TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 207, 1975

FUNCTIONS OF VANISHING MEAN OSCILLATION(1)

BY

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ABSTRACT. A function of bounded mean oscillation is said to have vanishing mean oscillation if, roughly speaking, its mean oscillation is locally small, in a uniform sense. In the present paper the class of functions of vanishing mean oscillation is characterized in several ways. This class is then applied to answer two questions in analysis, one involving stationary stochastic processes satisfying the strong mixing condition, the other involving the algebra $H^\infty + C$.

1. **Introduction.** C. Fefferman and E. M. Stein [4] have recently exhibited the basic importance in harmonic analysis of BMO, the space of functions of bounded mean oscillation. The purpose of the present paper is to call attention to a certain natural subspace of BMO, which, roughly speaking, occupies the same position in BMO as does the space of bounded uniformly continuous functions on $\mathbf{R}^N$ in the space L^∞. This subspace is called VMO, the space of functions of vanishing mean oscillation. The functions in BMO are characterized by the boundedness of their mean oscillations over cubes. The functions in VMO are those with the additional property that their mean oscillations over small cubes are small.

The precise definition of VMO and several alternative characterizations of it are given in the following section. For simplicity, the discussion there is limited to the one-dimensional case, the proofs being carried out in the context of the real line. These proofs involve only elementary measure theoretic considerations and a few known properties of BMO; they apply, with simplifications, to the case of the unit circle and, with only minor complications, to the case of $\mathbf{R}^N$, as well. The remainder of the paper is devoted to applications of VMO to certain questions in analysis. These applications provided the author's initial motivation for studying VMO. (The space VMO has been useful in other connections also; see [13] and [15].) §3 concerns weight functions on the unit circle whose corresponding stationary stochastic processes satisfy the strong mixing condition. The space VMO provides the link between two conditions such weight functions are known, to satisfy. §4 contains the answer to a question about the algebra $H^\infty + C$ raised

Received by the editors April 16, 1974.

AMS (MOS) subject classifications (1970). Primary 30A76, 34J10, 42A40; Secondary 60G10.

(1) Research supported in part by National Science Foundation Grant GP-30798X.

by R. G. Douglas. §5 contains an analogous result about a related algebra.

The author is indebted to Ronald Douglas and Charles Fefferman for very helpful discussions. He is especially greatful to Lynn Williams for giving invaluable aid at several stages of this investigation.

2. **The space VMO.** We shall be concerned in the present section with functions on the real line (except for a brief mention at the end of the section of the situation on the unit circle). The Lebesgue measure of a measurable subset E of $\mathbf{R}$ will be denoted by $|E|$. For f a locally integrable, complex valued function on $\mathbf{R}$ and I a finite interval, we let $f_I = |I|^{-1}\int_I f(x)\,dx$. For $0 < a < \infty$ we define

$$M_a(f) = \sup_{|I|\leq a} |I|^{-1}\int_I |f(x) - f_I|\,dx,$$

and we set

$$M_0(f) = \lim_{a\to 0} M_a(f), \qquad \|f\|_* = \lim_{a\to\infty} M_a(f).$$

The function f is said to have bounded mean oscillation, or to belong to BMO, if $\|f\|_* < \infty$. The space BMO is a Banach space under the norm $\|\cdot\|_*$, provided that two functions differing by a constant are identified.

A function f in BMO is said to have vanishing mean oscillation, or to belong to VMO, if $M_0(f) = 0$. An elementary argument establishes that VMO is a closed subspace of BMO. It is obvious that VMO contains all uniformly continuous functions in BMO. We let UC denote the space of complex valued, uniformly continuous functions on $\mathbf{R}$ and BUC the space of bounded functions in UC.

The spaces BMO and VMO are obviously translation invariant. For y a real number, we let T_y denote the operator of translation by y; that is, $(T_y f)(x) = f(x - y)$ for any function f on $\mathbf{R}$.

The following theorem offers several alternative descriptions of VMO.

THEOREM 1. *For f a function in* BMO, *the following conditions are equivalent*:

(i) *f is in* VMO;

(ii) *f is in the* BMO-*closure of* UC $\cap$ BMO;

(iii) $\lim_{y\to 0}\|T_y f - f\|_* = 0$;

(iv) *f can be written as $u + \tilde{v}$, where u and v belong to* BUC, *and $\tilde{v}$ denotes the conjugate function* (*or Hilbert transform*) *of v.*

Two lemmas are needed for the proof.

LEMMA 1. *If f is a function in* BMO *and φ is an integrable function of compact support, then $\varphi * f$ is in* BMO *and $\|\varphi * f\|_* \leq \|\varphi\|_1\|f\|_*$. If, in addition, φ is continuous, then $\varphi * f$ is in* UC.

LEMMA 2. *There is a positive constant A such that for all f in* BMO, $\operatorname{dist}(f, \mathrm{UC} \cap \mathrm{BMO}) \leqslant AM_0(f)$, *the distance being measured in the norm* $\|\cdot\|_*$.

The proof of Lemma 1 is omitted as it involves only standard manipulations with convolutions. To establish Lemma 2, let f belong to BMO, and let c be any positive real number exceeding $M_0(f)$. Choose $a > 0$ such that $M_a(f) < c$. Subdivide $\mathbf{R}$ into closed, nonoverlapping intervals of length $a/3$, and let $\mathcal{I}$ denote the collection of these intervals. Let h be the step function that, on the interval I in $\mathcal{I}$, takes the constant value f_I. We shall estimate the size of the discontinuities of h and the distance of h from f. Then, by taking a convolution of h, we shall produce a function in UC $\cap$ BMO which suitably approximates f.

Consider first any interval I_1 in $\mathcal{I}$, let I_2 and I_3 be the two adjacent intervals, and let $I' = I_1 \cup I_2 \cup I_3$. For $j = 1, 2, 3$ we have

$$|I_j|^{-1} \int_{I_j} |f(x) - f_{I'}|\, dx \leqslant 3|I'|^{-1} \int_{I'} |f(x) - f_{I'}|\, dx \leqslant 3c.$$

Consequently $|f_{I_j} - f_{I'}| \leqslant 3c$ $(j = 1, 2, 3)$, and so $|f_{I_j} - f_{I_k}| \leqslant 6c$ $(j, k = 1, 2, 3)$. From this we see that if $|x - y| \leqslant a/3$, then $|h(x) - h(y)| \leqslant 6c$.

To estimate $\|f - h\|_*$, consider first an interval I whose length does not exceed $a/3$. Then I is contained in an interval I' of the type considered above, and we have

$$\begin{aligned} &|I|^{-1} \int_I |f(x) - h(x) - (f-h)_I|\, dx \\ &\qquad \leqslant |I|^{-1} \int_I |f(x) - f_I|\, dx + |I|^{-1} \int_I |h(x) - h_I|\, dx \\ &\qquad \leqslant c + |h_I - h_{I'}| + |I|^{-1} \int_I |h(x) - h_{I'}|\, dx. \end{aligned}$$

By the estimate obtained above, h differs on I' from $h_{I'}$ by at most $3c$. Each of the last two terms in the preceding inequality is therefore at most $3c$, and we have

$$|I|^{-1} \int_I |f(x) - h(x) - (f-h)_I|\, dx \leqslant 7c.$$

Consider next an interval I of length greater than $a/3$. Let $\{I_1, I_2, \ldots, I_n\}$ be a minimal subcollection of $\mathcal{I}$ that covers I, and let $I' = \bigcup_1^n I_j$. Then $|I'|/|I| \leqslant 3$, and we have

$$\begin{aligned} |I|^{-1} \int_I |f(x) - h(x)|\, dx &= |I|^{-1} \sum_1^n \int_{I \cap I_j} |f(x) - f_{I_j}|\, dx \\ &\leqslant |I|^{-1} \sum_1^n \int_{I_j} |f(x) - f_{I_j}|\, dx \leqslant |I|^{-1} \sum_1^n c|I_j| = c|I'|/|I| \leqslant 3c. \end{aligned}$$

Consequently

$$|I|^{-1}\int_I |f(x) - h(x) - (f - h)_I|\,dx$$

$$\leqslant |I|^{-1}\int_I |f(x) - h(x)|\,dx + |(f - h)_I|$$

$$\leqslant 3c + 3c = 6c.$$

From the above estimates we see that $\|f - h\|_* \leqslant 7c$. In particular, we may conclude that h is in BMO.

Let φ be a positive continuous function with integral 1 and support contained in $(-a/6, a/6)$, and let $g = \varphi * h$. Our previous estimate on the size of the discontinuities of h guarantees that $\|h - g\|_\infty \leqslant 6c$. Hence $\|h - g\|_* \leqslant 12c$, so we have $\|f - g\|_* \leqslant 19c$. As g belongs to UC $\cap$ BMO by Lemma 1, we have established Lemma 2 with the constant $A = 19$.

The implication (i) $\Rightarrow$ (ii) of Theorem 1 is a corollary of Lemma 2. The implication (ii) $\Rightarrow$ (i) is trivial, as is the implication (ii) $\Rightarrow$ (iii). To establish the implication (iii) $\Rightarrow$ (ii), let f satisfy (iii), and for $n = 1, 2, \ldots$ let φ_n be a positive continuous function with integral 1 and support contained in $(-1/n, 1/n)$. Each function $\varphi_n * f$ is in UC $\cap$ BMO by Lemma 1. Either by a simple direct estimate or by referring to the theory of homogeneous Banach spaces [11], one can show that $\|f - \varphi_n * f\|_* \longrightarrow 0$, thereby verifying that f satisfies (ii).

The connection between property (iv) and the other properties depends upon results from [4]. Fefferman and Stein establish there the following basic theorems.

(I) *The conjugation operator is a bounded map of L^∞ into* BMO.

(II) *Every f in* BMO *can be written as $f = u + \tilde{v}$ with u and v in L^∞. Moreover, there is an absolute constant B such that, in the preceding representation, one may take $\|u\|_\infty \leqslant B\|f\|_*$ and $\|v\|_\infty \leqslant B\|f\|_*$.*

The implication (iv) $\Rightarrow$ (iii) is an immediate corollary of (I). To establish the reverse implication, and thereby complete the proof of Theorem 1, suppose f satisfies (iii) and, using (II), write $f = u_0 + \tilde{v}_0$, where u_0 and v_0 are in L^∞, $\|u_0\|_\infty \leqslant B\|f\|_*$, and $\|v_0\|_\infty \leqslant B\|f\|_*$. As in the proof of the implication (iii) $\Rightarrow$ (ii), there is a positive continuous function φ_1 with integral 1 and compact support such that $\|f - \varphi_1 * f\|_* < \frac{1}{2}\|f\|_*$. The functions $u_1 = \varphi_1 * u_0$ and $v_1 = \varphi_1 * v_0$ are then in BUC, with $\|u_1\|_\infty \leqslant B\|f\|_*$ and $\|v_1\|_\infty \leqslant B\|f\|_*$, and we have $\varphi_1 * f = u_1 + \tilde{v}_1$. Applying the same reasoning to the function $f - (u_1 + \tilde{v}_1)$ in place of f, we obtain functions u_2 and v_2 in BUC, with $\|u_2\|_\infty \leqslant \frac{1}{2}B\|f\|_*$ and $\|v_2\|_\infty \leqslant \frac{1}{2}B\|f\|_*$, such that $\|f - (u_1 + \tilde{v}_1) - (u_2 + \tilde{v}_2)\|_* < \frac{1}{4}\|f\|_*$. Iterating, we obtain sequences (u_n) and (v_n) in BUC with the following properties:

(a) $\|u_n\|_\infty \leq 2^{-n+1}B\|f\|_*$, $\|v_n\|_\infty \leq 2^{-n+1}B\|f\|_*$,

(b) $\|f - \Sigma_1^n (u_k + \tilde{v}_k)\|_* < 2^{-n}\|f\|_*$.

By (a), the functions $u = \Sigma_1^\infty u_n$ and $v = \Sigma_1^\infty v_n$ are in BUC, and from (b) and (I) it follows that $f = u + \tilde{v}$. The proof of Theorem 1 is now complete.

The author is indebted to Charles Fefferman for the proof of the last implication. The argument replaces one found by the author which has the disadvantage of not applying in higher dimensions.

We turn to another characterization of VMO, one which is relevant to the discussion in the following section. The characterization is analogous to and was suggested by a result on BMO due to R. Hunt, B. Muckenhoupt and R. Wheeden [8].

For w a positive measurable function on $\mathbf{R}$ and $a > 0$, we let $N_a(w)$ denote the supremum of $|I|^{-2}(\int_I w(x)\,dx)(\int_I w(x)^{-1}\,dx)$ as I ranges over all intervals satisfying $|I| \leq a$. We let $N_0(w) = \lim_{a\to 0} N_a(w)$. This quantity is obviously infinite unless both w and w^{-1} are locally integrable. An application of Schwarz's inequality shows that $N_0(w) \geq 1$ always.

THEOREM 2. *Let f be a real function in* BMO. *Then for f to belong to* VMO, *it is necessary and sufficient that $N_0(e^f) = 1$.*

The necessity of the condition in Theorem 2 is an immediate consequence of a corollary to the main lemma proved by F. John and L. Nirenberg in the original paper on functions of bounded mean oscillation [10, p. 415]. The relevant result can be stated, in the one dimensional case, as follows: There are positive constants b and B such that if f is any integrable function defined on an interval I, and if

$$|J|^{-1} \int_J |f(x) - f_J|\,dx \leq K < b$$

for every subinterval J of I, then

$$|I|^{-1} \int_I \exp|f(x) - f_I|\,dx \leq 1 + \frac{BK}{b - K}.$$

The sufficiency of the condition in Theorem 2 is elementary. It is an immediate consequence of the following simple measure theoretic lemma.

LEMMA 3. *Let (X, m) be a probability measure space and w a positive measurable function on X such that $(\int w\,dm)(\int w^{-1}\,dm) = 1 + c^3$, where $0 < c < \tfrac12$. Then*

$$\int \left| \log w - \int \log w\,dm \right| dm \leq 16c.$$

To establish this, we may assume without loss of generality that $\int w\,dm = 1$

and thus that $\int w^{-1}\, dm = 1 + c^3$. Let F be the set where $(1 + c)^{-1} < w < 1 + c$ and let $E = X - F$. We have

$$\begin{aligned} 2 + c^3 &= \int_E (w + w^{-1})\, dm + \int_F (w + w^{-1})\, dm \\ &\geqslant [1 + c + (1 + c)^{-1}]\, m(E) + 2m(F) \\ &= 2 + c^2(1 + c)^{-1} m(E). \end{aligned}$$

Hence $m(E) \leqslant c(1 + c) \leqslant 2c$, so $m(F) \geqslant 1 - 2c$. Therefore

$$\begin{aligned} \int_E w\, dm = 1 - \int_F w\, dm &\leqslant 1 - (1 + c)^{-1} m(F) \\ &\leqslant 1 - (1 - 2c)(1 + c)^{-1} \leqslant 3c, \end{aligned}$$

$$\begin{aligned} \int_E w^{-1}\, dm = 1 + c^3 - \int_F w^{-1}\, dm &\leqslant 1 + c^3 - (1 + c)^{-1} m(F) \\ &\leqslant 1 + c^3 - (1 - 2c)(1 + c)^{-1} \leqslant 4c. \end{aligned}$$

On F we have $|\log w| \leqslant \log(1 + c) \leqslant c$. Since $|\log w| \leqslant w + w^{-1}$ everywhere, we obtain

$$\int |\log w|\, dm \leqslant \int_E (w + w^{-1})\, dm + cm(F) \leqslant 3c + 4c + c = 8c.$$

Hence

$$\int \left| \log w - \int \log w\, dm \right| dm \leqslant 2 \int |\log w|\, dm \leqslant 16c,$$

and the lemma is proved.

A word is in order concerning the case of the unit circle. The situation on the unit circle is less complicated because one has no need to worry about behavior at ∞. Thus, Theorem 1 and its proof simplify slightly when they are translated to the unit circle. In particular, the space C of continuous functions on the unit circle plays the roles of both UC and BUC in the translation of Theorem 1.

3. **Strong mixing.** Let w be a nonnegative integrable function on the unit circle. Let $\mathcal{P}$ be the span in the Hilbert space $L^2(w)$ of the functions $e^{-ik\theta}$, $k = 1, 2, 3, \ldots$. For $n = 0, 1, 2, \ldots$, let $\mathcal{F}_n$ be the span in $L^2(w)$ of the functions $e^{ik\theta}$, $k = n, n + 1, \ldots$. We consider the problem of deciding when $\mathcal{P}$ and $\mathcal{F}_n$ are asymptotically orthogonal. More precisely, for $n = 0, 1, 2, \ldots$, we let ρ_n denote the supremum of $|(f, g)|$ as f and g range over the unit spheres of $\mathcal{P}$ and $\mathcal{F}_n$, respectively (the inner product being taken in the Hilbert space $L^2(w)$). The quantity ρ_n is then the cosine of the angle between $\mathcal{P}$ and $\mathcal{F}_n$. In case $\lim_{n\to\infty} \rho_n = 0$, we shall say that w belongs to the class W.

The problem of characterizing the functions in W is of interest in the theory of stationary stochastic processes. In case w arises as the spectral density function of such a process, it belongs to W if and only if the process satisfies the strong mixing condition of M. Rosenblatt; see [16] for a fuller explanation. A necessary and sufficient condition for w to belong to W was obtained by H. Helson and the author in [6] and improved by the author in [12]. The theorem from [12] states that w belongs to W if and only if it can be written as $|P|^2\exp(u + \tilde{v})$, where P is a polynomial, u and v belong to C (the space of continuous functions on the unit circle), and $\tilde{v}$ is the conjugate function of v. We let W_0 denote the class of functions $\exp(u + \tilde{v})$ with u and v in C.

Prior to the writing of [6], I. A. Ibragimov [9] obtained a necessary condition for w to belong to W, the statement of which requires some additional notation. For any subarc $I = \{e^{i\theta} : \theta_0 - a \leqslant \theta \leqslant \theta_0 + a\}$ of the unit circle, let

$$I_+ = \{e^{i\theta} : \theta_0 \leqslant \theta \leqslant \theta_0 + a\} \quad \text{and} \quad I_- = \{e^{i\theta} : \theta_0 - a \leqslant \theta \leqslant \theta_0\}.$$

Let W_1 be the class of functions w with the property that $(w_{I_+} - w_{I_-})/w_I$ tends uniformly to 0 as $|I|$ tends to 0. Ibragimov's theorem states that every function in W can be written as $|P|^2w$ where P is a polynomial and w is in W_1. We note that if w is bounded, then membership in W_1 implies that the indefinite integral of w is uniformly smooth, while the latter condition implies membership in W_1 for a w which is bounded away from 0.

Until now, the relation between the Helson-Sarason condition and Ibragimov's condition has not been well understood. Of course, the former condition implies the latter one, but a direct proof of this has not been available. The relation between the two conditions is clarified by Theorems 1 and 2, or, rather, by their analogues for the unit circle. From these theorems it follows that w belongs to W_0 if and only if $N_0(w) = 1$. Simple arithmetic shows that membership of w in W_1 is equivalent to the condition that w_{I_+}/w_{I_-} tends uniformly to 1 as $|I|$ tends to 0. The inclusion $W_0 \subset W_1$ follows from these descriptions of W_0 and W_1 via the following simple lemma.

LEMMA 4. *Let p, p', q, q' and c be positive numbers such that $pp' \geqslant 1$, $qq' \geqslant 1$, and $((p + q)/2)((p' + q')/2) \leqslant 1 + c$. Then*

$$1 + 4c - [(1 + 4c)^2 - 1]^{1/2} \leqslant p/q \leqslant 1 + 4c + [(1 + 4c)^2 - 1]^{1/2}.$$

The proof of the lemma is completely elementary and is therefore omitted. To derive the inclusion $W_0 \subset W_1$, one applies the lemma with $p = w_{I_+}$, $q = w_{I_-}$, $p' = (w^{-1})_{I_+}$, and $q' = (w^{-1})_{I_-}$.

It should be noted that W_0 is properly contained in W_1. In fact, by a construction which can be found in [3], there is a w which is bounded away from 0 and whose indefinite integral is uniformly smooth such that $\tilde{w}$ is not in L^1. Such a w is in W_1. However, it is not in L^p for any $p > 1$, by the M. Riesz projection theorem. Hence it is not in W_0, because the functions in W_0 belong to L^p for all finite p. We see, therefore, that Ibragimov's condition is not a sufficient one for strong mixing.

The author has been unable to produce a function in W_1 which is bounded and bounded away from 0 but does not belong to W_0.

4. **A characterization of $H^\infty + C$.** Let D denote the open disk and ∂D the unit circle in the complex plane. We shall regard integrable functions on ∂D as extended harmonically into D by means of Poisson's formula. Thus, the extension of the function f on ∂D is given by

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})P(r, \theta - t)\,dt, \qquad 0 \leq r < 1,$$

where P is Poisson's kernel:

$$P(r, t) = \frac{1 - r^2}{1 - 2r\cos t + r^2}.$$

For $0 < r < 1$ we let f_r denote the function on ∂D defined by $f_r(e^{it}) = f(re^{it})$.

Let C denote the space of continuous functions on ∂D and H^∞ the usual Hardy space of boundary functions for bounded analytic functions in D. The space $H^\infty + C$ is a closed subalgebra of L^∞ (of Lebesgue measure on ∂D); it is, in fact, the smallest closed subalgebra of L^∞ that contains H^∞ properly. These and other basic properties of $H^\infty + C$ can be found in [14]. One property of relevance to the present discussion is that the Poisson integral is asymptotically multiplicative on $H^\infty + C$: if f and g are any two functions in $H^\infty + C$, then $\lim_{r\to 1} \|f_r g_r - (fg)_r\|_\infty = 0$. This result follows from the multiplicativity of the Poisson integral on H^∞ and the observation that if g is in C, then the relation $\lim_{r\to 1} \|f_r g_r - (fg)_r\|_\infty = 0$ holds for every f in L^∞. The latter fact can be established by means of simple estimates of Poisson integrals.

The algebra of functions that belong together with their complex conjugates to $H^\infty + C$ is denoted by QC. A moment's thought reveals that a function in L^∞ belongs to QC if and only if it can be written as $u + \tilde{v}$ with u and v in C. Consequently, by the analogue for the circle of Theorem 1, we have the equality $QC = \mathrm{VMO} \cap L^\infty$. This relation forms the basis for the proof of the following theorem.

THEOREM 3. *Let f be a function in L^∞ with the property that $|f|$ is continuous in $\bar{D}$. Then f is in QC.*

This theorem resolves in the affirmative a conjecture of R. G. Douglas. The conjecture arose from a problem posed by Douglas about subalgebras of L^∞ which will be discussed later in this section.

Some additional notations are needed for the proof of Theorem 3. For f an integrable function on ∂D and I a subarc of ∂D, let $M(f, I)$ denote the supremum of $|J|^{-1}\int_J |f(e^{it}) - f_J|\,dt$ as J ranges over all subarcs of I. For z a point of ∂D, let $M(f, z)$ denote the infimum of $M(f, I)$ as I ranges over all subarcs centered at z. Thus, a necessary condition for f to belong to VMO is that $M(f, z) = 0$ for every z in ∂D. A standard covering argument shows that this condition is also sufficient for f to belong to VMO.

Suppose now that f is a function in L^∞ satisfying the condition of Theorem 3. In view of the preceding remark, to establish the theorem it will be enough to show that $M(f, z) = 0$ for every z in ∂D. The latter is obvious if $f(z) = 0$, so it will suffice to consider the case where $|f(z)| > 0$. In that case we can find a function g in C such that $|g(z)| = 1/|f(z)|$ and such that $\|fg\|_\infty = 1$. As noted above, we have $\lim_{r\to 1}\|f_r g_r - (fg)_r\|_\infty = 0$, and so fg also satisfies the condition of Theorem 3. Moreover, an elementary estimate shows that $M(fg, z) = |g(z)|M(f, z)$. It will thus be enough to show that $M(fg, z) = 0$, and we may obviously assume without loss of generality that $z = 1$. In other words, the problem of proving Theorem 3 is reduced to that of proving the following statement:

(*) *Let f be a function in L^∞ such that $\|f\|_\infty = |f(1)| = 1$ and such that $|f|$ is continuous in $\bar{D}$. Then $M(f, 1) = 0$.*

Once again, an elementary measure theoretic lemma is needed.

LEMMA 5. *Let (X, m) be a probability measure space and f a function in $L^\infty(m)$ such that $\|f\|_\infty \leqslant 1$ and $\int f\,dm = 1 - b^3$, where $0 < b < ½$. Let E be the set of points where $|1 - f| \geqslant b$. Then $m(E) \leqslant 2b$.*

Let $F = X - E$. We have

$$1 - b^3 = \int_E \frac{f+\bar{f}}{2}dm + \int_F \frac{f+\bar{f}}{2}dm \leqslant \int_E \frac{f+\bar{f}}{2}dm + m(F).$$

By an elementary calculation, if $|\lambda| \leqslant 1$ and $|1 - \lambda| \geqslant b$, then $(\lambda + \bar{\lambda})/2 \leqslant 1 - b^2/2$. Hence $\int_E (f+\bar{f})/2\,dm \leqslant (1 - b^2/2)\,m(E)$, so that

$$1 - b^3 \leqslant (1 - b^2/2)m(E) + m(F) = 1 - (b^2/2)m(E).$$

The desired inequality is now immediate.

To prove (*), fix b satisfying $0 < b < ½$, and choose $a > 0$ such that $|f(re^{i\theta})| \geqslant 1 - b^3$ whenever $|\theta| \leqslant a$ and $1 - a \leqslant r \leqslant 1$. Let I be the arc $\{e^{it}: -a \leqslant t \leqslant a\}$ and let $J = \{e^{it}: \theta_0 - a_0 \leqslant t \leqslant \theta_0 + a_0\}$ be any subarc of I. Then $a_0 \leqslant a$, so, letting $r_0 = 1 - a_0$, we have $|f(r_0 e^{i\theta_0})| \geqslant 1 - b^3$. Multiplying

f by a constant of unit modulus, we may assume without loss of generality that $f(r_0e^{i\theta_0}) > 0$, say $f(r_0e^{i\theta_0}) = 1 - b_0^3$ where $b_0 \leqslant b$. Let E be the set of points on the unit circle where $|1 - f| \geqslant b$. By Lemma 5,

$$\frac{1}{2\pi}\int_E P(r_0, \theta_0 - t)\,dt \leqslant 2b_0 \leqslant 2b.$$

By a simple estimate based on the expression

$$P(r, t) = \frac{1 - r^2}{(1 - r)^2 + 4r\sin^2(t/2)},$$

one can easily show that $P(r_0, \theta_0 - t) \geqslant 1/2a_0$ for e^{it} in J. Consequently

$$|J|^{-1}\int_{E\cap J} dt = \frac{1}{2a_0}\int_{E\cap J} dt \leqslant \int_E P(r_0, t)\,dt \leqslant 4\pi b.$$

We thus have

$$\begin{aligned} &|J|^{-1}\int_J |f(e^{it}) - 1|\,dt \\ &\quad = |J|^{-1}\int_{J\cap E} |f(e^{it}) - 1|\,dt + |J|^{-1}\int_{J-E} |f(e^{it}) - 1|\,dt \\ &\quad \leqslant 2|J|^{-1}\int_{J\cap E} dt + b|J|^{-1}\int_{J-E} dt \leqslant (8\pi + 1)b, \end{aligned}$$

and so

$$\begin{aligned} &|J|^{-1}\int_J |f(e^{it}) - f_J|\,dt \\ &\quad \leqslant |J|^{-1}\int_J |f(e^{it}) - 1|\,dt + |1 - f_J| \leqslant 2(8\pi + 1)b. \end{aligned}$$

We may conclude that $M(f, I) \leqslant 2(8\pi + 1)b$, and the proof of (*) is complete.

From Theorem 3 one can deduce the following characterization of $H^\infty + C$.

COROLLARY 1. *Let B be a closed subalgebra of L^∞ containing H^∞ properly on which the Poisson integral is asymptotically multiplicative. Then $B = H^\infty + C$.*

As noted above, $H^\infty + C$ is the smallest closed subalgebra of L^∞ containing H^∞ properly. Thus, the inclusion $H^\infty + C \subset B$ is automatic. To prove the inclusion is an equality, it will be enough to show that every invertible function in B belongs to $H^\infty + C$, because every function in B differs from an invertible one by a constant. Let g be an invertible function in B, and let h be the outer function such that $|h| = |g|$ on ∂D. Then $f = h^{-1}g$ is an invertible function in B, and $|f| = 1$ on ∂D. The inverse of f is, of course, $\bar f$. Because the Poisson integral is asymptotically multiplicative on B, we have $\lim_{r\to 1}\|f_r\bar f_r - 1\|_\infty = 0$, and consequently $|f|$ is continuous in $\bar D$. Therefore f belongs to $H^\infty + C$ by Theorem 3, and hence so does $g = hf$.

There is another version of Corollary 1 which relates more directly to the problem of Douglas referred to earlier in this section. For B a closed subalgebra of L^∞ containing H^∞, let $M(B)$ denote the Gelfand space (or space of multiplicative linear functionals) of B. In what follows we use the same symbol to denote a function and its Gelfand transform. The unit disk D embeds in $M(H^\infty)$ in the obvious way. The space $M(L^\infty)$ embeds naturally in $M(H^\infty)$ as the Shilov boundary of H^∞. (For these and most of the subsequent facts in this paragraph, see [7, Chapter 10].) Each functional φ in $M(H^\infty)$ has a unique representing measure m_φ on $M(L^\infty)$. One can extend any function f in L^∞ to a continuous function on $M(H^\infty)$ by defining $f(\varphi) = \int f\, dm_\varphi$. In D, this coincides with the ordinary harmonic extension of f. If B is a closed subalgebra of L^∞ containing H^∞, then $M(B)$ can be identified with the set of φ in $M(H^\infty)$ which are multiplicative on B, in other words, with the set of those φ whose representing measures are multiplicative on B. In particular, $M(H^\infty + C) = M(H^\infty) - D$.

Suppose B is a subalgebra of L^∞ containing H^∞ such that $M(B) = M(H^\infty) - D$. Then for f and g in B, the extension to $M(H^\infty)$ of fg coincides on $M(H^\infty) - D$ with the product of the extensions of f and g. By continuity, the same relation is approximately satisfied near $M(H^\infty) - D$, from which we conclude that the Poisson integral is asymptotically multiplicative on B. Combining this observation with Corollary 1, we obtain the following result.

COROLLARY 2. *If B is a closed subalgebra of L^∞ containing H^∞ such that $M(B) = M(H^\infty + C)$, then $B = H^\infty + C$.*

Douglas originally conjectured Theorem 3 (for unimodular functions) in an attempt to prove the above corollary; the argument going from the theorem to the corollary is his. He was promoted by a desire to gain information on the following question he had raised: Is every closed subalgebra of L^∞ containing H^∞ generated by H^∞ and the complex conjugates of inner functions? A recent discussion of Douglas' question can be found in [14]. It turns out that if the question has an affirmative answer, then each closed subalgebra of L^∞ containing H^∞ is uniquely determined by its Gelfand space. Corollary 2 can thus be viewed as a shred of evidence in support of an affirmative answer.

For z in ∂D, let X_z denote the fiber of $M(L^\infty)$ lying above z. K. Hoffman has pointed out that if f is a function in L^∞ with the property that $f|X_z$ is in $H^\infty|X_z$ for each z in ∂D, then f must be in $H^\infty + C$. This result is an immediate consequence of a theorem of E. Bishop on sets of antisymmetry [5]; it follows from that theorem and the evident fact that every set of antisymmetry of $H^\infty + C$ is contained in a single fiber. Corollary 2 leads to the following refinement of Hoffman's observation.

COROLLARY 3. *Let f be a function in L^∞ with the property that*

$f\,|\,\mathrm{supp}\, m_\varphi$ is in $H^\infty\,|\,\mathrm{supp}\, m_\varphi$ for each φ in $M(H^\infty) - D$. Then f is in $H^\infty + C$.

In fact, if B is the closed subalgebra of L^∞ generated by H^∞ and f, then each m_φ is obviously multiplicative on B, so the desired conclusion follows immediately from Corollary 2.

In connection with Corollary 3 it should be pointed out that no m_φ ($\varphi \in M(H^\infty) - D$) can have as its support an entire fiber. To see this, take a unimodular function g in QC which is discontinuous at the point z of ∂D. (See [14] for the existence of such a function.) Then $|g|$ is identically 1 on $M(H^\infty + C)$, but g is not constant on X_z. If φ is in the fiber of $M(H^\infty)$ above z, then $\int g\, dm_\varphi$ has modulus 1, and this can only happen if g is constant on the support of m_φ. Thus $\mathrm{supp}\, m_\varphi \neq X_z$.

It is natural to inquire whether some general theorem in function algebras lies at the root of Corollary 3. The author does not know the answer.

The finite Blaschke products are the only inner functions that are invertible in $H^\infty + C$. If Douglas' question has an affirmative answer, then every closed subalgebra of L^∞ containing $H^\infty + C$ properly must contain invertible inner functions other than finite Blaschke products. Kevin Clancey has pointed out that this much can be proved on the basis of Corollary 1. The author is indebted to Clancey for permission to reproduce his argument.

COROLLARY 4 (CLANCEY). *Let B be a closed subalgebra of L^∞ containing $H^\infty + C$ properly. Then there is an infinite Blaschke product which is invertible in B.*

By Corollary 1, there is a sequence (z_n) in D, with $|z_n| \longrightarrow 1$, along with two functions f and g in B, such that $\inf_n |f(z_n)g(z_n) - (fg)(z_n)| > 0$. Passing to a subsequence if necessary, we may assume that (z_n) is an interpolating sequence for H^∞; let h be the corresponding Blaschke product. Then h is nonzero at every point of $M(H^\infty + C)$ that is not a cluster point of the sequence (z_n). If φ is a cluster point of (z_n), then $f(\varphi)g(\varphi) - (fg)(\varphi)$ is a cluster point of the sequence $(f(z_n)g(z_n) - (fg)(z_n))$ and so is nonzero. Thus φ is not in $M(B)$. Consequently, h does not vanish on $M(B)$ and so is invertible in B, as desired.

Clancey also points out that the reasoning of [2] can now be used to obtain an affirmative answer to another question of Douglas.

COROLLARY 5. *Let B be a closed subalgebra of L^∞ containing $H^\infty + C$ properly. Then there is a closed subalgebra of L^∞ lying strictly between B and $H^\infty + C$.*

To establish this, let h be the Blaschke product constructed in the preceding proof. By an elementary argument, one can produce a factorization $h = h_1 h_2$, where h_1 and h_2 are infinite Blaschke products, and where $|h_1|$ tends to 1 along a subsequence of the zero sequence of h. Let B_1 be the closed subalgebra of L^∞ generated by H^∞ and $\bar{h}_1$. Then $M(B_1)$ consists of the set of points in $M(H^\infty + C)$ where $|h_1| = 1$. This set contains a cluster point of the zero sequence of h, so h is not invertible in B_1. Thus B_1 lies strictly between B and $H^\infty + C$.

5. H^∞ + BUC. We return to the real line, letting H^∞ now denote the space of boundary functions on **R** for bounded analytic functions in the open upper half-plane. The analogue on **R** of the space $H^\infty + C$ is the space H^∞ + BUC. It seems not to have been recognized before that H^∞ + BUC is actually a closed subalgebra of L^∞ of the line. We shall establish this and then indicate how some of the results of the preceding section translate to the present setting.

Henceforth, all spaces mentioned should be understood to live on **R**. Any function on **R** which is integrable with respect to the measure $(1 + t^2)^{-1}\,dt$ can be extended harmonically to the upper half-plane by means of Poisson's formula for the half-plane, and we shall regard such functions as so extended. The extension of the function f is given by

$$f(x + iy) = \frac{y}{\pi} \int_{-\infty}^{\infty} f(t)\,[(x - t)^2 + y^2]^{-1}\,dt, \qquad y > 0.$$

For $y > 0$ we let f_y denote the function on R defined by $f_y(x) = f(x + iy)$.

That H^∞ + BUC is closed in L^∞ is proved in the same way as the corresponding result on the circle. It depends upon the distance estimate $\operatorname{dist}(f, H^\infty) = \operatorname{dist}(f, H^\infty \cap \mathrm{BUC})$, valid for all f in BUC. To establish the distance estimate, let h be any function in H^∞. For $y > 0$ we have

$$\|f - h_y\|_\infty \leqslant \|f - f_y\|_\infty + \|f_y - h_y\|_\infty .$$

The second term on the right is majorized by $\|f - h\|_\infty$ and, assuming f is in BUC, the first term on the right tends to 0 with y. As h_y is in $H^\infty \cap$ BUC, we conclude that $\operatorname{dist}(f, H^\infty \cap \mathrm{BUC}) \leqslant \|f - h\|_\infty$. Taking the infimum over h yields the distance estimate.

It follows from the distance estimate that the natural map of BUC/$H^\infty \cap$ BUC into L^∞/H^∞ is an isometry and so has a closed range. As H^∞ + BUC is the inverse image of that range under the quotient map of L^∞ onto L^∞/H^∞, it also is closed.

The proof that H^∞ + BUC is an algebra is also the same as the proof of the corresponding result on the circle, except for technical complications. It is convenient to formulate the main step in the proof as a lemma.

LEMMA 6. *Let h be a function in H^∞, let a be a positive real number, and let $f(x) = e^{-iax}h(x)$. Then f is in H^∞ + BUC.*

Taking the lemma for granted temporarily, we see how it implies the desired conclusion. For $a > 0$ let A_a be the space of all functions of the form $e^{-iax}h(x)$ with h in H^∞. Then $\bigcup_{a>0} A_a$ is an algebra and hence so is its uniform closure. By a theorem of Kober [1, p. 249], the closure of $\bigcup_{a>0} A_a$ contains H^∞ + BUC. By Lemma 6, $\bigcup_{a>0} A_a$ is contained in H^∞ + BUC, and therefore so is its closure, since H^∞ + BUC is closed. Thus H^∞ + BUC equals the closure of $\bigcup_{a>0} A_a$ and so is an algebra.

To establish Lemma 6, let f and h be as in the statement, and let v be a C^∞ function of compact support such that $v = 1$ on $[-a, 0]$. Let u be the inverse Fourier transform of v, so that $v = \hat{u}$. We have $f = u * f + (f - u * f)$. As u belongs to the class of rapidly decreasing functions, the function $u * f$ is in BUC. It only remains to show that $f - u * f$ is in H^∞, for which it will be enough to show that $f - u * f$ annihilates H^1.

Let g belong to H^1. An application of Fubini's theorem gives

$$\int_{-\infty}^{\infty} (u * f)(x) g(x)\, dx = \int_{-\infty}^{\infty} (u_1 * g)(x) f(x)\, dx,$$

where $u_1(x) = u(-x)$. Hence

$$\int_{-\infty}^{\infty} [f(x) - (u * f)(x)]\, g(x) dx$$

$$= \int_{-\infty}^{\infty} [g(x) - (u_1 * g)(x)]\, e^{-iax} h(x) dx.$$

Now $\hat{u}_1(x) = v(-x) = 1$ for x in $[0, a]$. Therefore, the Fourier transform of $g - u_1 * g$ vanishes on $[0, a]$ and hence on $(-\infty, a]$ (since g and $u_1 * g$ belong to H^1). The function $e^{-iax}[g(x) - (u_1 * g)(x)]$ is therefore in H^1 and so annihilates H^∞. Consequently the integral on the right side of the above equality vanishes, and the proof of Lemma 6 is complete.

Let QBUC denote the algebra of functions that belong together with their complex conjugates to H^∞ + BUC. Then QBUC consists precisely of the functions in L^∞ that can be written as $u + \tilde{v}$ with u and v in BUC. Consequently QBUC = VMO $\cap\, L^\infty$. Using this equality one can establish the following analogue of Theorem 3.

THEOREM 4. *Let f be a function in L^∞ such that* $\lim_{y\to 0} |f_y|$ *exists uniformly. Then f is in* QBUC.

The proof will not be carried out here. It is substantially the same as the proof of Theorem 3, with slight complications due to the noncompactness of **R**.

On the basis of Theorem 4, one can obtain a characterization of $H^\infty +$ BCU analogous to that given by Corollary 1 for $H^\infty + C$. The Poisson integral for the upper half-plane is asymptotically multiplicative on H^∞ + BUC, in other words, for any two functions f and g in H^∞+ BUC one has $\lim_{y\to 0} \|f_y g_y - (fg)_y\|_\infty = 0$. The characterization states that H^∞ + BUC is the largest superalgebra of H^∞ with this property. More precisely: *Let B be a subalgebra of L^∞ containing H^∞ on which the Poisson integral for the upper half-plane is asymptotically multiplicative. Then B is contained in H^∞* + BUC. The analogue of Corollary 2 also holds for H^∞ + BUC and is proved analogously.

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