TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 207, 1975

THE SUBCLASS ALGEBRA ASSOCIATED WITH A FINITE GROUP AND SUBGROUP

BY

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ABSTRACT. Let G be a finite group and let H be a subgroup of G. If $g \in G$, then the set $E_g = \{hgh^{-1} \mid h \in H\}$ is the subclass of G containing g and $\Sigma_{x \in E_g} x$ is the subclass sum containing g. The algebra over the field of complex numbers generated by these subclass sums is called the subclass algebra (denoted by S) associated with G and H. The irreducible modules of S are demonstrated, and results about Schur algebras are used to develop formulas relating the irreducible characters of S to the irreducible characters of G and H.

Introduction. Throughout this article G will denote an arbitrary finite group and H a subgroup of G. We shall call two elements, g and g' of G, H-equivalent if there exists an element $h \in H$ such that $g = hg'h^{-1}$. Being H-equivalent is an equivalence relation on G and the equivalence classes under this relation are called subclasses of G. We shall denote the subclass containing the element $g \in G$ by E_g, and the subclass sum containing g is $B_g = \Sigma_{x \in E_g} x$. The algebra over the complex numbers K generated by these subclass sums is called the subclass algebra, denoted by S, associated with G and H. S is a subalgebra of the group algebra KG.

Subclasses were first studied by E. P. Wigner who developed formulas relating the restriction to H of the irreducible characters of G to the number of subclasses. The work done here was started in order to provide a more algebraic proof to Wigner's results. A paper by F. Roesler about Schur algebras provides an algebraic framework for working with the subclass algebra and proving Wigner's theorems. Roesler's results which are used in this paper are provided for the reader in §1.

Let $\{M_1 \ldots M_s\}$ be the irreducible KG-modules with M_j affording the irreducible character χ_j of G and let $\{N_1 \ldots N_t\}$ be the irreducible KH-modules with N_i affording the irreducible character Φ_i of H. Suppose $\{e_i\}_{i=1}^t$ is a set of orthogonal primitive idempotents of KH such that $N_i \simeq KHe_i$. Define the nonnegative integers $\{c_{ij}\}$ by $\chi_j|_H = \Sigma_{i=1}^t c_{ij}\Phi_i$. Then in §2 we prove:

Received by the editors April 23, 1973.

AMS(MOS) subject classifications (1970). Primary 20C05; Secondary 20C15.

[1] These results are a part of a Ph.D. thesis written under the direction of Professor R. L. Roth of the University of Colorado.

THEOREM (WIGNER). *$\Sigma_{i,j}(c_{ij})^2$ = the number of subclasses.*

THEOREM. *The irreducible S-modules are $\{e_iM_j\}$. The dimension of e_iM_j is c_{ij}.*

Let Ψ_{ij} be the irreducible character of S afforded by e_iM_j. In §3, we develop formulas relating the irreducible characters of S to the irreducible characters of G and H and give the relationship between the subclass algebra and the double coset algebra. The following theorem, proved in §3, may be considered a generalization of a well-known theorem of Clifford:

THEOREM. *For $c_{ij} \neq 0$,*

$$(\chi_j|_H)(h) = \frac{\deg \chi_j}{\deg \Phi_i |G|} \sum_{k \in G} \Phi_i^*(khk^{-1}),$$

where

$$\Phi_i^*(x) = \begin{cases} \Phi_i(x) & \text{if } x \in H, \\ \dfrac{\deg \Phi_i}{c_{ij}|E_x|} \Psi_{ij}(B_x) & \text{if } x \notin H. \end{cases}$$

In his paper, Wigner observed a number of rather simple but useful properties of subclasses:

(1) Each conjugacy class of G is a union of subclasses.

(2) If a subclass contains an element of H, then the subclass is a conjugacy class of H.

(3) Each subclass, as a set, commutes with every element of H.

(4) For any subclass E_g, the number of elements in E_g, denoted $|E_g|$, equals $|H|/|C_H(g)|$, where $C_H(g)$ is the subgroup of H consisting of all the elements of H which commute with g.

(5) The inverses of the elements of a subclass form a subclass.

(6) The product of two subclass sums is a sum of subclass sums (i.e. $B_{g_1}B_{g_2} = \Sigma_{g \in G} n_g B_g$ where the n_g are nonnegative integers).

1. Schur algebras.

DEFINITION 1.1. Let A be a subalgebra of KG. We shall call A a *Schur algebra* if there exists a partition, X, of G such that

(1) The elements $\Sigma_{g \in Y} g$ $(Y \in X)$ form a basis for A over K.

(2) If $Y \in X$, then $Y^* = \{g^{-1} | g \in Y\}$ is also in X.

(3) $\{e\} \in X$.

The element of X containing the element $g \in G$ will be denoted Y_g. We shall also use $\{Y_g\}$ (or just $\{Y\}$) to denote the basis elements of A. Evidently the subclass algebra is a Schur algebra. In this section A will always be an arbitrary Schur algebra.

A is semisimple [4, Satz 1] so if we let $\{\rho_i\}$ denote the irreducible characters of A, we may write $A = {}_i\bigoplus A_{\rho_i}$ where the A_{ρ_i} are simple two-sided ideals of A.

DEFINITION 1.2. $\zeta^A: A \longrightarrow A$ is defined by $\zeta^A(Z) = \Sigma_{Y \in X} c_Y YZY^*$ where $c_Y = v/|Y|$ and $v = \mathrm{LCM}\{|Y| \mid Y \in X\}$. $C_0^A = \zeta^A(Y_e)$ and $a_i = \rho_i(C_0^A)/\deg \rho_i$ for all irreducible characters ρ_i of A.

THEOREM 1.3 [4, Satz 4]. *The central idempotent e_i of A such that $A_{\rho_i} = Ae_i$ is*

$$e_i = \frac{\deg \rho_i}{a_i} \sum_Y c_Y \rho_i(Y^*) Y.$$

THEOREM 1.4 [4, p. 38]. *For any ρ_i there exists an irreducible character of χ of KG such that $\chi|_A$ contains ρ_i.*

DEFINITION 1.5. Let H' be a subgroup of G and let A' be a Schur algebra of KH' with X' a partition of H' such that $\{\Sigma_{g \in Y'} g\}$ $(Y' \in X')$ is a basis for A'. Then A' is called a *Schur subalgebra* of A if for each Y' there exists a subset V' of X such that $Y' = \bigcup_{Y \in V'} Y$.

Define $w = \mathrm{LCM}\{|Y'| \mid Y' \in X'\}$ and $c_{Y'} = w/|Y'|$ for $Y' \in X'$. $\{\gamma_j\}$ will denote the irreducible characters of A'. Define $\zeta^{A'}$ and $C_0^{A'}$ in the same manner as we defined ζ^A and C_0^A and let

$$b_j = \frac{\gamma_j(C_0^{A'})}{\deg \gamma_j} \quad \text{and} \quad \gamma_j^A = \frac{w}{v b_j} \widetilde{\gamma}_j \circ \zeta^A$$

where

$$\widetilde{\gamma}_j(Y) = \begin{cases} \gamma_j(Y')|Y|/|Y'| & \text{for } Y \subseteq Y' \in X', \\ 0 & \text{for } Y \not\subseteq Y' \quad \text{for all } Y' \in X'. \end{cases}$$

THEOREM 1.6 [4, Satz 6]. *γ_j^A is a character of A and if $\rho_i|_{A'} = \Sigma_j d_{ij} \gamma_j$, then $\gamma_j^A = \Sigma_i d_{ij} \rho_i$.*

Note that the above result is a Frobenius reciprocity theorem for Schur algebras.

THEOREM 1.7 [4, Satz 12]. (a) *If $C_0^A \in A'$, then for all ρ_i and γ_j such that* $d_{ij} \neq 0$,

$$\frac{\deg \gamma_j^A}{\deg \gamma_j} = \frac{w}{v} \cdot \frac{a_i}{b_j}.$$

(b) *All the ρ_i in the decomposition of γ_j^A have the same constant a.*

2. **The irreducible S-modules.**

PROPOSITION 2.1. (a) *Let y be an idempotent in KH and let M be a KG-module, then yM is an S-module.*

(b) *If e and d are primitive idempotents in KH such that $KHe \simeq KHd$ (as KH-modules), then $eM_j \simeq dM_j$ ($j = 1 \ldots s$) as S-modules.*

PROOF. (a) M is a KG-module so M is also an S-module. Let $x \in S$ and $m \in M$, then since every element of S commutes with every element of KH, we have $x(ym) = y(xm) \in yM$ and so yM is an S-module.

(b) Consider $P = \mathrm{Hom}_{KH}(KHe, M_i)$. P can be made into an S-module by defining for $a \in KHe$, $\Psi \in P$, and $x \in S$, $(x\Psi)(a) = x\Psi(a)$. Now we will show P is S-isomorphic to eM_i.

Define Φ: $P \longrightarrow eM_i$ by $\Phi(f) = f(e)$ for $f \in P$. $f(e) \in eM_i$ since $f(e) = f(e^2) = ef(e)$. Suppose $f, g \in P$ with $\Phi(f) = \Phi(g)$, then for $ke \in KHe$ ($k \in KH$) we have $f(ke) = kf(e) = kg(e) = g(ke)$ and so $f = g$. So Φ is one-to-one. Φ is onto since $\dim(P) = \dim(eM_i)$ [1, Theorem 54.15]. Let $x \in S$, then it is clear that $\Phi(xf) = x\Phi(f)$ and hence Φ is an S-isomorphism.

If $Q = \mathrm{Hom}_{KH}(KHd, M_i)$ where $KHe \simeq KHd$ (as KH-modules) then using an argument similar to the one above, $P \simeq Q$ as S-modules. So finally we have $eM_i \simeq P \simeq Q \simeq dM_i$ as S-modules. Q.E.D.

In the Introduction, the nonnegative integers c_{ij} were defined by $\chi_j|_H = \Sigma_{i=1}^t c_{ij}\Phi_i$ where $\{\chi_1 \ldots \chi_s\}$ are the irreducible characters of G and $\{\Phi_1 \ldots \Phi_t\}$ are the irreducible characters of H.

COROLLARY 2.2. $\dim(e_iM_j) = c_{ij}$.

PROOF. By the proof of the preceding proposition, $\dim(e_iM_j) = \dim(\mathrm{Hom}_{KH}(KHe_i, M_j)) = c_{ij}$ by Theorem 43.18 in [1]. Q.E.D.

Note that c_{ij} may equal 0 and in this case $e_iM_j = 0$. In the next proposition and elsewhere in this paper, we have the situation where a module M is isomorphic to a direct sum of a_1 copies of M_1, a_2 copies of $M_2, \ldots, a_n$ copies of M_n for some n and modules $\{M_i\}$. In this case, we shall write $M \approx \Sigma_{i=1}^n a_iM_i$.

If M is isomorphic to m copies of a single module N, we shall write $M \approx mN$.

PROPOSITION 2.3. $M_j|_S \approx \Sigma_{i=1}^t (\dim N_i) e_i M_j$.

PROOF. Let $C_1 \ldots C_t$ be a list of the simple two-sided ideals of KH with each $C_k = \oplus\Sigma_{l=1}^{\dim N_k} KHa_{l,k}$ where $\{a_{l,k}\}$ $(k = 1 \ldots t, l = 1 \ldots \dim N_k)$ is a list of orthogonal primitive idempotents. The S-modules $\{a_{l,k}M_j\}$ are submodules of $M_j|_S$. Suppose $a_{l_1,k_1}m_1 = a_{l_2,k_2}m_2$ with $l_1 \neq l_2$ or $k_1 \neq k_2$ and m_1, $m_2 \in M_j$. Then by multiplying by a_{l_1,k_1} we have $a_{l_1,k_1}m_1 = 0 = a_{l_2,k_2}m_2$. Hence $a_{l_1,k_1}M_j \cap a_{l_2,k_2}M_j = 0$. Similarly $a_{l_1,k_1}M_j \cap \Sigma_{(l,k)\neq(l_1,k_1)}a_{l,k}M_j = 0$. Hence $\Sigma_{l,k}a_{l,k}M_j$ is a direct sum of S-modules.

By Proposition 2.1(b), the primitive central orthogonal idempotents $\{e_i\}_{i=1}^t$ may be indexed so that $a_{l,i}M_j \simeq e_iM_j$ (as S-modules) for $1 \leq l \leq \dim N_i$. Therefore

$$\oplus\sum_{l,k} a_{l,k}M_j \approx \oplus\sum_{i=1}^{t} (\dim N_i)e_iM_j.$$

By Corollary 2.2, $\dim e_iM_j = c_{ij}$ and so we have that $\Sigma_{i=1}^t(\dim N_i)e_iM_j$ is a submodule of $M_j|_S$ of dimension $\Sigma_{i=1}^t(\dim N_i)c_{ij}$. But since $M_j|_{KH} \approx \Sigma_{i=1}^t c_{ij}N_i$, it follows that $\dim M_j = \Sigma_{i=1}^t(\dim N_i)c_{ij}$. Consequently we have $M_j|_S \approx \Sigma_{i=1}^t(\dim N_i)e_iM_j$. Q.E.D.

We are now ready to prove the first of Wigner's formulas in [7]. Let $z =$ the number of subclasses of G.

THEOREM 2.4. $\Sigma_{i,j}(c_{ij})^2 = z$.

In order to prove Theorem 2.4, the following lemma will be used:

LEMMA. $z = |H|^{-1}\Sigma_{h\in H}|C_G(h)|$.

PROOF OF THE LEMMA. Since $\Sigma_{g\in G}|C_H(g)| = \Sigma_{h\in H}|C_G(h)|$ we have

$$|H|^{-1}\sum_{h\in H}|C_G(h)| = \sum_{g\in G}\frac{|C_H(g)|}{|H|}$$

$$= \sum_{g\in E_1}\frac{1}{|E_1|} + \ldots + \sum_{g\in E_z}\frac{1}{|E_z|} = z,$$

where $\{E_1 \ldots E_z\}$ are the subclasses of G. Q.E.D.

PROOF OF THEOREM 2.4. Let U be the K-vector space of class functions of G and U' be the K-vector space of class functions of H. Following Winter [8], define $T_1\colon U \longrightarrow U'$ by $T_1(\theta) = \theta|_H$ and define $T_2\colon U' \longrightarrow U$ by $T_2(\Psi) = \Psi^G$. Then T_1 and T_2 are linear transformations. Let A_1 be the matrix for T_1 and A_2 be the matrix for T_2 taking for bases of U and U' the irreducible characters of

G and H respectively. Then $A_1 = (c_{ij})$ $(i = 1 \ldots t, j = 1 \ldots s)$ and A_2 = the transpose of A_1. Let $T\colon U \longrightarrow U$ be defined by $T(\theta) = (\theta\,|_H)^G = T_2 \circ T_1(\theta)$. So the matrix for T with respect to the basis of irreducible characters of G for U is

$$A = A_2A_1 = \begin{pmatrix} \Sigma_{i=1}^t c_{i1}^2 & \cdots & \vdots \\ \vdots & \Sigma_{i=1}^t c_{i2}^2 \ddots & \vdots \\ \cdots & & \Sigma_{i=1}^t c_{is}^2 \end{pmatrix}.$$

Another basis for U is $\{f_1 \ldots f_s\}$ defined by

$$f_i(g) = \begin{cases} 0 & \text{if } g \notin K_i, \\ 1 & \text{if } g \in K_i, \end{cases}$$

where $\{K_1 \ldots K_s\}$ is the set of conjugacy classes of G. Choose k such that $K_i \cap H \neq \emptyset$ $(1 \leqslant i \leqslant k)$ and $K_i \cap H = \emptyset$ for $i > k$. Let

$$r_i = |C_G(g_i)|\,|K_i \cap H|/|H|$$

where $g_i \in K_i$. It can be easily shown that the matrix for T with respect to the new basis $\{f_1 \ldots f_s\}$ of U is

$$B = \begin{pmatrix} r_1 & & & & & & 0 \\ & r_2 & & & & & \\ & & \ddots & & & & \\ & & & r_k & & & \\ & & & & 0 & & \\ & 0 & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}.$$

But

$$\begin{aligned} \sum_{i=1}^k r_i &= |H|^{-1} \sum_{i=1}^k |C_G(g_i)| \cdot |K_i \cap H| \\ &= |H|^{-1} \sum_{j=1}^t |C_G(h_j)| \cdot |C_j| \quad \text{where the conjugacy classes of } H \text{ are } \{C_1 \ldots C_t\} \text{ with } h_j \in C_j. \\ &= |H|^{-1} \sum_{h \in H} |C_G(h)| \\ &= z \quad \text{by the Lemma.} \end{aligned}$$

However, since T is a linear transformation we have trace(A) = trace(B) and therefore $\Sigma_{i,j}(c_{ij})^2 = z$. Q.E.D.

THEOREM 2.5. *The irreducible S-modules are* $\{e_iM_j\}$ $(i = 1 \ldots t, j = 1 \ldots s)$ *and if* $e_iM_j \neq 0$ *then* $e_iM_j \not\cong e_{i'}M_{j'}$ *for* $(i, j) \neq (i', j')$.

PROOF. Let $\{V_1 \ldots V_l\}$ be the irreducible S-modules. By Theorem 1.4 and Proposition 2.3, for each k $(1 \leqslant k \leqslant l)$ we may choose an $e_{i_k}M_{j_k}$ such that V_k is a submodule of $e_{i_k}M_{j_k}$. By Corollary 2.2, $c_{i_kj_k} = \dim e_{i_k}M_{j_k}$. If for some k, $e_{i_k}M_{j_k} = nV_k$ then $c_{i_kj_k} \geqslant \dim V_k$ and so $(c_{i_kj_k})^2 \geqslant (\dim V_k)^2$. And if for some k, $e_{i_k}M_{j_k} = \Sigma_p n_p V_p$ then $c_{i_kj_k} \geqslant \Sigma_p \dim V_p$ and so $(c_{i_kj_k})^2 \geqslant \Sigma_p(\dim V_p)^2$. Suppose there exists a k' such that $c_{i_{k'},j_{k'}} > \dim V_{k'}$. By Theorem 2.4, the number of subclasses $= \Sigma_{i,j}(c_{ij})^2 \geqslant \Sigma'(c_{i_kj_k})^2 > \Sigma_{k=1}^{l}(\dim V_k)^2 =$ the number of subclasses which is impossible (Σ' is summed only over those i_k, j_k's chosen above without repeating any). Hence $c_{i_kj_k} = \dim V_k$ $(1 \leqslant k \leqslant l)$ and $\{e_iM_j\}$ are exactly the irreducible S-modules. Q.E.D.

COROLLARY 2.6. $\chi_j|_S$ *is irreducible if and only if* $\chi_j|_H = n\Phi$ *for a one-dimensional character* Φ *of H.*

PROOF. Proposition 2.3 and Theorem 2.5. Q.E.D.

The next corollary was proved by Wigner as a main result of [7]. Let

$$c_{ij}^{\epsilon} = \begin{cases} 1 & \text{if } c_{ij} > 0, \\ 0 & \text{if } c_{ij} = 0. \end{cases}$$

Define a commutator of S to be an element of the form $xy - yx$ for $x, y \in S$.

COROLLARY 2.7. $\Sigma_{i,j}c_{ij}^{\epsilon}$ = *the number of subclasses minus the number of linearly independent commutators of these.*

PROOF. By Theorem 2.5, $\Sigma_{i,j}c_{ij}^{\epsilon}$ = the number of distinct irreducible characters of S. But as is well known, the number of distinct irreducible characters = $\dim(Z(S)) = \dim(S) - \dim(S')$ where S' is the subspace of S generated by commutators of S. Q.E.D.

3. **The irreducible characters of** S. In this section, we will use the information about Schur algebras discussed in §1. Recall that S and KG are Schur algebras of KG and $Z(KH)$ is a Schur algebra of KH. We may also view S as a Schur subalgebra of the Schur algebra KG, since each subclass is a union of basis elements of KG. Since the conjugacy classes of H are subclasses, $Z(KH)$ is a Schur subalgebra of S.

Theorem 2.5 identified the irreducible S-modules as $\{e_iM_j\}$ where $\{M_1 \ldots M_s\}$ are the irreducible KG-modules and $\{e_1 \ldots e_t\}$ is a set of orthogonal primitive idempotents of KH. It will be understood that e_iM_j appears in the list of irreducible S-modules if and only if $c_{ij} \neq 0$. Let $\{\Psi_{ij}\}$ be the irreducible characters of S with Ψ_{ij} afforded by e_iM_j. Proposition 2.3 implies that $\chi_j|_S = \Sigma_i(\deg \Phi_i)\Psi_{ij}$. So Ψ_{ij_0} is a component of $\chi_{j_0}|_S$ and is not a component of $\chi_k|_S$ for $k \neq j_0$.

Let $Z(KH)$ be the center of KH and $\{w_i\}_{i=1}^t$ the irreducible characters of $Z(KH)$ defined by $w_i(C_h) = |C_h|\Phi_i(h)/\deg \Phi_i$ ($i = 1 \ldots t$) where C_h is the conjugacy class of H containing the element h of H [2, p. 28]. The characters $\{\Phi_i\}$ are indexed so that Φ_i is the character afforded by the irreducible KH-module KHe_i.

THEOREM 3.1. $\Psi_{ij}|_{Z(KH)} = c_{ij}w_i$.

PROOF. Let

$$f_i = \frac{\deg \Phi_i}{|H|} \sum_{C_h} \Phi_i(h^{-1})C_h$$

[3, p. 483] be the orthogonal central idempotents of $Z(KH)$. So $Z(KH) \simeq \bigoplus\Sigma_{i=1}^t Z(KH)f_i$ where $\{Z(KH)f_i\}$ are the simple two-sided ideals of $Z(KH)$ and also the irreducible $Z(KH)$-modules since they are of dimension one.

Let $e_iM_j|_{Z(KH)} \approx \Sigma_{k=1}^t n_k^{ij}(Z(KH)f_k)$ where the n_k^{ij} are nonnegative integers. Since $f_ie_i = e_i$, it follows that $f_ie_iM_j \simeq e_iM_j$ ($i = 1 \ldots t$). Now, multiplication of both sides of the first equation in this paragraph by f_i yields $e_iM_j|_{Z(KH)} \approx n_i^{ij}(Z(KH)f_i)$. This implies

$$n_k^{ij} = \begin{cases} 0 & \text{if } k \neq i, \\ c_{ij} & \text{if } k = i, \end{cases}$$

since $\dim(e_iM_j) = c_{ij}$ and $\dim(Z(KH)f_k) = 1$. Hence $\Psi_{ij}|_{Z(KH)} = c_{ij}w_i$. Q.E.D.

$Z(KH)$ is a Schur subalgebra of S, so we may induce the irreducible characters w_i of $Z(KH)$ to characters w_i^S of S.

COROLLARY 3.2. *w_i^S is an irreducible character of S if and only if there exists a j such that $c_{ij} = 1$ and $c_{ik} = 0$ for $k \neq j$.*

PROOF. By Theorem 3.1, $\Psi_{ij}|_{Z(KH)} = c_{ij}w_i$ for all i and j. So by Theorem 1.6, $w_i^S = \Sigma_j c_{ij}\Psi_{ij}$ and w_i^S is irreducible if and only if one c_{ij} in the sum is nonzero and this c_{ij} has to be 1. Q.E.D.

In §1, integer constants v, c_y, and a_i were defined associated with a Schur algebra. Also the constant C_0^A of a Schur algebra A was defined. In Proposition 3.3, we give the values for these constants for the Schur algebra S. Note

$$v = \mathrm{LCM}\{|E_g| \mid g \in G\} \quad \text{and} \quad c_{B_g} = \frac{v}{|E_g|},$$

$$C_0^S = \sum_{B_g} c_{B_g} B_g B_g^* \qquad \text{and} \quad a_{ij} = \frac{\Psi_{ij}(C_0^S)}{\deg \Psi_{ij}}.$$

PROPOSITION 3.3.

(a)
$$C_0^S = \frac{v}{|H|} \sum_{h \in H; g \in G} ghg^{-1}h^{-1},$$

(b)
$$a_{ij} = \frac{v|G|c_{ij}}{\deg \Phi_i \deg \chi_j}.$$

PROOF. (a) C_0^S was defined to be $\Sigma_{B_g}(v/|E_g|)B_gB_g^*$. Choose a B_{g_1} and let $B_{g_1} = g_1 + \ldots + g_n$, then $B_{g_1}^* = g_1^{-1} + \ldots + g_n^{-1}$. For $g, g' \in B_{g_1}$ there exists an $h \in H$ such that $hC_H(g)h^{-1} = C_H(g')$. For each i with $1 \leqslant i \leqslant n$, there exists a subset $\{h_{i1}, \ldots, h_{in}\}$ of H such that $H = h_{i1}C_H(g_i) \cup \ldots \cup h_{in}C_H(g_i)$ and we may write

$$B_{g_1}^* = g_1^{-1} + h_{12}g_1^{-1}h_{12}^{-1} + \ldots + h_{1n}g_1^{-1}h_{1n}^{-1}$$

or

$$B_{g_1}^* = g_2^{-1} + h_{22}g_2^{-1}h_{22}^{-1} + \ldots + h_{2n}g_2^{-1}h_{2n}^{-1}$$
$$\vdots$$

or

$$B_{g_1}^* = g_n^{-1} + h_{n2}g_n^{-1}h_{n2}^{-1} + \ldots + h_{nn}g_n^{-1}h_{nn}^{-1}.$$

So

$$\begin{aligned} B_{g_1}B_{g_1}^* &= (1 + g_1h_{12}g_1^{-1}h_{12}^{-1} + \ldots + g_1h_{1n}g_1^{-1}h_{1n}^{-1}) \\ &\quad + (1 + g_2h_{22}g_2^{-1}h_{22}^{-1} + \ldots + g_2h_{2n}g_2^{-1}h_{2n}^{-1}) \\ &\quad + \ldots + (1 + g_nh_{n2}g_n^{-1}h_{n2}^{-1} + \ldots + g_nh_{nn}g_n^{-1}h_{nn}^{-1}) \\ &= \frac{1}{|C_H(g_1)|}\left[\sum_{h \in H} g_1hg_1^{-1}h^{-1} + \ldots + \sum_{h \in H} g_nhg_n^{-1}h^{-1}\right]. \end{aligned}$$

Hence $|E_g|^{-1}B_gB_g^* = |H|^{-1}\Sigma_{h\in H;k\in E_g}khk^{-1}h^{-1}$ for each $g \in G$ and so $C_0^S = v|H|^{-1}\Sigma_{h\in H;g\in G}ghg^{-1}h^{-1}$. Q.E.D.

(b) By the discussion in the beginning of this section, the only irreducible character of G with the property that Ψ_{ij} appears in its restriction to S is χ_j and Ψ_{ij} appears with multiplicity $\deg \Phi_i$. So $\Psi_{ij}^{KG} = (\deg \Phi_i)\chi_j$ by Theorem 1.6. This implies $\deg \Psi_{ij}^{KG} = (\deg \Phi_i)(\deg \chi_j)$. Letting $KG =$ the Schur algebra A in the definitions of § 1, we have $C_0^{KG} = |G|$ and the constants $b_j = \chi_j(C_0^{KG})/\deg \chi_j = |G|$. By Theorem 1.7(a),

$$\frac{\deg \Psi_{ij}^{KG}}{\deg \Psi_{ij}} = \frac{vb_j}{a_{ij}} \quad \text{or} \quad \frac{\deg \Phi_i \deg \chi_j}{\deg \Psi_{ij}} = \frac{v|G|}{a_{ij}}. \quad \text{Q.E.D.}$$

PROPOSITION 3.4.

$$\sum{}'(\deg \chi_j)\Psi_{ij}(B_g) = 0 \quad \textit{for} \quad B_g \cap H = \emptyset$$

and

$$\sum{}'(\deg \chi_j)\Psi_{ij}(B_g) = |G|\,|H|^{-1}\Phi_i(B_g) \quad \textit{for } B_g \cap H \neq \emptyset.$$

The sum Σ' means we sum over all j's such that $e_iM_j \neq 0$.

PROOF. As in the proof of Theorem 3.1, let

$$f_i = \frac{\deg \Phi_i}{|H|} \sum_{B_h \subseteq H} \Phi_i(h^{-1})B_h \quad \text{for } i = 1 \ldots t$$

be the central primitive idempotents of KH. (We may sum over the subclasses contained in H because they are exactly the conjugacy classes of H.) Since each element of S commutes with all the elements of KH, each f_i is also a central idempotent of S. Let $\{b_l\}$ be the primitive central idempotents of S indexed such that there exist disjoint sets $\{b_l\}_{l=1}^{n_1}, \{b_l\}_{l=n_1+1}^{n_2}, \ldots, \{b_l\}_{l=n_{t-1}+1}^{n_t}$ such that $f_i = \Sigma_{l=n_{i-1}+1}^{n_i} b_l$.

The simple two-sided ideals of S are $\{Sb_l\}$ and each Sb_l is the direct sum of isomorphic S modules. We will show that if $n_{i_0-1} < l \leq n_{i_0}$ for some i_0 and if $Sb_l \approx (\dim e_iM_j)e_iM_j$ for some i and j, then necessarily $KHf_{i_0} \approx (\dim KHe_i)KHe_i$. So suppose e_iM_j is a direct summand of Sb_l for $n_{i_0-1} < l \leq n_{i_0}$ and KHe_i is not a direct summand of KHf_{i_0}. Then $0 = S(f_{i_0}b_l)$ since in this case f_{i_0} annihilates e_iM_j. But this implies $f_{i_0}b_l = 0$ which is impossible.

Hence we may renumber the b_l's with two subscripts such that $Sb_{ij} \approx (\dim e_iM_j)e_iM_j$ and b_{ij} is a summand of f_i. Now by Theorem 1.3,

$$b_{ij} = \frac{\deg \Psi_{ij}}{a_{ij}} \sum_{B_g} c_{B_g} \Psi_{ij}(B_g^*)B_g$$

and hence

$$\frac{\deg \Phi_i}{|H|}\sum_{B_h\subseteq H}\Phi_i(h^{-1})B_h = \sum_j \frac{\deg \Psi_{ij}}{a_{ij}}\sum_{B_g} c_{B_g}\Psi_{ij}(B_g^*)B_g$$

$$= \sum_{B_g}\frac{1}{|E_g|}\left[\sum_j \frac{\deg \Phi_i \deg \chi_j}{|G|}\Psi_{ij}(B_g^*)\right]B_g$$

by Proposition 3.3(b). So by comparing coefficients, we have for $B_g \cap H = \varnothing$,

$$0 = \frac{1}{|E_g|}\frac{\deg \Phi_i}{|G|}\sum_j (\deg \chi_j)\Psi_{ij}(B_g),$$

therefore $\Sigma_j(\deg \chi_j)\Psi_{ij}(B_g) = 0$ and, for $B_g \cap H \neq \varnothing$,

$$\sum_j (\deg \chi_j)\Psi_{ij}(B_g) = |G|\,|H|^{-1}\Phi_i(B_g). \quad \text{Q.E.D.}$$

THEOREM 3.5. *For any c_{ij} that is not equal to zero we have for $g \in G$,*

$$\chi_j(g) = \frac{\deg \chi_j}{c_{ij}|K_g|}\Psi_{ij}(K_g)$$

where K_g is the conjugacy class of G containing g.

PROOF. By the discussion in the introduction to this section, Ψ_{ij} is a component of $\chi_k|_S$ if and only if $k = j$ and $c_{ij} \neq 0$ and if this is the case Ψ_{ij} occurs as a component of $\chi_j|_S$ with multiplicity $\deg \Phi_i$. Therefore, by Theorem 1.6,

$$(\deg \Phi_i)\chi_j = \Psi_{ij}^{KG} = (v/a_{ij})\tilde{\Psi}_{ij} \circ \zeta^{KG}$$

where

$$\tilde{\Psi}_{ij}(g) = |E_g|^{-1}\Psi_{ij}(B_g) \quad \text{and} \quad \zeta^{KG}(g) = \sum_{k\in G} kgk^{-1}.$$

So for $g \in G$ we have by Proposition 3.3(b),

$$(\deg \Phi_i)\chi_j(g) = \frac{v \deg \Phi_i \deg \chi_j}{v|G|c_{ij}}\tilde{\Psi}_{ij} \circ \zeta^{KG}(g)$$

and hence

$$\chi_j(g) = \frac{\deg \chi_j}{|G|c_{ij}}\sum_{k\in G}\frac{1}{|E_{kgk^{-1}}|}\Psi_{ij}(B_{kgk^{-1}})$$

$$= \frac{\deg \chi_j}{|G|c_{ij}}|C_G(g)|\sum_{k\in K_g}\frac{1}{|E_k|}\Psi_{ij}(B_k) = \frac{\deg \chi_j}{c_{ij}|K_g|}\Psi_{ij}(K_g). \quad \text{Q.E.D.}$$

COROLLARY 3.6. *For $h \in H$, and $c_{ij} \neq 0$,*

$$\chi_j|_H(h) = \frac{\deg \chi_j}{\deg \Phi_i |G|} \sum_{k \in G} \Phi_i^*(khk^{-1})$$

where

$$\Phi_i^*(x) = \begin{cases} \Phi_i(x) & \text{if } x \in H, \\ \dfrac{\deg \Phi_i}{c_{ij}|E_x|} \Psi_{ij}(B_x) & \text{if } x \notin H. \end{cases}$$

PROOF. By the proof of Theorem 3.5 and Theorem 3.1,

$$\chi_j(h) = \frac{\deg \chi_j}{|G| c_{ij}} \left[\sum_{k \in G; khk^{-1} \in H} c_{ij} \frac{\Phi_i(khk^{-1})}{\deg \Phi_i} + \sum_{k \in G; khk^{-1} \notin H} \frac{1}{|E_{khk^{-1}}|} \Psi_{ij}(B_{khk^{-1}}) \right]. \quad \text{Q.E.D.}$$

Note that if $H \lhd G$, Corollary 3.6 is Theorem 17.3(g) in [3].

THEOREM (CLIFFORD). *If χ is an irreducible character of G and N is a normal subgroup of G, then all the irreducible characters of N which are components of $\chi|_N$ are conjugate. That is, $\chi|_N = e\sum_{i=1}^{n} \Phi^{g_i}$ where $\Phi^x(g) = \Phi(xgx^{-1})$, $\{g_1 \ldots g_n\}$ are coset representatives of the subgroup $\{k \in G | \Phi^k = \Phi\}$ of G and $e = (\deg \chi)/n \deg \Phi$.*

So we may think of Corollary 3.6 as a generalization of this theorem of Clifford.

O. Tamaschke [5], [6] has done considerable work on Schur algebras, but unlike Roesler, he does not require $\{e\}$ to be in the partition of G which forms the basis elements for the algebra. An algebra Tamaschke has given special attention to is the double coset algebra which is the subalgebra of KG generated by all double cosets, HgH, with $g \in G$. By modifying a proof of Tamaschke, [5, p. 19] it can be shown that the double coset algebra is a two-sided ideal of the subclass algebra.

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