \(h = 0\) that \(s \geq k + 1\).

It follows that
\[
G_s(z) - W_k(z)F_h(z) = G_{k+m}(z)D_k(z)F_h(z) - D_k(z)F_h(z),
\]
where \(G_{k+m}(z) = G_s(z) - \lambda D_k(z)F_h(z)\) and \(k + m = \max(s, k + h)\).

If \(s \leq k + h\) there is one value of \(\lambda\) for which \(G_{k+m}(\lambda, z)\) has less than \(k + m\) roots, otherwise \(G_{k+m}(\lambda, z)\) is of exact degree \(k + m\) and satisfies the hypothesis of Proposition 1.1. Thus the Laurent series expansion (1.3) of \(f(z) - \lambda\) in \(A\) generates matrices \(T_n(f - \lambda)\) which are not triangular.

From the theory of algebraic functions [2, pp. 103–104] the set of values of \(\lambda\) for which \(G_{k+m}(\lambda, z) = 0\) has multiple roots, has \(z = 0\) as a root, or has less than \(k + m\) roots is a finite set. For all other \(\lambda\) we denote the roots of \(G_{k+m}(\lambda, z) = 0\) by \(r_i(\lambda), i = 1, \ldots, k + m\), and
\[
f(z) - \lambda = c(\lambda) \prod_{i=1}^{k+m} \frac{(z - r_i(\lambda))/D_k(z)F_h(z)}{D_k(z)F_h(z)},
\]
c(\(\lambda\)) = \(a, a - \lambda b,\) or \(-b\) according to whether \(s > k + h, s = k + h,\) or \(s < k + h.\) Note, \(a\) is the coefficient of \(z^s\) of \(G_s(z),\) and \(b\) is the coefficient of \(z^{k+h}\) of \(D_k(z) \cdot F_h(z).\)

By Theorem 3.1 and the first paragraph of \S 2

\[
(4.1) \quad D^n(f - \lambda) = [(-1)^m c(\lambda)]^{n+1} \sum_{I} \prod_{i \in I, j \in K} \frac{r_i(\lambda) - \delta_s (\rho_i - r_j(\lambda))}{r_i(\lambda) - r_j(\lambda)(\rho_i - \delta_s)},
\]
Assume that for fixed \(\lambda\) the roots \(r_i(\lambda)\) are indexed by increasing modulus, so that \(|r_1(\lambda)| \leq |r_2(\lambda)| \leq \cdots \leq |r_{k+m}(\lambda)|.\) We define the set \(C\) to be the set
$C = \{ \lambda : |r_k(\lambda)| = |r_{k+1}(\lambda)| \}$.

The analysis provided in [3] and [4] shows that the set $C$ is bounded, contains no isolated points, and consists of a finite union of closed analytic arcs. In addition the arguments of J. L. Ullman [5] may be employed to show that the set $C$ is connected. Since the set $C$ contains no isolated points and the set of $\lambda$'s for which the identity (4.1) does not hold is a finite set, the techniques of Schmidt and Spitzer allow us to make the identification of the limit set $B$ of the eigenvalues $\sigma_n$ with the set $C$ defined above.

A related question is the following. Define a sequence of measures $\alpha_n$,

$$\alpha_n(E) = (n + 1)^{-1} \sum_{\lambda_{ni} \in E} 1,$$

where $\lambda_{ni} \in \sigma_n$, and $E$ is an arbitrary set in the $\lambda$-plane. Let $\alpha$ be any weak limit of the measures $\alpha_n$. It will be shown in a later paper that the limit measure $\alpha$ is unique and has at most two atoms. The rational functions $f$ for which $\alpha$ has atoms will be characterized and the weight of the atoms determined.

REFERENCES


