ABSTRACT. The primary purpose of this paper is (1) to provide a "real" example of a regular first countable $T_1$-space which has no dense developable subspace and (2) to provide a new technique for producing Moore spaces which fail to have dense metrizable subspaces. Related results are established which produce new examples of noncompletable Moore spaces and which show that each regular hereditary $M$-space with a $G_δ$-diagonal has a dense metrizable subspace.

The existence of dense metrizable subspaces in a given space $S$ has been shown useful (1) to determine the equivalence of chain conditions and separability ([4], [14], and [16]), (2) to determine if $S$ satisfies the Baire property ([1], [18], and [24]), (3) to determine whether $S$ is densely embeddable in a space of the same type satisfying various completeness conditions ([1], [4], [16], and [18]), and (4) to determine if there exists a space $X$ the same type as $S$ in which each open set contains a copy of $S$ [23]. The inspiration for this work has been the considerable work done in [25], [4], [5], [11], [15], [16], [17], and [18] concerning the existence of dense metrizable subspaces in Moore spaces, i.e., regular developable spaces.

In [13] and [14], the author has investigated conditions under which first countable spaces (all spaces are to be $T_1$) have dense developable and dense metrizable subspaces. Surprisingly, the major problem in this investigation has been producing first countable spaces which fail to have such subspaces. The only known examples of such spaces are all nonseparable spaces which satisfy the countable chain condition. In [14], it was shown that hereditarily Lindelöf, nonseparable spaces have no dense developable subspaces. In [4], it was shown that nonseparable Moore spaces with the countable chain condition have no dense metrizable subspaces. These are the only techniques presently known for producing the required examples. Thus, other than the consistency of Souslin spaces, there is no known example of a regular first countable space which has no dense developable subspace. And there are very few known examples ([21], [22], and [10]) of Moore spaces which have no dense metrizable subspaces. It is the primary
purpose of this paper to provide a "real" example of a regular first countable space which has no dense developable subspace and to provide a new technique for producing Moore spaces which fail to have dense metrizable subspaces.

In Theorem 1, the author shows that a regular (in fact, hereditarily paracompact) first countable space due to Miščenko in [6] and Aull in [2] has no dense developable subspace. In [19] and [20] the author has developed a technique which associates a Moore space to each regular first countable space. In this paper it is shown that if a given regular first countable space has no dense developable subspace, then the associated Moore space has no dense metrizable subspace. Thus, by using Theorem 1, the author obtains a new example of a Moore space which fails to have a dense metrizable subspace. Furthermore neither the space of Theorem 1 nor its associated Moore space has the countable chain condition. Related results are established which (1) produce new examples of noncompletatable Moore spaces and (2) provide a partial answer to a question raised in [14] by showing that each regular hereditary $M$-space with a $G_6$-diagonal has a dense metrizable subspace.

**Preliminaries.** A development for a space $S$ is a sequence $G_1, G_2, \ldots$ of open coverings of $S$ such that for each $p \in S$ and each open set $D$ containing $p$, there exists an $n$ such that each element of $G_n$ containing $p$ is contained in $D$. A regular developable space is a Moore space. A Moore space is complete provided it has a complete development, i.e., a development $G_1, G_2, \ldots$ such that if $M_1, M_2, \ldots$ is a sequence of closed sets such that for each $i$, $M_i$ is contained in an element of $G_i$ and contains $M_{i+1}$, then $\bigcap M_i \neq \emptyset$. A Moore space is completatable provided it can be embedded in a complete Moore space. A space $S$ is screenable provided that for each open covering $G$ of $S$ there is an open covering $H = \bigcup_{i=1}^{\infty} H_i$ of $S$ which refines $G$ such that, for each $i$, $H_i$ is a collection of mutually exclusive open sets. In [3], it was shown that a Moore space is metrizable if and only if it is normal and screenable.

**Theorem 1.** There exists a hereditarily paracompact $T_2$-space $X$ with a point countable base which has no dense Moore subspace.

**Proof.** Define $(X, T)$ as follows: Denote by $A$ the set of all ordinal numbers which precede the first uncountable ordinal. For each $a \in A$, let $R(a)$ denote the set of all ordinal numbers which precede $a$ and let $X_a$ denote the set of all mappings $x$ of the set $R(a)$ into $N$, the set of all natural numbers. Now, let $X = \bigcup \{X_a | a \in A\}$. For each $a \in A$ and $x \in X_a$, call $a$ the length of $x$. Furthermore, say that the element $x$ of $X_a$ is an extension of the element $y$ of $X_b$ if $b < a$ and for $c < b$, $x(c) = y(c)$. Finally, for each $a \in A$, $x \in X_a$, and $n \in N$ denote by $u_n(x)$ the set consisting of the point $x$ and of all $y \in X$ such
that $y$ is an extension of $x$ and $y(a) \not\supseteq n$. It follows that $B = \{u_n(x) | x \in X$ and $n \in \mathbb{N}\}$ is a base for the topology $T$ on $X$. To see this, note that if $y \neq x$ and $y \in u_k(x)$, then $u_n(y) \subseteq u_k(x)$ for each $n \in \mathbb{N}$. Aull showed in [2] that $X$ is a hereditarily paracompact $T_2$-space with a point countable base.

Now, suppose that there exists a dense subspace $K$ of $X$ such that $K$ is a Moore space. Note that (1) if $x \in X$, then there exists $y \in K$ such that $y$ is an extension of $x$; and (2) if $x \in K$ and $D$ is an open set in $K$ containing $x$, then there exists $y \in D$ such that $y$ is an extension of $x$. Denote by $G_1, G_2, \ldots$ a development for $K$ such that if $g \in G_i$ for some $i$, then there exist $x \in X$ and $n \in \mathbb{N}$ such that $g \subseteq u_n(x)$. Observe that (3) if $\{x, y\} \subseteq g \in G_i$ for some $i$ and $y$ is an extension of $x$, then each $z \in K$ such that $z$ is an extension of $y$ is contained in $g$. Construct the sequences $x_1, x_2, \ldots$ and $g(x_1), g(x_2), \ldots$ as follows: Let $x_1 \in K$ and let $g(x_1) \in G_1$ such that $x_1 \in g(x_1)$. Let $x_2 \in g(x_1)$ such that $x_2$ is an extension of $x_1$ and let $g(x_2) \in G_2$ such that $x_2 \in g(x_2)$ and $g(x_2) \subseteq g(x_1)$. Continue this process such that for each $i > 2$, $x_i \in g(x_{i-1})$, $x_i$ is an extension of $x_{i-1}$, $x_i \in g(x_i) \subseteq g_i$, and $g(x_i) \subseteq g(x_{i-1})$. Then, for each $i$, let $a_i \in A$ such that $x_i \in X_{a_i}$ and let $a \in A$ such that for each $i$, $a_i < a$. Denote by $x$ an element of $X_a$ such that for each $i$, $x$ is an extension of $x_i$. By (1) and (3) above, there exists $y \in K$ such that $y$ is an extension of $x$ and $y \in g(x_i)$ for each $i$. Thus, for each $i$, $\{x, y\} \subseteq g(x_i) \subseteq g_i$ but $y$ is not a limit point of $\{x_1, x_2, \ldots\}$. This contradicts the assumption that $K$ is a Moore space.

**Theorem 2.** Suppose that $X$ is a first countable space which is the union of countably many subsets $X_i$ such that for each $i$, there exists a collection $U_i$ of mutually exclusive open sets in $X$ covering $X_i$ such that each element of $U_i$ contains only one point of $X_i$. Then $X$ has a dense screenable, developable subspace.

**Proof.** Let $K_1 = X_1$ and for each $i > 1$, let $K_i = X_i - (\bigcup_{j=1}^{i-1} X_j)$. It follows that $K = \bigcup_{i=1}^\infty K_i$ is dense in $X$. Consider $K$ as a subspace of $X$ and for each $i$, let $U'_i = \{u \cap K | u \in U_i$ and $u \cap K_i \neq \emptyset\}$. For each $i$ and each point $x \in K_i$, denote by $g_1(x), g_2(x), \ldots$ a nonincreasing sequence of open sets in $X$ which forms a local base at $x$ and is such that $g_1(x)$ is contained in the element of $U'_i$ which contains $x$. Finally, for each $j$ and each $i > 1$, let $K_{ij} = \{x \in K_i | x \notin g_j(y) \text{ for } y \in \bigcup_{n=1}^{i-1} K_n\}$. By the construction of $K$, it follows that $K = K_1 \cup \bigcup_{i=2}^\infty \bigcup_{j=1}^{i-1} K_{ij}$. Furthermore $K_1$ and each $K_{ij}$ are discrete subsets of $K$. Hence $K$ is the union of countably many discrete subsets and by [13, Lemmas 1.2 and 1.4] is developable. That $K$ is screenable follows immediately.

The following construction was developed by the author in [19] and [20].

**Construction of $S_0$.** Let $X_0$ be a regular first countable $T_1$-space. For each $x \in X_0$, denote by $u_1(x), u_2(x), \ldots$ a sequence of open sets in $X_0$ which
forms a local base at $x$ such that for each $i$, $u_{i+1}(x) \subset u_i(x)$. Now for each positive integer $m$, let $A_m = \{(n_1, n_2, \ldots, n_m) | n_1 = 1 \text{ and for } 1 \leq i \leq m, \text{ } n_i \text{ is a positive integer}\}$. Let $A = \bigcup_{m=1}^{\infty} A_m$. For each $a = (n_1, n_2, \ldots, n_m) \in A$, denote by $S_a$ a unique copy of $X_0$. And for each $x \in X_0$, denote by $x_a = (x_{n_1}, x_{n_2}, \ldots, x_{n_m})$ the element of $S_a$ which is identified with $x$. Let $S_0 = \bigcup\{S_a | a \in A\}$ and define a development for $S_0$ as follows: For each positive integer $j$, $a = (n_1, n_2, \ldots, n_m) \in A$, and $p = (y_{n_1}, y_{n_2}, \ldots, y_{n_m}) \in S_a$, let

$$g_j(p) = \{p\} \cup \{(x_{n_1}, x_{n_2}, \ldots, x_{n_m}, x_{k_1}, x_{k_2}, \ldots, x_{k_c}) | x \in X_0, c \text{ is a positive integer, for } 1 \leq i \leq c, k_i \geq j, \text{ and } x \in u_{k_1+j}(y) \text{ in } X_0\}.$$ 

It follows that $G_1, G_2, \ldots$, where for each $i$, $G_i = \{g_j(p) | p \in S \text{ and } j \geq i\}$, is a development for the Moore space $S_0$.

**Claims.** (1) If $D$ is open in $S_0$, then $\bar{U} = \{x \in X_0 | x_a \in D \text{ for some } a \in A\}$ is open in $X_0$. Furthermore, if $D'$ is an open set in $X_0$, then $D = \{x_a \in S_0 | x \in D' \text{ and } a \in A\}$ is open in $S_0$.

(2) If $K$ is dense in $S_0$, then $K' = \{x \in X_0 | x_a \in K \text{ for some } a \in A\}$ is dense in $X_0$.

(3) If $K'$ is dense in $X_0$, then $K = \{x_a \in S_0 | x \in K' \text{ and } a \in A\}$ is dense in $S_0$.

(4) For each $j$ and each $a \in A$, $g_j(x_a) \cap g_j(y_a) = \emptyset$ in $S_0$ if and only if $u_{2j}(x) \cap u_{2j}(y) = \emptyset$ in $X_0$.

(5) If $D'$ is an open set in $X_0$ and $d$ is an open set in $S_0$ such that $\bar{d} \subset D = \{x_a \in S_0 | x \in D' \text{ and } a \in A\}$, then $d' = \{x \in X_0 | x_a \in d \text{ for some } a \in A\}$ is open in $X_0$ and $\bar{d'} \subset D'$.

(6) Suppose $a = (n_1, n_2, \ldots, n_m) \in A, M \subset S_a$, and for some $i$, $D_i = \bigcup_{j=1}^{\infty} D_{ij}$, where for each $j \geq i$, $D_{ij} = \{(x_{n_1}, x_{n_2}, \ldots, x_{n_m}, x_{k_1}, x_{k_2}, \ldots, x_{k_c}) | x \in X_0, k_1 = j, c \text{ is a positive integer and, for } 1 \leq n \leq c, k_n \geq i, y_a \in M \text{ and } x \in u_{k_1+i}(y) \text{ in } X_0\}$. Furthermore, each $D_{ij}$ is open in $S_0$. $D_{ij} = \{x \in X_0 | x_b \in D_{ij} \text{ for some } b \in A\}$ is open in $X_0$, and if $d'$ is an open set in $X_0$ such that $\bar{d'} \subset D_{ij}$ then $d = \{x_b \in D_{ij} | b \in A \text{ and } x \in d'\}$ is open in $S_0$ and $\bar{d} \subset D_{ij}$.

In the following theorems, $X_0$ will denote a regular first countable space and $S_0$ will denote the Moore space associated to $X_0$ by the above construction.

Theorem 3. $S_0$ has a dense screenable subspace if and only if $X_0$ has a dense screenable Moore subspace.
Proof. Suppose that $S_0$ has a dense screenable subspace. Then by [15, Lemma 2.1 and the proof of Theorem 1.4] there exists a dense screenable subspace $K$ of $S$ such that $K = \bigcup_{i=1}^{\infty} K_i$, where for each $i$, $K_i$ is discrete in $S$ and there exists a collection $V_i$ of mutually exclusive open sets in $S$ covering $K_i$ such that each element of $V_i$ contains only one point of $K_i$. For each $i$ and each $a \in A$, let $K_i(a) = K_i \cap S_a$ and denote by $V_i(a)$ a collection of mutually exclusive open sets in $S_0$ covering $K_i(a)$ such that each element of $V_i(a)$ contains at most one point of $K_i(a)$. Now for each $i$, let $K_i(a) = \{p \in K_i(a)\}$ if $p \in g \in G_j$, then $g$ is contained in the element of $V_i(a)$ which contains $p$. Thus $K = \bigcup\{K_i(a)\}$. Finally, for each $K_i(a)$, consider $X_i(a) = \{x \in X_0 | x_a \in K_i(a)\}$. It follows from claim (4) that $H_i(a) = \{u_j(x) | x \in X_i(a)\}$ is a collection of mutually exclusive open sets in $X_0$ covering $X_i(a)$ such that each element of $H_i(a)$ contains only one point of $X_i(a)$. Hence $X = \bigcup\{X_i(a)\}$ satisfies the hypothesis of Theorem 2 and $X$ has a dense screenable Moore subspace. Furthermore, since $K$ is dense in $S_0$, $X$ is dense in $X_0$. Thus, $X_0$ has a dense screenable Moore subspace.

Suppose that $X_0$ has a dense screenable Moore subspace. As above let $X$ be such a subspace such that $X = \bigcup_{i=1}^{\infty} X_i$ where for each $i$, $X_i$ is discrete in $X$ and $H_i$ is a collection of mutually exclusive open sets in $X_0$ covering $X_i$ such that each element of $H_i$ contains only one element of $X_i$. For each $i$ and each $j$, let $X_ij = \{x \in X_i | U_j(x) \text{ is contained in the element of } H_i \text{ which contains } x\}$. Now for each $i$, each $j$, and each $a \in A$, let $K_i(a) = \{x_a \in S | x \in X_ij\}$. It follows that $U_i(a) = \{g(p) | p \in K_i(a)\}$ is a collection of mutually exclusive open sets in $S_0$ covering $K_i(a)$ such that each element of $U_i(a)$ contains only one point of $K_i(a)$. Thus, $K = \bigcup\{K_i(a)\}$ is a dense screenable subspace of $S_0$.

Theorem 4. If $S_0$ has a dense metrizable subspace, then $X_0$ has a dense screenable Moore subspace.

Proof. Each metrizable space is screenable. Thus if $S_0$ has a dense metrizable subspace, then by Theorem 3, $X_0$ has a dense screenable Moore subspace.

Corollary 5. If $X_0$ is the space of Theorem 1, then the associated Moore space $S_0$ has no dense metrizable subspace.

Theorem 6. If $X_0$ has a dense metrizable subspace, then $S_0$ has a dense metrizable subspace.

Proof. Let $K = \bigcup_{i=1}^{\infty} K_i$ denote a dense metrizable subspace of $X_0$ such that for each $i$, $K_i$ is discrete in $K$. For each $a = (n_1, n_2, \ldots, n_m) \in A$, let $S'_a = \{(x_{n_1}, x_{n_2}, \ldots, x_{n_m}) | x \in K\}$. It follows that $S' = \bigcup\{S'_a | a \in A\}$ is the required dense metrizable subspace of $S_0$. To see this, consider $K_i$ for each $i$. Since $K$ is metrizable, there exists a discrete collection $H_i$ of mutually exclusive
open sets in $K$ covering $K_i$ such that each element of $H_i$ contains only one point of $K_i$. For each $j$, denote by $K_{ij}$ the set of all $x \in K_i$ such that $u_j(x)$ is contained in the element of $H_i$ which contains $x$. Now, for each $a = (n_1, n_2, \ldots, n_m) \in A$ consider $S'a(i, j) = \{(x_1, x_2, \ldots, x_m) \in S'_a | x \in K_{ij}\}$. It follows that $V_a(i, j) = \{g_j(p) \cap S'| p \in S'_a(i, j)\}$ is a discrete collection of mutually exclusive open sets in $S'$ covering $S'_a(i, j)$ such that each element of $V_a(i, j)$ contains only one point of $S'_a(i, j)$. Thus, $S' = \bigcup_i S'_a(i, j)|i \in N, j \in N, a \in A|$ and by [13, Lemmas 1.3 and 1.4] is metrizable.

Remark. Is it true that $S_0$ has a dense metrizable subspace if and only if $X_0$ has a dense metrizable subspace? Or, more generally, must each screenable Moore space have a dense metrizable subspace? An affirmative answer to the latter question would generalize several results in [5], [11], and [15]. Recently, Tall and Przymusiński in [12] have very significantly shown that is consistent with set theory for there to exist a normal subspace of a nonseparable Moore space with the countable chain condition given in [10] which is also nonseparable and has the countable chain condition. Also, W. G. Fleissner has announced that it is consistent with set theory that each normal Moore space has a dense metrizable subspace. Thus, the proposition that each normal Moore space has a dense metrizable subspace is now known to be independent of set theory.

Identification. In [15], the author defined a space $S$ to be weakly densely normal (wd-normal) provided that if $D$ is an open set in $S$ and $H$ is a closed subset of $D$, then there exists a sequence $d_1, d_2, \ldots$ of open sets in $S$ such that $H \subset \bigcup_{i=1}^{\infty} d_i$ and for each $i$, $d_i \subset D$. A space $S$ is said to be perfectly wd-normal provided that for each open set $D$ in $S$ there exists a sequence $d_1, d_2, \ldots$ of open sets in $S$ such that $D \subset \bigcup_{i=1}^{\infty} d_i$ and for each $i$, $d_i \subset D$. It is easily seen that a wd-normal space in which closed sets are $G_{\delta}$ sets is perfectly wd-normal. Hence, the two properties are equivalent in Moore spaces. In [15], each Moore space with the countable chain condition was shown to be wd-normal.

Axiom C. A development $G_1, G_2, \ldots$ for a Moore space $S$ is said to satisfy Axiom C at the point $p$ of $S$ provided that for each open set $D$ in $S$ containing $p$ there exists an $n$ such that each element of $G_n$ intersecting an element of $G_n$ containing $p$ is contained in $D$. It follows from [8] and [25], that if $G$ is a development for the Moore space $S$, then $C(G)$, the set of all points in $S$ at which $G$ satisfies Axiom C, is, if nonempty, a metrizable $G_{\delta}$-subset of $S$. In [13], the author generalized this concept by defining a subset $M$ of the first countable space $S$ to be $C$-developable in $S$ provided there exists a sequence $G_1, G_2, \ldots$ of open covers of $S$ such that if $p \in M$, then for each open set $D$ in $S$ containing $p$ there exists an $n$ such that each element of $G_n$ intersecting an element of $G_n$ containing $p$ is contained in $D$. 
Theorem 7 ([15] and [18]). In a Moore space $S$, the following are equivalent:

1. $S$ has a development $G$ such that $C(G)$ is dense in $S$.
2. $S$ is wd-normal and has a dense screenable subspace.
3. $S$ can be densely embedded in a developable $T_2$-space which has the Baire property.

Theorem 8 [13]. In a regular first countable space, the following are equivalent:

1. $S$ has a dense subset which is $C$-developable in $S$.
2. $S$ is perfectly wd-normal and has a dense screenable Moore subspace.

Theorem 9. $S_0$ is perfectly wd-normal if and only if $X_0$ is perfectly wd-normal.

Proof. Suppose $X_0$ is perfectly wd-normal. Let $D$ be an open set in $S_0$. For each $a = (n_1, n_2, \ldots, n_m) \in A$ and each $i$, let $D_i(a) = \bigcup \{g_i(x_a) | x_a \in D \cap S_a$ and $g_i(x_a) \in D \}$. For each $j \geq i$, let $D_{ij} = \{(x_{n_1}, x_{n_2}, \ldots, x_{n_m}, x_{k_1}, x_{k_2}, \ldots, x_{k_c}) | x \in X_0, k_1 = j, c$ is a positive integer and for $1 \leq n \leq c, k_n \geq i, y_a \in D \cap S_a, and x \in u_{k_1+j}(y) \in X_0 \}$. For each $i$ and $j \geq i$, consider $D_{ij} = \{x \in X_0 | x_b \in D_{ij}$ for some $b \in A \}$. Since $X_0$ is wd-normal, there exists a sequence $d'_{ij}(1), d'_{ij}(2), \ldots$ of open sets in $X_0$ such that $D_{ij} \subset \bigcup_{m=1}^{\infty} d'_{ij}(m)$ and for each $m, d'_{ij}(m) \subset D_{ij}$. But by claim (6), for each $m, d_{ij}(m) = \{x_b \in D_{ij} | b \in A$ and $x \in d_{ij}(m) \}$ is open in $S$ and $d_{ij}(m) \subset D_{ij}$. It is easily seen from claim (3) that $D_{ij} \subset \bigcup_{m=1}^{\infty} d_{ij}(m)$. Hence, since $D_i(a) = \bigcup_{j=1}^{\infty} D_{ij}$ and $D = \bigcup_{a \in A} \bigcup_{i=1}^{\infty} D_i(a)$, it follows that $S_0$ is perfectly wd-normal.

Suppose $S_0$ is perfectly wd-normal. Let $D'$ be an open set in $X_0$. Let $D = \{x_a \in S_0 | x \in D'$ and $a \in A \}$. Denote by $d_1, d_2, \ldots$ a sequence of open sets in $S_0$ such that $D \subset \bigcup_{i=1}^{\infty} d_i$ and for each $i, \overline{d_i} \subset D$. But by claim (5) for each $i, d_{i} = \{x \in X_0 | x_a \in d_i$ for some $a \in A \}$ is open in $S_0$ and $\overline{d_i} \subset D'$. Furthermore by claim (2), $D' \subset \bigcup_{i=1}^{\infty} \overline{d_i}$. Hence, $X_0$ is perfectly wd-normal.

Corollary 10. $S_0$ has a dense subset which is $C$-developable in $S_0$ if and only if $X_0$ has a dense subset which is $C$-developable in $X_0$.

Corollary 11. If $X_0$ is not perfectly wd-normal, then $S_0$ is not completable. In fact, $S_0$ cannot be densely embedded in a developable $T_2$-space having the Baire property.

Examples. There are very few known examples of noncompletable Moore spaces. However, the above results make the production of such spaces much easier. For example, it follows immediately that neither the Michael Line nor the
space of countable ordinals with the order topology is perfectly \( wd \)-normal. Hence, their associated Moore spaces are noncompletable.

**Mspaces.** In [14], the author showed that each regular \( M \)-space with a \( G_\delta \)-diagonal in which closed sets are \( G_\delta \) sets has a dense metrizable subset. The following lemmas were the basis for that result.

**Lemma 12 [9].** If \( X \) is a regular \( M \)-space and \( H \) is a discrete subset of \( X \), then there exists a collection \( U \) of mutually exclusive open sets in \( X \) covering \( H \) such that each element of \( U \) contains only one point of \( H \).

**Lemma 13 [14].** If \( X \) is a regular \( M \)-space with a \( G_\delta \)-diagonal then there exists a dense subset \( K \) of \( X \) such that \( K = \bigcup_{i=1}^{\infty} K_i \) where for each \( i \), no point of \( K_i \) is a limit point of \( K_i \).

**Theorem 14.** If \( X \) is a regular hereditary \( M \)-space with a \( G_\delta \)-diagonal then \( X \) has a dense metrizable subspace.

**Proof.** Let \( K = \bigcup_{i=1}^{\infty} K_i \) be the dense subset of \( X \) from Lemma 13. For each \( i \), consider \( M = K - (K_i - K_i) \). Note that \( M \) is dense in \( K \) and \( K_i \) is discrete in \( M \). Thus, since \( X \) is hereditarily an \( M \)-space, by Lemma 12, there exists a collection \( U' \) of mutually exclusive open sets in \( M \) covering \( K_i \) such that each element of \( U' \) contains only one point of \( K_i \). And since \( M \) is dense in \( X \), there exists such a collection \( U \) in \( X \). Hence, by Theorem 2, \( X \) has a dense developable subspace \( Z \). But, \( Z \) is both a Moore space and an \( M \)-space, and is therefore metrizable.

**References**

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