TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 206, 1975

ON THE ACTION OF Θ^n. I

BY

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ABSTRACT. We prove two theorems about the inertia groups of closed, smooth, simply-connected n-manifolds. Theorem A shows that, in certain dimensions, the special inertia group, unlike the full inertia group, can never be equal to Θ^n; Theorem B shows, in dimensions $\equiv 3 \bmod 4$, how to construct explicit closed n-manifolds M^n such that $\Theta(\partial\pi)$ is contained in the inertia group of M^n.

Introduction. As is well known, in 1956 Milnor exhibited orientation preserving self-diffeomorphisms of certain $(n-1)$-spheres, $h: S^{n-1} \longrightarrow S^{n-1}$, which could not be extended to self-diffeomorphisms $H: D^n \longrightarrow D^n$ of the n-disk D^n, and it was natural to ask: given an orientation preserving self-diffeomorphism $h: S^{n-1} \longrightarrow S^{n-1}$, does there at least exist a smooth manifold M_0^n, with boundary $\partial M_0 = S^{n-1}$, and a self-diffeomorphism $H: M_0 \longrightarrow M_0$ such that $H \mid \partial M_0 = h$?

In [10] we answered this question in the affirmative; more explicitly, as an easy corollary of our Equator Theorem [10], now superseded by our Open Book Theorem [11], we proved:

THEOREM [10, THEOREM 2.10]. *In each dimension n, there exists a smooth, simply-connected n-manifold M_0^n with $\partial M_0 = S^{n-1}$ such that any self-diffeomorphism $h: S^{n-1} \longrightarrow S^{n-1}$ extends to a self-diffeomorphism $H: M_0 \longrightarrow M_0$; or, which is the same* (*see Proposition* 1.1 *below*): *for every n, there exists a smooth, closed, simply-connected n-manifold M^n such that the inertia group, $I(M)$, of M is equal to Θ^n.*

REMARK. Recall that the set of h-cobordism classes of oriented homotopy n-spheres is a finite abelian group under the operation # of connected sum, which is denoted by Θ^n; furthermore $\Theta^n(\partial\pi)$ denotes the subgroup of Θ^n consisting of homotopy spheres which bound parallelizable manifolds and one knows [5]: $\Theta^n(\partial\pi) = 0$, if n is even and $\Theta^n(\partial\pi)$ is cyclic for $n \equiv 1 \bmod 4$ and $n \equiv 3 \bmod 4$ and the generators are called, respectively, the Kervaire sphere and the Milnor sphere. The inertia group $I(M^n)$ (see, for example, [1]) of an orientable, smooth, closed n-manifold M^n is the subgroup of Θ^n, consisting of homotopy n-spheres

Received by the editors February 26, 1974.
AMS (MOS) subject classifications (1970). Primary 57D60, 57D55; Secondary 57D50.
Key words and phrases. Homotopy sphere, inertia group, h-cobordism, first exotic class.

Σ^n which "act trivially" on M^n, i.e. the set of $\Sigma^n \in \Theta^n$ such that the connected sum $M \# \Sigma$ is diffeomorphic to M by an orientation preserving diffeomorphism; if, furthermore, this diffeomorphism can be chosen to be homotopic to the "identity": $M \longrightarrow M \# \Sigma$, we say Σ lies in the *special* inertia group, $I_0(M)$, of M.

The object of this note is to answer the two additional natural questions:

(A) Is the above theorem still true if we require that H be homotopic to the identity; i.e. in each dimension n, does there exist a simply-connected, closed, smooth n-manifold with a maximal *special* inertia group, $I_0(M) = \Theta^n$?

(B) Given $h: S^{n-1} \longrightarrow S^{n-1}$, as above, can we find an explicit (well-known, familiar) manifold M_0, such that $\partial M_0 = S^{n-1}$ and h extends to $H: M_0 \longrightarrow M_0$? For example, in [2] Brown and Steer prove that if $h: S^{n-1} \longrightarrow S^{n-1}$ represents the Kervaire sphere Σ^n, then we can choose M_0 to be the familiar Stiefel manifold $V_{2m+1,2}$, with an open n-disk removed. (Here $n = 4m + 1$.)

We prove:

THEOREM A. *If $p > 2$ is prime, then in each dimension $n = 2p(p-1) - 2$ there exists a self-diffeomorphism $h: S^{n-1} \longrightarrow S^{n-1}$ such that, if h extends to a self-diffeomorphism $H: M_0 \longrightarrow M_0$, where $\partial M_0 = S^{n-1}$ and M_0 is simply-connected, then H is not homotopic to the identity. In other words (see Proposition* 1.1 *below), in these dimensions $I_0(M^n) \neq \Theta^n$ for any simply-connected, closed manifold.*

This theorem is a relatively easy consequence of rather strong theorems of Sullivan [7] and Gitler and Stasheff [3].

THEOREM B. *Let B^7 be any smooth, closed, simply-connected 7-manifold on which the Milnor 7-sphere Σ_0^7 acts trivially i.e. $\Sigma_0^7 \in I(B^7)$ (for example, let $B^7 = B_{2,0}$, the "explicit" 7-manifold of Tamura* [9]) *and let N^n $(n = 4(m-1))$ be any smooth, closed, simply-connected n-manifold with signature $\tau(N) = \pm 1$ ($N^n = CP^{2(m-1)}$, for example), then the Milnor sphere Σ_0^{4m+3} lies in the inertia group of $B^7 \times N^n$.*

Together with the result of Brown-Steer, Theorem B answers question (B) in the affirmative for all $\Sigma \in \Theta(\partial\pi)$.

We wish to thank Professor W. Browder for, among many other things, providing the basic idea for proving Theorem B.

1. **Proof of Theorem A.** Let $h: S^{n-1} \longrightarrow S^{n-1}$ represent the homotopy sphere Σ^n, i.e. Σ^n is diffeomorphic to $D^n \cup_h D^n$, two disjoint copies of D^n pasted together along S^{n-1} by h; let M^n be a smooth closed manifold and let M_0 denote M with an open n-disk removed so that $\partial M_0 = S^{n-1}$. The following

simple proposition allows us to state and prove our theorems in the more convenient language of inertia groups:

PROPOSITION 1.1. *$h: S^{n-1} \longrightarrow S^{n-1}$ can be extended to a self-diffeomorphism $H: M_0 \longrightarrow M_0$ if and only if $\Sigma \in I(M)$; H can be chosen to be homotopic to the identity if and only if $\Sigma \in I_0(M)$.*

PROOF. (1) If H exists then the map $H': M \longrightarrow M \# \Sigma$ defined as in Figure 1.2 is easily seen to induce a diffeomorphism $M \longrightarrow M \# \Sigma$.

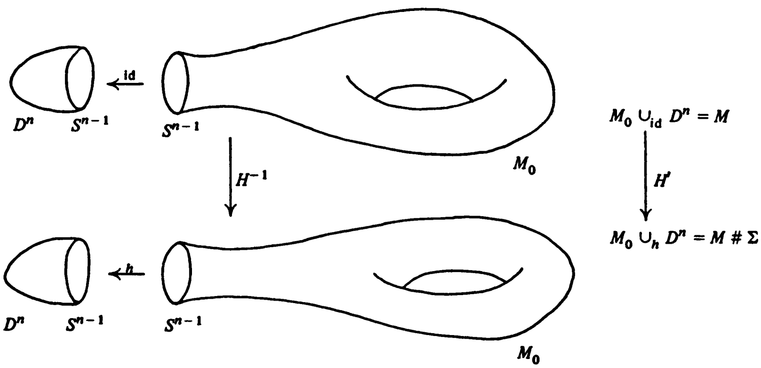

FIGURE 1.2

(2) Suppose $H': M \longrightarrow M \# \Sigma$ is a diffeomorphism.

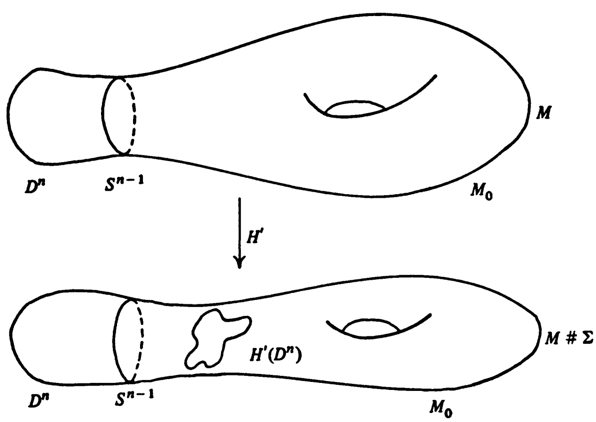

FIGURE 1.3

We apply the well-known Cerf-Palais lemma to the embeddings $i: D^n \longrightarrow D^n \subset M \# \Sigma$ and $j: H'|D^n: D^n \longrightarrow M \# \Sigma$, obtaining a diffeomorphism $G: M \# \Sigma \longrightarrow M \# \Sigma$, homotopic to the identity, and such that $Gj = i$; this implies that $GH'|M_0$ is a diffeomorphism of M_0 onto itself; since $GH'|D =$ identity:

$D^n \longrightarrow D^n$, $GH'|S^{n-1}: S^{n-1} \longrightarrow S^{n-1}$ is equal to h and $GH'|M_0$ is our H.

REMARK. Thus, to say $h: S^{n-1} \longrightarrow S^{n-1}$ can be extended to $H: M_0 \longrightarrow M_0$ is equivalent (by the h-cobordism theorem) to the existence of an "almost differentiable" h-cobordism.

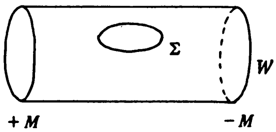

FIGURE 1.4

i.e. a differentiable cobordism W between M, $-M$ and Σ such that $W \cup$ (cone on Σ) is a h-cobordism.

PROOF OF THEOREM A. Recall that there exist classifying spaces BO, BPL and BF respectively for stable vector bundles, stable piecewise linear microbundles and stable spherical fibrations modulo fibre homotopy equivalences. Define PL/O, F/PL and F/O to be the fibres of the natural maps $BO \longrightarrow BPL$, $BPL \longrightarrow BF$ and $BO \longrightarrow BF$. There is a commutative diagram

$$\begin{tikzcd} & F/O \arrow[r] \arrow[d] & F/PL \arrow[d] \\ PL/O \arrow[ur, "j"] \arrow[r] & BO \arrow[r] \arrow[dr] & BPL \arrow[d] \\ & & BF \end{tikzcd}$$

where the rows and columns are fibrations. We also recall that it follows from theorems of Hirsch-Mazur and Smale that $\Theta^n = \pi_n(PL/O)$.

We can now state

LEMMA. *Let $n = 2p(p-1) - 2$ (p odd and prime), then there exists an element $[\alpha] \in \pi_n(PL/O)$ such that $hj_\#[\alpha] \neq 0$ where $j_\#: \pi_n(PL/O) \longrightarrow \pi_n(F/O)$ is induced by the j of the diagram and $h: \pi_n(F/O) \longrightarrow H_n(F/O)$ is the Hurewicz homomorphism.*

We prove Theorem A follows from the lemma.

In effect, we apply

THEOREM (SULLIVAN [7]). *Consider the diagram*

$$\begin{tikzcd} M^n \arrow[r, "f"] & S^n \arrow[r, "\alpha"] & PL/O \arrow[d, "j"] \\ & & F/O \end{tikzcd}$$

where f is a map of degree 1, *M is a smooth, closed, simply-connected manifold, j is the map of the diagram above and α represents the homotopy sphere Σ^n; then there exist a $\Sigma_0^n \in \Theta^n(\partial\pi)$ and an orientation preserving diffeomorphism $M \longrightarrow M \# \Sigma \# \Sigma_0$, homotopic to the "identity", if and only if $j\alpha f$: $M \longrightarrow F/O$ is homotopic to a constant.*

We apply this theorem: If we had a diffeomorphism H: $M \longrightarrow M \# \Sigma$, where Σ is represented by the α of the lemma and $H \cong$ "identity" then

$$(j\alpha)_* f_*([M]) = (j\alpha)_*[S^n] = 0$$

and α would not satisfy the hypothesis of the lemma.

PROOF OF THE LEMMA. We use

THEOREM (GITLER-STASHEFF [3, p. 258]). *Let n be as before; then there exist an element $e \in H^{n+1}(BF, Z_p)$ and a map β: $S^{n+1} \longrightarrow BF$ such that $\beta^*(e) \in H^{n+1}(S^{n+1}, Z)$ is $\neq 0$ (e is called the first exotic class of BF).*

Consider the commutative (up to sign at $H_n(F/O, Z_p)$), diagram:

$$\begin{tikzcd}
& \pi_{n+1}(BF) \arrow[r, "q\#", "\cong"'] \arrow[dr, "k"] & \pi_{n+1}(BO, F/O) \arrow[r, "\partial"] \arrow[d, "l"] & \pi_n(F/O) \arrow[d, "h"] \\
H_{n+1}(BO, Z_p) = 0 \arrow[rr] & & H_{n+1}(BO, F/O, Z_p) \arrow[r, "\delta"] \arrow[d, "q_*"] & H_n(F/O, Z_p) \\
& & H_{n+1}(BF, Z_p) &
\end{tikzcd}$$

where h, l and k are Hurewicz maps composed with the coefficient homomorphism $H_*(\ , Z) \longrightarrow H_*(\ , Z_p)$, $q_\#$ and q_* are induced by the fiber map q: $(BO, F/O) \longrightarrow (BF, \text{pt.})$ and the others belong to the homotopy and homology sequences of the pair $(BO, F/O)$. Since q is a fiber map, $q_\#$ is an isomorphism. Let $[\beta] \in \pi_{n+1}(BF)$ be the element defined by β, we claim $\gamma = \partial q_\#^{-1}([\beta]) \in \pi_n(F/O)$ is such that $h(\gamma) \neq 0$.

In effect, $k(\beta) \neq 0$ because, by Gitler and Stasheff, $\langle \beta^*(e), [S^n] \rangle = \langle e, k[\beta] \rangle \neq 0$ and so, by commutativity, $lq_\#^{-1}[\beta] \neq 0$. Since $H_{n+1}(BO, Z_p) = 0$, because $n + 1 \not\equiv 0 \bmod 4$, δ is a monomorphism and so $\delta l q_\#^{-1}(\beta) = \pm h(\gamma) \neq 0$. In order to obtain our α consider the map $j_\#$: $\pi_n(PL/O) \longrightarrow \pi_n(F/O)$; we claim it is a monomorphism with cokernel 0 or Z_2: Consider the homotopy sequence of the fibering

$$\begin{array}{ccc} PL/O & \xrightarrow{j} & F/O \\ & & \downarrow \\ & & F/PL \end{array}$$

$$\longrightarrow \pi_{n+1}(F/PL) \longrightarrow \pi_n(PL/O) \xrightarrow{j_\#} \pi_n(F/O) \longrightarrow \pi_n(F/PL) \longrightarrow$$

Sullivan [8] has computed $\pi_n(F/PL) = 0, Z_2, 0, Z$, for $n \equiv 1, 2, 3, 4 \bmod 4$ (we have $n = 2p(p-1) - 2 \equiv 2$ and $n + 1 \equiv 3 \bmod 4$) and so

$$0 \longrightarrow \pi_n(PL/O) \xrightarrow{j_\#} \pi_n(F/O) \longrightarrow Z_2$$

is exact. Therefore there exists an $\alpha \in \pi_n(FL/O)$ such that $j_\#(\alpha) = \gamma$ or 2γ; because p is an odd prime and $h(\gamma) \neq 0$, we also have $h(2\gamma) = 2h(\gamma) \neq 0$ ("any nonzero element of $H_*(\ , Z_p)$ has order at least p") and so α satisfies the requirements of the lemma and Theorem A is proven.

2. **Proof of Theorem B.** We need a

LEMMA. *Let Σ_0^7 and Σ_0^{4m+3} be Milnor spheres, let $N = N^{4(m-1)}$ be as above; then $(\Sigma_0^7 \times N) \# \Sigma_0^{4m+3}$ is diffeomorphic to $S^7 \times N$ by an orientation preserving diffeomorphism.*

PROOF OF THE LEMMA. Recall a famous theorem of Novikov (see [4]):

THEOREM (NOVIKOV). *Let W^{n+1} be a simply-connected manifold with simply-connected boundaries M_1^n and $-M_2^n$. Suppose there exists a map γ: $W \longrightarrow M_1$ such that*

(a) $\gamma|M_1 =$ *identity*: $M_1 \longrightarrow M_2$,

(b) $\gamma|M_2$: $M_2 \longrightarrow M_1$ *is a homotopy equivalence,*

(c) $\gamma^*(\nu(M_1)) = \nu(W)$ *where ν denotes the stable normal bundle.*

Then by doing surgery on W we can make it take the form $W' = V \cup_\Sigma H$

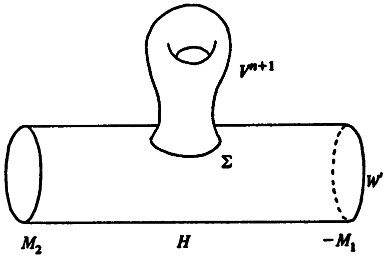

FIGURE 2.1

where H is an almost differentiable h-cobordism and V is a parallelizable manifold with $\partial V =$ a homotopy n-sphere Σ^n.

We also know (see [5]) that $\Sigma^n \in \Theta^n$ is a Milnor sphere if and only if it is cobordant to zero by a parallelizable manifold V of index ± 8. Let V^8 be such a (simply-connected) manifold for the Milnor 7-sphere Σ_0. If $W^8 = V^8 -$ open disk, i.e. $\partial W^8 = \Sigma^7 \cup (-S^7)$, there exists a map $q: W^8 \longrightarrow S^7 \times I$ ($I = [0, 1]$) such that $q|S^7 = \mathrm{id}: S^7 \longrightarrow S^7 \times \{0\}$ and $q|\Sigma^7$ is a homeomorphism $\Sigma^7 \longrightarrow S^7 \times \{1\}$ (this is true, since for any closed n-manifold M^n there exists a map $f: M^n \longrightarrow S^n$ of degree 1).

Since $\nu(W^8)$ is trivial, the map $\gamma: W^8 \longrightarrow S^7 \times \{0\}$ defined by pq, where $p: S^7 \times I \longrightarrow S^7 \times \{0\}$ is the projection, satisfies Novikov's theorem and therefore so does $\gamma \times \mathrm{id}: W^8 \times N \longrightarrow S^7 \times \{0\} \times N$. Hence, by surgery, we obtain $W' = V^{4(m+1)} \cup H^{4(m+1)}$:

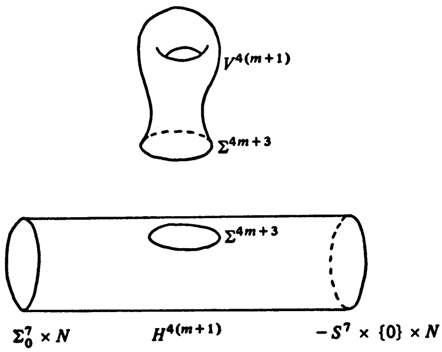

FIGURE 2.2

The index of V is ± 8 because the index of $W^8 \times N$ is ± 8 and is invariant under surgery, and the index of H is zero. Therefore Σ^{4m+3} is a Milnor sphere and the lemma is proven.

PROOF OF THEOREM B. By hypothesis and by the lemma there exist almost differentiable h-cobordisms H^8, $H^{4(m+1)}$:

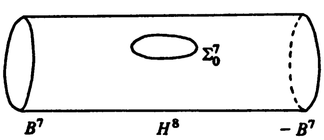

FIGURE 2.3

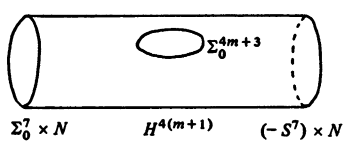

FIGURE 2.4

Now, it is easy to see that

$$H^8 \times N \underset{\Sigma_0^7 \times N}{\cup} H^{4(m+1)} \underset{S^7 \times N}{\cup} D^8 \times N$$

is an almost differentiable h-cobordism

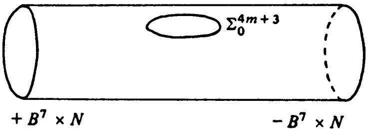

FIGURE 2.5

because pasting on $D^8 \times N$ has the same effect, piecewise linearly, as if we ignored Σ_0^7 in H^8, i.e., it is easy to see that our almost differentiable h-cobordism is just $H^8 \times N$ piecewise linearly, if we ignore the "holes" bounded by Σ_0^7 and Σ_0^{4m+3}. Therefore, Σ_0^{4m+3} acts trivially on $B^7 \times N$ and Theorem B is proven.

REMARK. Rohlin [6] found a smooth, closed, almost-parallelizable, simply-connected 4-manifold of signature = 16. Using this manifold as we used W^8 above, one proves that $2\Sigma_0^{4m+3} \in I(S^3 \times N^{4m})$ in the same manner. We conjecture that in fact $\Sigma_0^{4m+3} \in I(S^3 \times N^{4m})$, since otherwise, by the above method, we would obtain a somewhat curious proof of a fundamental theorem of Rohlin.

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