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ON THE EXTENSION OF MAPPINGS IN STONE-WEIERSTRASS SPACES(1)

BY

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Dedicated to Orrin Frink, Jr.

ABSTRACT. N. Veličko generalized the well-known result of A. D. Taĭmanov on the extension of continuous functions by showing that Taĭmanov's theorem holds when Y (the image space) is H-closed and Urysohn and the mapping f is weakly θ-continuous. We obtain, in a more direct fashion, an even stronger generalization of this theorem.

We proceed to show that the class of all SW spaces is not reflective in the category of all completely Hausdorff spaces and continuous mappings. However, an epi-reflective situation is achieved by suitably enlarging the class of admissible morphisms.

We conclude by establishing a number of results about SW extension spaces.

1. Preliminaries. A subset A of a space X is said to be a zero-set of X if there exists a function f in $C(X)$ (the set of all continuous, real-valued functions on X) such that $A = f^{-1}(\{0\}) = \{x \in X: f(x) = 0\}$. Complements of zero-sets are called cozero-sets. If $C^*(X)$ is the set of all bounded functions in $C(X)$, the subset A of X is said to be C^*-embedded in X if every function in $C^*(A)$ can be extended to a function in $C^*(X)$ [11]. If X and Y are spaces, let $C(X, Y)$ denote the set of all continuous functions from X to Y.

A mapping f of the space X into the space Y is said to be θ-continuous (weakly θ-continuous) if for an arbitrary point $x \in X$ and an arbitrary open set V of Y containing $y = f(x)$, there exists an open set U of X with $x \in U$ and $f(\mathrm{cl}_X U)(f(U)) \subseteq \mathrm{cl}_Y V$ ([8], [19]). The point x is a member of the θ-closure of the set S in X if and only if $(\mathrm{cl}_X G) \cap S \neq \emptyset$ for all open sets G containing x. The set S is said to be θ-closed if it is equal to its θ-closure [25].

A space X is said to be quasicompact if every family of zero-sets of X with the finite intersection property has a nonempty intersection [9]. The space X is said to be Urysohn if distinct points of X are contained in disjoint closed neighborhoods, and is said to be completely Hausdorff if for

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every pair x, y of distinct points there is a function f in $C(X)$ such that $f(x) \neq f(y)$. The word *space*, unqualified, shall henceforth mean a completely Hausdorff space. As in [21], the space X is said to be a Stone-Weierstrass space (or briefly, an SW space) if every point-separating subalgebra of $C^*(X)$ which contains the constants is uniformly dense in $C^*(X)$.

A mapping $f: X \to Y$, where X, Y are arbitrary spaces, is said to be cozero-set continuous if $f^{-1}(C)$ is open for all cozero-sets C of Y. Cozero-set continuous functions will be referred to as c-maps. If X is a space, let $\tilde{X}$ be the space obtained by taking on the same set X the weak topology relative to $C(X)$. As noted in [3], X is SW if and only if $\tilde{X}$ is compact. Moreover, a map f from an arbitrary space X to Y is a c-map if and only if $\gamma_Y f: X \to \tilde{Y}$ is continuous, where $\gamma_Y: Y \to \tilde{Y}$ is the identity map.

A Hausdorff space X is said to be absolutely closed, or simply H-closed, if it is closed in every Hausdorff space in which it can be embedded. This concept is a generalization of a property of compact Hausdorff spaces, and was introduced in 1924 by Alexandroff and Urysohn [1]. In [15], Katětov showed that any Hausdorff space X could be densely embedded in an H-closed space κX, now referred to as the Katětov extension of X, having the property that X is a C^*-embedded subset. For a construction of κX, the reader is referred to [18].

A filter $\mathfrak{F}$ on a space X is said to be completely regular if $\mathfrak{F}$ has a base $\mathfrak{B}$ of open sets such that for each set $A \in \mathfrak{B}$ there exist a set $B \in \mathfrak{B}$ contained in A and a function $f \in C(X)$ which is equal to 0 on B and 1 on $X \setminus A$ [5]. The filter $\mathfrak{F}$ is said to be free or fixed according as the intersection of all its members is empty or nonempty.

The space Y is said to be an extension of the space X if there exists a homeomorphism h from X into Y such that $h(X)$ is dense in Y. If h is the identity map, the reference to h is omitted. The extensions Y and Z of X are said to be isomorphic if there is a homeomorphism of Y onto Z which leaves X pointwise fixed.

An arbitrary topological space X is said to be realcompact if every real maximal ideal in $C(X)$ is fixed [6].

An open filter is a filter in the lattice of open sets. The open filter $\mathfrak{A}$ is said to have the countable closure intersection property (abbreviated c.c.i.p.) provided that for each countable subset $\mathfrak{C}$ of $\mathfrak{A}$, $\bigcap\{\mathrm{cl}_X C: C \in \mathfrak{C}\} \neq \emptyset$. An open ultrafilter is a maximal open filter.

A Hausdorff space X is said to be almost realcompact if every open ultrafilter with the c.c.i.p. converges.

2. Extension of maps. The following theorem is proved by Taĭmanov in [23].

(2.1) *Let A be a dense subspace of an arbitrary space X, and let f: $A \to Y$ be a continuous mapping of A into the compact Hausdorff space Y. The mapping f has a continuous extension from X to Y if and only if, for every pair F_1, F_2 of closed disjoint subsets of Y, we have* $\mathrm{cl}_X f^{-1}(F_1) \cap \mathrm{cl}_X f^{-1}(F_2) = \emptyset$.

It is easily verified that in the above theorem we may replace "closed disjoint subsets" by "disjoint zero-sets".

Lemma 2.2. *Let $f: X \to Y$ be a map from an arbitrary space X to an H-closed Urysohn space Y. Then the following conditions are equivalent:*

(a) *f is θ-continuous,*

(b) *f is weakly θ-continuous, and*

(c) *f is a c-map.*

Proof. That (a) implies (b) is obvious. A weakly θ-continuous function is always a c-map and therefore it remains to prove that (c) implies (a).

Suppose $p \in X$ and that $f(p) \in V$, where V is open in Y. An H-closed Urysohn space is an SW space [18] and hence completely Hausdorff. There is for each $q \in Y \setminus \{f(p)\}$ a function h_q in $C(Y)$ with $h_q(q) = 0$ and $h_q(f(p)) = 1$. It is clear that $C_q = \{y \in Y: h_q(y) < \frac{1}{2}\}$ is open in Y, $D_q = \{y \in Y: h_q(y) \leq \frac{1}{2}\}$ is a zero-set of Y, $f(p) \notin D_q$, and $Y \subseteq \bigcup\{C_q: q \in Y \setminus \{f(p)\}\} \cup V$. The space Y is H-closed, and so there exist elements $q_1, q_2, q_3, \cdots, q_n$ of $Y \setminus \{f(p)\}$ with $Y \subseteq \bigcup_{i=1}^n \mathrm{cl}_Y C_{q_i} \cup \mathrm{cl}_Y V = \bigcup_{i=1}^n D_{q_i} \cup \mathrm{cl}_Y V$ [16]. Since $\bigcup_{i=1}^n f^{-1}(D_{q_i}) = E$ is a zero-set of X and $p \in X \setminus E$, there is an open set U of X with $p \in U \subseteq \mathrm{cl}_X U \subseteq X \setminus E$. Clearly $f(\mathrm{cl}_X U) \subseteq f(X \setminus E) \subseteq \mathrm{cl}_Y V$ and thus f is θ-continuous.

The following theorem generalizes and follows from (2.1).

Theorem 2.3. *Let A be a dense subspace of an arbitrary space X, and let f: $A \to Y$ be a c-map from A to the SW space Y. The mapping f has a c-extension from X to Y if and only if, for every pair F_1, F_2 of disjoint zero-sets of Y, we have* $\mathrm{cl}_X f^{-1}(F_1) \cap \mathrm{cl}_X f^{-1}(F_2) = \emptyset$.

Proof. That the condition is necessary follows from the inclusion $\mathrm{cl}_X f^{-1}(F_i) \subseteq g^{-1}(F_i)$ where $g: X \to Y$ is a c-extension of f. To see that

the condition is sufficient, we note that disjoint zero-sets of Y are disjoint zero-sets of $\tilde{Y}$ and hence $\gamma_Y f: A \to \tilde{Y}$ extends to a continuous map $l: X \to \tilde{Y}$ by (2.1). The function $h: X \to Y$, defined by $\gamma_Y h = l$, is a c-map and $h|A = f$.

Corollary 2.4. *Let A be a dense subset of an arbitrary space X, and let $f: A \to Y$ be weakly θ-continuous where Y is H-closed and Urysohn. Then the following conditions are equivalent:*

(a) *f has a weakly θ-continuous extension from X to Y,*

(b) *for any pair F_1, F_2 of θ-closed disjoint subsets of Y, we have $\mathrm{cl}_X f^{-1}(F_1) \cap \mathrm{cl}_X f^{-1}(F_2) = \emptyset$,*

(c) *for any pair F_1, F_2 of disjoint zero-sets of Y, we have $\mathrm{cl}_X f^{-1}(F_1) \cap \mathrm{cl}_X f^{-1}(F_2) = \emptyset$.*

Proof. (a) $\Rightarrow$ (b): Suppose g is the weakly θ-continuous extension of f. If $p \in \mathrm{cl}_X f^{-1}(F_i)$, then $g(p)$ is, easily, a member of the θ-closure of F_i.

(b) $\Rightarrow$ (c): We need merely observe that zero-sets are θ-closed.

(c) $\Rightarrow$ (a): The function f is a c-map by Lemma 2.2 and therefore has a c-extension g from X to Y by Theorem 2.3. Again, by Lemma 2.2, g is weakly θ-continuous.

In the above corollary, there are simple examples showing that we may not replace "weakly θ-continuous" by "continuous". On the other hand, by Lemma 2.2, "weakly θ-continuous" may be replaced by "θ-continuous" or "c-map".

Veličko generalized Taĭmanov's theorem by showing that, under the hypothesis of Corollary 2.4, (a) is equivalent to (b) [25]. Stephenson [21] has shown that there exists a noncompact regular SW space Y. Since a regular absolutely closed space is compact, Y cannot be absolutely closed. Thus, Y is seen to be an example of an SW space which is not H-closed and so Theorem 2.3 covers a wider class of spaces than Veličko's result.

Almost realcompact spaces were defined and studied by Frolík in [10], where he showed that in many instances they behave much like realcompact spaces. In [7], Engelking gave the analogue of Taĭmanov's theorem for completely regular realcompact spaces: Let A be a dense subspace of an arbitrary topological space X, and let $f: A \to Y$ be a continuous function of A into the completely regular realcompact space Y. The mapping f has a continuous extension from X to Y if and only if, for any sequence $\{F_i\}_{i=1}^{\infty}$ of closed subsets of Y such that $\bigcap_{i=1}^{\infty} F_i = \emptyset$, we have $\bigcap_{i=1}^{\infty} \mathrm{cl}_X f^{-1}(F_i) = \emptyset$.

It is a simple matter to establish the counterpart of Theorem 2 for completely Hausdorff realcompact spaces. In the same vein, we have

Theorem 2.5. *Suppose Y is almost realcompact, κY is an SW space, A is dense in the arbitrary space X, and $f: A \to Y$ is weakly θ-continuous. The mapping f has a weakly θ-continuous extension from X to Y if and only if for any sequence $\{F_i\}_{i=1}^{\infty}$ of θ-closed subsets of Y such that $\bigcap_{i=1}^{\infty} F_i = \emptyset$, we have $\bigcap_{i=1}^{\infty} \mathrm{cl}_X f^{-1}(F_i) = \emptyset$.*

Proof. To prove that the condition is necessary, we need merely argue as in (a) $\Rightarrow$ (b) of Corollary 2.4. We now establish the sufficiency of the condition and first note that f may be regarded as a weakly θ-continuous function from A to κY. If H_1 and H_2 are disjoint θ-closed subsets of κY, then $H_1 \cap Y$ and $H_2 \cap Y$ are disjoint θ-closed subsets of Y, and therefore $\mathrm{cl}_X f^{-1}(H_1) \cap \mathrm{cl}_X f^{-1}(H_2) = \mathrm{cl}_X f^{-1}(H_1 \cap Y) \cap \mathrm{cl}_X f^{-1}(H_2 \cap Y) = \emptyset$. By virtue of Corollary 2.4, f has a weakly θ-continuous extension g from X to κY. It remains to show that $g(X) \subseteq Y$.

Suppose, on the contrary, that there is a point p of $X \setminus A$ such that $g(p) = \mathfrak{A} \in \kappa Y \setminus Y$. Now $\mathfrak{A}$ is an open ultrafilter on Y which does not have the c.c.i.p., and so there exists a countable subset $\mathcal{C} = \{G_i\}_{i=1}^{\infty}$ of $\mathfrak{A}$ with $\bigcap_{i=1}^{\infty} \mathrm{cl}_Y G_i = \emptyset$. Since κY is H-closed and Urysohn, $\mathrm{cl}_Y G_i$ is θ-closed for all positive integers i [24], and letting $F_i = \mathrm{cl}_Y G_i$ it follows that $\bigcap_{i=1}^{\infty} \mathrm{cl}_X f^{-1}(F_i) = \emptyset$. There exists a positive integer j such that $p \in X \setminus \mathrm{cl}_X f^{-1}(F_j)$, and we observe that $G_j \cup \{\mathfrak{A}\}$ is open in κY and contains $\mathfrak{A}$. If U is an arbitrary open subset of X which contains p, choose a point s of $[X \setminus \mathrm{cl}_X f^{-1}(F_j)] \cap U \cap A$. Now $s \notin \mathrm{cl}_X f^{-1}(F_j)$ and hence $s \notin f^{-1}(\mathrm{cl}_Y G_j)$ so that $g(s) = f(s) \notin \mathrm{cl}_Y G_j$. It follows that $g(s) \in Y \setminus \mathrm{cl}_Y G_j$, an open subset of κY which does not intersect $G_j \cup \{\mathfrak{A}\}$. Hence $g(U) \not\subseteq \mathrm{cl}_{\kappa Y}[G_j \cup \{\mathfrak{A}\}]$, and so g is not weakly θ-continuous at p.

Remarks. Porter and Thomas have given necessary and sufficient conditions on X for κX to be an SW space [18].

It is easily verified that Lemma 2.2 is still valid if it is required merely that κY be Urysohn; thus in the above theorem, "weakly θ-continuous" may be replaced by "θ-continuous" or "c-map".

3. Reflectiveness of SW spaces. Many extensions such as the Stone-Čech compactification, the Hewitt realcompactification, and the Banaschewski zero-dimensional compactification have, on account of their similar

mapping properties, been studied from a categorical standpoint and classified as epi-reflections in appropriate categories. For a thorough discussion of this theory, the reader is referred to [13].

We will not distinguish among isomorphic objects of any category, and for any category, 1_X will denote the identity morphism for the object X. For categorical notions not specifically defined, the reader should consult Mitchell [17].

Definition. If $\mathfrak{A}$ is a full subcategory of a category $\mathfrak{B}$ and if for each object X in $\mathfrak{B}$ there exist an object $X_{\mathfrak{A}}$ in $\mathfrak{A}$ and a morphism (resp. epimorphism) $r: X \to X_{\mathfrak{A}}$ such that for each object Y in $\mathfrak{A}$ and each morphism f: $X \to Y$, there exists a unique morphism $\bar{f}: X_{\mathfrak{A}} \to Y$ such that the diagram

$$\begin{tikzcd} X \arrow[rr, "r"] \arrow[dr, "f"'] & & X_{\mathfrak{A}} \arrow[dl, dashed, "\bar{f}"] \\ & Y & \end{tikzcd}$$

is commutative, then $\mathfrak{A}$ is said to be a reflective (resp. an epi-reflective) subcategory of $\mathfrak{B}$ and r is called a reflection morphism (resp. epimorphism) from X to $X_{\mathfrak{A}}$.

In establishing our next theorem, the techniques employed in Theorem 1 of [12] were most useful. We will furthermore lean heavily upon the following modification of Niemytski's classic example [11, 3K]. Let $X = I^2 = \{(x, y): 0 \leq x \leq 1, 0 \leq y \leq 1\}$ be the unit square with the usual topology τ_1 and let $A = \{(x, 0): (x, 0) \in X\}$. To each $(x, 0) \in A$ define $V_x = \{(x, 0)\} \cup \{(u, v) \in X: v > 0 \text{ and } (u - x)^2 + v^2 < (1/4)^2\}$. Let τ_2 be the topology on X generated by the collection of sets $\tau_1 \cup (V_x)_{x \in I}$ as a subbase. It is easily verified that (X, τ_2) is H-closed and Urysohn and hence SW, and that A with the induced topology is discrete.

Theorem 3.1. *Let $\mathfrak{B}$ be the category of all completely Hausdorff spaces and continuous functions, and let $\mathfrak{A}$ be its full subcategory of all SW spaces. Then $\mathfrak{A}$ is not reflective in $\mathfrak{B}$.*

Proof. Suppose, on the contrary, that $\mathfrak{A}$ is reflective in $\mathfrak{B}$. Consider the space (X, τ_2) described above, and let r be the reflection morphism from A to the SW space $A_{\mathfrak{A}}$. If $i: A \to X$ is the identity map, there is a unique morphism $\bar{i}: A_{\mathfrak{A}} \to X$ such that $i = \bar{i} \circ r$. Since i is a homeomorphism into, it follows readily that r is a homeomorphism into.

Let D_2 be the discrete space composed of two elements, 1 and 2, let $P = X \times D_2$, and for $n = 1, 2$, let $j_n: X \to P$ be the map defined by $j_n(y) = (y, n)$. For each $y \in i(A)$ identify $j_1(y)$ and $j_2(y)$, let Q be the corresponding quotient space, and let χ be the quotient map from P to Q. Now X and D_2 are H-closed and it follows that P and hence Q is H-closed. One can easily check that Q is Urysohn and therefore SW.

It is now possible to proceed exactly as Herrlich and Strecker have done in [12] to show that A is homeomorphic to the SW space $A_{\mathfrak{A}}$: this is a contradiction since A is not quasicompact.

Let α_X be the set of all nonisomorphic SW extensions Y of X such that X is C^*-embedded in Y, and let e_X denote the set of all those members of α_X with the property that each trace filter is completely regular.

Stephenson [21] introduced and studied a particular member of e_X, which we denote by σX. Specifically, if $\mathfrak{M}$ is the set of all free maximal completely regular filters on X, then σX is the space whose points are the elements of $X \cup \mathfrak{M}$ and whose topology is generated by all sets V^* of the form $V \cup \{\mathcal{F} \in \mathfrak{M} | V \in \mathcal{F}\}$ for V open in X. The space σX enjoys many of the properties of the Stone-Čech compactification of a Tychonoff space and is, in fact, homeomorphic to βX in case X is completely regular.

In [20], Raha has described an extension of a space X, which we denote by δX, whose points are again the elements of $X \cup \mathfrak{M}$ and whose topology is similar, in construction, to the topology of the Katětov extension. In particular, any set, open in X, is also open in δX, and if $\mathcal{F} \in \mathfrak{M}$, basic neighborhoods of $\mathcal{F}$ are sets of the form $G \cup \{\mathcal{F}\}$ for $G \in \mathcal{F}$. It is easily verified that $\delta X \in e_X$.

Lemma 3.2. *If $Y \in \alpha_X$ and $f: X \to Z$ is a c-map from X to the SW space Z, then there exists a unique c-map $g: Y \to Z$ with $g|X = f$.*

Proof. Let F_1 and F_2 be disjoint zero-sets of Z. The sets $A_1 = f^{-1}(F_1)$ and $A_2 = f^{-1}(F_2)$ are disjoint zero-sets of X, and so there is an element l of $C^*(X)$ which is 0 on A_1 and 1 on A_2 [11]. Now l can be extended to a function in $C^*(Y)$ which implies that $\mathrm{cl}_Y A_1 \cap \mathrm{cl}_Y A_2 = \emptyset$. By Theorem 2.3, there is a c-map $g: Y \to Z$ with $g|X = f$. The uniqueness of g follows from the uniqueness of $\gamma_Z g$.

Theorem 3.3. *Let $\mathfrak{B}^*$ be the category of all spaces and c-maps. If $\mathfrak{A}^* \subseteq \mathfrak{B}^*$ is the full subcategory of all SW spaces, then the natural mappings $r: X \to \sigma X$ and $r_1: X \to \delta X$ are reflection epimorphisms for $\mathfrak{A}^*$.*

Proof. The composition of c-maps is a c-map, the identity function is a c-map, and therefore $\mathfrak{B}^*$ is a category. Since X is C^*-embedded in σX and δX ([21], [20]), the proof now follows directly from Lemma 3.2.

4. Projective extrema. If Y is an extension space of X, the trace filters of Y are the filters $\mathfrak{N}(y)$, $y \in Y \setminus X$, where $\mathfrak{N}(y)$ is the filter on X generated by the traces $U \cap X$ of the open sets U of Y which contain y. If X is Tychonoff, then the trace filters $\mathfrak{N}(y)$ of βX are precisely the free maximal completely regular filters on X ([2], [5]).

The extension Y of X is said to be projectively larger than the extension Z of X, denoted $Y \geq Z$, if there exists a continuous surjection $f: Y \to Z$ which leaves X pointwise fixed. If η is a class of extensions of X, an element Y of η is said to be a projective maximum (resp. projective minimum) if $Y \geq Z$ (resp. $Z \geq Y$) for all Z in η. Projective maximums (resp. projective minimums), if they exist, are unique [4].

We are now in a position to give a simple proof of the following theorem due to Stephenson [21, Theorem 4(vii)].

Theorem 4.1 (Stephenson). *The projective minimum of e_X is σX.*

Proof. If $Y \in e_X$, let $\tilde{\eta}(y)$ denote the trace filter of $\tilde{Y}$ corresponding to $y \in \tilde{Y} \setminus \tilde{X}$. Noting that $\beta\tilde{X} = \tilde{Y}$ and that $\tilde{\eta}(y)$ is a subset of the completely regular filter $\eta(y)$, we must have $\eta(y) = \tilde{\eta}(y)$. Let $g: Y \to \sigma X$ be the function defined by $g(x) = x$ for $x \in X$ and $g(y) = \eta(y) \in \mathfrak{M}$ for $y \in Y \setminus X$. If $V \cup \{\mathcal{G} \in \mathfrak{M}: V \in \mathcal{G}\} = T$ is a basic open set of σX, then $g^{-1}(T) = V \cup \{y \in Y \setminus X: V \in \eta(y)\}$ which is open in Y by Lemma 4.1 of [18]. Thus, g is continuous.

It is clear that $g(Y)$ is quasicompact and thus SW [3]. It follows that $g(Y)$ is closed in σX [21], and therefore g is onto.

Stephenson also proved that the function g in the above theorem is 1-1. From the manner in which we have defined g, this follows immediately from the fact that Y is completely Hausdorff.

Porter and Thomas [18] and Liu [16] have shown that the Katětov extension is a projective maximum in the class of H-closed extensions of a Hausdorff space X. In view of its affinity with the Katětov extension, it is natural to inquire about the role of δX as a projective maximum.

Theorem 4.2. *If $Y \in \alpha_X$, then $\delta X \geq Y$ if and only if $Y \in e_X$.*

Proof. If $\delta X \geq Y$, then for any $y \in Y \setminus X$ there is an $\mathcal{A} \in \mathfrak{M}$ such that

$\eta(y) \subseteq \mathfrak{A}$. Since $\tilde{\eta}(y)$ is a maximal completely regular filter and $\tilde{\eta}(y) \subseteq \eta(y)$, it follows that $\eta(y) = \mathfrak{A}$. Hence $Y \in e_X$.

On the other hand, suppose $Y \in e_X$. The identity map $i: X \to Y$ is a c-map, δX is a reflection epimorphism for $\mathfrak{A}^*$, and therefore i can be extended to a c-map $f: \delta X \to Y$. We claim that $f(\mathfrak{M}) \subseteq Y \setminus X$. For if $f(\mathfrak{A}) = x \in X$, let $\mathfrak{D}$ be the class of all cozero-sets of X which contain x. Since f is a c-map and X is C^*-embedded in Y, it follows that $\mathfrak{D} \subseteq \mathfrak{A}$. Now $\mathfrak{D}$ is a base for a fixed maximal completely regular filter, and so $\mathfrak{A}$ is fixed which is a contradiction.

It is clear that f is continuous at each point of x. If $f(\mathfrak{A}) = y \in Y \setminus X$, let U be an open set Y which contains y. Since $\eta(y)$ is completely regular, $\tilde{\eta}(y) = \eta(y)$ which entails the existence of a cozero-set C of Y containing y with $C \cap X \subseteq U$. Since f is a c-map, there is a member G of $\mathfrak{A}$ with $f(G \cup \{\mathfrak{A}\}) \subseteq C$. Clearly, $f(G \cup \{\mathfrak{A}\}) \subseteq U$ and so f is continuous at y. That f is onto follows as in Theorem 4.1. Thus $\delta X \geq Y$.

The following corollary is analogous to a result of Banaschewski in [4].

Corollary 4.3. *If X is a space, then $\sigma X \leq Y \leq \delta X$ for all Y in e_X.*

Remarks. The set e_X (and hence also α_X) may have substantial cardinality. We shall make use of the fact that any Hausdorff extension $Y \supseteq X$ such that $\sigma X \leq Y \leq \delta X$ belongs to e_X. Let N be the space of positive integers with the discrete topology, let $Y = N \cup \mathfrak{M}$, and if $\mathfrak{A} \in \mathfrak{M}$, let $\tau_{\mathfrak{A}}$ be the topology on Y generated by the topology of σN together with $\{N \cup \{\mathfrak{A}\}\}$. Clearly $\sigma N \leq (Y, \tau_{\mathfrak{A}}) \leq \delta N$ and so $(Y, \tau_{\mathfrak{A}}) \in e_N$. If $\mathfrak{A}$ and $\mathfrak{B}$ are distinct members of $\mathfrak{M}$, a routine argument shows that $(Y, \tau_{\mathfrak{A}})$ and $(Y, \tau_{\mathfrak{B}})$ are nonisomorphic extensions of N. Finally, since σN is the Stone-Čech compactification of N, it follows that $2^c = \text{card } \mathfrak{M} \leq \text{card } e_N$ [11].

It is natural to ask if σX is a projective minimum in α_X. We shall answer this question negatively, but we will first need to describe another SW extension of a given completely Hausdorff space X. Let $\mathfrak{W}$ be the set of all free zero-set ultrafilters (see [11]) on X, and let $\pi X = X \cup \mathfrak{W}$. We define a topology for πX by taking as a base for the open sets the family of all sets of the form $G \cup \{\mathfrak{A} \in \mathfrak{W}: \exists A \in \mathfrak{A} \text{ with } A \subseteq G\}$ where G is any open set of X. It is readily verified that πX is SW and that X is a dense, C^*-embedded subset.

Let $I = [0, 1]$, let τ be the usual topology on I, let J be the subset of I consisting of all irrational numbers, and choose disjoint dense subsets J_1,

J_2 of (I, τ) such that $J = J_1 \cup J_2$. Let $I_1 = I \setminus J_2$, let τ_1 be the topology on I_1 induced by τ, let $\mathcal{S}_1$ be the topology on I_1 generated by $\tau_1 \cup \{J_1\}$, and denote the space $(I_1, \mathcal{S}_1)$ by P. A routine argument shows that (I_1, τ_1) and P have the same continuous functions, and so τ_1 is the collection of cozero-sets of both (I_1, τ_1) and P.

Suppose $g \in C(\pi P, \sigma P)$ and $g|X$ is the identity map. Since no cozero-set of P is contained in J_1, the set J_1 is open in σP. However, if $p \in U \subseteq J_1$ where $U \in \mathcal{S}_1$, then U contains a member of a free zero-set ultrafilter on P, and hence $g^{-1}(J_1) = J_1$ is not open in πP. This is a contradiction, and it is now evident that σP is not a projective minimum in α_P.

Clearly $\pi P \in \alpha_P \setminus e_P$, and it follows from Theorem 4.2 that $\delta P \not\geq P$. Therefore δX is not necessarily the projective maximum of α_X, and this disproves a theorem of Raha [20].

Although σX is not in general a projective minimum in α_X, we do have the following result.

Theorem 4.4. *If X is Tychonoff, then σX is the projective minimum in α_X.*

Proof. If $Y \in \alpha_X$, then $\tilde{Y} = \beta X = \sigma X$; hence $\gamma_Y\colon Y \to \tilde{Y}$ is the desired map.

However, even if X is Tychonoff, δX need not be the projective maximum of α_X. For in [21], Stephenson has described a Tychonoff space X with a one-point noncompact SW extension Y. Moreover, X is C^*-embedded in Y. If $Y = X \cup \{y\}$ and $\eta(y)$ were completely regular, then one could check the various cases to show that Y would be Tychonoff and hence compact ([3], [14]). Hence $Y \in \alpha_X \setminus e_X$.

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