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TOPOLOGICAL EXTENSION PROPERTIES

BY

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ABSTRACT. It is known that if a topological property P of Tychonoff spaces is closed-hereditary, productive, and possessed by all compact P-regular spaces, then each P-regular space X is a dense subspace of a space $\gamma_P X$ with P such that if Y has P and $f: X \longrightarrow Y$ is continuous, then f extends continuously to $f^\gamma: \gamma_P X \longrightarrow Y$. Such topological properties are called extension properties; $\gamma_P X$ is called the maximal P-extension of X. In this paper we study the relationships between pairs of extension properties and their maximal extensions. A basic tool is the concept of P-pseudocompactness, which is studied in detail (a P-regular space X is P-pseudocompact if $\gamma_P X$ is compact). A classification of extension properties is attempted, and several means of constructing extension properties are studied. A number of examples are considered in detail.

1. **Introduction.** Topologists have long been fascinated by the problem of determining when a continuous function from a topological space X to a space Y can be continuously extended to a space T containing X as a subspace. If Y is the space of real numbers, this leads to the study of C^*-embedding and C-embedding; these concepts are discussed in detail in the Gillman-Jerison text [12]. If, for each normal space T and each closed subspace X of T, each continuous function from X to Y can be continuously extended to T, then Y is called an absolute retract. Finally, we may require Y to belong to a restricted class of topological spaces, and ask if we can find a space T, belonging to this class and containing X as a dense subspace, such that each continuous function from X to Y can be extended continuously to T. The earliest result of this latter sort (see [4] and [21]) asserts the existence of the Stone-Čech compactification of a completely regular Hausdorff space (henceforth called a Tychonoff space); each such space X is contained densely in a compact space βX, its Stone-Čech compactification, and if f maps X continuously to a compact space, then f can be continuously extended to βX. Here the "restricted class" of spaces is the class of compact spaces. Another such "restricted class" is the class of realcompact spaces; Hewitt

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[15] proved that each Tychonoff space X is a dense subspace of a realcompact space υX, its Hewitt realcompactification, and if f maps X continuously to a realcompact space, then f can be continuously extended to υX (see [12, Chapter 8] for details). One might ask what it is about compactness and realcompactness that gives rise to the constructions βX and υX with their parallel properties. The best answer seems to have been given by Herrlich and van der Slot [14] as follows. Let $\mathcal{P}$ be a topological property of Tychonoff spaces. Call a space $\mathcal{P}$-regular if it is homeomorphic to a subspace of a product of spaces each of which has $\mathcal{P}$. The following theorem appears in [14, Theorem 1].

1.1. THEOREM. *Let $\mathcal{P}$ be a topological property of Tychonoff spaces. The following are equivalent*:

(i) *$\mathcal{P}$ is closed-hereditary and productive.*

(ii) *Each $\mathcal{P}$-regular space X is a dense subspace of a Tychonoff space $\gamma_{\mathcal{P}}X$ with the following properties*: *$\gamma_{\mathcal{P}}X$ has $\mathcal{P}$ and if f is a continuous map from X into a space Y with $\mathcal{P}$, then there exists a continuous function f^{γ}: $\gamma_{\mathcal{P}}X \longrightarrow Y$ such that $f^{\gamma}|X = f$.*

(In the language of category theory, the conditions in 1.1 are equivalent to the condition that the category of spaces with $\mathcal{P}$ and continuous maps is a reflective subcategory of the category of $\mathcal{P}$-regular spaces and continuous maps.)

Let us call a closed-hereditary, productive topological property $\mathcal{P}$ (of Tychonoff spaces) such that each $\mathcal{P}$-regular space has a $\mathcal{P}$-regular compactification an *extension property*. In this paper we undertake a systematic study and classification of extension properties. In particular we investigate the relationship between pairs of extension properties and describe several ways of producing large collections of extension properties. Our chief tool in this study is the concept of $\mathcal{P}$-pseudocompactness (a $\mathcal{P}$-regular space X is $\mathcal{P}$-pseudocompact if $\gamma_{\mathcal{P}}X$ is compact).

The remainder of this section is devoted to a discussion of the notation, terminology, and known results to be used in subsequent sections. Henceforth all topological spaces discussed are assumed to be Tychonoff spaces. The notation and terminology of the Gillman-Jerison text [12] will be used whenever appropriate.

I am grateful to the referee for pointing out that some of the results discussed in this section are also dealt with in [23].

The space $\gamma_{\mathcal{P}}X$ of 1.1 is called the *maximal $\mathcal{P}$-extension* of X. If S and T are two spaces containing X as a dense subspace, we say that S and T are *equivalent extensions* of X if there is a homeomorphism from S onto T whose restriction to X is the identity. We identify equivalent extensions of a space; evidently any two maximal $\mathcal{P}$-extensions of X are equivalent, so $\gamma_{\mathcal{P}}X$ is uniquely determined by the description given in 1.1.

The following properties of extension properties are proved in [14, Lemma 2 and Proposition 2] and in [13, §3, p. 234 and 4.3, p. 237].

1.2. THEOREM. *Let $\mathcal{P}$ be an extension property.*

(a) *If X is $\mathcal{P}$-regular and if $(Y_\alpha)_{\alpha\in\Sigma}$ is a collection of subspaces of X having $\mathcal{P}$, then $\bigcap_{\alpha\in\Sigma} Y_\alpha$ has $\mathcal{P}$ (in fact this holds for any closed-hereditary productive property; see* [23]).

(b) *If f is a perfect map from the $\mathcal{P}$-regular space X onto a space Y with $\mathcal{P}$, then X has $\mathcal{P}$. (A continuous function f from X onto Y is perfect if it is closed and if inverse images of points of Y are compact subsets of X.)*

1.3. THEOREM. *Let $\mathcal{P}$ be an extension property. Each $\mathcal{P}$-regular space X has a maximal $\mathcal{P}$-regular compactification $\beta_{\mathcal{P}}X$ in the sense that if K is a compact $\mathcal{P}$-regular space and if $f\colon X \longrightarrow K$ is continuous, then f extends continuously to $\bar{f}\colon \beta_{\mathcal{P}}X \longrightarrow K$. If equivalent extensions of X are identified, then $\beta_{\mathcal{P}}X$ is uniquely determined by the above description; furthermore $\gamma_{\mathcal{P}}X$ is equivalent to (and hence can be identified with) the intersection of all subspaces of $\beta_{\mathcal{P}}X$ that contain X and have $\mathcal{P}$.*

(The final assertion of 1.3 is a simple generalization of the final paragraphs of [14].)

Two extension properties $\mathcal{P}$ and $\mathcal{Q}$ will be called *coregular* if the class of $\mathcal{P}$-regular spaces and the class of $\mathcal{Q}$-regular spaces are the same. Note that if $\mathcal{P}$ and $\mathcal{Q}$ are coregular then $\beta_{\mathcal{P}}X$ and $\beta_{\mathcal{Q}}X$ are equivalent for each $\mathcal{P}$-regular space X, and hence can be identified.

We shall identify a topological property with the class of spaces possessing that property. The usual set-theoretic notation will be used to denote relationships between classes of spaces and formation of new classes. Thus if $\mathcal{P}$ is a topological property and a space X has this property, we shall express this by writing $X \in \mathcal{P}$; if $\mathcal{P}$ and $\mathcal{Q}$ are two classes of spaces, then $\mathcal{P} \cap \mathcal{Q}$ denotes the class of spaces possessing both $\mathcal{P}$ and $\mathcal{Q}$; and so on.

Next we show that if $\mathcal{P}$ is a nontrivial topological property, then each zero-dimensional space is $\mathcal{P}$-regular (a space is zero-dimensional if its open-and-closed (clopen) subsets form a base for its open sets).

1.4. PROPOSITION. *Let $\mathcal{P}$ be a topological property enjoyed by (Hausdorff) spaces other than the one-point space. Then each zero-dimensional space is $\mathcal{P}$-regular.*

PROOF. If $X \in \mathcal{P}$ and $|X| > 1$, then let p and q be distinct points of X. As X is Hausdorff, the two-point space $\{p, q\}$ is discrete. Hence any subspace of any product of copies of the two-point discrete space is $\mathcal{P}$-regular; but the

class of zero-dimensional spaces is precisely the class of spaces homeomorphic to some subspace of such a product. □

We conclude this section by giving some examples of extension properties; numerous other examples are constructed in later sections. If E is any topological space, a space X is called *E-compact* if X is homeomorphic to a closed subspace of a product of copies of E (see [16] for basic information on E-compactness). Let $\langle E\rangle$ denote the class of E-compact spaces. If E is Tychonoff and has an $\langle E\rangle$-regular compactification, then $\langle E\rangle$ is an extension property. Three important examples of this are $\langle [0, 1]\rangle$ (= compactness), $\langle \mathbf{R}\rangle$ (= realcompactness; $\mathbf{R}$ is the space of reals), and $\langle \mathbf{N}\rangle$ (= N-compactness; $\mathbf{N}$ is the countable discrete space). In the first two cases the $\langle E\rangle$-regular spaces are just the Tychonoff spaces, and in the last case the $\langle E\rangle$-regular spaces are the zero-dimensional spaces.

Not all extension properties are E-compactness for some suitably chosen E. Recall that a Hausdorff space X is *almost realcompact* if each open ultrafilter $\mathcal{U}$ on X such that $\bigcap\{\mathrm{cl}_X U\colon U \in \mathcal{U}\} = \varnothing$ contains a countable subset $(U_i)_{i\in N}$ such that $\bigcap\{\mathrm{cl}_X U_i\colon i \in N\} = \varnothing$ (see [7]). The property "almost realcompact Tychonoff" is an extension property (see [27]) but Frolik and Mrowka [8] have shown that it is not E-compactness for any E. As another example, let m be an infinite cardinal and call a Tychonoff space X *m-bounded* if each subset of X of cardinality no greater than m has compact X-closure. Then m-boundedness is an extension property (see [24]), and in §4 we prove that $\aleph_0$-boundedness is not E-compactness for any E.

Finally, note that if $\mathcal{P}$ is an extension property, then $\mathcal{P}_0$, the class of zero-dimensional spaces with $\mathcal{P}$, is also an extension property. This follows from the fact that each zero-dimensional space has a maximal zero-dimensional compactification $\beta_0 X$; $\beta_0 X$ can be identified with the Stone space of the Boolean algebra of clopen sets of X, and X can be identified with the subspace of $\beta_0 X$ consisting of ultrafilters of clopen sets of X with nonempty intersection.

2. $\mathcal{P}$-pseudocompact spaces.

2.1. Definition. Let $\mathcal{P}$ be an extension property. A $\mathcal{P}$-regular space X is *$\mathcal{P}$-pseudocompact* if $\gamma_{\mathcal{P}} X = \beta_{\mathcal{P}} X$. The class of $\mathcal{P}$-pseudocompact spaces will be denoted by $\mathcal{P}'$.

If $\mathcal{P}$ is realcompactness, then $\mathcal{P}$-pseudocompactness is just pseudocompactness (see [12, 8A.4]); this is the motivation for our terminology. Note that X is $\mathcal{P}$-pseudocompact iff $\gamma_{\mathcal{P}} X$ is compact. $\mathcal{P}$-pseudocompactness has many of the properties possessed by pseudocompactness, as the next result shows.

2.2. Proposition. *Let $\mathcal{P}$ and $\mathcal{Q}$ be extension properties and let X and T be $\mathcal{P}$-regular spaces. Then*:

(a) *If $X \in \mathcal{P}'$ and X is dense in T, then $T \in \mathcal{P}'$.*

(b) *If X is the union of two $\mathcal{P}$-pseudocompact spaces then X is $\mathcal{P}$-pseudocompact.*

(c) *$X \in \mathcal{P} \cap \mathcal{P}'$ iff X is compact.*

(d) *Continuous $\mathcal{P}$-regular images of $\mathcal{P}$-pseudocompact spaces are $\mathcal{P}$-pseudocompact.*

(e) *If $\mathcal{P}$ and $\mathcal{Q}$ are coregular and $\mathcal{P} \subseteq \mathcal{Q}$ then $\mathcal{Q}' \subseteq \mathcal{P}'$.*

(f) *$X \in \mathcal{P}'$ iff given $S \in \mathcal{P}$ and a continuous map f from X to S, then $\mathrm{cl}_S f[X]$ is compact.*

PROOF. (a) $\gamma_{\mathcal{P}} T \in \mathcal{P}$ so by 1.1 the inclusion map $j\colon X \longrightarrow \gamma_{\mathcal{P}} T$ extends continuously to $j^{\gamma}\colon \gamma_{\mathcal{P}} X \longrightarrow \gamma_{\mathcal{P}} T$. But $\gamma_{\mathcal{P}} X$ is compact so the compact space $j^{\gamma}[\gamma_{\mathcal{P}} X]$ is a subspace of $\gamma_{\mathcal{P}} T$ containing the dense subspace X of $\gamma_{\mathcal{P}} T$. Hence $\gamma_{\mathcal{P}} T$ is compact and $T \in \mathcal{P}'$.

(c) If X is compact obviously $X \in \mathcal{P} \cap \mathcal{P}'$. If $X \in \mathcal{P} \cap \mathcal{P}'$ then $\gamma_{\mathcal{P}} X = X$ and $\gamma_{\mathcal{P}} X = \beta_{\mathcal{P}} X$ so X is compact.

(b) Let $X = A \cup B$ where $A, B \in \mathcal{P}'$. Then $\gamma_{\mathcal{P}} X = \mathrm{cl}_{\gamma_{\mathcal{P}} X} A \cup \mathrm{cl}_{\gamma_{\mathcal{P}} X} B$. As $\mathcal{P}$ is closed-hereditary, $\mathrm{cl}_{\gamma_{\mathcal{P}} X} A \in \mathcal{P}$. By (a) above, $\mathrm{cl}_{\gamma_{\mathcal{P}} X} A \in \mathcal{P}'$; hence by (c) $\mathrm{cl}_{\gamma_{\mathcal{P}} X} A$ is compact. Similarly $\mathrm{cl}_{\gamma_{\mathcal{P}} X} B$ is compact, so $\gamma_{\mathcal{P}} X$ is compact.

(d) Let Y be $\mathcal{P}$-regular and let $f\colon X \longrightarrow Y$ be continuous and onto. Then f maps into $\gamma_{\mathcal{P}} Y$ and so can be continuously extended to $f^{\gamma}\colon \gamma_{\mathcal{P}} X \longrightarrow \gamma_{\mathcal{P}} Y$. If $X \in \mathcal{P}'$ then $f^{\gamma}[\gamma_{\mathcal{P}} X]$ is compact and $Y \subseteq f^{\gamma}[\gamma_{\mathcal{P}} X] \subseteq \gamma_{\mathcal{P}} Y$; hence $\gamma_{\mathcal{P}} Y$ is compact.

(e) Let $X \in \mathcal{Q}'$. As $\mathcal{P} \subseteq \mathcal{Q}$, $\gamma_{\mathcal{P}} X \in \mathcal{Q}$. Since $\gamma_{\mathcal{Q}} X$ is the intersection of all spaces between $\beta_{\mathcal{P}} X$ $(= \beta_{\mathcal{Q}} X)$ and X that have $\mathcal{Q}$, it follows that $X \subseteq \gamma_{\mathcal{Q}} X \subseteq \gamma_{\mathcal{P}} X \subseteq \beta_{\mathcal{P}} X$. But $\gamma_{\mathcal{Q}} X = \beta_{\mathcal{Q}} X = \beta_{\mathcal{P}} X$ so $\gamma_{\mathcal{P}} X = \beta_{\mathcal{P}} X$. Hence $X \in \mathcal{P}'$.

(f) Let $X \in \mathcal{P}'$. By (d) above $f[X] \in \mathcal{P}'$ so $\mathrm{cl}_S f[X] \in \mathcal{P}'$ by (a). But $\mathrm{cl}_S f[X] \in \mathcal{P}$ as $\mathcal{P}$ is closed-hereditary, so $\mathrm{cl}_S f[X]$ is compact by (c). Conversely, $\gamma_{\mathcal{P}} X \in \mathcal{P}$ and $\mathrm{cl}_{\gamma_{\mathcal{P}} X} X = \gamma_{\mathcal{P}} X$; hence if the condition is fulfilled, $X \in \mathcal{P}'$. □

2.3. THEOREM. *Let $\mathcal{P}$ and $\mathcal{Q}$ be coregular extension properties such that $\mathcal{Q} \subseteq \mathcal{P}'$. Then $\gamma_{\mathcal{Q}} X - \gamma_{\mathcal{P}} X$ is dense in $\beta_{\mathcal{P}} X - \gamma_{\mathcal{P}} X$.*

PROOF. Since $\gamma_{\mathcal{Q}} X$ can be thought of as the intersection of all subspaces of $\beta_{\mathcal{P}} X$ $(= \beta_{\mathcal{Q}} X)$ with $\mathcal{Q}$ (see 1.3), $\gamma_{\mathcal{Q}} X - \gamma_{\mathcal{P}} X$ is a subset of $\beta_{\mathcal{P}} X - \gamma_{\mathcal{P}} X$.

Suppose the theorem is false. Find $p \in \beta_{\mathcal{P}} X - \gamma_{\mathcal{P}} X$ and V open in $\beta_{\mathcal{P}} X$ such that $p \in V - \gamma_{\mathcal{P}} X$ and $(V - \gamma_{\mathcal{P}} X) \cap (\gamma_{\mathcal{Q}} X - \gamma_{\mathcal{P}} X) = \varnothing$. Let A be a regular closed $\beta_{\mathcal{P}} X$-neighborhood of p contained in V. (Thus $A = \mathrm{cl}_{\beta_{\mathcal{P}} X}(\mathrm{int}_{\beta_{\mathcal{P}} X} A)$.) Then $(A - \gamma_{\mathcal{P}} X) \cap (\gamma_{\mathcal{Q}} X - \gamma_{\mathcal{P}} X) = \varnothing$, and so $A \cap \gamma_{\mathcal{Q}} X \subseteq \gamma_{\mathcal{P}} X$. But

$$A \cap X \subseteq A \cap \gamma_{\mathcal{Q}} X \subseteq A \cap \gamma_{\mathcal{P}} X = \mathrm{cl}_{\gamma_{\mathcal{P}} X}(A \cap X);$$

the last equality follows from the fact that A is a regular closed subset of $\beta_{\mathcal{P}}X$ and X is dense in $\gamma_{\mathcal{P}}X$. As $A \cap \gamma_{\mathcal{Q}}X \in \mathcal{Q}$, by hypothesis $A \cap \gamma_{\mathcal{Q}}X \in \mathcal{P}'$. Thus by 2.2(a) $\mathrm{cl}_{\gamma_{\mathcal{P}}X}(A \cap X) \in \mathcal{P} \cap \mathcal{P}'$; hence $\mathrm{cl}_{\gamma_{\mathcal{P}}X}(A \cap X)$ is compact. Thus $\mathrm{cl}_{\gamma_{\mathcal{P}}X}(A \cap X) = A$; and so $A \subseteq \gamma_{\mathcal{P}}X$, contradicting the fact that $p \in A - \gamma_{\mathcal{P}}X$. The theorem follows. □

2.4. EXAMPLE. Let $\mathcal{P}$ be realcompactness and let $\mathcal{Q}$ be $\aleph_0$-boundedness. Then $\mathcal{P}$ and $\mathcal{Q}$ are coregular extension properties; all Tychonoff spaces are $\mathcal{P}$-regular. Evidently each $\aleph_0$-bounded space is countably compact and hence pseudocompact, so $\mathcal{Q} \subseteq \mathcal{P}'$. Here $\gamma_{\mathcal{P}}X$ is the Hewitt realcompactification υX, $\beta_{\mathcal{P}}X$ is the Stone-Čech compactification βX, and $\gamma_{\mathcal{Q}}X$ consists of all points of βX that are in the βX-closure of some countable subset of X (see [24, Theorem 1.3]). Theorem 2.3 says that $\gamma_{\mathcal{Q}}X - \upsilon X$ is dense in $\beta X - \upsilon X$; i.e. the set of points of $\beta X - \upsilon X$ that are in the βX-closure of some countable subset of X forms a dense subset of $\beta X - \upsilon X$. This can be deduced directly, of course, and is well known. A less elementary application of 2.3 appears in 4.6.

2.5. LEMMA. *Let $\mathcal{P}$ be an extension property and let B be a clopen subset of the $\mathcal{P}$-regular space X. Then*:

(a) $\mathrm{cl}_{\gamma_{\mathcal{P}}X}B = \gamma_{\mathcal{P}}B$.

(b) *$B \in \mathcal{P}'$ iff $\mathrm{cl}_{\gamma_{\mathcal{P}}X}B$ is compact.*

(c) *If $X \in \mathcal{P}'$ then $B \in \mathcal{P}'$.*

PROOF. (a) Let $Y \in \mathcal{P}$ and let $f\colon B \longrightarrow Y$ be continuous. Pick $y_0 \in Y$ and define $g\colon X \longrightarrow Y$ as follows: $g|B = f$, and $g[X - B] = \{y_0\}$. Then g is continuous since B is clopen. Extend g to $g^{\gamma}\colon \gamma_{\mathcal{P}}X \longrightarrow Y$ and put $f^{\gamma} = g^{\gamma}|\mathrm{cl}_{\gamma_{\mathcal{P}}X}B$. Then f^{γ} is a continuous extension of f to $\mathrm{cl}_{\gamma_{\mathcal{P}}X}B$, so $\mathrm{cl}_{\gamma_{\mathcal{P}}X}B$ is equivalent to (and hence can be identified with) $\gamma_{\mathcal{P}}B$.

(b) If $B \in \mathcal{P}'$ then $\mathrm{cl}_{\gamma_{\mathcal{P}}X}B \in \mathcal{P}'$ by 2.2(a). But $\mathrm{cl}_{\gamma_{\mathcal{P}}X}B \in \mathcal{P}$, so $\mathrm{cl}_{\gamma_{\mathcal{P}}X}B$ is compact by 2.2(c). Conversely if $\mathrm{cl}_{\gamma_{\mathcal{P}}X}B$ is compact then $\gamma_{\mathcal{P}}B$ is compact by (a) above, so $B \in \mathcal{P}'$.

(c) This follows immediately from (b). □

If $\mathcal{P}$ is realcompactness, then 2.5(b) holds if B is a regular closed subset of X (see [5, Theorem 4.1]). We have been unable to prove this for an arbitrary extension property $\mathcal{P}$, which is why we have restricted ourselves to zero-dimensional spaces in the following theorem.

2.6. THEOREM. *Let $\mathcal{P}$ and $\mathcal{Q}$ be coregular extension properties and let each $\mathcal{P}$-regular space be zero-dimensional. The following are equivalent*:

(a) $\mathcal{P}' = \mathcal{Q}'$.

(b) *$\beta_{\mathcal{P}}X - \gamma_{\mathcal{P}}X$ and $\beta_{\mathcal{P}}X - \gamma_{\mathcal{Q}}X$ are dense in $\beta_{\mathcal{P}}X - (\gamma_{\mathcal{P}}X \cap \gamma_{\mathcal{Q}}X)$ for each zero-dimensional space X.*

PROOF. As remarked earlier, $\gamma_Q X$ can be regarded as a subspace of $\beta_P X$, which in this case is the maximal zero-dimensional compactification $\beta_0 X$ of X.

(a) $\Rightarrow$ (b) Let A be a clopen subset of $\beta_0 X$ and assume that $A \subseteq \gamma_P X$. By 2.5(b) $A \cap X \in \mathcal{P}'$ and hence $A \cap X \in \mathcal{Q}'$. Thus $A \subseteq \gamma_Q X$ by 2.5(b). As $\beta_0 X$ is zero-dimensional, it follows that $\beta_0 X - \gamma_P X$ is dense in $\beta_0 X - (\gamma_P X \cap \gamma_Q X)$.

(b) $\Rightarrow$ (a) Suppose without loss of generality that $X \in \mathcal{P}'$ but $X \notin \mathcal{Q}'$. Then $\beta_P X - \gamma_P X = \varnothing$ but $\beta_P X - (\gamma_P X \cap \gamma_Q X) = \beta_P X - \gamma_Q X \neq \varnothing$, and (b) fails. □

2.7. EXAMPLE. Let $\mathcal{P}$ be the class of N-compact spaces, $\mathcal{Q}$ the class of realcompact zero-dimensional spaces, and $\mathcal{R}$ the class of almost realcompact zero-dimensional spaces. Then $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$ are all extension properties of zero-dimensional spaces (see §1) and $\mathcal{P} \subsetneqq \mathcal{Q} \subsetneqq \mathcal{R}$ (obviously $\mathcal{P} \subseteq \mathcal{Q}$, and Nyikos [18] provides an example of a space in $\mathcal{Q} - \mathcal{P}$; that $\mathcal{Q} \subseteq \mathcal{R}$ follows from [7, Theorem 10] and Mrowka [17] provides an example of a space in $\mathcal{R} - \mathcal{Q}$). Hence $X \subsetneqq \gamma_R X \subsetneqq \gamma_Q X \subsetneqq \gamma_P X \subsetneqq \beta_0 X$ in general (for a zero-dimensional space X). It follows from [1, 2.3 and 2.4] and [19, 1.9.4] that $\gamma_P X$ can be identified with the subspace of $\beta_0 X$ consisting of the ultrafilters of clopen subsets of X with the countable intersection property, and that $\gamma_P X = \beta_0 X$ iff X is pseudocompact. Since $\gamma_Q X$ is compact if X is pseudocompact, it follows that $\mathcal{P}' = \mathcal{Q}'$. Since each almost realcompact pseudocompact space is compact (this follows immediately from [12, 9.13]) we conclude that $\mathcal{R}' = \mathcal{Q}'$. Hence 2.7 implies that $\beta_0 X - \gamma_P X$ is dense in $\beta_0 X - \gamma_Q X$ which in turn is dense in $\beta_0 X - \gamma_R X$. □

We next show that if $\mathcal{P}$ and $\mathcal{Q}$ are coregular extension properties such that $\mathcal{P}$-regularity is complete regularity, then $\mathcal{P}' \subseteq \mathcal{Q}$ only in trivial situations. I would like to thank the referee for pointing out an error in the original formulation of Theorem 2.8 below.

2.8. THEOREM. *Let $\mathcal{P}$ and $\mathcal{Q}$ be coregular extension properties such that $\mathcal{P}$-regularity is complete regularity. If $\mathcal{P}' \subseteq \mathcal{Q}$ then $\mathcal{P}$ or $\mathcal{Q}$ is the class of all Tychonoff spaces.*

PROOF. Recall [12, Problem 6J] that a Tychonoff space X is *almost compact* if $|\beta X - X| \leqslant 1$. Evidently if X is almost compact, and $\mathcal{S}$ is any extension property of completely regular spaces, then $X \in \mathcal{S}$ or $X \in \mathcal{S}'$.

Note that $\beta_P X = \beta X$. We claim that either $\mathcal{P}$ contains all almost compact spaces, or else $\mathcal{Q}$ does. For if not, let X and Y be almost compact spaces such that $X \notin \mathcal{P}$ and $Y \notin \mathcal{Q}$. Then $|\beta X - X| = |\beta Y - Y| = 1$, so let $\beta X - X = \{x_0\}$ and $\beta Y - Y = \{y_0\}$. Put $S = \beta X \times \beta Y - \{(x_0, y_0)\}$. Now $X \times \beta Y$ is pseudocompact [12, 9.14 and 6J] so by [11, Theorem 1] $\beta(X \times \beta Y) = \beta X \times \beta Y$. Thus $X \times Y \subseteq S \subseteq \beta(X \times Y)$ so $\beta S = \beta X \times \beta Y$ (see [12, 6.7]). Hence S is almost compact. Then $S \in \mathcal{P}$ or $S \in \mathcal{P}'$. The former cannot occur as $X \times \{y_0\}$ is a closed

subspace of S not in $\mathcal{P}$ and the latter cannot occur as $\mathcal{P}' \subseteq \mathcal{Q}$ and $\{x_0\} \times Y$ is a closed subspace of S not in $\mathcal{Q}$. This contradiction verifies our claim, so assume (without loss of generality) that $\mathcal{P}$ contains all almost compact spaces.

Now let X be any Tychonoff space. If $p \in \beta X - X$, then the space $\beta X - \{p\}$ is almost compact (see [12, 6.7]). Hence as $X = \bigcap\{\beta X - \{p\}: p \in \beta X - X\}$, by 1.2(a) $X \in \mathcal{P}$. Hence $\mathcal{P}$ contains all Tychonoff spaces. Similarly if $\mathcal{Q}$ had contained all almost compact spaces, then $\mathcal{Q}$ would have contained all Tychonoff spaces. □

We can generalize 2.8 to the case where $\mathcal{P}$-regularity is zero-dimensionality. The nontrivial facts that are needed are: (1) if X is zero-dimensional and $|\beta_0 X - X| = 1$, then X is pseudocompact; (2) if $X \times Y$ is pseudocompact and zero-dimensional, then $\beta_0(X \times Y) = \beta_0 X \times \beta_0 Y$. The first assertion follows from [19, 1.9.4], and a modification of the proof of Glicksberg's theorem [11, Theorem 1] yields the second.

We conclude this section by dividing extension properties into two broad classes and deriving some consequences of this classification.

2.9. THEOREM. *Let $\mathcal{P}$ be an extension property. Then either*

(a) *$\mathcal{P}$ is contained in the class of countably compact spaces, or*

(b) *$\mathcal{P}$ contains the class of* N*-compact spaces.*

PROOF. If (a) is true then $\mathbf{N} \notin \mathcal{P}$ as $\mathbf{N}$ is not countably compact; hence (b) is false. If (a) is false then there exists a space X in $\mathcal{P}$ such that X is not countably compact. As X is Hausdorff, it has a subspace S that is countably infinite and has no limit points in X. Hence S is homeomorphic to $\mathbf{N}$ and closed in X. It follows that $\mathbf{N} \in \mathcal{P}$, so each $\mathbf{N}$-compact space is in $\mathcal{P}$. □

We note in passing that $\aleph_0$-boundedness is an extension property contained in the class of countably compact spaces.

2.10. COROLLARY. *Let $\mathcal{P}$ be an extension property such that $\mathcal{P}$-regularity is zero-dimensionality. Then either*

(a) *$\mathcal{P}$ is the class of compact zero-dimensional spaces, or*

(b) *$\mathcal{P}'$ does not properly contain the class of pseudocompact zero-dimensional spaces.*

PROOF. If (a) is true then $\mathcal{P}'$ is the class of all zero-dimensional spaces and (b) is false. Suppose (a) is false. Either $\mathbf{N} \in \mathcal{P}$ or $\mathbf{N} \notin \mathcal{P}$; if $\mathbf{N} \in \mathcal{P}$ then (b) is true by 2.2(e) and 2.7. If $\mathbf{N} \notin \mathcal{P}$ then by 2.9 each space in $\mathcal{P}$ is countably compact and hence pseudocompact. Since (a) is false, $\mathcal{P}$ contains noncompact spaces; hence by 2.2(c), (b) must be true. □

We do not know if 2.10 is valid for extension properties $\mathcal{P}$ for which $\mathcal{P}$-regularity is not zero-dimensionality.

The following corollary to 2.9 somewhat limits the scope of 2.3.

2.11. COROLLARY. *Let $\mathcal{P}$ and $\mathcal{Q}$ be coregular extension properties such that $\mathcal{P} \subseteq \mathcal{Q}'$. If $\mathcal{P}$ is not contained in countable compactness, then each $\mathcal{P}$-regular space of nonmeasurable cardinal is in $\mathcal{Q}'$.*

PROOF. If $\mathcal{P}$ is not contained in countable compactness, then $\mathbf{N} \in \mathcal{P}$ by 2.9. Thus each discrete space of nonmeasurable cardinal, being N-compact, is in $\mathcal{P}$ (see [12, Chapter 12] for details). If X is $\mathcal{P}$-regular and $|X|$ is nonmeasurable, then X is the continuous image of a discrete space D of nonmeasurable cardinal. As $D \in \mathcal{Q}'$, by 2.2(d) $X \in \mathcal{Q}'$. □

We conclude this section by presenting a table giving a representative list of extension properties $\mathcal{P}$, together with characterizations (if available) of the corresponding pseudocompactness property $\mathcal{P}'$ and maximal $\mathcal{P}$-extension $\gamma_{\mathcal{P}}X$. When appropriate, references either to sections of this paper or items in the bibliography are included in the table. The frequent occurrence of question marks indicates that much remains to be learned about all but the best-known extension properties. Of course, Table 1 is far from being an exhaustive list of extension properties.

I would like to thank the referee for suggesting the inclusion of this table.

3. Construction of new extension properties from old. If $\mathcal{P}$ is an extension property, there are several ways of constructing new extension properties from $\mathcal{P}$. In this section we discuss several of these constructions and study how the new extension property is related to the original one.

3.1. DEFINITION. Let $\mathcal{P}$ be a topological property of Tychonoff spaces. Then $A\mathcal{P}$ is defined to be the class of all $\mathcal{P}$-regular spaces that are images under a perfect map of some space in $\mathcal{P}$.

In [6, 1.7] Dykes shows that if $\mathcal{P}$ is the class of realcompact spaces, then $A\mathcal{P}$ is the class of (Tychonoff) almost realcompact spaces.

3.2. PROPOSITION. *If $\mathcal{P}$ is an extension property then $A\mathcal{P}$ is an extension property.*

PROOF. We must show that $A\mathcal{P}$ is closed-hereditary, productive, and that each $A\mathcal{P}$-regular space has an $A\mathcal{P}$-regular compactification. Let (Y_α) be a collection of spaces with $A\mathcal{P}$. Then there exist, for each index α, a space $X_\alpha \in \mathcal{P}$ and a perfect map $k_\alpha: X_\alpha \longrightarrow Y_\alpha$. The product map $\Pi_\alpha k_\alpha: \Pi_\alpha X_\alpha \longrightarrow \Pi_\alpha Y_\alpha$ is perfect; as $\Pi_\alpha X_\alpha \in \mathcal{P}$, $\Pi_\alpha Y_\alpha \in A\mathcal{P}$ (as $\Pi_\alpha Y_\alpha$ is $\mathcal{P}$-regular). Hence $A\mathcal{P}$ is productive. If $Y \in A\mathcal{P}$ and A is a closed subset of Y, let $f: X \longrightarrow Y$ be a perfect map, with

TABLE I. Examples of extension properties

$\mathcal{P}$	$\mathcal{P}$-regularity	$\mathcal{P}'$	$\gamma_{\mathcal{P}}X$
compactness	complete regularity	complete regularity	βX (Stone-Čech [12, Chapter 6])
complete regularity	complete regularity	compactness	X
realcompactness	complete regularity	pseudocompactness	υX (Hewitt [12, Chapter 8])
almost realcompactness	complete regularity	pseudocompactness	aX (Woods [27])
m-boundedness	complete regularity	?	mX (Woods [24])
$\mathcal{D}$-compactness (§4)	complete regularity	?	$\gamma_{[\mathcal{D}]}X$ (4.3)
compact zero-dimensional	zero-dimensionality	zero-dimensionality	$\beta_0 X$
zero-dimensionality	zero-dimensionality	compact zero-dimensional	X
realcompact zero-dimensional	zero-dimensionality	pseudocompact zero-dimensional	?
N-compact	zero-dimensionality	pseudocompact zero-dimensional [1], [19]	set of cluster points in $\beta_0 X$ of clopen ultrafilters on X with C.I.P. [1], [19]
all connected components are compact (3.6)	complete regularity	?	?
all pseudocompact subspaces are relatively compact (3.6)	complete regularity	pseudocompactness (3.9)	?

$X \in \mathcal{P}$. Then $f|f^{\leftarrow}[A]: f^{\leftarrow}[A] \longrightarrow A$ is a perfect map and $f^{\leftarrow}[A] \in \mathcal{P}$, so $A \in \mathcal{A}\mathcal{P}$. Thus $\mathcal{A}\mathcal{P}$ is closed-hereditary. Since each space with $\mathcal{A}\mathcal{P}$ is $\mathcal{P}$-regular, it has a compactification with $\mathcal{P}$, and hence with $\mathcal{A}\mathcal{P}$. Thus $\mathcal{A}\mathcal{P}$ is an extension property. □

Let $\mathcal{R}(X)$ denote the collection of regular closed subsets of the space X. It is well known that $\mathcal{R}(X)$, partially ordered by inclusion, is a complete Boolean algebra under the following operations:

(a) $\bigvee_\alpha A_\alpha = \mathrm{cl}_X[\bigcup_\alpha A_\alpha]$.

(b) $\bigwedge_\alpha A_\alpha = \mathrm{cl}_X \mathrm{int}_X[\bigcap_\alpha A_\alpha]$.

(c) $A' = \mathrm{cl}_X(X - A)$ (A' denotes the Boolean-algebraic complement of A).

Let $\mathcal{P}$ be an extension property. We will show that the members of $\mathcal{R}(X)$ having $\mathcal{A}\mathcal{P}$ form an ideal Δ of $\mathcal{R}(X)$; our goal is to obtain a representation for the Stone space of the factor algebra $\mathcal{R}(X)/\Delta$. Recall [23, §1] that the Stone space $S(U)$ of a Boolean algebra U is the collection of all ultrafilters on U, topologized as follows: if $u \in U$, put $\lambda(u) = \{\alpha \in S(U): u \in \alpha\}$ and let $\{\lambda(u): u \in U\}$ serve as an open base for a topology on $S(U)$. So topologized, $S(U)$ becomes a compact totally disconnected Hausdorff space and $u \longmapsto \lambda(u)$ is a Boolean algebra isomorphism from U onto the Boolean algebra of clopen subsets of $S(U)$. If $U = \mathcal{R}(X)$ there is a perfect irreducible map k: $S(\mathcal{R}(X)) \longrightarrow \beta X$ defined as follows: $k(\alpha) = \bigcap\{A \in \mathcal{R}(X): A \in \alpha\}$ (see [10, Theorem 3.2]) (a closed surjection is *irreducible* if images of proper closed subsets of the domain are proper subsets of the range). The restriction of k to $k^{\leftarrow}[X]$ is also perfect and irreducible; since $S(\mathcal{R}(X))$ is extremally disconnected (i.e. its open sets have open closures) and $k^{\leftarrow}[X]$ is dense in $S(\mathcal{R}(X))$, $k^{\leftarrow}[X]$ is also extremally disconnected and is C^*-embedded in $S(\mathcal{R}(X))$ (see [12, 6M]). We call $S(\mathcal{R}(X))$ the projective cover of βX and denote it by $E(\beta X)$; similarly $k^{\leftarrow}[X]$ is the projective cover of X and is denoted by $E(X)$. The abovementioned C^*-embedding implies that $\beta E(X) = E(\beta X)$, and we shall henceforth make this identification. See [25, §1] for a more detailed discussion of the above matters.

We now return to extension properties.

3.3. LEMMA. *Let $\mathcal{P}$ be an extension property. If $X, Y \in \mathcal{P}$ then the free union $X \dot{\cup} Y$ has $\mathcal{P}$.*

PROOF. Assuming that X contains distinct points x_0 and x_1, and that $y_0 \in Y$, we see that $X \dot{\cup} Y$ is homeomorphic to the closed subspace $\{(x, x_0, y_0): x \in X\} \cup \{(x_0, x_1, y): y \in Y\}$ of $X \times X \times Y$. □

3.4. LEMMA. *Let $\mathcal{P}$ be an extension property. If X is an extremally disconnected member of $\mathcal{A}\mathcal{P}$ then $X \in \mathcal{P}$.*

PROOF. Find $Y \in \mathcal{P}$ and a perfect map f: $Y \longrightarrow X$. By [22, Lemma 5] there is a closed subset F of Y such that $f|F$ is an irreducible perfect map onto X. Since X is extremally disconnected, by [22, Lemma 7] $f|F$ is a homeomorphism of F onto X. As $F \in \mathcal{P}$, it follows that $X \in \mathcal{P}$. □

We can now prove our main result.

3.5. THEOREM. *Let $\mathcal{P}$ be an extension property and X a $\mathcal{P}$-regular space. Then:*

(a) *If A and B are closed subspaces of X and $A, B \in \mathcal{AP}$, then $A \cup B \in \mathcal{AP}$.*

(b) *The family Δ of all members of $\mathcal{R}(X)$ with $\mathcal{AP}$ is an ideal of the Boolean algebra $\mathcal{R}(X)$.*

(c) *The Stone space of the factor algebra $\mathcal{R}(X)/\Delta$ is homeomorphic to $\mathrm{cl}_{\beta E(X)}[\gamma_{\mathcal{P}} E(X) - E(X)]$.*

PROOF. (a) There exist spaces C and D with $\mathcal{P}$, and perfect maps $f\colon C \longrightarrow A$ and $g\colon D \longrightarrow B$. The map $k\colon C \dot{\cup} D \longrightarrow A \cup B$ defined by $k|C = f$, $k|D = g$, is easily verified to be perfect. Since $C \dot{\cup} D \in \mathcal{P}$ by 3.3, it follows that $A \cup B \in \mathcal{AP}$.

(b) Obviously $\varnothing \in \mathcal{AP}$ trivially; if $A, B \in \mathcal{R}(X)$ and A and B have $\mathcal{AP}$, then $A \vee B = A \cup B \in \mathcal{AP}$ by (a). If $A \in \mathcal{R}(X)$, $A \in \mathcal{AP}$, and $B \in \mathcal{R}(X)$ with $B \subseteq A$, then $B \in \mathcal{AP}$ as $\mathcal{AP}$ is closed-hereditary. Thus Δ is an ideal of $\mathcal{R}(X)$.

(c) Let I be an ideal of a Boolean algebra U. According to [20, p. 31], $S(U/I)$ is homeomorphic to $S(U) - \bigcup\{\lambda(u)\colon u \in I\}$, where λ is the isomorphism mentioned previously. Thus

$$S(\mathcal{R}(X)/\Delta) \cong \beta E(X) - \bigcup\{\lambda(A)\colon A \in \Delta\}.$$

Thus we must prove that

$$\bigcup\{\lambda(A)\colon A \in \Delta\} = \beta E(X) - \mathrm{cl}_{\beta E(X)}[\gamma_{\mathcal{P}} E(X) - E(X)]. \tag{1}$$

Let $p \in \beta E(X) - \mathrm{cl}_{\beta E(X)}[\gamma_{\mathcal{P}} E(X) - E(X)]$. As $\{\lambda(B)\colon B \in \mathcal{R}(X)\}$ is a base for the open sets of $\beta E(X)$, we can find $B \in \mathcal{R}(X)$ such that $\mathcal{P} \in \lambda(B)$ and $\lambda(B) \cap [\gamma_{\mathcal{P}} E(X) - E(X)] = \varnothing$. Thus $\lambda(B) \cap E(X) = \lambda(B) \cap \gamma_{\mathcal{P}} E(X)$, so $\lambda(B) \cap E(X) \in \mathcal{P}$ (as $\lambda(B)$ is compact). A straightforward computation (whose details appear in [26, Lemma 2.6]) shows that $k|E(X) \cap \lambda(B)$ is a perfect map from $\lambda(B) \cap E(X)$ onto B. Hence $B \in \mathcal{AP}$ and so $p \in \bigcup\{\lambda(A)\colon A \in \Delta\}$.

Conversely, let $p \in \bigcup\{\lambda(A)\colon A \in \Delta\}$. Find $A \in \mathcal{R}(X)$ such that $A \in \mathcal{AP}$ and $p \in \lambda(A)$. In [26, Lemma 2.6] it is shown that $\lambda(A) \cap E(X)$ is homeomorphic to $E(A)$. Now $E(A)$ is the inverse image of A under a perfect map, so by 1.2(b) since $E(A)$ is $\mathcal{P}$-regular (being zero-dimensional), we have $E(A) \in \mathcal{AP}$. But $E(A)$ is extremally disconnected, so $E(A) \in \mathcal{P}$ by 3.4; thus $\lambda(A) \cap E(X) \in \mathcal{P}$. It follows from 2.5(a) that $\mathrm{cl}_{\gamma_{\mathcal{P}} E(X)}[\lambda(A) \cap E(X)] = \lambda(A) \cap E(X)$; thus $\lambda(A) \cap [\gamma_{\mathcal{P}} E(X) - E(X)] = \varnothing$. Thus $p \notin \mathrm{cl}_{\beta E(X)}[\gamma_{\mathcal{P}} E(X) - E(X)]$ and so $p \in \beta E(X) - \mathrm{cl}_{\beta E(X)}[\gamma_{\mathcal{P}} E(X) - E(X)]$. $\square$

We now examine a second method of generating extension properties.

3.6. THEOREM. *Let $\mathcal{Q}$ be a topological property such that*

(a) *each $\mathcal{Q}$-regular space has a $\mathcal{Q}$-regular compactification,*

(b) *$\mathcal{Q}$-regular continuous images of spaces with $\mathcal{Q}$ have $\mathcal{Q}$.*

Let $\hat{\mathcal{Q}}$ denote the class of $\mathcal{Q}$-regular spaces X whose subspaces with $\mathcal{Q}$ have compact X-closure. Then $\hat{\mathcal{Q}}$ is an extension property.

PROOF. Each $\hat{\mathcal{Q}}$-regular space is $\mathcal{Q}$-regular, and hence has a $\mathcal{Q}$-regular compactification, which has $\hat{\mathcal{Q}}$ and hence is $\hat{\mathcal{Q}}$-regular. If $X \in \hat{\mathcal{Q}}$, A is a closed subset of X, and $S \subseteq A$, then $\mathrm{cl}_X S = \mathrm{cl}_A S$; hence $A \in \hat{\mathcal{Q}}$. If (X_α) is a collection of spaces with $\hat{\mathcal{Q}}$ and if $S \subseteq \Pi_\alpha X_\alpha$ and $S \in \hat{\mathcal{Q}}$, then $p_\alpha[S] \in \mathcal{Q}$ by hypothesis (p_α is the αth projection map). As $X_\alpha \in \hat{\mathcal{Q}}$, $\mathrm{cl}_{X_\alpha} p_\alpha[S]$ is compact. Hence $\Pi_\alpha \mathrm{cl}_{X_\alpha} p_\alpha[S]$ is compact and contains S, so the $\Pi_\alpha X_\alpha$-closure of S is compact. Hence $\Pi_\alpha X_\alpha \in \hat{\mathcal{Q}}$ and so $\hat{\mathcal{Q}}$ is an extension property. □

We have already considered an extension property of this type: if $\mathcal{Q}$ is "being separable", then $\hat{\mathcal{Q}}$ is $\aleph_0$-boundedness. Other properties $\mathcal{Q}$ satisfying the hypotheses of 3.6 are σ-compactness, connectedness, and the Lindelöf property; each of these will give rise to an extension property. The most important example of an extension property of this type is discussed in Theorem 3.9. First we give a definition and state a lemma (whose proof is obvious).

3.7. DEFINITION. Two extension properties $\mathcal{P}$ and $\mathcal{Q}$ are *copseudocompact* if they are coregular and $\mathcal{P}' = \mathcal{Q}'$. The *copseudocompactness class* of an extension property $\mathcal{P}$ is the class of all extension properties that are copseudocompact with $\mathcal{P}$.

3.8. LEMMA. *If $\mathcal{P}$ and $\mathcal{Q}$ are coregular topological properties satisfying the hypotheses of* 3.6, *then $\mathcal{P} \subseteq \mathcal{Q}$ implies $\hat{\mathcal{Q}} \subseteq \hat{\mathcal{P}}$.*

3.9. THEOREM. *Let $\mathcal{P}$ be an extension property. Then $\widehat{\mathcal{P}'}$ is an extension property copseudocompact with $\mathcal{P}$, and $\widehat{\mathcal{P}'}$ contains any other extension property that is copseudocompact with $\mathcal{P}$.*

PROOF. That $\widehat{\mathcal{P}'}$ is an extension property coregular with $\mathcal{P}$ follows from 2.2(d) and 3.6. If $\mathcal{Q}$ is an extension property copseudocompact with $\mathcal{P}$, let $X \in \mathcal{Q}$ and let S be a subset of X with $\mathcal{P}'$. Then $\mathrm{cl}_X S \in \mathcal{P}'$ by 2.2(a) so $\mathrm{cl}_X S \in \mathcal{Q}'$. But $\mathrm{cl}_X S \in \mathcal{Q}$ as $X \in \mathcal{Q}$, so $\mathrm{cl}_X S$ is compact. Thus $X \in \widehat{\mathcal{P}'}$ and so $\mathcal{Q} \subseteq \widehat{\mathcal{P}'}$. In particular $\mathcal{P} \subseteq \widehat{\mathcal{P}'}$, so $(\widehat{\mathcal{P}'})' \subseteq \mathcal{P}'$ by 2.2(e). Conversely if $Y \in \mathcal{P}'$, by 2.2(a) $\gamma_{\hat{\mathcal{P}}'} Y \in \mathcal{P}'$ also. But subsets of $\gamma_{\hat{\mathcal{P}}'} Y$ with $\mathcal{P}'$ have compact $\gamma_{\hat{\mathcal{P}}'} Y$-closure. As Y is such a subset, $\gamma_{\hat{\mathcal{P}}'} Y$ is compact and so $Y \in (\widehat{\mathcal{P}'})'$. Hence $\mathcal{P}$ and $\widehat{\mathcal{P}'}$ are copseudocompact. □

Theorem 3.9 tells us that every copseudocompactness class of extension properties has a largest member; if $\mathcal{P}$ is in the class, then $\widehat{\mathcal{P}'}$ is this largest member. On the other hand, if $\mathcal{Q}$ is a topological property satisfying the hypotheses of 3.6, then $\hat{\mathcal{Q}}$ is the largest member of its copseudocompactness class, as we see below.

3.10. PROPOSITION. *Let $\mathcal{Q}$ be a topological property satisfying the hypotheses of* 3.6. *Then*

(a) $\widehat{(\hat{\mathcal{Q}})'} = \hat{\mathcal{Q}}$,

(b) $\mathcal{Q} = \mathcal{P}'$ *for some extension property* $\mathcal{P}$ *iff* $(\hat{\mathcal{Q}})' = \mathcal{Q}$.

PROOF. (a) By 3.9 $\hat{\mathcal{Q}} \subseteq \widehat{(\hat{\mathcal{Q}})'}$. Also $\mathcal{Q} \subseteq (\hat{\mathcal{Q}})'$; for if $X \in \mathcal{Q}$ and $X \subseteq T \subseteq \beta_{\mathcal{Q}}X$, then subsets of T with $\mathcal{Q}$ have compact T-closure only if $T = \beta_{\mathcal{Q}}X$, and so $X \in (\hat{\mathcal{Q}})'$. Thus $\widehat{(\hat{\mathcal{Q}})'} \subseteq \hat{\mathcal{Q}}$ by 3.8.

(b) If $\mathcal{Q} = \mathcal{P}'$ for an extension property $\mathcal{P}$ then $(\hat{\mathcal{Q}})' = \widehat{(\mathcal{P}')}' = \mathcal{P}' = \mathcal{Q}$ by 3.9. The converse is obvious. □

It is worth noting that if $\mathcal{Q}$ satisfies the hypotheses of 3.6 together with properties (a) and (b) of Proposition 2.2 (with "$\mathcal{P}$-pseudocompactness" replaced by "$\mathcal{Q}$"), it does not necessarily follow that $\mathcal{Q} = (\hat{\mathcal{Q}})'$. For example, if $\mathcal{Q}$ has separability then $\mathcal{Q} \neq (\hat{\mathcal{Q}})'$ because not every compact space is separable.

We briefly consider the copseudocompactness class of realcompactness.

3.11. EXAMPLE. If $\mathcal{P}$ is the class of realcompact Tychonoff spaces, then $\widehat{\mathcal{P}'}$ is the class of Tychonoff spaces whose pseudocompact closed subsets are compact. Recall [12, Problem 4J] that a Tychonoff space is a *P-space* if all its zero-sets are clopen. In [12, Problem 4K] it is shown that each pseudocompact P-space is compact and each subspace of a P-space is a P-space. Hence $\widehat{\mathcal{P}'}$ contains all P-spaces. Thus $\widehat{\mathcal{P}'}$ properly contains the class of almost realcompact spaces (an example of a P-space that is not almost realcompact is given in [12, Problem 9L]). Hence realcompactness, almost realcompactness, and $\widehat{\mathcal{P}'}$ are three distinct copseudocompact extension properties. □

If $\mathcal{P}$ is an extension property that is preserved by continuous maps, then $\hat{\mathcal{P}}$ has several special properties.

3.12. THEOREM. *Let $\mathcal{P}$ be an extension property such that if $X \in \mathcal{P}$ and Y is a continuous $\mathcal{P}$-regular image of X, then $Y \in \mathcal{P}$. Then*:

(a) *Either $\mathcal{P}$ contains all $\mathcal{P}$-regular spaces of nonmeasurable cardinal or else each space having $\mathcal{P}$ is countably compact.*

(b) *If each space having $\mathcal{P}$ is countably compact then each $\mathcal{P}$-regular space whose pseudocompact closed subsets are compact has $\hat{\mathcal{P}}$; hence $(\hat{\mathcal{P}})'$ is contained in the class of pseudocompact $\mathcal{P}$-regular spaces.*

(c) $A(\hat{\mathcal{P}}) = \hat{\mathcal{P}}$.

PROOF. (a) By 2.9 either each space having $\mathcal{P}$ is countably compact or else $\mathbf{N} \in \mathcal{P}$; then each discrete space of nonmeasurable cardinal, being $\mathbf{N}$-compact, has $\mathcal{P}$. As each Tychonoff space is a continuous image of a discrete space of the same cardinality, the result follows.

(b) Let $\mathcal{Q}$ be the class of $\mathcal{P}$-regular spaces whose pseudocompact closed subsets are compact. Then $\mathcal{Q}$ is an extension property coregular with $\mathcal{P}$. If $X \in \mathcal{Q}$ and $S \subseteq X$ such that $S \in \mathcal{P}$, then S is pseudocompact so $\mathrm{cl}_X S$ is compact. Thus $X \in \hat{\mathcal{P}}$, and $\mathcal{Q} \subseteq \hat{\mathcal{P}}$. By 2.2(e) $(\hat{\mathcal{P}})' \subseteq \mathcal{Q}'$ and by 3.9 $\mathcal{Q}'$ is the class of pseudocompact $\mathcal{P}$-regular spaces.

(c) Let $X \in \mathcal{P}$ and let k be a perfect map from X onto the $\mathcal{P}$-regular space Y. Let $S \subseteq Y$ and $S \in \mathcal{P}$. By 1.2(b) $k^{\leftarrow}[S] \in \mathcal{P}$; thus $\mathrm{cl}_X k^{\leftarrow}[S]$ is compact. Thus $k[\mathrm{cl}_X k^{\leftarrow}[S]]$ is compact, and $k[\mathrm{cl}_X k^{\leftarrow}[S]] = \mathrm{cl}_Y S$ as k is a closed map. Hence $Y \in \hat{\mathcal{P}}$, and so $\mathcal{A}(\hat{\mathcal{P}}) = \hat{\mathcal{P}}$. □

We briefly mention another way of constructing new extension properties from old. Since extension properties are classes rather than sets, and since the class of all extension properties is not a set, we cannot ask if the class of all extension properties forms a lattice. However, it is evident that when "partially ordered by inclusion", this family behaves like a complete lattice.

3.13. PROPOSITION. *Let $(\mathcal{P}_\alpha)_{\alpha\in\Sigma}$ be a collection of coregular extension properties (indexed by a set Σ). Define $\bigwedge\{\mathcal{P}_\alpha\colon \alpha \in \Sigma\}$ to be the class of spaces belonging to each $\mathcal{P}_\alpha$, and $\bigvee\{\mathcal{P}_\alpha\colon \alpha \in \Sigma\}$ to be the class of spaces homeomorphic to a closed subspace of a product of the form $\Pi\{X_\alpha\colon \alpha \in \Sigma\}$, where $X_\alpha \in \mathcal{P}_\alpha$ for each $\alpha \in \Sigma$. Then $\bigwedge\{\mathcal{P}_\alpha\colon \alpha \in \Sigma\} \subseteq \mathcal{P}_\gamma \subseteq \bigvee\{\mathcal{P}_\alpha\colon \alpha \in \Sigma\}$ for each $\gamma \in \Sigma$, and if $\mathcal{Q}$ and $\mathcal{R}$ are extension properties (coregular with each $\mathcal{P}_\alpha$) such that $\mathcal{Q} \subseteq \mathcal{P}_\alpha \subseteq \mathcal{R}$ for each $\alpha \in \Sigma$, then $\mathcal{Q} \subseteq \bigwedge\{\mathcal{P}_\alpha\colon \alpha \in \Sigma\}$ and $\bigvee\{\mathcal{P}_\alpha\colon \alpha \in \Sigma\} \subseteq \mathcal{R}$.*

The proof of 3.13 is immediate and is not included.

4. Extension properties contained in countable compactness. In this section we consider the first of the two classes of extension properties described in 2.9(a). We begin with a discussion of $\mathcal{D}$-compactness, a concept defined by A. R. Bernstein [2]. Although Bernstein defines $\mathcal{D}$-compactness for arbitrary topological spaces, we shall, as before, consider only Tychonoff spaces. Throughout §4 the symbol $\mathcal{D}$ will denote a free ultrafilter on the positive integers.

4.1. DEFINITION. Let (x_n) be a sequence of points of a space X. A point $p \in X$ is a *$\mathcal{D}$-limit point* of (x_n) if, given any neighborhood V of p, the set $\{n \in N\colon x_n \in V\}$ belongs to $\mathcal{D}$.

A space is called *$\mathcal{D}$-compact* if each sequence of points of the space has a $\mathcal{D}$-limit point.

Bernstein proves the following facts about $\mathcal{D}$-compactness:

4.2. PROPOSITION (BERNSTEIN). *Let X be a space. Then:*

(a) *Each sequence in X has at most one $\mathcal{D}$-limit point; if X is compact each sequence has exactly one $\mathcal{D}$-limit point.*

(b) *Each $\mathcal{D}$-compact space is countably compact.*

(c) *X is $\aleph_0$-bounded iff X is $\mathcal{E}$-compact for every free ultrafilter $\mathcal{E}$ on N (see* [2, 3.3, 3.4, *and* 3.5]).

(d) *$\mathcal{D}$-compactness is productive.*

We are interested in $\mathcal{D}$-compactness because, roughly speaking, it is the largest extension property contained in countable compactness. We make this more precise in 4.3 and 4.5 below. Part of 4.3 has been obtained independently by V. Saks; see [9] for details.

4.3. THEOREM. *Let $\mathcal{P}$ be an extension property and let $[\mathcal{D}]$ denote the class of $\mathcal{P}$-regular $\mathcal{D}$-compact spaces. Then*:

(a) *$[\mathcal{D}]$ is an extension property contained in countable compactness.*

(b) *If X is $\mathcal{P}$-regular, define $\gamma_0 X = X$. For each countable ordinal α, assuming that $\gamma_\eta X$ has been defined for each ordinal η less than α, put $\gamma_\alpha X = \{p \in \beta_p X$: p is a $\mathcal{D}$-limit point in $\beta_p X$ of some sequence in $\bigcup\{\gamma_\eta X: \eta < \alpha\}$. Then the maximal $\mathcal{D}$-compact extension $\gamma_{[\mathcal{D}]} X$ of X is just $\bigcup\{\gamma_\alpha X: \alpha < \omega_1\}$, where ω_1 is the first uncountable ordinal.*

PROOF. (a) Obviously $\mathcal{D}$-compactness is closed-hereditary, and 4.2(a) implies that each compact space is $\mathcal{D}$-compact. As $[\mathcal{D}]$ is productive by 4.2(d), it follows that $[\mathcal{D}]$ is an extension property; $[\mathcal{D}]$ is contained in countable compactness by 4.2(b).

(b) Let (x_n) be a sequence of points in $\bigcup\{\gamma_\alpha X: \alpha < \omega_1\}$. There exists a countable ordinal α_n such that $x_n \in \gamma_{\alpha_n} X$ for each n; find a countable ordinal α_0 such that $\alpha_n < \alpha_0$ for each n. Then $\gamma_{\alpha_0} X$ contains a $\mathcal{D}$-limit point (in $\beta_p X$) of (x_n) (see 4.2(a)). Hence $\bigcup\{\gamma_\alpha X: \alpha < \omega_1\}$ is $\mathcal{D}$-compact.

Now let Y be a $\mathcal{P}$-regular $\mathcal{D}$-compact space and let $f: X \longrightarrow Y$ be continuous. Then f extends continuously to $\bar{f}: \beta_p X \longrightarrow \beta_p Y$, and it suffices to show that $\bar{f}[\gamma_\alpha X] \subseteq Y$ for each $\alpha < \omega_1$ in order to show that $\bigcup\{\gamma_\alpha X: \alpha < \omega_1\} = \gamma_{[\mathcal{D}]} X$. Obviously $\bar{f}[\gamma_0 X] \subseteq Y$. Suppose that $\alpha < \omega_1$ and that $\bar{f}[\gamma_\eta X] \subseteq Y$ for each $\eta < \alpha$. If $p \in \gamma_\alpha X - \bigcup\{\gamma_\eta X: \eta < \alpha\}$, there is a sequence $(x_n) \subseteq \{\gamma_\eta X: \eta < \alpha\}$ such that p is a $\mathcal{D}$-limit point (in $\beta_p X$) of (x_n). As $\bar{f}$ is continuous, it is easy to check that $\bar{f}(p)$ is a $\mathcal{D}$-limit point (in $\beta_p Y$) of (y_n), where $y_n = \bar{f}(x_n)$ for each n. But $(y_n) \subseteq Y$ by assumption, and hence (y_n) has a $\mathcal{D}$-limit point in Y. By 4.2(a) this means that $\bar{f}(p) \in Y$, and the proof is finished. □

The following theorem is due to Ginsburg and Saks [9].

4.4. THEOREM. *Let X be a T_1 topological space. The following are equivalent*:

(a) *X^m is countably compact for each cardinal m.*

(b) *X^{2^c} is countably compact.*

(c) *There exists a free ultrafilter $\mathcal{E}$ on N such that X is $\mathcal{E}$-compact.*

An immediate consequence is

4.5. THEOREM. *Let $\mathcal{P}$ be an extension property contained in countable compactness. Then*:

(a) *There exists a free ultrafilter $\mathcal{E}$ on N such that $\mathcal{P} \subseteq [\mathcal{E}]$, where $[\mathcal{E}]$ is the class of $\mathcal{P}$-regular $\mathcal{E}$-compact spaces.*

(b) *The class of $\mathcal{P}$-regular $\aleph_0$-bounded spaces is the "intersection" as $\mathcal{D}$ varies over the set of free ultrafilters on N, of the classes of spaces with $[\mathcal{D}]$.*

PROOF. (a) Suppose not; then for each free ultrafilter $\mathcal{U}$ on N, there is a space $X(\mathcal{U}) \in \mathcal{P}$ such that $X(\mathcal{U})$ is not $\mathcal{U}$-compact. The product $X = \Pi\{X(\mathcal{U})$: $\mathcal{U}$ is a free ultrafilter on $N\}$ has $\mathcal{P}$, but is not $\mathcal{U}$-compact for any $\mathcal{U}$ as $\mathcal{U}$-compactness is closed-hereditary. Thus by 4.4 X^{2^c} is not countably compact, although it has $\mathcal{P}$. This contradiction shows that (a) holds.

(b) This follows from 4.2(c). □

Theorem 4.4 says that for each free ultrafilter $\mathcal{E}$ on N, $[\mathcal{E}]$ is a maximal element in the "lattice" of extension properties coregular with $\mathcal{P}$ and contained in countable compactness, and the class of $\aleph_0$-bounded $\mathcal{P}$-regular spaces is the "inf" of all such maximal elements.

4.6. APPLICATION. By 2.3 $\gamma_{[\mathcal{D}]}X - \upsilon X$ is dense in $\beta X - \upsilon X$ where $[\mathcal{D}]$ here is the class of $\mathcal{D}$-compact Tychonoff spaces. This should be compared with Example 2.4. More generally, if $\mathcal{P}$ is an extension property such that $\mathcal{P}'$ is the class of $\mathcal{P}$-regular pseudocompact spaces (see 3.11 for examples), then $\gamma_{[\mathcal{D}]}X - \gamma_{\mathcal{P}}X$ is dense in $\beta_{\mathcal{P}}X - \gamma_{\mathcal{P}}X$ (where $[\mathcal{D}]$ here is the class of $\mathcal{P}$-regular $\mathcal{D}$-compact spaces). □

We next show that $\aleph_0$-boundedness, and a number of other extension properties contained in countable compactness, are not E-compactness for any topological space E. We modify a proof used by Blefko [3, Theorem 3].

As usual, for each ordinal α let ω_α be the smallest ordinal of cardinality $\aleph_\alpha$. Let $W(\omega_\gamma)$ denote, for a given ordinal γ, the set of all ordinals less than ω_γ, equipped with the order topology. Let $\mathcal{W}(\omega_\alpha)$ denote the class of ordinal spaces $W(\omega_\gamma)$, where γ is any nonlimit ordinal not less than α.

4.7. THEOREM. *Let $\alpha > 0$. If $\mathcal{P}$ is an extension property containing the class $\mathcal{W}(\omega_\alpha)$ then $\mathcal{P}$ is not E-compactness for any space E.*

PROOF. We will show that if we are given an ordinal $\alpha > 0$ and a topological space E, then there is a space in the class $\mathcal{W}(\omega_\alpha)$ that is not E-compact; this will suffice to prove the theorem.

Find a nonlimit ordinal γ such that $\gamma \geqslant \alpha$ and $|E| < \aleph_\gamma$. Then $W(\omega_\gamma) \in \mathcal{W}(\omega_\alpha)$ and it suffices to show that $W(\omega_\gamma)$ is not E-compact. To show this it is enough to show that if f: $W(\omega_\gamma) \longrightarrow E$ is continuous, then f extends continuously

to f^{γ}: $W(\omega_\gamma + 1) \longrightarrow E$ (see [16, Theorem 4.10]). (Note that $W(\omega_\gamma + 1)$ is a compact zero-dimensional space and hence is E-compact; also $W(\omega_\gamma + 1)$ contains $W(\omega_\gamma)$ is a dense subspace.)

Suppose that $f^{\leftarrow}(x)$ is bounded in $W(\omega_\gamma)$ for each $x \in E$; say $\eta(x) = \sup f^{\leftarrow}(x)$. As no subset of $W(\omega_\gamma)$ of cardinality less than $\aleph_\gamma$ is cofinal (see [12, 9K]), it follows that $\sup\{\eta(x)\colon x \in E\}$ exists in $W(\omega_\gamma)$; call it η. Then $\eta + 1 \in W(\omega_\gamma)$ but $f(\eta + 1)$ cannot be defined, which is absurd.

Here there exists $x_0 \in E$ such that $f^{\leftarrow}(x_0)$ is unbounded in $W(\omega_\gamma)$. Furthermore x_0 is unique, for if $x_1 \in E$, $x_1 \neq x_0$, and if $f^{\leftarrow}(x_1)$ were unbounded, then $f^{\leftarrow}(x_0)$ and $f^{\leftarrow}(x_1)$ would be disjoint closed unbounded subsets of $W(\omega_\gamma)$, which cannot happen (see [12, 9K.2]). As in the preceding paragraph we see that $\{\eta(x)\colon x \in E - \{x_0\}\}$ is bounded above in $W(\omega_\gamma)$ by μ, say; then $\theta \geqslant \mu$ implies that $f(\theta) = x_0$. Evidently f can be continuously extended to $W(\omega_\gamma + 1)$ by putting $\bar{f}(\omega_\gamma) = x_0$. □

4.8. Corollary. *If $\mathcal{P}$ is an extension property, then the class of $\aleph_0$-bounded $\mathcal{P}$-regular spaces, and the class of $\mathcal{D}$-compact $\mathcal{P}$-regular spaces, are not the class of E-compact spaces for any space E.*

Proof. If α is a nonlimit ordinal then $W(\omega_\alpha)$ is $\aleph_0$-bounded. □

We next show that, unlike $\aleph_0$-boundedness, the class $[\mathcal{D}]$ of $\mathcal{D}$-compact spaces is not the largest extension property in its copseudocompactness class; i.e. we show that $[\mathcal{D}] \neq \widehat{[\mathcal{D}]'}$.

4.10. Proposition. *The space* $\mathbf{R}$ *of real numbers has* $\widehat{[\mathcal{D}]'}$. *Hence* $\widehat{[\mathcal{D}]'}$ *contains realcompactness.*

Proof. We must show that if $A \subseteq \mathbf{R}$ and $A \in [\mathcal{D}]'$, then A has compact $\mathbf{R}$-closure. If A itself is not compact, then A is a realcompact, noncompact space. By [12, Theorem 9.11] $|\beta A - A| \geqslant 2^c$. Hence if $A \in [\mathcal{D}]'$, then $|\gamma_{[\mathcal{D}]}A - A| \geqslant 2^c$. But A contains only c sequences, so an examination of the construction of $\gamma_{[\mathcal{D}]}X$ given in 4.3 reveals that $|\gamma_{[\mathcal{D}]}A - A| \leqslant c|A| = c$. This contradiction shows that if $A \in [\mathcal{D}]'$, then A is compact. Hence $\mathbf{R} \in \widehat{[\mathcal{D}]'}$, and so each realcompact space has $\widehat{[\mathcal{D}]'}$. □

Since $\widehat{[\mathcal{D}]'}$ contains both the realcompact and the $\mathcal{D}$-compact spaces, it is evidently a "large" class of spaces. Therefore we note in passing that the "Tychonoff plank" does not belong to $\widehat{[\mathcal{D}]'}$ (see [12, 8.20] for a detailed discussion of the Tychonoff plank).

We conclude this section by examining the pseudocompactness classes $(\hat{\mathcal{P}})'$ and $(\widehat{[\mathcal{D}]})'$, where $\mathcal{P}$ is the class of $\aleph_0$-bounded zero-dimensional spaces.

4.11. THEOREM. *Let $\mathcal{S}$ denote the class of pseudocompact zero-dimensional spaces. Then $\mathcal{P} \subset (\hat{\mathcal{P}})' \subset (\widehat{[\mathcal{D}]})' \subset \mathcal{S}$, and each of the inclusions is proper.*

PROOF. In 3.10 it is shown that $\mathcal{P} \subseteq (\hat{\mathcal{P}})'$. Since $\mathcal{P} \subseteq [\mathcal{D}] \subseteq \mathcal{S}$, it follows from 2.2(e) and 3.8 that $(\hat{\mathcal{P}})' \subset (\widehat{[\mathcal{D}]})' \subset \mathcal{S}$ (note that $(\hat{\mathcal{S}})' = \mathcal{S}$ by 3.10(b)). It remains to show that the inclusions are proper.

First we show that the Tychonoff plank T belongs to $(\hat{\mathcal{P}})' - \mathcal{P}$. Evidently $T \notin \mathcal{P}$ as its "right edge" $\{\omega_1\} \times \mathbf{N}$ is a countable noncompact closed subspace of T. Since $|\beta T - T| = 1$, either $T \in \hat{\mathcal{P}}$ or $T \in (\hat{\mathcal{P}})'$. But $W \times \{\omega_0\}$ is an $\aleph_0$-bounded subset of T whose T-closure is noncompact; thus $T \notin \hat{\mathcal{P}}$. Hence $T \in (\hat{\mathcal{P}})' - \mathcal{P}$ (see [12, 8.20] for our notation and terminology concerning T).

Next we show that $\gamma_{[\mathcal{D}]}\mathbf{N} \in (\widehat{[\mathcal{D}]})' - (\hat{\mathcal{P}})'$. Since $\gamma_{[\mathcal{D}]}\mathbf{N} \in [\mathcal{D}]$, it follows from 3.10 that $\gamma_{[\mathcal{D}]}\mathbf{N} \in (\widehat{[\mathcal{D}]})'$. To show that $\gamma_{[\mathcal{D}]}\mathbf{N} \notin (\hat{\mathcal{P}})'$ it suffices, since $\gamma_{[\mathcal{D}]}\mathbf{N}$ is not compact, to show that $\gamma_{[\mathcal{D}]}\mathbf{N} \in \hat{\mathcal{P}}$ (see 2.2(c)). To prove this latter assertion, it suffices to show that each $\aleph_0$-bounded subset of $\gamma_{[\mathcal{D}]}\mathbf{N}$ is compact. If $D = (d_n)$ is a countably infinite subset of $\gamma_{[\mathcal{D}]}\mathbf{N}$, it is C^*-embedded in $\beta\mathbf{N}$ (see [12, 60.6]), so $\mathrm{cl}_{\beta\mathbf{N}} D$ is homeomorphic to βD (see [12, 6.9]). As $\beta\mathbf{N}$ contains no infinite closed subsets of cardinality less than 2^c (see [12, 9.12]), D cannot be pseudocompact. Hence βD has cardinality at least 2^c [12, 9.12] and so $\mathrm{cl}_{\gamma_{[\mathcal{D}]}\mathbf{N}} D$ cannot be compact as $|\mathrm{cl}_{\beta\mathbf{N}} D| = 2^c$ and $|\gamma_{[\mathcal{D}]}\mathbf{N}| = c$. Thus $\gamma_{[\mathcal{D}]}\mathbf{N} \in \hat{\mathcal{P}}$ as claimed.

Finally we show that the space Ψ described in [12, Problem 5I] lies in $\mathcal{S} - (\widehat{[\mathcal{D}]})'$. Recall that to construct Ψ, one finds an infinite maximal family $\mathcal{E}$ of infinite subsets of $\mathbf{N}$ such that the intersection of any two is finite. Let $D = \{\omega(E) : E \in \mathcal{E}\}$ be a new set of distinct points and put $\Psi = \mathbf{N} \cup D$, topologized as follows: each point of $\mathbf{N}$ is isolated in Ψ, and a subset A of Ψ is a neighborhood of $\omega(E)$ iff $E - A$ is finite. It is indicated in [12, Problem 5I], that Ψ is a pseudocompact zero-dimensional completely regular Hausdorff space that is not countably compact. Hence to show that $\Psi \notin (\widehat{[\mathcal{D}]})'$ it suffices to show that each countably compact subset of Ψ is finite. Since D is an infinite discrete closed subset of Ψ, any infinite countably compact subspace of Ψ contains only finitely many points of D, and hence is of the form $S = N_0 \cup \{\omega(E_i) : i = 1 \text{ to } k\}$, where N_0 is an infinite subset of $\mathbf{N}$. Now $N_0 - \bigcup_{i=1}^k E_i = A$ is a closed discrete subspace of S and hence is finite if S is countably compact. In this case $S = [\bigcup_{i=1}^k E_i \cup \{\omega(E_i)\}] \cup A$, which is a finite union of compact spaces and thus is compact. Our result follows. □

5. Some unanswered questions. Many interesting questions about the classification of extension properties, and the properties of $\mathcal{P}$-pseudocompact spaces, remain unanswered. We list some of these below.

5.1. If $\mathcal{P}$ is an extension property, is the product of a space in $\mathcal{P}'$ with a compact $\mathcal{P}$-regular space necessarily in $\mathcal{P}'$? This is true if $\mathcal{P}'$ is pseudocompactness (see [12, 9.14]).

5.2. If $\mathcal{P}$ is an extension property and if $X \in \mathcal{P}'$, is the projective cover $E(X)$ in $\mathcal{P}'$? This is true if $\mathcal{P}'$ is pseudocompactness (see [26, 2.5]).

5.3. What is the characterization of those topological properties $\mathcal{Q}$ that are $\mathcal{P}$-pseudocompactness for some extension property $\mathcal{P}$? Evidently $\mathcal{Q}$ must satisfy the properties of $\mathcal{P}$-pseudocompactness discussed in 2.2; if it does, then $\mathcal{Q} = \mathcal{P}'$ iff $\mathcal{Q} = (\hat{\mathcal{Q}})'$ (see 3.10). When does this happen?

5.4. What is the extension property "generated" by the class of P-spaces (see 3.11)? In other words, what zero-dimensional spaces are homeomorphic to a closed subspace of a product of P-spaces? According to 3.11, this extension property is copseudocompact with N-compactness.

5.5. Let $\mathcal{W}(\omega_\alpha)$ denote the class of all ordinal spaces $W(\omega_\gamma)$, where γ is any nonlimit ordinal not less than α. If $W(\omega_\gamma) \in \mathcal{W}(\omega_{\alpha+1})$ then by [12, 9K.1] no subset of $W(\omega_\gamma)$ of cardinality less than or equal to $\aleph_\alpha$ is cofinal in $W(\omega_\gamma)$. Thus $W(\omega_\gamma)$ is $\aleph_\alpha$-bounded and zero-dimensional. Our question is whether the extension property generated by $\mathcal{W}(\omega_{\alpha+1})$ is precisely the property of being $\aleph_\alpha$-bounded and zero-dimensional. In other words, is a zero-dimensional space $\aleph_\alpha$-bounded iff it is homeomorphic to a closed subspace of a product of spaces in the class $\mathcal{W}(\omega_{\alpha+1})$?

5.6. Let $\mathcal{P}$ be an extension property. How can the points of $\beta_{\mathcal{P}}X - X$ that lie in $\gamma_{\hat{\mathcal{P}}}X$ be characterized? In particular, what points of $\beta X - X$ lie in $\gamma_{\hat{\mathcal{P}}}X$ if $\mathcal{P}$ is realcompactness?

ADDED IN PROOF. S. Broverman has shown that the answer to 5.1, 5.2, and 5.5 is "no".

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