ABSTRACT. If G is a group of automorphisms of a C*-algebra A with identity, then G acts in a natural way as a transformation group on the state space S(A) of A. Moreover, this action is uniformly almost periodic if and only if G has compact pointwise closure in the space of all maps of A into A. Consideration of the enveloping semigroup of (S(A), G) shows that, in this case, this pointwise closure G is a compact topological group consisting of automorphisms of A. The Haar measure on G is used to define an analogue of the canonical center-valued trace on a finite von Neumann algebra. If A possesses a sufficiently large group G₀ of inner automorphisms such that (S(A), G₀) is uniformly almost periodic, then A is a central C*-algebra. The notion of a uniquely ergodic system is applied to give necessary and sufficient conditions that an approximately finite dimensional C*-algebra possess exactly one finite trace.

Introduction. The purpose of this paper is to apply some ideas from topological dynamics to the study of C*-algebras. If X is a compact Hausdorff space and (X, Γ) is a topological transformation group, then Γ has a natural representation as a group of automorphisms of the commutative C*-algebra C(X): for t ∈ Γ and f ∈ C(X) put

\((tf)(x) = f(xt), \quad x \in X.\)

It is often possible to express properties of (X, Γ) in terms of the system (Γ, C(X)); for example, (X, Γ) is uniformly almost periodic iff for each f ∈ C(X), the set \(\{tf: t \in \Gamma\}\) is relatively compact in C(X). If A is an arbitrary C*-algebra with identity and G is a group of automorphisms of A, we may view the pair (G, A) as a noncommutative version of (Γ, C(X)). We shall see that some of the relationships between (X, Γ) and (Γ, C(X)) have noncommutative analogues, and that these analogues can be used to obtain information about the structure of certain C*-algebras.
1. Preliminaries. We shall generally follow the terminology of [6] for topological dynamics, that of [5] for C*-algebras, and that of [12] for uniform spaces and topologies on function spaces. We shall however translate Dixmier’s “morphism” by “*-homomorphism”, and we define “trace” below.

**Definition 1.1.** Let $G$ be a semigroup with identity $e$, and let $X$ be a set. A right action of $G$ on $X$ is a mapping 

$$
\pi: X \times G \to X: (x, \alpha) \mapsto x\alpha = \pi(x, \alpha)
$$

such that

1. $xe = x$ for all $x \in X$, and
2. $(x\alpha)\beta = x(\alpha\beta)$ for all $x \in X$ and all $\alpha, \beta \in G$.

When there is no danger of confusion, we shall write $x\alpha$ for $\pi(x, \alpha)$.

A left action of $G$ on $X$ is a mapping $(\alpha, x) \mapsto \alpha x$ of $G \times X$ into $X$ such that $ex = x$ and $\alpha(\beta x) = (\alpha\beta)x$ for all $x \in X$ and all $\alpha, \beta \in G$. We make the convention that the term “action” will mean “right action” unless we specify otherwise. An action is continuous if it is continuous from the product topology on $X \times G$.

**Remark 1.2.** If $X$ and $Y$ are sets, we shall write $Y^X$ for the set of all mappings of $X$ into $Y$. When $Y = X$, we may use composition of mappings to provide $X^X$ with two natural semigroup structures: Let $(p, q)$ be an ordered pair of elements of $X^X$. If we write our mappings on the right, we shall define $pq$ by 

$$
x(pq) = (xp)q, \quad x \in X.
$$

If we write our mappings on the left, we shall compose in the opposite order:

$$
(pq)x = p(qx), \quad x \in X.
$$

These definitions give actions of $X^X$ on $X$ on the right and left respectively. We shall find it convenient to write maps of C*-algebras on the left and maps of their state spaces on the right.

**Remark 1.3.** Let $A$ be a C*-algebra with identity. We write $S(A)$ for its state space, and we give $S(A)$ the weak* topology. Then the set $S(A)$ is convex, and the topological space $S(A)$ is compact and Hausdorff, hence is a uniform space in a unique way. We note that the uniformity on $S(A)$ is determined by the family of all pseudo-norms of the form $(p, q) \mapsto |p(a) - q(a)|$, where $a$ is a positive element of $A$. It follows that a net $\rho_{\gamma}$ in $S(A)^{S(A)}$ converges to $\rho \in S(A)^{S(A)}$ in the topology of uniform convergence iff for each positive $a \in A$ we have

$$
\sup_{p \in S(A)} |(p\rho_{\gamma})(a) - (pp)(a)| \to 0
$$

[12, pp. 226--227]. We note also that as $S(A)$ is compact, a family of maps in $S(A)^{S(A)}$ is equicontinuous iff it is uniformly equicontinuous.
We write $ES(A)$ for the set of pure states of $A$ with the weak* topology.

**Remark 1.4.** Let $(X, G)$ be a transformation group with compact Hausdorff phase space $X$. For each $t \in G$, let $\pi^t$ denote the map $x \mapsto xt$, $x \in X$. The pointwise closure in $X^X$ of the set $\{\pi^t: t \in G\}$ is a semigroup, called the enveloping semigroup of $(X, G)$ [6, 3.2]. The following are equivalent:

1. $(X, G)$ is uniformly almost periodic.
2. $\{\pi^t: t \in G\}$ is an equicontinuous family.
3. The enveloping semigroup of $(X, G)$ is a group of continuous maps.
4. If $f \in C(X)$, then $f$ is almost periodic, i.e. $\{f \circ \pi^t: t \in G\}$ has compact closure in $C(X)$ [6, 4.4 and 4.15].

(The proof given in [6, 4.15] for real functions applies equally well to $C(X)$.)

Let $A$ be a C*-algebra with identity. If $p \in S(A)$, we write $L^p$ for the representation of $A$ obtained by applying the Gelfand-Naimark-Segal construction to $p$, and we say that $L^p$ is associated to $p$. The *left kernel* of $p$ is the left ideal $\{a \in A: p(a^*a) = 0\}$. A state $\tau$ of $A$ is a *trace* on $A$ if $\tau$ is invariant under the inner automorphisms of $A$, i.e. $\tau(a) = \tau(uau^*)$ for all $a \in A$ and all unitary $u \in A$. Since every element of $A$ is a linear combination of unitaries, a state $\tau$ of $A$ is a trace iff $\tau(ab) = \tau(ba)$ for all $a, b \in A$. We denote the set of all traces on $A$ by $T(A)$, and we write $ET(A)$ for the set of extremal traces of $A$, i.e. extreme points of $T(A)$. A trace $\tau$ is extremal iff $L^\tau$ is a factor representation [5, 6.7.3 and 6.8.5].

A *face* of a compact convex set $K$ is a convex subset $F$ of $K$ such that if $p, q \in K$ and $\frac{1}{2}p + \frac{1}{2}q \in F$, then $p$ and $q$ are in $F$. An extreme point of a face of $K$ is also an extreme point of $K$, and the inverse image of an extreme point under an affine map is a face.

If $A$ is a C*-algebra, we denote by $\text{Max}(A)$ the space of all maximal ideals of $A$ equipped with the relative topology from the Jacobson topology on $\text{Prim}(A)$ [5, 3.1.1]. We write $ZA$ for the center of $A$. Suppose $A$ has an identity. Then there is a mapping $p$ of $\text{Prim}(A)$ onto $\text{Max}(ZA)$ given by $p: P \mapsto P \cap ZA$. This mapping is continuous, and since $\text{Prim}(A)$ is compact and $\text{Max}(ZA)$ is Hausdorff, it is also closed. If $p$ is one-to-one (i.e. a homeomorphism), then $A$ is said to be a *central C*-algebra [1].

**Remark 1.5.** Let $A$ be a C*-algebra with identity $I$, and let $p$ be a state of $A$ such that $L^p$ is a factor representation (e.g. a pure state or an extremal trace). We identify the center of $L^p(A)'$ with $C$. Then $p$ coincides with $L^p$ on $ZA$, hence is multiplicative on $ZA$. Suppose moreover that $\ker L^p$ is a primitive ideal. Then the character of $ZA$ which corresponds to $\ker L^p \cap ZA$ is $p|_{ZA}$. For if the value of this character on $z$ is $\lambda$, then $\lambda I \in \ker L^p \cap ZA \subseteq \ker L^p$, so $p(z - \lambda I) = L^p(z - \lambda I) = 0$, and $p(z) = \lambda$. 

2. Uniformly almost periodic groups of automorphisms. In this section \( A \) will denote a C*-algebra with identity \( I \). An automorphism of \( A \) is an invertible \(*\)-homomorphism of \( A \) onto \( A \), and we write \( \text{Aut}(A) \) for the group of all automorphisms of \( A \). We shall characterize those subgroups of \( \text{Aut}(A) \) which act uniformly almost periodically on \( S(A) \).

Let \( G \) be a subgroup of \( \text{Aut}(A) \). We say that an element \( a \) of \( A \) is \( G \)-invariant if \( \alpha(a) = a \) for all \( \alpha \in G \). A state \( p \) of \( A \) is \( G \)-invariant if \( p \circ \alpha = p \) for all \( \alpha \in G \). We denote the algebra of all \( G \)-invariant elements of \( A \) by \( Z_G A \) and the set of all \( G \)-invariant states of \( A \) by \( S_G(A) \). Then \( Z_G A \) is a C*-subalgebra of \( A \), and \( S_G(A) \) is a compact convex subspace of \( S(A) \).

Let \( A^A \) have the pointwise (product) topology. Then the set \( \text{Aut}(A) \) is not in general closed in \( A^A \), since a net of automorphisms may converge pointwise to a map which is not onto. It will therefore be convenient for us to consider a slightly larger subset of \( A \): Let \( H(A) \) be the set of all \(*\)-homomorphisms \( \alpha \) of \( A \) into \( A \) such that \( \alpha(I) = I \). Then \( H(A) \) is pointwise closed, the elements of \( H(A) \) are norm-decreasing positive maps, and an element of \( H(A) \) is an automorphism iff it is an invertible mapping. Moreover, \( H(A) \) is closed under composition of mappings, hence is a subsemigroup of \( A^A \). We note that a net \( \{ \alpha_{\gamma} \} \) converges to \( \alpha \) in \( H(A) \) iff \( \alpha_{\gamma}(a) \to \alpha(a) \) for each positive \( a \in A \).

**Lemma 2.1.** \( H(A) \) is a topological semigroup, and \( \text{Aut}(A) \) is a topological group.

**Proof.** Suppose \( (\alpha_{\gamma}, \beta_{\gamma}) \to (\alpha, \beta) \) in \( H(A) \times H(A) \) and let \( a \in A \). Then

\[
\|\alpha_{\gamma} \beta_{\gamma}(a) - \alpha \beta(a)\| \leq \|\alpha_{\gamma}\| \|\beta_{\gamma}(a) - \beta(a)\| + \|\alpha_{\gamma}(\beta(a)) - \alpha(\beta(a))\|. 
\]

As \( \alpha_{\gamma} \to \alpha, \beta_{\gamma} \to \beta, \) and \( \|\alpha_{\gamma}\| \leq 1 \) for all \( \gamma \), this tends to zero, so \( \alpha_{\gamma} \beta_{\gamma} \to \alpha \beta \). Thus \( H(A) \) is a topological semigroup.

To show that \( \text{Aut}(A) \) is a topological group we suppose that \( \alpha_{\gamma} \to \alpha \) in \( \text{Aut}(A) \). Let \( a \in A \). Automorphisms of C*-algebras are isometric, so

\[
\|\alpha_{\gamma}^{-1}(a) - \alpha^{-1}(a)\| = \|a - \alpha_{\gamma} \alpha^{-1}(a)\| = \|\alpha^{-1}(a)\| - \alpha_{\gamma}(\alpha^{-1}(a))\| \to 0.
\]

Thus inversion is continuous on \( \text{Aut}(A) \), and \( \text{Aut}(A) \) is a topological group.

If \( \alpha \in H(A) \) and \( p \) is a state of \( A \), then \( p \circ \alpha \) is again a state of \( A \). Thus there is a natural action of \( H(A) \) on \( S(A) \) defined by

\[
(p, \alpha) \mapsto p \circ \alpha = p \alpha, \quad p \in S(A), \quad \alpha \in H(A).
\]

This action is continuous: if \( (p_{\gamma}, \alpha_{\gamma}) \to (p, \alpha) \) in \( S(A) \times H(A) \) and \( a \in A \), then

\[
|p_{\gamma} \circ \alpha_{\gamma}(a) - p \circ \alpha(a)| \leq \|\alpha_{\gamma}(a) - \alpha(a)\| + |p_{\gamma}(\alpha(a)) - p(\alpha(a))| \to 0.
\]
It follows that if $G$ is any subgroup of $\text{Aut}(A)$, then the restriction of this action to $S(A) \times G$ makes $(S(A), G)$ into a transformation group.

Since $H(A)$ is closed in $A^A$, we have for any subset $G$ of $H(A)$ that the closures of $G$ in $H(A)$ and in $A^A$ coincide. We shall find the following theorem very useful in providing examples of uniformly almost periodic actions on state spaces.

**Theorem 2.2.** Let $G$ be a subset of $H(A)$, and let $S$ be any subset of $A$ such that the linear span of $S$ is dense in $A$. Then the closure of $G$ in $H(A)$ is compact iff for every $a \in S$, the set $G[a] = \{\alpha(a) : \alpha \in G\}$ has compact closure in $A$.

**Proof.** Let $\bar{G}$ be the closure of $G$. As $\alpha \rightarrow \alpha(a)$ is continuous, we have $\bar{G}[a] = \{\alpha(a) : \alpha \in \bar{G}\} \subseteq \bar{G}[a]$ for every $a \in A$.

If $\bar{G}$ is compact, then for each $a \in A$ we have $\bar{G}[a]$ compact, whence $\bar{G}[a] = \bar{G[a]}$. In particular, $\bar{G}[a]$ is then compact for every $a \in S$.

Conversely, suppose that for every $a \in S$, the set $\bar{G}[a]$ is compact. The elements of $\bar{G}$ are linear and norm-decreasing, so the restriction mapping $r : \alpha \rightarrow \alpha|_S$ is a one-to-one map of $\bar{G}$ into $A^S$. Let $A^S$ have the product topology. Then $\alpha_\gamma \rightarrow \alpha$ in $\bar{G}$ iff $r(\alpha_\gamma) \rightarrow r(\alpha)$ in $A^S$. For if $r(\alpha_\gamma) \rightarrow r(\alpha)$, then $\alpha_\gamma \rightarrow \alpha$ pointwise on $S$, hence pointwise on the linear span of $S$. As this span is dense, and as $\bar{G}$ consists of maps uniformly bounded in norm, an $\epsilon/3$-argument shows that $\alpha_\gamma(a) \rightarrow \alpha(a)$ for all $a \in A$.

Thus $r$ is a homeomorphism of $\bar{G}$ onto its image in $A^S$. To see that this image is closed in $A^S$, suppose $r(\alpha_\gamma)$ is a net in the image such that $r(\alpha_\gamma) \rightarrow \theta$ in $A^S$. It is enough to show that $\theta$ has an extension to a map $\alpha$ in $A^A$ such that $\alpha_\gamma \rightarrow \alpha$ in $A^A$. We extend $\theta$ to the linear span of $S$ by $\theta(\sum_{i=1}^n \lambda_i a_i) = \sum_{i=1}^n \lambda_i \theta(a_i)$. We have

$$\sum_{i=1}^n \lambda_i \theta(a_i) = \sum_{i=1}^n \lambda_i \lim_{\gamma} \alpha_\gamma(a_i) = \lim_{\gamma} \alpha_\gamma \left( \sum_{i=1}^n \lambda_i a_i \right).$$

It follows that $\theta$ is well defined and linear and that $\alpha_\gamma$ converges pointwise to $\theta$ on the span of $S$. Since the $\alpha_\gamma$ are norm-decreasing, $\theta$ is also norm-decreasing on this span. Thus $\theta$ has an extension to a linear, norm-decreasing map $\alpha$ of $A$ into $A$. By the same $\epsilon/3$-argument as above, $\alpha_\gamma \rightarrow \alpha$ in $A^A$.

To complete the proof we observe that the image of $\bar{G}$ is contained in $X_\alpha \subseteq X_\alpha$ and apply the Tychonoff Theorem.

We shall need to consider more closely the maps of $S(A)$ into itself which are induced by the elements of $H(A)$. For each $\alpha \in H(A)$, let $i(\alpha)$ be the mapping $p \rightarrow p\alpha$. Then $i$ is an injection of $H(A)$ into $S(A)^S(A)$, for if $p(\alpha(a)) = p(\beta(a))$ for all $p \in S(A)$ and all $a \in A$, then $\alpha(a) = \beta(a)$ for all $a \in A$. Moreover,
\(i\) has the following additional properties:

1. \(i\) takes the identity of the semigroup \(H(A)\) onto that of the semigroup \(S(A) S(A)\) and \(i\) is a homomorphism of semigroups: \(i(\alpha \beta) = i(\alpha)i(\beta)\).

2. For each \(\alpha \in H(A)\), \(i(\alpha)\) is weak*-continuous and affine.

3. If \(\alpha \in H(A)\) and \(\alpha\) is invertible, then \(i(\alpha)\) is invertible and \(i(\alpha)^{-1} = i(\alpha^{-1})\).

**Lemma 2.3.** The map \(i\) is bicontinuous from \(H(A)\) into \(S(A) S(A)\) when \(S(A) S(A)\) is given the topology of uniform convergence.

**Proof.** If \(a \in A\) is positive and \(\alpha, \beta, \alpha \in H(A)\), then \(\alpha a = \alpha a\) is selfadjoint. Hence \(\|\alpha a - \alpha a\| = \sup_{p \in S(A)} \|p(\alpha a) - p(\alpha a)\|\).

**Lemma 2.4.** Let \(K\) be a convex subset of the dual of a Banach space \(B\), and let \(K\) have the weak* topology. Let \(F\) be a family of affine maps of \(K\) into \(K\). Then the pointwise closure of \(F\) in \(K^K\) is again a family of affine maps.

**Proof.** Let \(\lambda \in [0, 1]\), let \(p, q \in K\), and suppose \(\beta\) converges pointwise in \(K^K\). Then the functionals \(\lim_{\gamma} p(\beta(\lambda p + (1 - \lambda)q))\) and \(\lim_{\gamma} \beta(p) + (1 - \lambda) \lim_{\gamma} \beta(q)\) agree on each element of \(B\).

**Theorem 2.5.** Let \(G\) be a subgroup of \(\text{Aut}(A)\). Then the following are equivalent:

1. The transformation group \((S(A), G)\) is uniformly almost periodic.

2. The closure \(\overline{G}\) of \(G\) in \(H(A)\) (or in \(A^A\)) is compact.

Under these conditions \(\overline{G}\) is a group, and \(i\) is a homeomorphism and a group isomorphism of \(\overline{G}\) onto the enveloping semigroup \(E\) of \((S(A), G)\).

**Proof.** Let \(T\) be the topology of uniform convergence. We use below without comment Remark 1.4 and some topological results which can be found in [12, pp. 227, 232–233].

Suppose \(\overline{G}\) is compact. Then \(i(\overline{G})\) is \(T\)-compact by Lemma 2.3. The topology \(T\) is jointly continuous on the family of all continuous maps of \(S(A)\) into \(S(A)\), so \(i(\overline{G})\) is equicontinuous. In particular, the subfamily \(i(G)\) is equicontinuous, so \((S(A), G)\) is uniformly almost periodic.

Conversely, suppose \((S(A), G)\) is uniformly almost periodic. Let \(A_R\) be the selfadjoint part of \(A\). By Theorem 2.2 it suffices to show that for each \(a \in A_R\), the orbit \(G[a]\) has compact closure (in \(A_R\) or in \(A\)). For such an \(a\), let \(\hat{a}\) be the map \(p \rightarrow p(a)\) of \(S(A)\) into the real numbers. Then \(a \rightarrow \hat{a}\) is an isometric linear map of \(A_R\) into \(C(S(A))\). Thus it suffices to show that for each \(a \in A_R\), \(\{a(a): \alpha \in G\}\) has compact closure in \(C(S(A))\). Since \(\hat{a}(p(a)) = p(a) = \hat{a}(p\alpha)\), we need only show that each \(\hat{a}\) is an almost periodic function, which follows from uniform almost periodicity of \((S(A), G)\).
Now suppose that (1) and (2) are satisfied. Since $i$ is continuous into the topology $T$, it is also continuous into the pointwise topology. Thus $i(G)$ is pointwise compact. Since $S(A)^S(A)$ is pointwise Hausdorff, $i$ is a homeomorphism of $\tilde{G}$ onto $i(G)$. But then $i(G)$ is pointwise dense in both $i(G)$ and $E$, so $i(G) = E$.

Now $E$ is a group, and $i$ is an isomorphism of the semigroup $\tilde{G}$ onto $E$ which takes the identity of $\tilde{G}$ to the identity of $E$. If $\alpha \in \tilde{G}$, then there exists $\beta \in \tilde{G}$ such that $i(\beta) = i(\alpha)^{-1}$. But then $\beta$ will be an inverse for $\alpha$, so $\tilde{G}$ is a group.

**Corollary 2.6.** If $\tilde{G}$ is compact, then $\tilde{G}$ is a subgroup of Aut$(A)$. In particular, $\tilde{G}$ is a compact topological group.

**Corollary 2.7.** The closure of $G$ in $A^A$ is compact iff the closure of $G$ in Aut$(A)$ is compact, and in this case the two closures coincide.

**Corollary 2.8.** If $\tilde{G}$ is compact, then every element of $E$ maps the set of pure states of $A$ into itself.

**Proof.** As $(S(A), G)$ is uniformly almost periodic, the elements of $E$ are invertible maps. By Lemma 2.4, they are affine. Thus each $\alpha \in E$ must take extreme points to extreme points.

**Remark.** The methods of this section can also be used to obtain analogous results for groups of $C^*$-automorphisms as defined in [11].

3. Uniformly almost periodic $C^*$-algebras. In this section we use uniform almost periodicity of $(S(A), G)$ to obtain information about the traces and the ideal structure of the algebra $A$. We remark that our discussion of centrality is based on that in [13], in which Mosak obtained most of the results of this section for certain group $C^*$-algebras.

We continue to assume that $A$ is a $C^*$-algebra with identity $I$. Moreover, we assume that $G$ is a group of automorphisms of $A$ such that $(S(A), G)$ is uniformly almost periodic. Let $\mu$ be normalized Haar measure on $\tilde{G}$, and let $a \in A$. As $\alpha \rightarrow \alpha(a)$ is continuous on $\tilde{G}$, it is weakly $\mu$-measurable. The image of $\tilde{G}$ is a compact metric space, hence is separable, so the Bochner integral $\int_{\tilde{G}} \alpha(a) d\mu(\alpha)$ exists [20, pp. 131-133]. We may thus define a mapping $\#$ of $A$ into $A$ by $a^\# = \int_{\tilde{G}} (\alpha(a)) d\mu(\alpha), \ a \in A$.

**Lemma 3.1.** The mapping $\#$ is a positive, linear, idempotent mapping of $A$ onto $Z_G A$. It is norm-decreasing and takes no nonzero positive element of $A$ to zero.

**Proof.** The first statement is proved in [18, Example 1.1]. That $\#$ is norm-decreasing follows from $\|a^\#\| \leq \int_{\tilde{G}} \|\alpha(a)\| d\mu(\alpha)$. If $(a^*a)^\# = 0$, then for every $p \in S(A)$ we have $\int_{\tilde{G}} p(\alpha(a^*a)) d\mu(\alpha) = 0$. As $\alpha \rightarrow p(\alpha(a^*a))$ is positive and con-
tinuous, it follows that \((a^*a)^\# = 0\) iff \(p(\alpha(a^*a)) = 0\) for all \(p \in S(A)\) and all \(\alpha \in \bar{G}\). Thus \((a^*a)^\# = 0\) implies \(a^*a = 0\). (See [13, 3.6].)

Let \(r: S_G(A) \rightarrow S(ZG_A)\) be restriction to \(ZG_A\).

**Lemma 3.2.** The mapping \(r\) is an affine homeomorphism of \(S_G(A)\) onto \(S(ZG_A)\).

**Proof.** \(r\) is the inverse of the mapping \(\Phi^*\) in [18, Example 1.1].

If we wish to study the ideals or traces of \(A\), it is natural to consider the group \(I(A)\) of all inner automorphisms of \(A\). This group is generally too large to act uniformly almost periodically on \(S(A)\). For suppose \(A\) is a UHF-algebra (not finite dimensional), and let \(p\) be a pure state of \(A\). Then the set of all states of the form \(b \rightarrow p(ubu^*)\), \(u\) unitary in \(A\), is weak*-dense in \(S(A)\). If the action of \(I(A)\) were uniformly almost periodic, then \(S(A)\) would be a minimal set [6, 2.5]. But this contradicts the existence of a trace on \(A\). (I am indebted to Erling Størmer for pointing out this counterexample.)

**Definition 3.3.** Let \(A\) be a C*-algebra with identity. We say that \(A\) is uniformly almost periodic if

1. every state of \(ZA\) is the restriction of some trace of \(A\), and
2. there exists a group \(G\) of inner automorphisms of \(A\) such that \((S(A), G)\) is uniformly almost periodic and \(ZGA = ZA\).

**Remark 3.4.** Let \(U_0\) be a group of unitary elements of \(A\) such that the linear span of \(U_0\) is dense in \(A\), and let \(G_0\) be the group of all inner automorphisms of \(A\) induced by the elements of \(U_0\). Suppose \((S(A), G_0)\) is uniformly almost periodic. Then \(A\) is uniformly almost periodic. For if \(a \in A\) commutes with every \(u \in U_0\), then \(a \in ZA\), so \(ZG_0A = ZA\). By Lemma 3.2, restriction takes \(S_{G_0}(A)\) onto \(S(ZA)\). If \(\tau\) is a \(G_0\)-invariant state, then for every \(a \in A\) and every \(u \in U_0\), we have \(\tau(ua - au) = 0\), whence \(\tau(ab) = \tau(ba)\) for all \(a, b \in A\). Thus \(S_{G_0}(A) = T(A)\). It follows that \(A\) is uniformly almost periodic.

We give examples of uniformly almost periodic C*-algebras in the last section. We assume for the remainder of this section that \(A\) is uniformly almost periodic and that \(G\) is a group of inner automorphisms of \(A\) which satisfies condition (2) of Definition 3.3.

**Theorem 3.5.** The sets \(S_G(A)\) and \(T(A)\) coincide, and \(ET(A)\) is a weak*-closed subset of \(T(A)\). (That is, \(T(A)\) is a Bauer simplex.) Moreover, \(\tau \in T(A)\) is extremal iff \(\tau\big|_{ZA}\) is a character, and \(r\) restricted to \(ET(A)\) is a homeomorphism onto \(ES(ZA)\).

**Proof.** Since \(G\) consists of inner automorphisms, \(T(A) \subseteq S_G(A)\). As each \(\psi \in S(ZA)\) is the restriction of a trace of \(A\), \(r(T(A)) = S(ZA)\). Then \(T(A) = \)
$S_G(A)$, since $r$ is one-to-one. By Remark 1.5, the restriction of an extremal trace to $ZA$ is a character. If $\tau \in T(A)$ and $r(\tau)$ is pure in $S(ZA)$, then $\tau$ must be extremal, since $r$ is affine and one-to-one. Thus $r$ restricts to a bijection of $ET(A)$ and $ES(ZA)$. It follows from Lemma 3.2 that this bijection is a homeomorphism and that $ET(A) = r^{-1}(ES(ZA))$ is weak* closed in $T(A)$.

**Lemma 3.6.** If $\tau \in ET(A)$, then the left kernel of $\tau$ is a primitive ideal of $A$.

**Proof (after Mosak).** Let $\tau|_{ZA} = \psi$. Then $\psi$ is an irreducible representation of $ZA$, and we can find an irreducible representation $\pi$ of $A$ on some Hilbert space $H_{\pi}$ such that $\pi$ is an extension of $\psi$. Then

$$\tau(a*a) = \tau((a*a)^\#) = \psi((a*a)^\#) = \pi((a*a)^\#), \quad a \in A.$$ 

Thus it suffices to show that if $\pi$ is a representation of $A$, then $\pi(a) = 0$ iff $\pi((a*a)^\#) = 0$, or equivalently that $\pi(a*a) = 0$ iff $\pi((a*a)^\#) = 0$.

If $x \in H_{\pi}$, let $\omega_x(b) = (bx, x)$, $b \in B(H_{\pi})$. Then

$$\pi((a*a)^\#) = 0 \iff \omega_x \circ \pi((a*a)^\#) = 0 \quad \forall x \in H_{\pi} \iff$$

$$\int_G \omega_x \circ \pi(\alpha(a*a))d\mu(\alpha) = 0 \quad \forall x \in H_{\pi} \iff$$

$$\omega_x \circ \pi \circ \alpha(a*a) = 0 \quad \forall x \in H_{\pi}, \forall \alpha \in \overline{G} \iff \pi \circ \alpha(a*a) = 0 \quad \forall \alpha \in \overline{G}.$$

Let $K$ be the kernel of $\pi$. Since $G$ consists of inner automorphisms and $K$ is a closed ideal, each $\alpha$ in $\overline{G}$ maps $K$ into $K$. It follows that $\pi \circ \alpha(a*a) = 0 \forall \alpha \in \overline{G}$ iff $\pi(a*a) = 0$.

Let $\theta$ be the mapping of $ET(A)$ into $\text{Prim}(A)$ defined by sending an extremal trace into its left kernel, and let $\rho$ be the mapping $P \rightarrow P \cap ZA$ of $\text{Prim}(A)$ onto $\text{Max}(ZA)$. We shall identify a maximal ideal of $ZA$ with the corresponding character. With this identification the mapping $r$ restricted to $ET(A)$ is a homeomorphism of $ET(A)$ and $Max(ZA)$. By Remark 1.5 this homeomorphism factors into $\rho \circ \theta$, i.e. $\tau|_{ZA} = \ker L^T \cap ZA$ when $\tau \in ET(A)$. Since $r$ is one-to-one, $\theta$ is also one-to-one from $ET(A)$ into $\text{Prim}(A)$.

**Lemma 3.7.** If $\tau \in ET(A)$, then its left kernel is a maximal ideal and $\theta$ is a homeomorphism of $ET(A)$ onto $\text{Max}(A)$. Moreover, $\rho$ restricts to a homeomorphism of $\text{Max}(A)$ onto $\text{Max}(ZA)$.

**Proof.** Let $M \in \text{Max}(A)$, and let $p$ be a state of $A$ such that $L^p$ has kernel $M$. Then $p^\#: a \rightarrow p(a^\#)$ is a trace on $A$. Let $a \in M$. Since the elements of $\overline{G}$ map $M$ into $M$, $p$ vanishes on each $\alpha(a)$, $\alpha \in \overline{G}$. Thus $p(a^\#) = 0$, and $p^\#$ is a trace which vanishes on $M$. The set of all traces which vanish on $M$ is a weak*
closed face of $T(A)$. Let $r$ be any extreme point of this face. Then $\ker L^r \supseteq M$, and by maximality of $M$ we have $\ker L^r = M$. Thus $\theta$ maps $ET(A)$ onto a subspace of $\text{Prim}(A)$ which contains $\text{Max}(A)$.

Let $P \in \text{Prim}(A)$, and suppose there exists $\sigma \in ET(A)$ such that $\ker L^\sigma = P$. Choose $M_P \in \text{Max}(A)$ such that $P \subseteq M_P$, and choose $\tau \in ET(A)$ such that $\ker L^\tau = M_P$. By Remark 1.5, the characters corresponding to $P \cap ZA$ and $M_P \cap ZA$ are $\sigma|_{ZA}$ and $\tau|_{ZA}$ respectively. Since $M_P \cap ZA = P \cap ZA$, these characters are equal. As $r$ is one-to-one, we have $\sigma = \tau$, hence $M_P = P$. Thus $\theta$ is a bijection of $ET(A)$ and $\text{Max}(A)$.

Now $\rho$ is one-to-one on $\text{Max}(A)$, since $r = \rho \circ \theta$, $r$ is one-to-one, and $\theta$ is a bijection. As $\text{Max}(A)$ is compact and $\rho$ is continuous, $\rho$ restricts to a homeomorphism $\rho_0$ of $\text{Max}(A)$ onto $\text{Max}(ZA)$. It follows that $\theta = \rho_0^{-1} \circ r$ is also a homeomorphism.

**Lemma 3.8.** Every primitive ideal of $A$ is maximal. In particular, $\theta$ is a homeomorphism of $ET(A)$ and $\text{Prim}(A)$.

**Proof (after Mosak).** We define a mapping of $\hat{A}$ into $ET(A)$ as follows. If $\pi$ is an irreducible representation of $A$, then $\pi^\# : a \rightarrow \pi(a^\#)$ is a trace on $A$. Suppose $\pi^\# = \frac{1}{2} \tau_1 + \frac{1}{2} \tau_2$ with $\tau_1$ and $\tau_2$ in $T(A)$. Then, as $\pi^\#|_{ZA}$ is a pure state of $ZA$, $\pi^\# = \tau_1 = \tau_2$ on $ZA$. Since restriction is one-to-one, we have $\tau_1 = \tau_2 = \pi^\#$ on all of $A$. If $\pi$ is unitarily equivalent to $\pi_0$, let $p$ and $p_0$ be states associated with $\pi$ and $\pi_0$ respectively. Then there exists a unitary $u$ in $A$ such that $p(uau^*) = p_0(a)$ for all $a \in A$. Hence $\pi^\# = p^\# = p_0^\# = \pi_0^\#$, and $\pi \rightarrow \pi^\#$ is well defined.

Now $\pi \rightarrow \pi^\#$ maps onto $ET(A)$. For if $\tau \in ET(A)$, let $p$ be a pure state of $A$ which agrees with $\tau$ on $ZA$. Let $\pi = L^p$, and then $\pi^\# = \pi = p = \tau$ on $ZA$, so $\pi^\# = \tau$.

If we can show that the mapping $\pi \rightarrow \ker \pi$ of $\hat{A}$ onto $\text{Prim}(A)$ is the composition of $\pi \rightarrow \pi^\#$ and $\theta$, then $\theta$ must map onto $\text{Prim}(A)$, and hence $\text{Max}(A) = \text{Prim}(A)$. So we must show that if $\pi \in \hat{A}$, then the kernel of $\pi$ is $\{a : \pi((a^*a)^\#) = 0\}$. But we verified this in the proof of Lemma 3.6.

**Theorem 3.9.** If $A$ is uniformly almost periodic, then $A$ is a central $C^*$-algebra.

**Proof.** Combine Lemmas 3.7 and 3.8.

4. Uniquely ergodic $C^*$-algebras. We turn now to uniquely ergodic systems and approximately finite $C^*$-algebras. If $X$ is a compact metric space and $T$ is a homeomorphism of $X$ onto $X$, then by [15, 2.1] there exists at least one normalized $T$-invariant Borel measure on $X$. The system $(X, T)$ is said to be uniquely
ergodic if there exists exactly one such measure, or equivalently if \( C(X) \) has exactly one \( T \)-invariant state. By analogy we define a \( C^* \)-algebra to be **uniquely ergodic** if it possesses exactly one trace.

A \( C^* \)-algebra \( A \) with identity \( I \) is said to be **approximately finite** if there exists an increasing sequence \( \{A_n\} \) of finite dimensional \( C^* \)-subalgebras of \( A \), each \( A_n \) containing \( I \), such that \( A = \bigcup_{n=1}^{\infty} A_n \) [3]. We shall see below that every approximately finite \( C^* \)-algebra possesses at least one trace, and we shall characterize those which are uniquely ergodic. We assume in this section that \( A \) is approximately finite with \( \{A_n\} \) and \( I \) as above. We note that \( A \) is separable, and hence that \( S(A) \) is metrizable.

For each \( n \geq 1 \), the unitary group \( U_n \) of \( A_n \) is compact, so there exists a map \( \varphi_n \) of \( A \) into \( A' = \{a \in A: ab = ba \text{ for all } b \in A_n\} \). Each \( A_n \) is a \( C^* \)-subalgebra of \( A \) and \( \bigcap_{n=1}^{\infty} A_n = ZA \). If \( a, b \in A_n \), then \( \varphi_n(ab) = \varphi_n(ba) \).

**Lemma 4.1.** For each \( n \geq 1 \), \( \varphi_n \) is a norm-decreasing, idempotent, positive, linear map of \( A \) onto \( A'_n = \{a \in A: ab = ba \text{ for all } b \in A_n\} \). Each \( A_n' \) is a \( C^* \)-subalgebra of \( A \) and \( \bigcap_{n=1}^{\infty} A_n' = ZA \). If \( a, b \in A_n \), then \( \varphi_n(ab) = \varphi_n(ba) \).

**Proof.** If \( a \in \bigcap_{n=1}^{\infty} A_n' \), then \( a \) commutes with every element of \( \bigcup_{n=1}^{\infty} A_n \), hence with every element of \( A \). It is trivial that \( \varphi_n \) is norm-decreasing, and the rest of the lemma follows from [18, Example 1.1].

**Lemma 4.2.** Let \( \{p_n\} \) be a sequence of states of \( A \). Then \( \{p_n \circ \varphi_n\} \) is a sequence of states of \( A \) and has at least one limit point in \( S(A) \). Every limit point is a trace of \( A \).

**Proof.** Clearly \( \{p_n \circ \varphi_n\} \) is a sequence of states, and it has a limit point by compactness of \( S(A) \). Let \( p_{n_1} \circ \varphi_{n_1} \to \tau \) in \( S(A) \). If \( a, b \in \bigcup_{n=1}^{\infty} A_n \), then for all sufficiently large \( n \) we have \( \varphi_n(ab - ba) = 0 \), whence \( \tau(ab - ba) = 0 \). The map \( (a, b) \to ab - ba \) is continuous on \( A \times A \), so \( \tau(ab - ba) = 0 \) for all \( a, b \in A \).

**Corollary 4.3.** If \( \psi \) is a state of \( ZA \), then there exists a trace \( \tau \) of \( A \) whose restriction to \( ZA \) is \( \psi \). If \( \psi \) is a character, then \( \tau \) can be chosen to be extremal.

**Proof.** Let \( p \) be a state of \( A \) which extends \( \psi \), and let \( \tau \) be a weak* limit point of \( \{p \circ \varphi_n\} \). Then \( \tau = \psi \) on \( ZA \). Suppose now \( \psi \) is a character. The set \( F = \{\tau \in T(A): \tau|_{ZA} = \psi \} \) is a nonempty closed face of \( T(A) \). Any extreme point of \( F \) is an extremal trace which extends \( \psi \).
Remark. The following proposition describes the approximately finite C*-algebras which possess a centering map analogous to the map # of the last section. We note however that the map \( a \mapsto \varphi(a) \) below may annihilate some nonzero positive elements of \( A \).

**Proposition 4.4.** The following are equivalent:

1. For each \( a \in A \) the sequence \( \{\varphi_n(a)\} \) converges in norm to an element \( \varphi(a) \) of \( A \).
2. The mapping \( r: \tau \mapsto \tau|_{ZA} \) of \( T(A) \) onto \( S(ZA) \) is one-to-one. If these conditions are satisfied, then for each \( a \in A \) we have \( \varphi(a) \in ZA \), and for each \( p \in S(A) \) the mapping \( a \mapsto p(\varphi(a)) \) is a trace.

**Proof.** (1) \( \Rightarrow \) (2): Let \( a \in A \). If \( \sigma \in T(A) \), then for each \( n \) we have \( \sigma(\varphi_n(a)) = \sigma(a) \), so \( \sigma(a) = \sigma(\varphi(a)) \). Suppose \( \sigma \) and \( \tau \) are in \( T(A) \) and \( \sigma = \tau \) on \( ZA \). As \( \varphi(a) \in \bigcap_{n=1}^{\infty} A_n' = ZA \), \( \sigma(a) = \tau(a) \). Thus \( r \) is one-to-one. That \( a \mapsto p(\varphi(a)) \) is a trace follows from the fact that \( \varphi \) is positive and linear and vanishes on \( ab - ba \) for all \( a, b \in \bigcup_{n=1}^{\infty} A_n \).

(2) \( \Rightarrow \) (1): If \( r \) is one-to-one, then it is an affine homeomorphism of \( T(A) \) onto \( S(ZA) \), and its restriction to \( ET(A) \) is a homeomorphism onto \( ES(ZA) \). We use this homeomorphism and the Gelfand transform \( ^* \) to identify \( ZA \) with \( C(ET(A)) \). For each \( a \in A \), put \( a^#(r) = \tau(a), \tau \in ET(A) \). Then \( a^# \in ZA \), and the mapping \( a \mapsto a^# \) is linear, norm-decreasing, positive, and invariant under the inner automorphisms of \( A \). As \( z^#(r) = \tau(r(z)) \), \( z^# = z \) for \( z \in ZA \). To show that \( \varphi_n(a) \) is convergent for each \( a \in A \), it suffices to show that for each positive \( a \in A \), \( \| \varphi_n(a) - a^# \| \to 0 \), i.e.

\[
\sup_{p \in S(A)} |p \circ \varphi_n(a) - p(a^#)| \to 0.
\]

If this is false, then there exist \( a_0 \geq 0 \) in \( A \), a subsequence \( \{\varphi_{n_i}\} \) of \( \{\varphi_n\} \), and \( p_i \in S(A) \) such that

\[
|p_i \circ \varphi_{n_i}(a_0) - p_i(a_0^#)| \geq \epsilon > 0 \quad \text{for all } i \geq 1.
\]

By passing to a subsequence we may assume \( p_i \circ \varphi_{n_i} \to \tau \) and \( p_i \to p_0 \) in \( S(A) \). Then \( \tau \) and \( p_0^# \): \( a \to p_0(a^#) \) are traces of \( A \). For \( z \in ZA \) we have

\[
\tau(z) = \lim_{i \to \infty} p_i \circ \varphi_{n_i}(z) = \lim_{i \to \infty} p_i(z) = p_0(z) = p_0(z^#),
\]

so \( r(\tau) = r(p_0^#) \). But then \( \tau = p_0^# \), which contradicts (1).

If \( a \in A \), let \( \hat{a} \) be the mapping \( p \to p(a) \) of \( S(A) \) into \( C \). The following is a C*-algebraic analogue of [15, 5.3].

**Theorem 4.5.** If \( A \) is an approximately finite C*-algebra, then the following are equivalent:
(1) $A$ is uniquely ergodic.

(2) For each $a \in A$ the sequence $\{\varphi_n(a)\}$ converges uniformly on $S(A)$ to a constant function.

(3) For each $a \in A$ there exists a subsequence of $\{\varphi_n(a)\}$ which converges pointwise on $S(A)$ to a constant function.

If these conditions are satisfied, then the constant function of conditions (2) and (3) has the value $\tau(a)$, where $\tau$ is the trace of $A$.

Proof. (1) $\Rightarrow$ (2): Let $a \in A$. By Corollary 4.3, $Z_A = CI$, and by the last proposition $\varphi_n(a)$ converges in norm to $K_aI$ for some complex number $K_a$. Thus $\sup_{p \in S(A)} |p \circ \varphi_n(a) - K_a| \rightarrow 0$.

(2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (1): Let $a \in A$, $\sigma \in T(A)$, and suppose $\{\varphi_{n_t}(a)\}$ converges pointwise to the constant function $K_a$. Then $\sigma(a) = \sigma(\varphi_{n_t}(a)) \rightarrow K_a$, so $a \rightarrow K_a$ is the only trace on $A$.

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5. Examples.

Example 5.1. Let $X$ be a compact Hausdorff space and $A = C(X)$. Let $(X, \Gamma)$ be a transformation group. As in the introduction we let $t_f$ be the function $(t_f)(x) = f(tx)$, $x \in X$, where $t \in \Gamma$ and $f \in C(X)$. Let $G$ be the group of all automorphisms of $A$ which have the form $f \rightarrow tf$, $t \in \Gamma$. It follows from Remark 1.4 and Theorems 2.2 and 2.5 that $(X, \Gamma)$ is uniformly almost periodic iff $(S(A), G)$ is uniformly almost periodic. It is not difficult to show that if these two transformation groups are uniformly almost periodic, then their enveloping semigroups are homeomorphic and isomorphic.

Example 5.2. Let $A$ be a UHF-algebra. We may write $A = \bigcup_{n=1}^{\infty} A_n$, where $A_n = M_1 \otimes \ldots \otimes M_n$ and for each $i \geq 1$, $M_i$ is a finite dimensional factor. Let $U$ be the group of all unitaries in $\bigcup_{n=1}^{\infty} A_n$ which have the form $u_1 \otimes \ldots \otimes u_k$, where $k \geq 1$ and $u_i$ is a unitary element of $M_i$, $i = 1, 2, \ldots, k$. If $G$ is the group of all inner automorphisms of $A$ induced by elements of $U$, then we claim that $(S(A), G)$ is uniformly almost periodic. By Theorems 2.2 and 2.5, it suffices to show that if $a \in \bigcup_{n=1}^{\infty} A_n$, then the set $\{uau^*: u \in U\}$ has compact closure in $A$. Now $U$ leaves the generating set $\{a_1 \otimes \ldots \otimes a_n: a_i \in M_i, i = 1, \ldots, n\}$ of $A_n$ invariant, hence leaves $A_n$ invariant, $n \geq 1$. It follows that $\{uau^*: u \in U\}$ lies in the closed ball of radius $\|a\|$ in some $A_n$, hence has compact closure, since $A_n$ is finite dimensional.

Example 5.3. Let $\Gamma$ be a discrete group, and let $A$ be the group $C^*$-algebra
of $\Gamma$ [5, 13.9.1]. We identify $L^1(\Gamma)$ with a dense $*$-subalgebra of $A$, and for each $g \in \Gamma$ we write $\delta_g$ for the function which is one at $g$ and zero elsewhere on $\Gamma$. Then $\Gamma$ is isomorphic to the subgroup $\{\delta_g: g \in \Gamma\}$ of the unitary group of $A$, and we also identify these groups. Then $\Gamma$ has dense linear span in $A$.

Let $\text{Aut}(\Gamma)$ be the group of all automorphisms of $\Gamma$. Each $\alpha \in \text{Aut}(\Gamma)$ has a unique extension to an automorphism $\tilde{\alpha}$ of $A$, and $\alpha \rightarrow \tilde{\alpha}$ is a one-to-one group homomorphism from $\text{Aut}(\Gamma)$ into $\text{Aut}(A)$.

A group is said to be class-finite if every conjugacy class in the group is a finite set, i.e. every element has a finite orbit under the action of the inner automorphisms. Let $G$ be the group of all inner automorphisms of $\Gamma$. By Theorems 2.2 and 2.5, $(S(A), \tilde{G})$ is uniformly almost periodic iff the orbit of each $\delta_g$ has compact closure. Since $\Gamma$ is a discrete subset of $A$, $(S(A), \tilde{G})$ is uniformly almost periodic iff $\Gamma$ is class-finite.

We remark that class-finite groups are precisely the discrete $[FIA]^-$-groups studied by Mosak in [13]. Thoma studied harmonic analysis on class-finite groups in [19], and Neumann gave a structure theory for such groups in [14].

**Remark 5.4.** The algebras given in these three examples are uniformly almost periodic $C^*$-algebras: In 5.1 put $U_0$ equal to the unitary group of $A$, in 5.2 put $U_0 = U$, and in 5.3 put $U_0 = \{\delta_g: g \in \Gamma\}$. In all three cases $U_0$ has dense linear span and we may apply Remark 3.4.

**References**


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